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Progress of Theoretical Physics, Vol. 21, No. 1, January 1959

On the Unstable States in Quantum Field Theory

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The unstable state in quantum field theory is investigated. Its properties are studied in detail by making use of Lee's model. The renormalization procedure is proposed and the corresponding Z-factor has values only between zero and unity. In the limit of g (the coupling constant)$=0$, the Z-factor becomes unity in contrast to Naitō's Z-factor which becomes one-half.

§ 0. Introduction

Recently many authors[1)-5)] have investigated the properties of an unstable particle. Araki et al.[1)] proposed two methods for defining the mass and lifetime of an unstable particle. The corresponding Z-factor, however, can take values larger than unity, or negative or complex values and this causes the well-known difficulty of renormalization procedures proposed by them. Naitō[4)] investigated the production and the decay of an unstable particle in the stationary treatment and defined its physical state. The physical state of an unstable particle defined by Naitō is essentially identical with the "approximate eigenstate" proposed by Glaser and Källén. The Z-factor cannot take any value except $0 \leq Z \leq 1$ and it is interpreted as the dissociation probability. However, Naitō's unstable state has a curious property. In the state defined by him, the cloud of θ-particles leaks out appreciably to the region where the distance from the centre is larger than $1/\mu$ (μ: mass of a θ-particle). The region in which a θ-particle is found becomes larger as the coupling constant becomes smaller. This curious fact causes the curious properties of the Z-factor which tends to one-half instead of unity at the weak coupling limit. Nakanishi[5)] treated the problem of an unstable particle by an interesting method. However, his treatment is a mathematical generalization of quantum field theory and the physical meaning is not clear. We, therefore, investigate the unstable state in quantum field theory by a method the physical meaning of which is clear.

A physical stationary state is defined as an eigenstate of the total Hamiltonian with an appropriate boundary condition. For simplicity, we employ Lee's model. The physical state of a stable V-particle is obtained from the stationary Schrödinger equation $H|\Psi_\nu> = E|\Psi_\nu>$ with the boundary condition that there are no θ-particles at sufficiently large distance. The Schrödinger equation with this boundary condition

becomes the equation to determine an eigenvalue of energy E which corresponds to the mass of the clothed V-particle. With the boundary condition that at sufficiently large distance there is a θ-particle which is a plane wave or an outgoing wave, the Schrödinger equation is not the equation to determine an eigenvalue of energy but only the equation to decide the behaviour of a state which has an arbitrary energy. The state is called a "scattering state". It is well known that boundary coonditions are determined according to physical reasons. An unstable V-particle decays into an N- and a θ-particle. A θ-particle which is emitted from an unstable V-particle is composed of only outgoing waves. Thus we expect that the Schrödinger equation with an appropriate boundary condition which is determined according to such a physical reason describes an unstable state.

In § 1, the general properties of asymptotic behaviours are discussed in stable and also in unstable states. The relation between the pole of the one particle propagator and the eigenvalue of the total Hamiltonian is also considered. In the later sections we employ Lee's model[6] and study in detail the properties of an unstable state. In § 2, we shall derive the equations which determine the energy eigenvalue for an unstable V-particle and it will be shown that the equation will be identical with that given by Araki et al. In § 3, the properties of the state vector for the unstable state derived in § 2 are discussed and hermitian and non-hermitian properties of the total Hamiltonian are considered. It is shown that the state of the unstable particle does not belong to the so-called "Hilbert space" spanned by ordinary states (scattering states etc.)

The renormalization procedures are proposed in § 4 and the Z-factor is considered and it is shown that the Z-factor is interpreted as the dissociation probability of the physical unstable state. Our Z-factor is essentially identical with that proposed by Nakanishi. The arguments of this section are analogous to those in the theory of nuclear disintegration. To understand the nature of the physical unstable state, we briefly study its time development in § 5.

§ 1. General properties of "asymptotic behaviours"

We will begin to examine the eigenvalue problems and corresponding boundary conditions in quantum field theory. For the sake of simplicity, we consider the case in which a neutral scalar field ϕ interacts with a Dirac field ψ. The total Hamiltonian H is decomposed into two parts, the free part H_0 and the interaction part H_1;

$$H=H_0+H_1. \qquad (1\cdot1)$$

H_1 contains the mass renormalization terms $\delta\mu^2\,\phi^2$ and $\delta m\,\bar{\psi}\psi$. Now let us consider the eigenvalue problem of the Schrödinger equation

$$H|\Psi\rangle=E|\Psi\rangle \qquad (1\cdot2)$$

i) The case of a stable boson state. (Pole of one particle propagator and energy eigenvalue.)

At first, we consider the one particle state of a stable boson. In this case, the state vector is given by

$$|\Psi\rangle = U(0, -\infty)|\varphi_0\rangle \tag{1·3}$$

where

$$H_0|\varphi_0\rangle = E|\varphi_0\rangle \tag{1·4}$$

and the matrix element $\langle\varphi_0|H_1|\Psi\rangle$ is easily calculated and the result is

$$\langle\varphi_0|H_1|\Psi\rangle = 0, \tag{1·5}$$

i.e.

$$\langle\varphi_0|H_1 U(0,-\infty)|\varphi_0\rangle = 0. \tag{1·6}$$

From (1.6), it is easily proved that

$$\langle\varphi_0|H_1(t)U(t,-\infty)|\varphi_0\rangle = 0 \tag{1·7}$$

where the operators are expressed in the interaction representation. Integrating (1.7) from $t=-\infty$ to $t=+\infty$, we obtain

$$\langle\varphi_0|\int_{-\infty}^{+\infty} dt H_1(t)U(t, -\infty)|\varphi_0\rangle = 0,$$

i.e.

$$\langle\varphi_0|(S-1)|\varphi_0\rangle = 0. \tag{1·8}$$

From this equation, we can determine the renormalization constant $\delta\mu^2$ and it is clear that this procedure is equivalent to the procedure to determine $\delta\mu^2$ from the pole of a one particle propagator.

To investigate the asymptotic behaviour of the state, we expand the state $|\Psi\rangle$ into free particle states, i.e.

$$|\Psi\rangle = c^*(0)|0\rangle + \int dx\, dy\ \varphi(x, y)a^*(x)b^*(y)|0\rangle \tag{1·9}$$

$$+ \cdots\cdots$$

where

$$a^*(x) = \int dk\ e^{ikx}\ a^*(k)$$

$$b^*(x) = \int dk\ e^{ikx}\ b^*(k),$$

and $a^*(k)$, $b^*(k)$ and $c^*(k)$ are creation operators of a fermion, an antifermion and a boson of momentum k, respectively. $|0\rangle$ is the free vacuum state and $\varphi(x, y)$ is given by the following equation:

$$\varphi(x, y) = \langle 0|a(x)b(y)|\Psi\rangle \tag{1·10}$$

where $a(x)$ and $b(y)$ are hermitian conjugates of $a^*(x)$ and $b^*(y)$.

According to the formal solution of $|\Psi\rangle$, equation (1.10) is written as follows:

$$\varphi(\boldsymbol{x},\ \boldsymbol{y}) = \langle 0|a(\boldsymbol{x})b(\boldsymbol{y})\frac{1}{E-H_0+i\epsilon}H_1|\Psi\rangle$$

$$= \int d\boldsymbol{k}\ e^{-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})}\frac{1}{E-2\sqrt{\boldsymbol{k}^2+m^2}}\langle 0|a(\boldsymbol{k})b(-\boldsymbol{k})H_1|\Psi\rangle \qquad (1\cdot 11)$$

where we assume that the total momentum is zero.
Integrating the angular part, we obtain

$$\varphi(\boldsymbol{x},\ \boldsymbol{y}) \sim \int k\ dk\ e^{+ikr}\frac{1}{E-2\sqrt{k^2+m^2}}\ F(k)$$

and from this expression we can gain information about the asymptotic behaviour of $\varphi(\boldsymbol{x}, \boldsymbol{y})$. For the stable boson, $E=\mu<2m$ and thus $\varphi(\boldsymbol{x}, \boldsymbol{y})$ vanishes as $|\boldsymbol{x}-\boldsymbol{y}|=r$ goes to sufficiently large values. (Of course, we assume $F(k)$ has no singularity on the real axis of the k-plane. This assumption is very plausible.)

ii) Unstable case. ($\mu>2m$)

The stable one particle state is a solution of the Schrödinger equation (1·2) with the boundary condition that at an infinite distance from the centre of gravity fermions, antifermions and bosons do not exist. (This statement is not exact, for the true vacuum state contains fermions, antifermions and bosons and thus as the distance goes to infinity the state contains fermions, antifermions and bosons which belong to the vacuum state. Therefore, the boundary condition should be correctly stated as follows: as the distance goes to infinity, the state goes to the vacuum state.)

For the unstable boson, we decompose the total Hamiltonian H into two parts, i.e. H_0 and H_1. By analogy with the stable case, H_0 contains a complex mass and H_1 has the complex mass renormalization term. However, to preserve hermiticity of the total Hamiltonian in the ordinary sense, we have to relate the imaginary parts of the mass and the mass renormalization term. That is

$$\text{Im}\ \mu^2 = -\text{Im}\ \delta\mu^2.$$

Now, we intend to solve the Schrödinger equation (1·2) with the following boundary condition: As the distance from the centre of gravity goes to infinity, the state contains only outgoing fermions and antifermions. The eigenvalue cannot be real, that is, the total Hamiltonian is not hermitian because the boundary condition is such that probability is not conserved. For the unstable, just as for the stable particle, we can derive the equation (1·5), i.e.

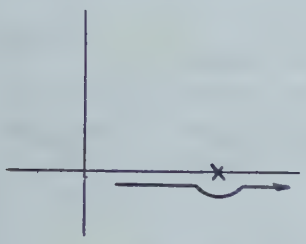
$$\langle\varphi_0'|H_1|\Psi\rangle = 0 \qquad (1\cdot 5)$$

where $H_0^{\dagger}|\varphi_0'\rangle = E^*|\varphi_0'\rangle$.

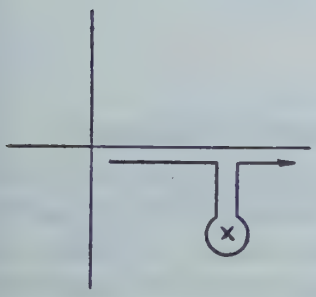
The formal solution of (1·2) is given by the equation

$$|\Psi\rangle = |\varphi_0\rangle + \frac{1}{E-H_0}H_1|\Psi\rangle. \qquad (1\cdot 13)$$

The singularity of (1·13) must be determined by boundary conditions.



a) ordinary case



b) unstable case

Fig. 1. Integration path of ω

For ordinary cases, i.e. for boundary conditions which preserve hermiticity of the Hamiltonian, the singularity is determined as follows:

$$\frac{1}{E-H_0} \to \frac{1}{E-H_0+i\epsilon}.$$

In the case of an unstable particles, E is complex and hence there is no singularity on the real energy axis. However, there exists a pole in the lower half plane of E and we must take account of this pole. (See Fig. 1)

This is just equivalent to taking into account the boundary condition for the decaying state. To see this fact, let us consider the component of the state which contains only a fermion and its anti-particle. Its amplitude is

$$\varphi(\boldsymbol{x}, \boldsymbol{y}) = \langle 0|a(\boldsymbol{x})b(\boldsymbol{y})|\Psi\rangle$$

$$\sim \int k dk e^{ikr} \frac{1}{E-2\sqrt{m^2+k^2}} F(k)$$

$$\sim \int_c d\omega \; e^{ir\sqrt{\omega^2-m^2}} \frac{1}{E-2\omega} F(\omega)$$

where the integration path is given in Fig. 1 b).

Thus
$$\varphi(r) \sim Ce^{ip_0 r} + \int_{-\infty}^{+\infty} k dk \; e^{ikr} \frac{1}{E-2\sqrt{k^2+m^2}} F(k) \qquad (1\cdot14)$$

where $p_0 = p_R - ip_I$ and C is a constant.
The second term vanishes and the first term increases exponentially as $r \to \infty$. As is seen, there are only outgoing waves of a fermion and an antifermion.

Now, it is clear that the formal solution (1·13) with the specification of the singularity is obtained by analytical continuation of the ordinary case. (See Fig. 1) Hence we can determine the renormalization constant $\delta\mu^2$ with the pole of one particle propagator which is continuated analytically to the lower half plane as a function of energy. Then the second method proposed by Araki et al. can be extended to more general field theory. In this paper, however, we do not intend to discuss the general case, but study in detail the structure of the unstable state by making use of a special model, i.e. Lee's model.

§ 2. The eigenvalue problem for an unstable V-particle in Lee's model

In this section we consider the eigenvalue problem in Lee's model. To make

the boundary condition clear, we use operators represented in configuration space. The Hamiltonian of Lee's model is

$$H=m_0\psi_V^+\psi_V+m_N\psi_N^+\psi_V+m_N\psi_N^+\psi_N+\int d\boldsymbol{x}\,\alpha^+(\boldsymbol{x})\,(\mu^2-\Delta^2)^{1/2}\alpha(\boldsymbol{x})$$

$$-\frac{\pi}{2}g\int d\boldsymbol{x}\,d\boldsymbol{x}'\,\rho(\boldsymbol{x})\;Y(\boldsymbol{x}-\boldsymbol{x}')[\alpha(\boldsymbol{x}')\psi_V^+\psi_N+\text{herm. conj.}] \qquad (2\cdot1)$$

where $\alpha(\boldsymbol{x})=\int d\boldsymbol{k}\;\alpha(\boldsymbol{k})\;e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$

$\alpha^+(\boldsymbol{x})=$hermitian conjugate of $\alpha(\boldsymbol{x})$

$Y(\boldsymbol{x})=1/(2\pi)^{3/2}\int d\boldsymbol{k}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}(\mu^2+k^2)^{-1/4}$

$\rho(\boldsymbol{x})$: a cut off factor

$\alpha(\boldsymbol{k})$: an annihilation operator of a θ-particle of momentum $\boldsymbol{k}$.

It is clear that $\alpha(\boldsymbol{x})$ and $\alpha^+(\boldsymbol{x})$ are annihilation and creation operators of a θ-meson at the point $\boldsymbol{x}$. m_0, m_N and μ are the "bare" masses of V-, N- and θ-particles respectively and g is an unrenormalized coupling constant. To avoid ambiguity in the calculations, we introduce the cut off factor $\rho(\boldsymbol{x})$ which suppresses divergences at small distances. In our case, there is no eigenstates whose eigenvalue is smaller than $m_N+\mu$. (We consider the sector $(N,\,\theta:\,V)$ in which a bare-V particle or an N-and a θ-particle are present). State vectors can be expressed in general as follows:

$$|\Psi\rangle=C|V\rangle+\int d\boldsymbol{x}\;\varphi(\boldsymbol{x})\alpha^+(\boldsymbol{x})|N\rangle \qquad (2\cdot2)$$

where $|V\rangle=\psi_V^+|0\rangle$, $|N\rangle=\psi_N^+|0\rangle$ and $|0\rangle$ is the vacuum state. Operating with the total Hamiltonian H on this state, we obtain

$$H|\Psi\rangle=[m_0C-\frac{\pi}{2}g\int d\boldsymbol{x}\;F(\boldsymbol{x})\varphi(\boldsymbol{x})]|V\rangle \qquad (2\cdot3)$$

$$+\int d\boldsymbol{x}[(m_N+(\mu^2-\Delta^2)^{\frac{1}{2}})\varphi(\boldsymbol{x})-\frac{\pi}{2}CgF(\boldsymbol{x})]\alpha^+(\boldsymbol{x})|N\rangle$$

where $F(\boldsymbol{x})=\int d\boldsymbol{x}'\;\rho(\boldsymbol{x}')\;Y(\boldsymbol{x}-\boldsymbol{x}')$.

Now, we study the Schrödinger equation

$$H|\Psi\rangle=E|\Psi\rangle \qquad (2\cdot4)$$

with the boundary condition that as $r=|\boldsymbol{x}|$ goes to infinity, there is only an outgoing θ-particle, or more precisely,

$$\lim \varphi(r) \sim \frac{1}{r} e^{iKr} \tag{2·5}$$

where $\quad \mathrm{Re}\ K>0.$

The physical meaning of this condition is that at large distances there is only a decay product emitted by the unstable V-particle which was located at the origin.

From equations (2·3) and (2·4), the equations for $\varphi(\boldsymbol{x})$ and C are derived as follows:

$$C(E-m_0)+\frac{\pi}{2}g\int F(\boldsymbol{x})\varphi(\boldsymbol{x})\,d\boldsymbol{x}=0,$$

$$[E-m_N-(\mu^2-\Delta^2)^{1/2}]\varphi(\boldsymbol{x})=-\frac{\pi}{2}g\,CF(\boldsymbol{x})$$

or

$$[(E-m_N)^2-\mu^2+\Delta^2]\varphi(\boldsymbol{x})=-\frac{\pi}{2}gC\,[E-m_N+(\mu^2-\Delta^2)^{1/2}]F(\boldsymbol{x}) \tag{2·6}$$

and what we have to do is to solve equations (2·6) under the condition (2·5).

Setting

$$\varphi(\boldsymbol{x})=u(r)/r$$

$$F(\boldsymbol{x})=f(r)/r$$

$$=1/r\cdot i\pi\int_{-\infty}^{+\infty}dk\ \frac{k\rho(k)}{(\mu^2+k^2)^{1/4}}e^{ikr},$$

we can write the third of eqs. (2·6) as below:

$$[(E-m_N)^2-\mu^2+\frac{\partial^2}{\partial r^2}]\ u(r)=-\frac{\pi g}{2}Cg(r)$$

where

$$g(r)=i\pi\int_{-\infty}^{+\infty}dk\,e^{ikr}\frac{k\rho(k)}{(\mu^2+k^2)^{1/4}}[E-m_N+\sqrt{\mu^2+k^2}], \tag{2·7}$$

The solution of (2·7) which satisfies the boundary conditions

i) $\quad u(0)=0$

ii) $\quad \lim_{r\to\infty} u(r)\sim e^{Ar} \qquad \mathrm{Im}\ A>0$

is easily obtained and the result is

$$u(r)=-\frac{\pi gC}{4A}\Big[e^{Ar}\left\{\int_0^r dr'\ e^{-Ar'}g(r')-\int_0^\infty dr'\ e^{Ar'}g(r')\right\}$$

$$+e^{-Ar}\int_r^\infty dr'\ e^{Ar'}\ g(r')$$

where $$-A^2=(E-m_N)^2-\mu^2. \tag{2·8}$$

Substituting this solution in the first of eqs. (2·6), we have the equation to determine the energy eigenvalue, i.e.

$$C(E-m_0)+2\pi g\int_0^\infty dr\ u(r)f(r)=0. \tag{2·9}$$

Here, we have to keep in mind the following fact: the eigenvalue equation (2·9) is meaningless if the integral $\int_0^\infty dr\ u(r)f(r)$ diverges. The convergence condition for this integral restricts the magnitude of Re A. The function $f(r)$ decreases as $e^{-\mu r}$ where $u(r)$ behaves as e^{Ar} for large r. Then the convergence condition for the integral is

$$\text{Re } A<\mu.$$

If Re $A>\mu$, eq. (2·9) has no meaning and hence the eigenvalue problem does not hold.

We intend to rewrite eq. (2·9) by means of the Laplace transform. The Laplace transform and its inverse transform of $u(r)$ are given as follows:

$$u(s)=\int_0^\infty dr\ e^{-sr}\,u(r) \qquad \text{Re } s>s_0$$

$$u(r)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\ u(s)\ e^{sr} \qquad C>s_0.$$

The function $f(r)$ can be written as follows:

$$f(r)=i\pi\int_{-\infty}^{+\infty} dk\ \frac{k\rho(k)}{(\mu^2+k^2)^{1/4}}e^{ikr}$$

$$=i\pi\int_{-\infty+i\mu}^{+\infty+i\mu} dk\frac{k\rho(k)}{(\mu^2+k^2)^{1/4}}e^{ikr}.$$

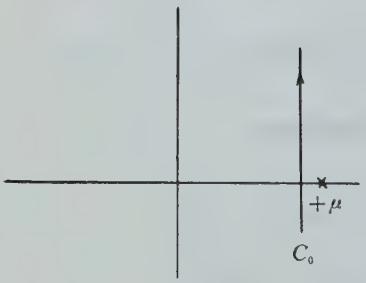


Fig. 2. Integration path C_0

Setting $k=is$, we obtain

$$f(r)=i\pi\int_{-i\infty+\mu}^{+i\infty+\mu} ds\frac{s\rho(is)}{(\mu^2-s^2)^{1/4}}e^{-sr}$$

$$\equiv i\pi\int_{C_0} ds\frac{s\rho(is)}{(\mu^2-s^2)^{1/4}}e^{-sr}.$$

Substituting these formulae into eq. (2·9), we have the following equation,

$$(E-m_0)+2\pi^3\ i\frac{g}{C}\int_0^\infty dr\ u(r)\int_{C_0} ds\frac{s\rho(is)}{(\mu^2-s^2)^{1/4}}e^{-sr}=0$$

or

$$(E-m_0)+2\pi^3 i\frac{g}{C}\int_{C_0} ds\frac{s\rho(is)}{(\mu^2-s^2)^{1/4}}u(s)=0. \qquad (2\cdot10)$$

(For the path C_0, see Fig. 2)

By a similar method, we can obtain the expression of $u(s)$:

$$u(s)=\frac{i\pi^2 gC}{2}\int_{C_0} d\sigma\frac{\sigma\rho(i\sigma)}{(\mu^2-\sigma^2)^{1/4}}(E-m_N+\sqrt{\mu^2-\sigma^2})\times$$

$$\times\frac{1}{(\sigma-A)(\sigma+s)(s-A)} \qquad (2\cdot11)$$

where $$-A^2=(E-m_N)^2-\mu^2.$$

From (2·10) and (2·11) we have the following equation which determines the energy E:

$$(E-m_0)-\pi^5 g^2\int_{C_0} ds\int_{C_0} d\sigma\frac{s\rho(is)}{(\mu^2-s^2)^{1/4}}\frac{\sigma\rho(i\sigma)}{(\mu^2-\sigma^2)^{1/4}}(E-m_N+\sqrt{\mu^2-\sigma^2})\times$$

$$\times\frac{1}{(\sigma-A)(\sigma+s)(s-A)}=0. \qquad (2\cdot12)$$

As is proved in the Appendix, it is possible to represent (2·12) as follows:

$$E-m_0-\pi^6 g^2 i\int_\Gamma ds\frac{s^2\rho^2(is)}{(\mu^2-s^2)^{1/2}}(E-m_N+\sqrt{\mu^2-s^2})\frac{1}{s^2-A^2}=0,$$

or by virtue of the definition of A

$$(E-m_0)-\pi^6 g^2 i\int_\Gamma ds\frac{s^2\rho^2(is)}{\sqrt{\mu^2-s^2}}\frac{1}{(E-m_N-\sqrt{\mu^2-s^2})}=0 \qquad (2\cdot13)$$

where the path Γ is shown in Fig. 3.

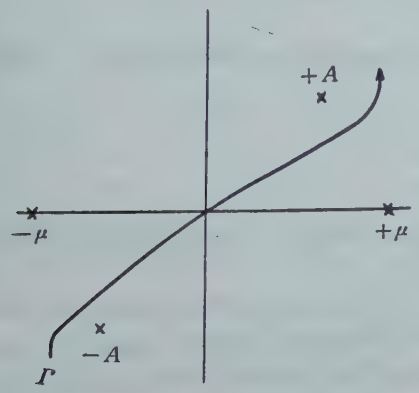


Fig. 3. The path Γ

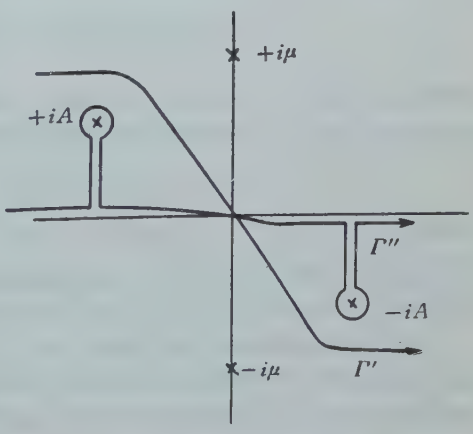


Fig. 4 The pathes Γ' and Γ''

By putting $s \to ik$, (2·13) becomes

$$(E-m_0)+\pi^6 g^2 \int_{\Gamma'} dk \frac{k^2 \rho^2(k)}{\sqrt{k^2+\mu^2}} \frac{1}{E-m_N-\sqrt{k^2+\mu^2}}=0 \qquad (2\cdot14)$$

where Γ'' is shown in Fig. 4.
Deforming the path Γ'' to Γ''', we can obtain the same formulae given by Arakī et al. (See Ref. 1) eq. (3·6)), that is,

$$m_0-m_s=2\pi^6 g^2 \int_0^\infty \frac{k^2 dk}{\sqrt{\mu^2+k^2}} \rho^2(k) \frac{\sqrt{\mu^2+k^2}-m_s+m_N}{(\sqrt{\mu^2+k^2}+m_N-m_s)^2+\gamma_s^2/4}+4\pi^7 g^2 \eta\,,$$

$$\frac{\gamma_s}{2}\left\{1+2\pi^6 g^2 \int_0^\infty dk \frac{k^2 \rho^2(k)}{\sqrt{\mu^2+k^2}} \frac{1}{(\sqrt{\mu^2+k^2}+m_N-m_s)^2+\gamma_s^2/4}\right\}=4\pi^7 g^2 \xi$$

where

$$\rho^2(\sqrt{(E-m_N)^2-\mu^2})\sqrt{(E-m_N)^2-\mu^2}=\xi-i\eta\,, \qquad (2\cdot15)$$

$$E=m_s-\frac{i}{2}\gamma_s.$$

§ 3. The state vector for the unstable state in Lee's model

The state vector obtained in the last section cannot be normalized in the ordinary sense, since the amplitude $\varphi(r)$ increases exponentially as r goes to infinity. This is inherent not only in quantum field theory but also in all stationary treatments of unstable states in wave theories. For example, in the case of the spontaneous α-disintegration of nuclei, the same situation appears. At $r \to \infty$ only the decay product emitted by unstable states is present. Thus we obtain the knowledge of decay products by studying the state vector at $r \to \infty$. For example, we will study the momentum distribution of decay products. For the measurement of momentum we have to make observations in a large region of space and hence the momentum distribution of the decay product is given as follows:

$$f(p)=\left|\int_{R_1}^{R_1} dr\, u(r) e^{-ipr}\right|^2 / \int_{R_1}^{R_2} dr\, u^*(r) u(r) \qquad (3\cdot1)$$

where R_1 and R_2-R_1 go to infinity after the integrations are performed. The evaluation of (3·1) is easy and the result is

$$f(p)=2A_R/[(A_I-p)^2+A_R^2] \qquad (3\cdot2)$$

where $A=A_R+iA_I$.

As is easily seen from (3·2), the momentum distribution of decay products has a maximum at $p=A_I$ and its width is given by A_R.

We shall consider the reason why Glaser and Källén could not obtain the complex eigenvalue. As the unstable state obtained in the last section cannot be normalizable, it is not contained in the space spanned by ordinary, normalizable states. Though the total Hamiltonian is hermitian in the so-called "Hilbert space" of

ordinary states, our unstable state does not preserve hermiticity of total Hamiltonian, To derive the eigenvalue equation, Glaser and Källén expanded the eigenstate $|\Psi\rangle$ into only normalizable states and hence they took account of only the ordinary Hilbert space which preserved hermiticity of the total Hamiltonian. This is just the reason why they could not obtain the complex eigenvalue of the total Hamiltonian. Contrary to the result of Källén, ours is the same one as that given by Araki et al. who obtained it by analytical continuation of the S-matrix. This fact corresponds to the arguments given by Møller[7] in quantum mechanics.

Our state has the main part at large distances, only a small portion is found near the origin. This is interpreted as follows: At the beginning, the main part of the particles is near the origin and then it decreases as the time elapses. As the V-particle probability decreases the decay products increase and run away far from the origin. At last, after infinite time, there are only decay products at infinite distances from the origin and there is nothing near the origin. Our state corresponds to this fact and represents the asymptotic expression of the state after infinite time. In section 5, we will study the time dependent development of the unstable state and the above statement will be shown more concretely. In such a treatment of the unstable state, it is sufficient to take account of the ordinary Hilbert space and then hermiticity of the total Hamiltonian is preserved.

§ 4. Renormalization procedures and the clothed state of an unstable V-particle

Now, we shall consider the procedure to remove the divergences which appear in the limit of $\rho(\boldsymbol{x})$ going to $\delta(\boldsymbol{x})$. For this purpose, let us study the eigenvalue equation (2·15), i.e.

$$\delta m=m_0-m_s=2\pi^6 g^2\int_0^\infty \frac{k^2dk}{\omega_k}\rho^2(k)\frac{\omega_k+m_N-m_s}{(\omega_k+m_N-m_s)^2+r_s^2/4}+4\pi^7 g^2\eta$$

$$\frac{\gamma_s}{2}\left\{1+2\pi^6 g^2\int_0^\infty \frac{k^2dk}{\omega_k}\rho^2(k)\frac{1}{(\omega_k+m_N-m_s)^2+\gamma_s^2/4}\right\}=4\pi^7 g^2\xi \qquad (2\cdot15)$$

where $\omega_k=\sqrt{k^2+\mu^2}$

$$\xi-i\eta=\rho^2(\sqrt{(E-m_N)^2-\mu^2})\sqrt{(E-m_N)^2-\mu^2},\quad E=m_s-(i/2)\gamma_s.$$

It is natural to define the mass renormalization constant δm by the first of eqs. (2·15). The main problem of renormalization procedures is how to define the charge renormalization constant Z. As the interaction strength for the decay process is closely related to the decay constant γ_s, a clue to the charge renormalization procedure may be contained in the equation to determine the decay constant γ_s.

In eq. (2·15) ξ is interpreted as a mean value of the momentum of an outgoing meson, we can rewrite the right-hand side of the second of eqs. (2·15) as follows:

$$4\pi^7 g^2 \tilde{\xi} \simeq 4\pi^7 g^2 \rho^2(\bar{\omega}) \bar{\omega} \bar{v} \tag{4·1}$$

where $\bar{\omega}$ is a mean energy of an outgoing θ-partible and $\bar{v}$ is its velocity. (Of course, we assume that $\rho((E-m_N)^2-\mu^2) \sim \rho(\bar{\omega})$.)

Now, let us consider the behaviour of θ-particles. As is seen from the expression for $u(r)$, i.e. eq. (2·8), the incoming wave is present only in the region $r \lesssim 1/\mu$ and in the region $r > 1/\mu$ there is only the outgoing wave. Thus we can interpret that the wave function $u(r)$ in the region $r > 1/\mu$ represents decay products only. And then, defining the wave function in the region $r \lesssim 1/\mu$ as the clothing of an unstable V-particle, we will calculate $4\pi|u(r_c)|^2 (r_c \sim 1/\mu)$ which is the probability of finding θ-particles on the surface of a clothed V-particle. Assuming $\mu \gg \mathrm{Re}\, A$, we obtain an approximate expression for $u(r_c)$:

$$u(r_c) \sim \frac{\pi g C}{4A} [g(-A) - g(A)] e^{Ar_c} \tag{4·2}$$

where $$g(A) = \int_0^\infty dr\ e^{-Ar}\ g(r)$$

and we have used the following approximation :

$$\int_0^{r_c} dr\ e^{\pm Ar}\ g(r) \sim \int_0^\infty dr\ e^{\pm Ar}\ g(r)$$

$$\int_{r_c}^\infty dr\ e^{Ar}\ g(r) \sim 0,$$

since $g(r) \sim e^{-\mu r}$ as r is large. From (4·2) and the expression for $g(r)$, it is easily found that

$$4\pi|u(r_c)|^2 \sim 4\pi^7 g^2 \rho^2(\bar{\omega}) \bar{\omega} C^2. \tag{4·3}$$

(Of course, $\rho(E) \sim \rho(\bar{\omega})$, etc., are assumed.)
By comparing (4·3) and (4·1), the following relation is obtained :

$$4\pi^7 g^2 \tilde{\xi} \simeq 4\pi|u(r_c)|^2\, \bar{v} \frac{1}{C^2}, \tag{4·4}$$

and hence the decay constant γ_s is expressed as follows :

$$\gamma_s = 4\pi|u(r_c)|^2 \bar{v} \frac{2}{C^2\left[1 + 2\pi^6 g^2 \int_0^\infty \frac{k^2 dk}{\omega_k} \rho^2(k) \frac{1}{(m_s - m_N - \omega_k)^2 + \gamma_s^2/4}\right]} \tag{4·5}$$

It is natural to interpret the decay constant γ_s as the product of the probability $4\pi|u(r_c)|^2$ of finding the θ-particle on the surface and its escape velocity $\bar{v}$.
Then, setting

$$\frac{2}{C^2\left[1 + 2\pi^6 g^2 \int_0^\infty \frac{k^2 dk}{\omega_k} \rho^2(k) \frac{1}{(\omega_k + m_N - m_s)^2 + \gamma_s^2/4}\right]} = 1, \tag{4·6a}$$

we have

$$\gamma_s = 4\pi |u(r_c)|^2 \bar{v}. \tag{4·6b}$$

In order to interpret $4\pi|u(r_c)|^2$ as probability, we have to determine C^2 as the normalization constant of a clothed unstable state and then it is given by the following equation:

$$C^{-2} = 1 + 4\pi C^{-2} \int_0^{r_c} dr\ u^*(r) u(r). \tag{4·7}$$

From the above discussion, it seems to be natural to identify C^2 as the charge renormalization constant Z. Then the charge renormalization procedure is as follows:

$$g_r^2 = Z g^2 = C^2 g^2, \tag{4·8}$$

where g_r is a renormalized coupling constant.

By considering (4·6a), it is seen that

$$Z = \frac{2}{1 + 2\pi^6 g^2 \int \frac{k^2 dk}{\omega_k} \rho^2(k) \frac{1}{(m_N + \omega_k - m_s)^2 + \gamma_s^2/4}}, \tag{4·9}$$

and hence it is clear that the Z-factor satisfies the following inequality,

$$0 \leq Z \leq 1. \tag{4·10}$$

Contrary to Naitō's definition, ours goes to unity in the weak coupling limit.

To conclude this section, we study the relation between the Z-factor ($\equiv C^2$) and the factor $(1/1 + 2\pi^6 g^2 \int dk \cdots) \equiv Z_0$
The state in which Z_0 is interpreted as the dissociation probability is expressed as follows:

$$|\Psi_0\rangle = Z_0^{1/2} \left[|V\rangle + \pi g \int dk \left(\frac{2\pi}{\omega_k} \right)^{1/2} \rho(k) \frac{1}{m_s - \omega_k - m_N - (i/2)\gamma_s} \alpha_k^+ |N\rangle \right]. \tag{4·11}$$

This state is identical with Naitō's physical state of an unstable V-particle. The region of this state in which θ-particles are found is $r \lesssim 1/A_R$ and as the coupling decreases this region increases. To compare the clothed unstable state with the state (4·11), we decompose $u(r) \cdot \theta(r_c - r)$ into its Fourier components. By virtue of $g(r) \sim 0$ for $r \gg 1/\mu$, the Fourier component $v(k)$ of $u(r_c)\theta(r_c - r)$ is as follows:

$$v(k) \sim \frac{-\pi g C}{2A} \frac{1}{k^2 + A^2} \left[2A \int_0^\infty dr g(r) \sin kr \right.$$
$$\left. - \int_0^\infty g(r) \sin iAr\, dr \left\{ (ik + A) e^{-(ik - A) r_c} + (ik - A) e^{(ik + A) r_c} \right\} \right] \tag{4·12}$$

The second term of (4·12) is zero for $k \ll |A|$ and it rapidly oscillates and then becomes effectively zero for $k \gg \mu$. For $k = \pm iA$, however, the second term is equal to the first one, but the sign of the second term is opposite to that of the first one. Then, $v(k)$ has no pole at $k = \pm iA$. As the first term is identical with the amplitude of a θ-particle of momentum k in the state (4·11), the state of the unstable particle is nearly equal to the state (4·11) excepting the region near the momentum $k = \pm iA$. Thus we can write the unstable state as below:

$$|\Psi\rangle = C\left[|V\rangle + \pi g \int dk \left(\frac{2\pi}{\omega_k}\right)^{\frac{1}{2}} \rho(k) \frac{1-f(k)}{m_s - m_N - \omega_k - i\gamma_s/2} \alpha_k^+ |N\rangle \right]$$

where $f(k) \simeq 1$ near the point $k = \pm iA$,

$\simeq 0$ otherwise. (4·13)

This state is identical with the physical unstable state proposed by Nakanishi. Then we can write the Z-factor as follows:

$$Z = \frac{1}{1 + 2\pi^6 g^2 \int_0^\infty dk \frac{k^2 \rho^2(k)}{\omega_k} \frac{|1-f(k)|^2}{(m_s - m_N - \omega_k)^2 + \gamma_s^2/4}} \tag{4·14}$$

§ 5. The time development of unstable states[3)]

In the foregoing sections, we have studied the stationary treatment of unstable states. The stationary Schrödinger equation for unstable states has been treated as an eigenvalue problem. We shall study, in this section, the time development of the unstable state and the time-dependent Schrödinger equation will be considered as an initial value problem. The basic equation is

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{5·1}$$

where $|\Psi(t)\rangle = \int d\boldsymbol{k}\ a(\boldsymbol{k},t)\ \alpha^+(\boldsymbol{k})|N\rangle + b(t)|V\rangle.$

In this case, the state $|\Psi(t)\rangle$ is normalizable and we can expand the state $|\Psi(t)\rangle$ into ordinary eigenstates of the free Hamiltonian, which are normalizable and preserve hermiticity of the total Hamiltonian. From (5·1), we obtain the simultaneous differential equations for $a(\boldsymbol{k}, t)$ and $b(t)$:

$$\begin{aligned} i\frac{\partial a(\boldsymbol{k}, t)}{\partial t} &= (m_N + \omega_K) a(\boldsymbol{k}, t) - g\frac{\rho(\boldsymbol{k})}{\sqrt{2\omega_K}} b(t), \\ i\frac{\partial b(t)}{\partial t} &= m_0\, b(t) - g\int d\boldsymbol{k}\, \frac{\rho(\boldsymbol{k})}{\sqrt{2\omega_K}}\, a(\boldsymbol{k}, t). \end{aligned} \tag{5·2}$$

With the aid of the Laplace transformation, the general solutions of (5·2) are easily obtained:

$$b(E)=\frac{b_0+ig\int d\boldsymbol{k}\ \frac{\rho(\boldsymbol{k})}{\sqrt{2\omega_k}}\ \frac{a^0(\boldsymbol{k})}{E+i(m_N+\omega_k)}}{E+im_0+g^2\int d\boldsymbol{k}\ \frac{\rho^2(\boldsymbol{k})}{2\omega_k}\ \frac{1}{E+i(m_N+\omega_k)}}$$

$$a(\boldsymbol{k},\ E)=\frac{a^0(\boldsymbol{k})}{E+i(m_N+\omega_k)}+ig\ \frac{\rho(\boldsymbol{k})}{\sqrt{2\omega_k}}\ \frac{1}{E+i(m_N+\omega_k)}b(E) \qquad (5\cdot3)$$

where $b(E)$ and $a(\boldsymbol{k},\ E)$ are Laplace transforms of $b(t)$ and $a(\boldsymbol{k},\ t)$ respectively, i.e.

$$b(E)=\int_0^\infty dt\ e^{-Et}\ b(t)$$

$$a(\boldsymbol{k},E)=\int_0^\infty dt\ e^{-Et}\ a(\boldsymbol{k},\ t)$$

and b_0 and $a^0(\boldsymbol{k})$ are initial values of $b(t)$ and $a(\boldsymbol{k},\ t)$, i.e.

$$b(t=0)=b_0$$

$$a(\boldsymbol{k},\ t=0)=a^0(\boldsymbol{k}).$$

From (5·3), we can obtain $a(\boldsymbol{k},\ t)$ and $b(t)$ by means of the inverse Laplace transform, that is,

$$b(t)==\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon}dE\ e^{Et}\ b(E)$$

$$a(\boldsymbol{k},\ t)=\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon}dE\ e^{Et}\ a(\boldsymbol{k},\ E) \qquad \epsilon>0. \qquad (5\cdot4)$$

To perform the integration of (5·4), it is useful to take advantage of the analytical properties of $b(E)$ and $a(\boldsymbol{k},\ E)$. The denominator of $b(E)$ is identical with the one particle propagator S_V' which has already been investigated by Araki et al. When the state (4·13) is taken to be an initial state $|\Psi(0)\rangle$, it is seen that the time variation of this state is given by $e^{-im_s\,t-(\tau_s/2)t}|\Psi(0)\rangle$ and the wave front of θ-mesons is placedat $r=\bar{v}t$ for sufficiently large time, where the velocity $\bar{v}$ is given by

$$\sim\bar{v}[(m_s-m_N)^2-\mu^2]^{1/2}/m_s-m_N.$$

The physical meaning of the above fact is clear and the norm of the state vector $|\Psi(t)\rangle$ is conserved, hence the total Hamiltonian is hermitian. That is, the decrease of the norm of $|\Psi(0)\rangle$ is compensated by the increase of the norm of the state composed of an N-and an outgoing θ-particle.

§ 6. Discussions

Nakanishi's exact state of the unstable V-particle has zero norm contrary to

our state which is not normalizable. Though we are very much interested in his treatment, it seems to us that the physical meaning of his state is not clear. Under what boundary condition did he solve Schrödinger equation $H|\Psi\rangle=E|\Psi\rangle$? What kind of relations are there between the exact and physical states of the unstable V-particle? It seems to us that the procedure to construct the physical state from the exact state is only artificial.

Our eigenstate of the total Hamiltonian is composed of the decay products and the physical unstable state. The separation of the physical V-particle from the eigenstate is not unique. But the meaning of the separation is clear. The uncertainty presented in the treatment of the decaying state may be inherent in the fundamental nature of quantum theory. In the preceding sections, we have supposed that the surface of the unstable state is sharp, but this is not essential in our treatment. We need to assure only that the cloud of θ-particles decreases rapidly near the point $r\sim r_c$. We are unable to define the behaviour of the cloud precisely because the interaction region has a tail. In the theory of nuclear reactions the same situation occurs. (See Ref. 8) and 9)). Our treatment is the natural application of the method used in particle mechanics to the field theory. There is no drastic reformulation of the field theory.

The author wishes to thank Professor R. Utiyama for his kind guidance and encouragement.

Appendix

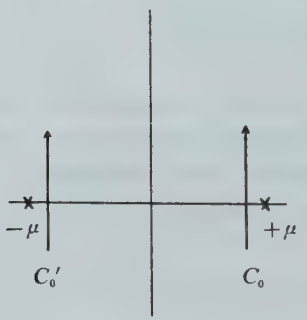


Fig. A1.

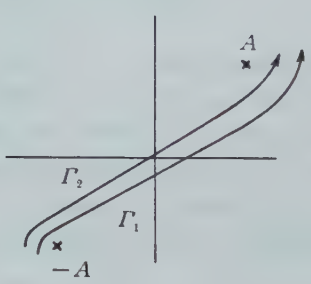


Fig. A2.

We intend to calculate the integral (2·12)

$$I=\int_{C_0}d\sigma\int_{C_0}ds\ F(s)G(\sigma)\frac{1}{(\sigma+s)(\sigma-A)(s-A)}$$

where $F(s)$ and $G(s)$ are odd functions of s.

Setting $\sigma=-\sigma$, we obtain

$$I=\int_{C_0'}d\sigma\int_{C_0}ds\ F(s)G(\sigma)\frac{1}{(s-\sigma)(s-A)(\sigma+A)} \qquad (A\cdot1)$$

where the path C_0' is shown in Fig. (A·1).

By deforming the path C_0 and C_0' into Γ_1 and Γ_2, the integral I is written as follow:

$$I=\int_{\Gamma_2} d\sigma \int_{\Gamma_1} ds\ F(s)G(\sigma)\frac{1}{(s-\sigma)(s-A)(\sigma+A)} \qquad (A\cdot 2)$$

where Γ_1 and Γ_2 are shown in Fig. A·2.

We can write (A·2) as follows:

$$I=2\pi i\int_{\Gamma} ds\ F(s)G(s)\frac{1}{s^2-A^2}+\int_{\Gamma_1} d\sigma \int_{\Gamma_2} ds\ F(s)\,G(\sigma)\frac{1}{(s-\sigma)(s-A)(\sigma+A)}. \qquad (A\cdot 3)$$

From (A·2) and (A·3), the integral I becomes as follows:

$$I=i\pi\int_{\Gamma} ds\ F(s)\,G(s)\frac{1}{s^2-A^2}$$

$$+\tfrac{1}{2}\int_{\Gamma} ds\ d\sigma\frac{1}{(s^2-A^2)(\sigma^2-A^2)}\Big[\frac{1}{s-\sigma}(\sigma s-A^2)(F(\sigma)G(s)-F(s)G(\sigma))$$

$$+A\ (F(\sigma)G(s)+F(s)\ G(\sigma))\Big],$$

where we choose the path $\Gamma_1=\Gamma_2\equiv\Gamma$ since the integrand has no pole on the integral path. It is easy to see that the second term becomes zero because $F(s)$ and $G(s)$ are odd functions. Then we obtain

$$I=i\pi\int_{\Gamma} ds\ F(s)\,G(s)\frac{1}{s^2-A^2}.$$

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A Note on the Nuclear Reactions in Stars†

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Building up of the elements in stars by helium capture, neutron capture and proton capture is considered. A possible model for the formation of the iron peak is presented. A new neutron source is suggested.

§ 1. Introduction

A whole evolutionary scheme of the universe (including the stars and the interstellar gas) was proposed by Taketani, Hatanaka and Obi[1] among others. They explained the differences between the characteristics of stars belonging to Populations I and II. We believe that their evolutional scheme is well-founded and very interesting; hence it would be worth-while to investigate the role of nuclear reactions in stars according to their scheme.

The theories of nuclear reactions in stars have been extensively developed by many authors.[2] Recently, detailed calculations on the rates of the α-process reactions have also been given by Nakagawa et al.[3] and by Hayakawa et al.[4] It is the purpose of this note to investigate the subsequent reactions of α-process. In § 2, the α-capture processes are discussed up to the formation of the iron peak together with the conditions under which the capturing process takes place. The explanation of the iron peak of Hoyle et al. is undoubtedly interesting, but their calculation is much oversimplified. The first point is that the abundances of the proton and the neutron should be much larger than the ones deduced from the equilibrium equation. The second: "the freezing temperature" should differ from one nucleus to another, etc. A possible model for the formation of the iron peak is presented in § 2. In § 3 and § 4, proton and neutron capture processes are discussed respectively. New neutron sources are suggested.

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† The outline of the contents of the present paper was read by one of the authors (S. N.) at the International Congress on Theoretical Physics held at the University of Washington in Seattle, September 1956.

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§ 2. Formation of the iron peak

a) Preliminaries.

It seems rather difficult to explain in a clear cut way the formation of the iron peak[5] in the abundance of the elements due to the fact that the abundance of iron group is greater than one of the neighbouring elements by the order of 10^4. One may expect that the binding energies of the iron group nuclei are so large that the thermal equilibrium theory may provide a reasonable ratio of the abundance in conformity with the observed results by suitably choosing several parameters which describe the conditions of internal space of the stars. The latest attempts along this line are those by Burbidge, Burbidge, Fowler and Hoyle,[2] after several earlier[2a] works. Since our results presented at the Seatle conference (1956) were somewhat different from theirs, we will briefly describe our procedures and calculations.

Firstly, we will consider many possibilities other than the equilibrium theory. Is there any particularity in the nuclear reactions which lead to the anomaly of the observed value for iron? Unfortunately we do not have favourable phenomena for this purpose. But we believe that it is impossible to decide definitely that there is no anomaly in nuclear reactions, because we do not have detailed information about the comparatively low energy region.

To proceed further, it seems desirable to observe the situation under which the formation of the heavy element takes place. Let the temperature of that situation be 2×10^8 °K or more. In this case we may observe that the nuclear reaction by helium capture can proceed to those nuclei farther than the iron group. The energy released by α-capture reactions is as follows,

$$Ne^{20}+\alpha\rightarrow Mg^{24}+9.33\ \text{MeV},\ \text{(abbreviated in } Ne^{20}\rightarrow(9.33)Mg^{24}).$$

$$Mg^{24}\rightarrow(10.00)Si^{28}\rightarrow(6.94)S^{32}\rightarrow(6.7)A^{36}\rightarrow(7.3)Ca^{40}$$

$$\rightarrow(5.29)Ti^{44}\xrightarrow{\beta^+}(0.18)Sc^{44}\xrightarrow{\beta^+}(3.64)Ca^{44}\rightarrow(9.23)Ti^{48}$$

$$\rightarrow(9.36)Cr^{52}\rightarrow(7.45)Fe^{56}\rightarrow(6.13)Ni^{60}\rightarrow(4.5)Zn^{64}$$

$$\rightarrow(2.55)Ge^{68}\xrightarrow{\beta^+}(0.7)Ga^{68}\xrightarrow{\beta^+}(1.88)Zn^{68}\rightarrow(5.26)Ge^{72}$$

$$\rightarrow(4.51)Se^{76}\rightarrow(5.26)Kr^{80}\rightarrow(5.32)Sr^{84}\rightarrow(4.9)Zr^{88}$$

$$\xrightarrow{\beta^+}(0.7)Y^{88}\xrightarrow{\beta^+}(2.73)Sr^{88}\rightarrow(2.9)Zr^{92}\rightarrow\cdots\cdots$$

Thus the reaction can proceed to Zr^{92} at least. The subsequent elements are Mo and Ru whose masses are hardly determined; hence we have no empirical data to give the released value of the energy. Then, if the semiempirical mass formula by Weizsächer and Bethe is used to infer the energy release, the results are

$$Zr^{92}\rightarrow(3.7)Mo^{96}\rightarrow(3.2)Ru^{100}\rightarrow(2.6)Pb^{104}\rightarrow(2.1)Cd^{108}$$

$$\rightarrow(1.5)Sn^{112}\rightarrow(1.0)Te^{116}\xrightarrow{\beta^+}Sb^{116}\xrightarrow{\beta^+}Sn^{116}\rightarrow\cdots\cdots$$

The exothermic reactions can take place beyond the mass number 110. The exact formula used here is

Mass $=993.94\,A-0.86\,Z+13.35\,A^{2/3}+0.627\,Z^2A^{-1/3}+83(Z-A/2)^2A^{-1}+\delta$, in mmu. The mass of an α-particle is 4003.86 mmu. The equilibrium theory can be applied only to the case in which the abundances of the α-particles as well as the protons and the neutrons, etc., are much reduced to the values expected from the statistical mechanics. This seems to be unrealistic.

b) Does the α-capture reaction stop at Ne^{20}?

According to the calculations performed recently by the Tokyo group[3] and the Kyoto group,[4] the reaction $Ne^{20}+\alpha\rightarrow Mg^{24}$ is so slow that the helium 4 is completely converted into neon 20 before the formation of Mg^{24}, consequently it is impossible to build elements heavier than Mg^{24}. Is it true? To consider this question, it will be necessary to reexamine the assumptions made in their papers; the constancy of the ratio of α to C^{12}, O^{16}, Ne^{20} in the convection core. It is equivalent to a good mixing assumption in the core. If the mixing rate is rather low compared with the reaction rate, the supply of He^4 is deficient in the region where C, O, and Ne are being produced. Then the rate of the reaction $3\alpha\rightarrow C^{12}$ is reduced, because the rate of this reaction is proportional to the cube of the helium density. In order to maintain the observed radiation from the star, higher temperature is needed; then the reaction $Ne^{20}+\alpha\rightarrow Mg^{24}$ may proceed. We now assume that this is the case and the heavier elements are produced subsequently. Let us estimate the condition under which the α-capture reaction can proceed.

c) Temperature necessary for the formation of the iron group

Since the internal structure of stars in the later stage of the steller evolution is not well established, we will make a very rough approximation for inferring the real condition. Our chief interest lies in the order of magnitude consideration. Let an element A' (mass number $4n+4$) be formed by an α-capture of an element A (mass number $4n$). Since the mixing is not great, the reaction rate of $3\alpha\rightarrow C^{12}$ equals to the reaction rate of $A+\alpha\rightarrow A'$ approximately, because the reaction rate of forming the element A from C^{12} is very large compared with both reactions mentioned above. In other words, the reaction is restricted essentially by the reaction rate of $3\alpha\rightarrow C^{12}$, which should be determined by the observed radiation intensity from the stars. This reaction rate is

$$p_{3\alpha}\sim 10^{22}\,\rho_\alpha{}^3(10^5)\,e^{-43.2/T}\ \mathrm{cm^{-3}\ sec^{-1}},$$

where $\rho_\alpha(10^5)$ is the density of α-particles measured in $10^5\mathrm{g/cm^3}$, T is the temperature measured in $10^8\,^\circ\mathrm{K}$. This leads to the fact that $\rho_\alpha{}^3$ is proportional to $e^{43.2/T}$. Since this formula is not correct for low temperature, we put

$$\rho_\alpha=\rho_0\,e^{14.4/(T+T_0)}$$

and choose T_0 and ρ_0 suitably. If we assume $T_0\sim 1.8$ and $\rho_0\sim 10^{-5.5}$, $p_{3\alpha}$ is approximately in conformity with the observed radiation. On the other hand, the reaction rate for $A+\alpha\rightarrow A'$ may be given by

$$p_A\sim 10^{42}\rho_\alpha(10^5)\rho_A(10^5)\,T^{-3/2}\exp(-23.4Z^{2/3}/T^{1/3})\exp(3.73Z^{1/2}A^{1/6})\ \mathrm{cm^{-3}\ sec^{-1}}$$

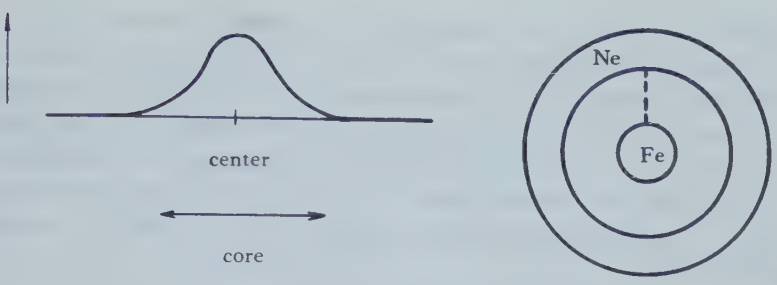

Fig. 1

where A is the mass number and Z is the charge number of the element A. Putting $p_{3\alpha}=p_A$, and $p'_A=10^{-2}$, we can know the temperature T under which the element A' is produced. The results are as follows: Mg^{24} begins to be produced at about $T=1.5(\times 18^8\,^\circ K)$, and Fe^{56} at $T=7.4$, Ni^{60} at $T=8.2$. The calculations hitherto described are based on the assumption that there is only one resonance level at the most favourable energy value. If the reaction rate of α-capture is reduced by the factor of 10^{-5}, Mg^{24} is produced at temperature as high as $T=2.9$, and Ni^{60} at $T=12$. Since the exothermic α reaction can proceed to go beyond the mass number 100, the reaction chain does not end at Fe^{56}. But, if some endothermic reaction begings to take place with a comparative rate at the temperature at which the production of Fe^{56} proceeds, a considerable part of C^{12} may be converted to Fe^{56} only.

d) Urca-process

Let us consider the Urca-process of Gamow-Schönberg[7] as one of such endothermic reactions. This is, for example,

$$Fe^{56}+e^-\rightarrow Mn^{56}+\nu,\ Mn^{56}\rightarrow Fe^{56}+e^-+\bar{\nu}.$$

The energy escapes in the form of neutrino (and antineutrino) by these β-decay and inverse β-decay, so that the raising of the temperature T may be prevented. In general, the reaction rate of the following type

$$(A,\ Z)+e^-\rightarrow(A,\ Z-1)+\nu,\ (A,\ Z-1)\rightarrow(A,\ Z)+e^-+\bar{\nu},$$

is

$$p\sim(1+7\,kT/Q)\,(n_{z-1}/t),$$

where t: the mean life of the decay of the nucleus $(A,\ Z-1)$, Q: the maximum electron energy for the β-decay of the $(A,\ Z-1)$ nucleus, $n_{z-1}=(C_z\rho/m_H)[1+0.98\times10^{-3}(\hbar c)^{-3}m_H\rho^{-1}Q^{9/2}(kT)^{-3/2}e^{|Q/kT|}]^{-1}$, the notation of which follows that of reference 7). It is assumed that the electron is extremely relativistic, while the nucleus is non-relativistic. As to $(Fe^{56},\ Mn^{56})$, this reaction rate turns out to be equal to the reaction rate of $3\alpha\rightarrow C^{12}$ at $T\sim20\times10^8\,^\circ K$. Therefore the Urca process is not important so far as the energy production is concerned. The α-capture reaction is not stopped at Fe by the Urca process. One of the other endothermic reaction possible in such a situation is the $(\gamma,\ p)$ reaction. This possibility was suggested

to the authors by Professor C. Hayashi of the Kyoto University.

e) (γ, p) reaction, (γ, n) reaction

The threshold energy of a (γ, p) reaction in the vicinity of Fe^{56} is so great (~ 10 MeV) that one may infer that (γ, p) does not take place in a low energy region ($\sim 10^8$ °K). But since the Coulomb barrier for a proton is considerably lower than that for He^4-nucleus, the reaction rate of (γ, p) turns out to be not negligible as shown below. The energy density for the photons is assumed to be expressed by the Planck formula.

$$\rho_\nu d\nu = (8\pi h\nu^3/c^3)\, d\nu/(\exp(h\nu/kT)-1).$$

The cross section of the reaction $A+\gamma$ (photon) $\rightarrow (A-1)+$proton is

$$\sigma_\gamma = (\pi/k^2)\Gamma_p\Gamma_\gamma[(E_\gamma - E_r)^2 + (\Gamma/2)^2]^{-1},\quad E_\gamma = \hbar\omega = \hbar kc,$$

where we assume the reaction proceeds through the resonance level higher than the ground state of A-nucleus by the energy E_r as shown in Fig. 2. Hence the reaction rate of a (γ, p) process is

$$p_\gamma = (4n\Gamma_p\Gamma_\gamma/\Gamma h)\,(\exp(E_r/kT)-1)^{-1}$$

where n is the number of nucleus A per cm^3,

$$n = 6\times 10^{28} A^{-1}\rho_A(10^5) \sim 6\times 10^{26} A^{-1},$$

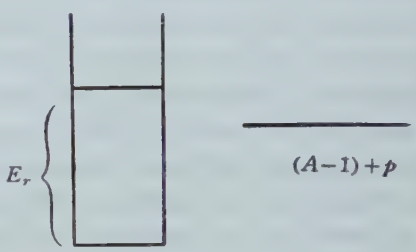

Fig. 2

and Γ_p, Γ_γ and Γ are the partial widths for a proton and a photon and the total width, respectively. Then we get

$$p_\gamma \sim 6\times 10^{43}\rho_A(10^5)\Gamma_p(\mathrm{eV})\Gamma_\gamma(\mathrm{eV})/[A\Gamma(\mathrm{eV})\,(\exp(E_r/kT)-1)],$$

$$\Gamma_p(\mathrm{eV}) \sim 10^5 \exp[-G],\quad G \sim \frac{Z_A}{\sqrt{E(\mathrm{MeV})}} - 1.32\, Z_A^{1/2} A^{1/6}.$$

Let the density of the resonance level be $(100\ \mathrm{eV})^{-1}$ and the threshold energy be 10 MeV, we get

$$p_\gamma \sim 5\times 10^{45}\rho_A(10^5)\Gamma_\gamma(\mathrm{eV})\, A^{-1} T(10^8)\exp[-1160/T(10^8)].$$

If we put $\Gamma_\gamma(\mathrm{eV}) \sim 10^2$, $\rho_A(10^5) \sim 10^{-2}$, the reaction rate becomes

$$p_\gamma \sim 10^{44} T(10^8)\, 10^{-[505/T(10^8)]}$$

If p_γ becomes approximately equal to p_A described in c) of this section, the α-capture process does not proceed to synthesize heavier elements by balancing the building up by the α-capture and the breaking down by the (γ, p) reaction. This temperature is calculated to be about 13×10^8 °K. We have assumed the Planck formula for the energy distribution of photons. Actually the high energy photons of non-thermal equilibrium may present more than expected by the Planck formula; the temperature may be reduced somewhat. Thus the (γ, p) reaction will have a

function by which the iron peak may be produced.

Sometimes the (γ, n) reaction can take place instead of the (γ, p) reaction. For example, we have in the vicinity of Fe^{56} and Ni^{60}

$$Ni^{60}+\gamma\rightarrow\begin{cases}Co^{59}+p-\ 9.52\ \text{MeV}\\ Ni^{59}+n-11.38\ \text{MeV}\end{cases}$$

$$Co^{59}+\gamma\rightarrow\begin{cases}Fe^{58}+p-\ 7.15\ \text{MeV}\\ Co^{58}+n-10.23\ \text{MeV}\end{cases}$$

$$Fe^{58}+\gamma\rightarrow\begin{cases}Mn^{57}+p-12.1\ \text{MeV}\\ Fe^{57}+n-10.1\ \text{MeV}\end{cases}$$

$$Fe^{57}+\gamma\rightarrow\begin{cases}Mn^{56}+p-10.56\ \text{MeV}\\ Fe^{56}+n-\ 7.64\ \text{MeV}\,.\end{cases}$$

If Ni^{60} is formed by the reaction $Fe^{56}+\alpha\rightarrow Ni^{60}+6.13$ MeV, Ni^{60} is converted to Fe^{56} by emitting two protons and two neutrons through the above reactions. These neutrons are possibly one of the neutron sources of neutron capture processes discussed in § 4.

§ 3. Proton capture process

The subject of the present section is to discuss the proton reactions by $4n$-nuclei formed by the α-capture process. For lighter $4n$ nuclei, it is well known that the C−N cycle, the Ne−Na cycle, etc., can take place. The cyclic reactions including $4n$ nuclei with smaller mass numbers than 40 have been studied by Imoto, Ogawa and Yamaguchi.[8] The reactions are more complicated if the heavier elements are concerned, as will be shown in Appendix I.

The branches of reactions are illustrated in the diagram in § 4 together with the neutron capture processes. It is observed in series of reactions that the proton capture reactions tend to build heavier elements in general. The cyclic reaction is possible only for (Fe−Co) chain. This fact is favourable to the formation of the iron peak in the element abundance.

§ 4. Neutron capture process

The neutron source in the nuclear reactions of stars is of importance for the synthesis of the elements heavier than the iron group. A typical reaction[9] is $C^{13}(\alpha, n)O^{16}+2.2$ MeV which was proposed by Cameron and Greenstein. Fowler et al.[10] added $Ne^{21}(\alpha, n)Mg^{24}$, etc. This reaction can take place only when the proton is absorbed by C^{12} (or Ne^{20}) and subsequently the α-particle must be absorbed by C^{13} (or Ne^{21}) with a reasonable reaction rate. But, according to a model of nuclear burning of stars, protons may be almost exhausted in a region where He^4 nuclei are produced. It will be more promising if neutrons can be produced by

proton collisions only. The exothermic reactions of such kind are found as follows.

$$K^{40}(p, n)Ca^{40}+0.57 \text{ MeV},$$

$$V^{50}(p, n)Cr^{50}+0.40 \text{ MeV},$$

$$Te^{98}(p, n)Ru^{98}+0.5 \text{ MeV},$$

$$Cr^{36}(p, n)A^{36}-0.14 \text{ MeV}.$$

The elements of the left-hand side of (p, n) are necessarily β-active, because an element having the maximum disintegration energy smaller than 0.78 MeV gives an endothermic reaction. 0.78 MeV equals the neutron mass (in MeV) subtracted by the proton mass and the electron mass. On the other hand, if the maximum disintegration energy is much larger than 0.78 MeV, this element should be quite unstable, so that it is not useful as a neutron source. This is the reason why we have only four elements for this purpose. Among these four reactions, the second reaction $V^{50}(p, n)Cr^{50}$, will be the most effective one due to a comparatively large abundance of V^{50}. It does not seem impossible to supply the necessary quantity of neutrons. This neutron source may be added to the (α, n) reactions discussed by Cameron,[9] Greenstein,[9] and Fowler et al.[10] as well as the reactions between two heavy elements (for example, $C^{12}+Ne^{20}\rightarrow n+S^{31}+4.2$ MeV) discussed in reference 3). The chain of nuclear reactions induced by neutron capture is given in Appendix II. The series of reactions in Appendix I and II is summarized in Fig. 3.

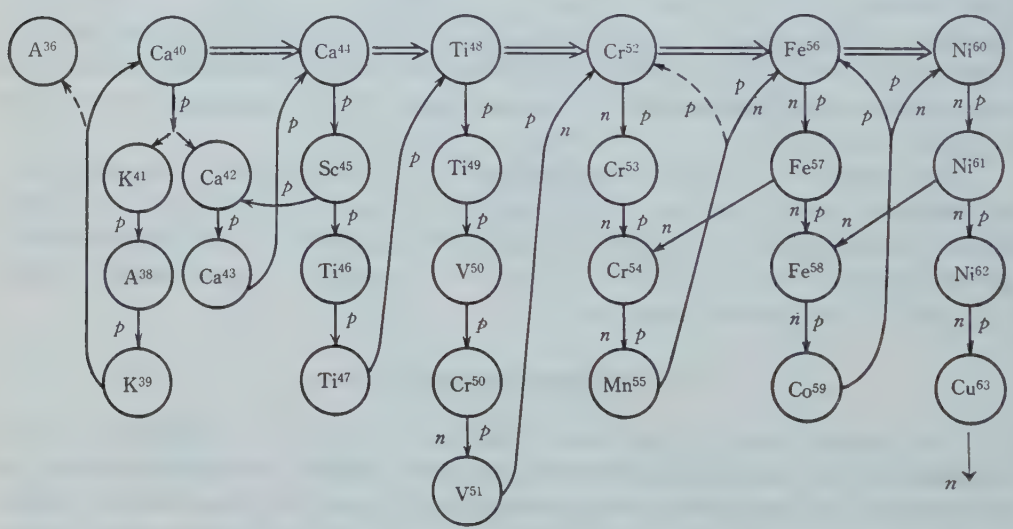

Fig. 3

We have discussed under the assumption that the heavy elements are synthesized from hydrogen by nuclear reactions. If this is true, and if the mixing rate in the stars is small, the observational existence of heavy elements in population II stars

should be explained by a non-nuclear origin. The effect of accretion of heavy elements of globular cluster stars and other causes may not be sufficient[11] to account for the existence of heavy elements. If we assume that this is true, it seems probable at present that a primordial state of the universe already contained some heavy elements in addition to hydrogen. We may arbitrarily consider a special state including nucleons, hyperons, mesons and others as an initial condition of the universe, which may bring about an improvement on the theory of element synthesis.

We are deeply indebted to Professor T. Hatanaka, Professor S. Obi, Dr. H. Takebe and Mr. M. Shimoda of the University of Tokyo, and Professor C. Hayashi of the Kyoto University for very helpful discussions.

Appendix I

The exothermic reactions induced by proton capture are given. One may regard this table as a data concerning the interaction between the heavier elements formed in an inner core and protons surrounding this core of stars. Several abbreviations are made, for example, $K^{41}+p\rightarrow(3.85)A^{38}(\alpha)$ means $K^{41}+p\rightarrow A^{38}+He^4+3.85$ MeV in the center of mass system. The other example is $Ca^{41}+p\rightarrow(4+6.5)Ca^{42}(\beta^+)+p\rightarrow(0.23)K^{39}(\alpha)$, which means that 1). $Ca^{41}+p\rightarrow(Sc^{42}+4\text{ MeV})$, 2). $(Sc^{42})\rightarrow Ca^{42}+\beta^++6.5$ MeV, 3). $Ca^{42}+p\rightarrow K^{39}+He^4+0.23$ MeV, 4). the lifetime of the intermediate nucleus (Sc^{42}) is shorter than 1 day, 5). the production of K^{39} will be negligible compared with other processes for the very low energy (say, <0.5 MeV) proton. 6). the maximum energy of β^+ emitted from Sc^{42} is 6.5 MeV minus twice the electron rest energy.

$$Ca^{40}+p\rightarrow(1.64+5.96)Ca^{41}(\beta^+)\rightarrow(0.44,\ 10^5y)K^{41}(K),$$

$$Ca^{41}+p\rightarrow(4+6.5)Ca^{42}(\ \ )+p\rightarrow\begin{cases}(4.96+2.2)Ca^{43}(\beta^+)\begin{cases}\nearrow(1.1)A^{36}(\alpha),\\ \searrow(8.34)Ca^{40},\end{cases}\\ (0.23)K^{39}(\alpha)+p\longrightarrow(8.34)Ca^{40},\end{cases}$$

$$K^{41}+p\begin{cases}\nearrow(10.12)Ca^{42}\\ \rightarrow(3.85)A^{38}(\alpha)+p\rightarrow(6.5)K^{39},\end{cases}$$

$$Ca^{43}+p\rightarrow(6.87+3.6)Ca^{44}(\beta^++\gamma)+p\rightarrow(6.89)Sc^{45}+p\begin{cases}\nearrow(2.14)Ca^{42}(\alpha),\\ \searrow(10.47)Ti^{46},\end{cases}$$

$$\searrow(0.07)K^{40}(\alpha)\begin{cases}\nearrow(1.33)Ca^{40}(\beta^-,\ 89\%),\\ \searrow(1.46)A^{40}(\beta^+,\ 11\%),\end{cases}\quad K^{40}:\ \text{life}=1.3\times10^9y.$$

$$K^{40}+p\rightarrow(1.64)A^{37}(\alpha)\rightarrow(0.8)Cl^{37}+p\begin{cases}\rightarrow(7)S^{34}(\alpha),\\ \searrow(10.2)A^{38}+p\rightarrow(6.5)K^{39},\end{cases}$$

$$Ti^{46}+p\rightarrow(2.91+4.99)Ti^{47}(\beta^+)+p\rightarrow(4.0+6.7)Ti^{48}(\beta^+)+p\rightarrow(6.73)V^{49},$$

$$V^{49}\rightarrow(6.73,\ 330\,d)Ti^{49}+p\rightarrow(7.74)V^{50}+p\begin{cases}\nearrow(10.40)Cr^{50}(n),\\ \rightarrow(9.55)Cr^{51},\\ \searrow(0.92)Ti^{47}(\alpha),\end{cases}$$

$$Cr^{50}+p\rightarrow(5.3+3.1)\,Cr^{51}(\beta^+)\rightarrow(0.75,\ 28\,d)\,V^{51}+p\rightarrow(10.70)\,Cr^{52},$$
$$\searrow(0.62)\,Ti^{48}(\alpha),$$

$$Cr^{52}+p\rightarrow(6.55)\,Mn^{53}\rightarrow(0.6,\ 140\,y)\,Cr^{53}+p\rightarrow(7.56)\,Mn^{54}$$
$$\rightarrow(0.54,\ 290\,d,\ 0.84\gamma)\,Cr^{54}(K),$$

$$Cr^{54}+p\rightarrow(7.9)\,Mn^{55}+p\rightarrow(2.73)\,Cr^{52}(\alpha),$$
$$\searrow(10.18)\,Fe^{56}+p\rightarrow(6.2)\,Co^{57}\rightarrow(0.7,\ 270\,d)\,Fe^{57},$$

$$V^{49}+p\begin{cases}\nearrow(9.53)\,Cr^{50},\\ \searrow(1.37)\,Ti^{46}(\alpha),\end{cases}$$

$$Mn^{53}+p\rightarrow(8.70)\,Fe^{54}+p\rightarrow(5.06+3.46)\,Fe^{55}(\beta^+)\rightarrow(0.22,\ 3\,y)\,Mn^{55}(K),$$
$$\searrow(0.35)\,Cr^{50}(\alpha),$$

$$Fe^{57}+p\begin{cases}\nearrow(0.40)\,Mn^{54}(\alpha)\rightarrow(1.38,\ 290\,d)\,Cr^{54},\\ \searrow(7.02)\,Co^{58}\rightarrow(2.30,\ 72\,d)\,Fe^{58}+p\rightarrow(7.15)\,Co^{59}+p\begin{cases}\nearrow(3.39)\,Fe^{56}(\alpha),\\ \searrow(9.52)\,Ni^{60}\end{cases}\end{cases}$$

$$Ni^{60}+p\rightarrow(5.52)\,Cu^{61}\rightarrow(2.23)\,Ni^{61}+p\rightarrow(5.9+3.93)\,Ni^{62}(\beta^+)$$
$$\searrow(0.23)\,Co^{57}\rightarrow(0.7,\ 270\,d)\,Fe^{57},$$

$$Ni^{62}+p\rightarrow(5.94)\,Cu^{63}+p\begin{cases}\nearrow(3.2)\,Ni^{60}(\alpha),\\ \rightarrow(7.70)\,Zn^{64}+p\begin{cases}\nearrow(1.02+2.23)\,Ni^{61}(\alpha,\ \beta^+),\\ \rightarrow(4.0+3.1)\,Zn^{65}(\beta^+)\end{cases}\end{cases}$$

$$Zn^{65}\rightarrow(1.35,\ 245\,d)\,Cu^{65}+p\begin{cases}\rightarrow(4.58)\,Ni^{62}(\alpha),\\ \searrow(8.8)\ Zn^{66}+p\begin{cases}\nearrow(1.67)\,Cu^{63}(\alpha),\\ \rightarrow(5.1+1.00)\,Zn^{67}(K),\end{cases}\end{cases}$$

$$Zn^{67}+p\begin{cases}\rightarrow(6.54+2.9)\,Zn^{68}(\beta^+)+p\begin{cases}\nearrow(2.26)\,Cu^{65}(\alpha),\\ \rightarrow(6.6)\,Ga^{69}+p\begin{cases}\rightarrow(4.64)\,Zn^{66}(\alpha),\\ \searrow(8.55)\,Ga^{70},\end{cases}\end{cases}\\ \searrow(2.73+1.68)\,Ni^{64}(\alpha,\ \beta^+)+p\rightarrow(7.39)\,Cu^{65},\end{cases}$$

$$Ga^{70}+p\rightarrow(4.36+2.01)\,Ge^{71}(\beta^+)\rightarrow(0.24,\ 11\,d)\,Ga^{71}+p\rightarrow(5.04)\,Zn^{68}(\alpha),$$

$$Ge^{72}+p\begin{cases}\rightarrow(5.2)\,As^{73}\rightarrow(0.37,\ 76\,d)\,Ge^{73}+p\begin{cases}\nearrow(2.54+1.65)\,Ge^{70}(\alpha,\ \beta^-),\\ \searrow(6.80)\,As^{74}\begin{cases}\nearrow(1.36,\ 32\%)\,Se^{74},\\ \searrow(2.56,\ 68\%)\,Ge^{74},\end{cases}\end{cases}\\ \searrow(1.28)\,Ga^{69}(\alpha),\end{cases}$$

$$As^{74}:\ \text{life}=18\,d,$$

$$Se^{74}+p\rightarrow(4.36+7.7)\,Se^{75}(\beta^+)\rightarrow(0.87,\ 127\,d)\,As^{75}+p\begin{cases}\rightarrow(5.0)\,Ge^{72}(\alpha),\\ \searrow(9.5)\,Se^{76},\end{cases}$$

$$Se^{76}+p\begin{cases}\rightarrow(5.28+1.36)\,Se^{77}(\beta^+)+p\begin{cases}\rightarrow(1.27)\,As^{74}(\alpha),\\ \searrow(6.42+3.3)\,Se^{78}(\beta^+)+p\begin{cases}\nearrow(6.4)\,Br^{79},\\ \searrow(0.88)\,As^{75}(\alpha),\end{cases}\\ \rightarrow(6.42+0.6)\,Kr^{78}(\beta^-),\end{cases}\\ \searrow(0.71)\,As^{73}(\alpha),\end{cases}$$

$$Ge^{74}+p\begin{cases}\nearrow(1.44)\,Ga^{71}(\alpha),\\ \searrow(6.75)\,As^{75},\end{cases}$$

$$Kr^{78}+p\rightarrow(3.8+3.6+1.61)\,Br^{79}(2\beta^+)+p\begin{cases}\nearrow(9.20)\,Kr^{80},\\ \rightarrow(3.84)\,Se^{76}(\alpha),\end{cases}$$

$$Kr^{80}+p\rightarrow(4.8+2.2)\,Kr^{81}(\beta^+)+p\begin{cases}\nearrow(0.88+3.5)\,Se^{78}(\alpha,\ \beta^+),\\ \rightarrow(6.2+4.1)\,Kr^{82}(\beta^+),\\ \searrow(0.3,\ 2\times10^5\,y)\,Br^{81}+p\rightarrow(3.8)\,Se^{78}(\alpha),\\ \qquad\qquad\qquad\qquad\qquad\qquad\searrow(10.1)\,Kr^{82},\end{cases}$$

$$Kr^{82}+p\rightarrow(6.0)\,Rb^{83}\rightarrow(0.77,\ 83\,d)\,Kr^{83}+p\begin{cases}\nearrow(2.65,\ 95\%)\,Kr^{84}(\beta^+),\\ \rightarrow(7.0)\,Rb^{84}\rightarrow(1.0,\ 5\%)\,Sr^{84}(\beta^-),\\ \searrow(0.65+2.0,\ 92\%)\,Kr^{80}(\beta^-),\\ \rightarrow(0.65+1.9,\ 8\%)\,Se^{80}(\beta^+),\end{cases}$$

Rb^{84}: life $=34\,d$,

$$Sr^{84}+p\rightarrow(4.4+3.5)\,Sr^{85}(\beta^+)\rightarrow(1.1,\ 65\,d)\,Rb^{85}(K)+p\begin{cases}\rightarrow(3.52)\,Kr^{82}(\alpha),\\ \searrow(9.5)\,Sr^{86},\end{cases}$$

$$Se^{80}+p\rightarrow(7.36+3.8+2.2)\,Kr^{81}(2\beta^+),$$
$$\qquad\searrow(0.75+0.68)\,Se^{77}(\beta^-),$$

$$Kr^{84}+p\rightarrow(6.83)\,Rb^{85},$$

$$Sr^{86}+p\rightarrow(5.9+1.7)\,Sr^{87}(\beta^+)+p\rightarrow(6.6)\,Y^{88}\rightarrow(3.74,\ 105\,d)\,Sr^{88},$$

$$Sr^{88}+p\rightarrow(7.2)\,Y^{89}+p\rightarrow(8.10)\,Zr^{90}$$
$$\qquad\searrow(1.55)\,Sr^{86}(\alpha),$$

Appendix II

It has been suggested in § 4 that $V^{50}(p,\ n)Cr^{50}$ may be one of the neutron sources for the formation of heavy elements. In this connection exothermic neutron reactions are given in the same notation used in Appendix I.

$$Cr^{50}+n\rightarrow(9.15)\,Cr^{51}\rightarrow(0.75,\ 28\,d,\ 4.18)\,Cr^{52}(K,\ \beta^-),$$

$$Cr^{52}+n\rightarrow(7.93)\,Cr^{53}+n\rightarrow(9.72)\,Cr^{54}+n\rightarrow(5.8+2.85)\,Mn^{55}(\beta^-),$$
$$\qquad\searrow(1.75)\,Ti^{50}(\quad)+n\rightarrow(6.23+2.45)\,V^{51}(\beta^-),$$

$$Mn^{55}+n\rightarrow(7.26+3.70)\,Fe^{56}(\beta^-)+n\rightarrow(7.64)\,Fe^{57}+n\rightarrow(10.10)\,Fe^{58},$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\searrow(2.56)\,Cr^{54}(\alpha),$$

$$Fe^{58}+n\rightarrow(6.37)\,Fe^{59}\rightarrow(1.56,\ 45\,d)\,Co^{59}+n\rightarrow(7.49)\,Co^{60}\rightarrow(2.81,\ 5.2\,y)\,Ni^{60}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\searrow(0.47+3.70)\,Fe^{56},$$

$$Ni^{60}+n\rightarrow(8.53)\,Ni^{60}+n\rightarrow(3.08)\,Fe^{58}(\alpha),$$
$$\qquad\searrow(1.51)\,Fe^{57}(\alpha),\quad\searrow(10.6)\,Ni^{62}+n\rightarrow(6.66)\,Ni^{63}\rightarrow(0.063,\ 85\,y)\,Cu^{63},$$

$$Cu^{63}+n\begin{cases}\rightarrow(7.91+1.68,\ 62\%)\,Ni^{64}+n\rightarrow(6.07+2.10)\,Cu^{65}(\beta^-),\\ \searrow(7{\cdot}91+0.57,\ 38\%)\,Zn^{64}+n\rightarrow(7.93)\,Zn^{65}\rightarrow(1.35,\ 245\,d)\,Cu^{65}(\beta^+),\\ \rightarrow(1.18)\,Co^{60}(\alpha),\qquad\qquad\searrow(4.04)\,Ni^{61}(\alpha),\end{cases}$$

$$\mathrm{Cu}^{65}+n\rightarrow(7.0+2.63)\,\mathrm{Zn}^{66}(\beta^-)+n\rightarrow(6.9)\,\mathrm{Zn}^{67},$$
$$\searrow(2.44)\,\mathrm{Ni}^{63}(\alpha)\rightarrow(0.063,\ 85\,y)\,\mathrm{Cu}^{63}(\beta^-),$$

$$\mathrm{Zn}^{67}+n\rightarrow(5.2)\,\mathrm{Ni}^{64}(\alpha),$$
$$\searrow(10.22)\,\mathrm{Zn}^{68}+n\rightarrow(6.4+0.90)\,\mathrm{Ga}^{69}(\beta^+),$$
$$\searrow(0.47+2.10)\,\mathrm{Cu}^{65}(\alpha,\ \beta^-),$$

$$\mathrm{Ga}^{69}+n\begin{cases}\nearrow(2.66+2.63)\,\mathrm{Zn}^{66}(\alpha,\ \beta),\\ \rightarrow(7.68+1.65)\,\mathrm{Ge}^{70}(\beta^-)\\ \searrow(7.68+0.67)\,\mathrm{Zn}^{70}(?)\end{cases}$$

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A Nucleonic Cascade Theory and an Analysis of Extensive Air Showers

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A model of extremely high-energy nuclear collisions which describes Landau's model approximately or satisfies the constancy of transverse momenta of secondary particles as special cases is applied to the nucleonic cascade process in the atmosphere. The model contains some free parameters which specify the dynamical character of the nuclear collisions. The fact that the diffusion equation is solved analytically in a rather simple form on certain assumptions makes it easy to find the best values of the parameters by comparing with experimental information if available. Applications to extensive air showers are presented. The main results obtained are as follows. (i) If the energy spectrum of secondary charged pions at a high-energy collision is well approximated by a power law spectrum $E^{-n}dE$, $2 \geq n \geq 1.5$ is consistant with the lateral distribution of high-energy muons far underground. (ii) Any model in which the energy spectrum, the inelasticity, the ratio of charged-to neutral pions are independent of an incident energy in a wide range of 10^{12}~10^{17} eV seems unable to explain the altitude variation of the electron component. (iii) The lateral distribution of the electron component is insensitive to the energy spectrum of neutral pions at a high-energy collisions and also to the primary energy. Further discussions on our model and the solution of the diffusion equation are given.

§ 1. Introduction

The knowledge about the interactions between elementary particles in the extremely high-energy region is very important for the fundamental ground of field theory. Information on the high-energy interaction is being given by the study of jet showers and extensive air showers (EAS). These phenomena, though so complicated that we are frequently confused when interpreting experimental results, provide only clues to give us information on such interactions for the time being.

Jet showers and EAS have a rather complemental character to each other concerning the behavior of high-energy collisions. The former, though they are rather lower-energy phenomena than the latter, seems to give us more direct information on the angular distribution of secondary particles. And the latter will give us information on the energy flow carried by secondary particles, that is, the inelasticity of a high-energy collision.

In order to get the true situation concerning high-energy collisions, it is very

important to pick up those measurable quantities in EAS which are well influenced by the dynamical character of the high-energy collision. We have obtained some important clues which are essentially different from what we had before and force us to build a new kind of nucleonic cascade theory to analyse EAS. There are two important clues, one concerning the angular distribution of secondary particles and the other concerning the inelasticity at a high-energy collision. Two years ago, J. Nishimura[1] pointed out by analysing the data available at that time that transverse momenta of secondary particles P_T were nearly constant ($\sim\mu c$, μ the pion rest mass). Recently, this fact has been confirmed by the analysis of jet showers by the Japanese group[2] and the Bristol group.[3] That is, the transverse momentum of a secondary pion is almost independent of energies of an incident particle and of the secondary pion, its value being about 400 MeV/c, and its fluctuation is rather small. This result is the best established fact among the studies of extremely high-energy phenomena in recent years. From this result, if we assume one to one correspondence between angle and momentum of a secondary pion, we can obtain the energy distribution from the angular distribution, giving the result dE/E^n, n being nearly equal to 2, at least for secondary pions.[3),4)]

As another character of high-energy collisions, the importance of inelasticity has been pointed out. From an analysis of EAS carried by the Russian group,[5] it is asserted that the average value of the inelasticity observed in EAS is about 0.3. However, this value is not acceptable for the following reasons. Recently, S. Miyake[6] has pointed out that, if we took into account the fluctuations in depth at which the first collision of a primary cosmic ray particle with an air nucleus occurs, the interpretation of the results on EAS would be much modified. And he has suggested that the average value of the inelasticity may be larger than 0.3. Even if the fluctuations are negligible, the inelasticity obtained by the Russian group is not consistent with our previous calculation.[7] Experimental data concerning the attenuation length of the electron component, if the fluctuations are negligible, give a nearly constant value of about 200 gcm^{-2} over a wide range of primary energies, $10^{14}\sim10^{17}$ eV. In our previous paper, therefore, we suggested that in order to explain this fact it is necessary to introduce a model having increasing inelasticity or a stronger energy dependence of multiplicity with increasing incident energy. On the other hand, there is no plausible theoretical interpretation except Landau's hydrodynamical theory.[8] In his theory an incident particle after a collision can not be discriminated from secondary particles, and the elasticity will be defined as a ratio of the energy carried by a particular secondary particle with the largest energy to the energy of the incident particle. This "apparent" elasticity decreases slightly with increasing energy. Considering these theoretical and experimental situations concerning inelasticity we think it necessary to extend further the analysis both theoretically and experimentally.

Apart from the direct connection of the structure of EAS with the elementary interactions in the high-energy region, EAS themselves have a striking feature that

the lateral distribution of the electron component is independent of shower size and altitude at which observations are made, i. e. mountain elevation and sea level, and is in agreement with the Nishimura-Kamata function of age $\cong 1.3$.[9]

In our previous paper[7] (hereafter quoted as A) we analysed the structure of EAS with use of a rather crude model (hereafter quoted as the model I) and pointed out that the inelasticity strongly affects the altitude variation of the electron component. We also showed that the lateral distribution of the electron component is insensitive to a model concerning the high-energy collision and also to primary energy.[10]

In this paper we shall develop a somewhat improved nucleonic cascade theory and its application to EAS with use of another model (hereafter quoted as the model II) on the high-energy nuclear collision, taking into account the experimental situation as stated above, the constancy of the transverse momenta and the effect of inelasticity to EAS, and comparing with the result from the analysis based on the model I. The main characteristics of the model II are as follows:

(1) The model II contains some free parameters which characterize the dynamical property of the high-energy collision. Landau's model is included approximately by properly fixing the values of the free parameters which are introduced in the model.

(2) The requirement $P_T \cong$ const. is also satisfied by fixing the parameters properly.

(3) After a collision, most part of the incident energy is carried away by a few secondary particles and the residual energy is shared mainly with secondary particles having rather low energies.

The model II was derived through the analysis[11] * of Landau's model. His model was derived with use of the relativistic hydrodynamical description. Since it is not clear whether the model with varying values of the parameters can be derived or not on the basis of the same hydrodynamical description, the model II should be regarded as a phenomenological one including his model as a special one. The characteristic feature of his model was the introduction of those interactions between secondary particles, whose existence has been confirmed by the pion-pion interaction in a GeV region, and from which the characteristic (3) was derived. In the model II this feature plays in some sense a role of inelasticity, and in connection with it the "apparent" inelasticity is introduced, which will be described in detail in the next section.

The advantages of our nucleonic cascade theory based on the model II are as follows. The exact and rather simple solution of the diffusion equation can be derived without fixing the parameters. This fact makes it easy to investigate other unknown properties of the high-energy collision. That is, our parameters could also be applied to any other theories by changing their values, and our results would

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* Hereafter reference 11) is quoted as B.

be rather simply modified by estimating the effects of the change of the dynamical aspects of the high-energy collision in our model. A similar treatment was done by S. Olbert and R. Stora[12] using Fermi's[13] and Landau's[8] models. However, their work is rather mathematical and much more complicated than ours and it does not seem to be very useful for further applications. Further, our results may be interesting in the sense that they somewhat resemble the electron-photon cascade theory and the cascade function is so simple that their change with the value of the primary energy, inelasticity and/or other parameters can be read from the graphs by changing the units of the scales of the ordinate and abscissa by some simple factors. We hope that our results will give a useful tool for the development of cosmic ray physics as well as a fundamental key to the future theory of elementary particles.

In the next section, the features of the model II will be given. First, the assumptions introduced in the model will be explained and then the angular and number distributions of secondary particles in the center of momentum system will be given. Further, a comment on the "apparent" inelasiticity in the model II will be made. Finally the energy spectrum of secondary particles in the laboratory system will be derived. In § 3 the recurrence formula of the diffusion equation will be derived and general expressions of nucleonic, electron and muon components will be given. The analytic solution of the diffusion equation will be derived in §4. In § 5 some general properties of our nucleonic cascade theory will be described. In § 6 the results of numerical calculations will be given by fixing the parameters so as to satisfy the constancy of the transverse momenta, and comparison of our results with the model I and experimental data will be discussed. Finally, our main results will be summarized in § 7.

§ 2. The features of the model

2·1. *Assumptions in our model*

We introduce the following assumptions for the model of extremely high-energy collisions :

(1) The collision cross section is independent of the energy of a colliding particle and also independent of the kind of a colliding particle, i. e. a nucleon or a charged pion.

(2) The number of secondary particles produced by a nucleon (or a pion) of energy E_0 in the lab. system colliding with a nucleon is given by

$$n=2(bE_0)^d, \qquad (2\cdot1)^*$$

where b and d are constant parameters.

(3) Nucleons and charged pions are not distinguished from each other. The

* In the case of nucleon-(or pion) nucleus collisions, some modification is necessary. But as far as pions emitted to the forward direction in the center of momentum system are concerned, the model does not depend on whether a target is a nucleon or a nucleus. On this point see the next subsection.

only difference between them is the decay effect of charged pions which may be neglected for energies above 100 GeV. The decay effect will be estimated in Appendix II in an improved approximation.

(4) The fraction ν of secondary particles are nucleons and charged pions and the rest of secondaries are neutral pions. The production of nucleon pairs and strange particles are disregarded. ν is a constant parameter.

(5)* The angular distributions of the energy and the number of secondary particles are given by power functions of an angle between the directions of the incident particle and a secondary particle in the center of momentum system (C. M. S.). The values of the powers do not depend on the incident energy.

The assumptions (1)～(4) are similar to those in the model I. The essential difference between the models I and II is based on the assumption (5) and its details will be described in the next subsection.

2·2. *The angular and energy distributions of secondary particles in the center of momentum system*

We assume the angular distributions of the energy and the number of secondary particles, when a nucleon (or a pion) collides with a *nucleon* in the C. M. S., as follows:

$$E^*=\mathcal{A}\theta^{*-\alpha}, \quad dn/d\theta^*=\mathcal{B}\theta^{*-\beta}. \tag{2·2}$$

Here θ^* is an angle between the directions of an incident and a secondary particles in the C. M. S.. dn is the number of pions emitted at an angle between θ^* and $\theta^*+d\theta^*$, and E^* is the energy of the pion emitted at the angle θ^*. $\mathcal{A}$ and $\mathcal{B}$ are normalization constants determined by requirements:

$$\int_0^{\pi/2}\frac{dn}{d\theta^*}d\theta^*=\frac{n}{2}, \qquad \int_{\pi/2}^{\pi}\frac{dn}{d\theta^*}d\theta^*=\frac{n}{2},$$

$$\int_{\theta^*_{\min}}^{\pi/2}E^*\frac{dn}{d\theta^*}d\theta^*=E_0^*, \qquad \int_{\pi/2}^{\pi-\theta^*_{\min}}E^*\frac{dn}{d\theta^*}d\theta^*=E_0^*, \tag{2·3}$$

where E_0^* is an incident energy in the C. M. S..

When a nucleon (or a pion) collides with a *nucleus*, eq. (2·3) must be modified as follows:

$$\int_0^{\pi/2}\frac{dn}{d\theta^*}d\theta^*=n_f, \qquad \int_{\pi/2}^{\pi}\frac{dn}{d\theta^*}d\theta^*=n_b, \tag{2·4}$$

$$\int_{\theta^*_{\min}}^{\pi/2}E^*\frac{dn}{d\theta^*}d\theta^*=E_{0f}^*, \qquad \int_{\pi/2}^{\pi-\theta^*_{\min}}E^*\frac{dn}{d\theta^*}d\theta^*=E_{0b}^*,$$

and parameters α and β in eq. (2·2) may now be modified respectively to different

* See the footnote on the previous page.

ones, α' and β'. Here n_f, E_{0f}^* and n_b, E_{0b}^* are the numbers and the total energies of secondary particles emitted in the forward and backward directions respectively in the C. M. S. for an incident particle and a target nucleon in the target nucleus. According to Landau's theory[9] or the jet model,[14] n_f and E_{0f}^* are not too much different respectively from $n/2$ and E_0^*, and n_b and E_{0b}^* are approximately given respectively by $\bar{\zeta}\cdot n/2$ and $\bar{\zeta}\cdot E_0^*$ where $\bar{\zeta}$ is the number of nucleons in the jet tunnel of the target nucleus. The parameters α' and β' in the collision with a nucleus are different for the forward and backward directions, but as far as we are concerned with the forward direction, α' and β' may be the same as those of the collision with a nucleon. Thus eq. (2·3) is modified only for the backward direction. We assume that the parameters α and β, which are closely connected to the model, are constants not depending on the energy of an incident particle.

$\theta_{\min}^*$ is the minimum angle of secondary pions which is necessary to avoid the divergence of the total energy. In Landau's theory, for example, α, β and $\theta_{\min}^*$ are well approximated by the following expressions.

$$\alpha=1-\frac{1}{3\sqrt{3}}, \qquad \beta=\frac{1}{\sqrt{3}}, \quad \theta_{\min}^* \propto E_0^{*-\sqrt{3}/2}. \tag{2·5}$$

The comparison of eq. (2·2) with Landau's theory is made in Figs. 1a and 1b. If we require $P_T=\text{const.}\simeq\mu c$, α and $\theta_{\min}^*$ are given by

$$\alpha=1, \qquad \theta_{\min}^* \propto (E_0^*/n)^{-1/\beta}. \tag{2·6}$$

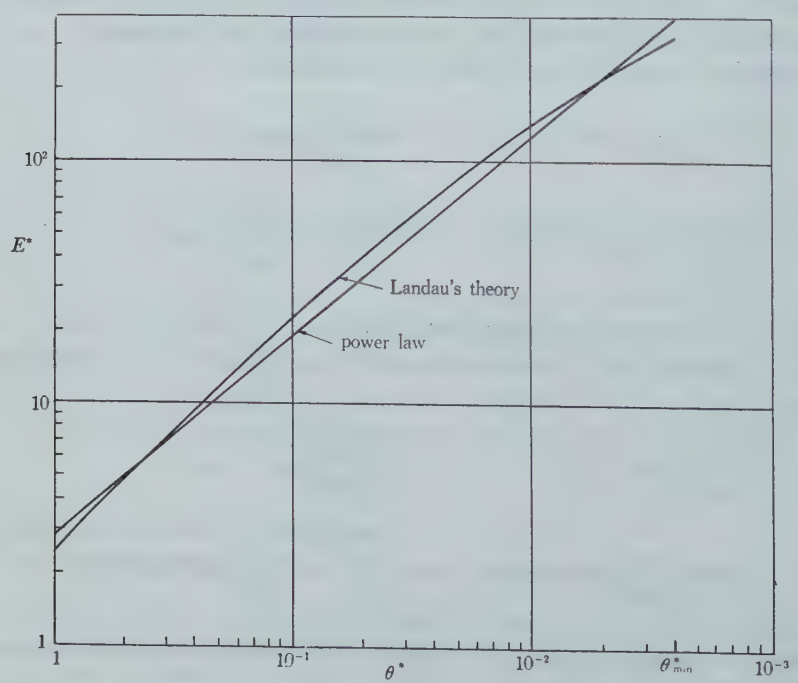

Fig. 1a. Angular dependence of an emitted pion energy for incident energy $E_0^*=10^3$ GeV.

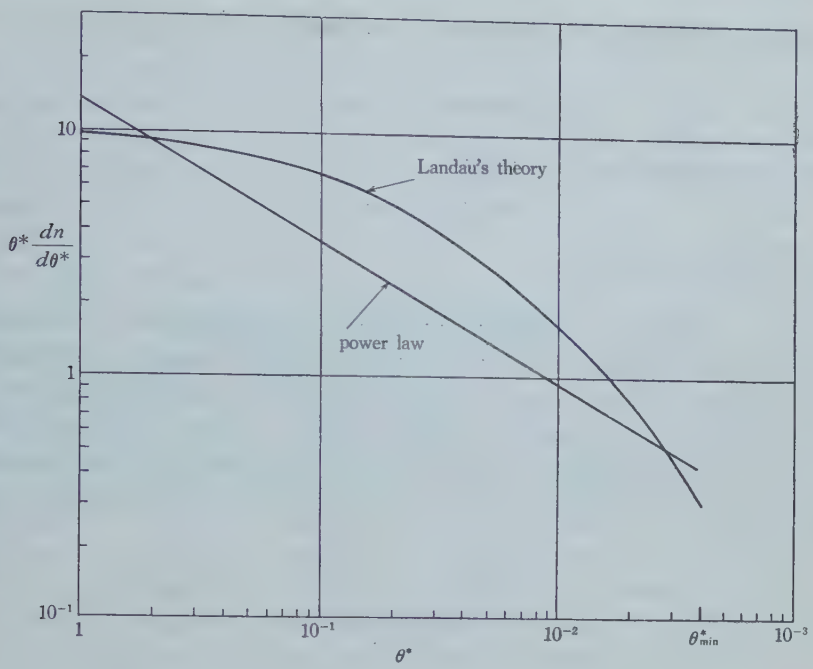

Fig. 1b. Angular dependence of pion number for $E_0^*=10^3$ GeV.

Hereafter we do not fix α, β and θ^*_{min}, but regard them as free parameters. Only restrictions for α and β are

$$\alpha+\beta>1, \qquad 1>\beta>0. \tag{2·7}$$

Eq. (2·7) means that most of particles belong to a low-energy region, and great part of the energy is carried by a few particles which are emitted in small angles ($\theta^*\sim\theta^*_{min}$). The maximum value of the energy E^*_{max} in this distribution is from eqs. (2·2) and (2·3) given by

$$E^*_{max}=\frac{E_0^*}{n}-\frac{\alpha+\beta-1}{1-\beta}\theta^{*\beta-1}_{min}, \tag{2·8}*$$

and it is proportional to $E_0^{*\sqrt{3}/2}$ for Landau's theory. E^*_{max} is rather large and comparable with an incident energy E_0^*, and their ratio varies slowly with energy. Hence it may be more convenient to introduce a parameter "a" which is the ratio of E^*_{max} to E_0^*, i. e.,

$$E^*_{max}=aE_0^*, \qquad a\leqq 1. \tag{2·9}$$

E^*_{max} does not necessarily mean the most probable value of the largest energy carried by one secondary particle; this is somewhat smaller than E^*_{max}. Hereafter we use α, β and a instead of α, β and θ^*_{min} as a set of parameters.

* To derive eq. (2·8), the upper limit of the integral (2·3) is put equal to 1.

The energy distribution of secondary pions in the C. M. S. is from eq. (2·2) given by

$$dn/dE^*=C(1/E^*)^{1+\delta}, \tag{2·10}$$

where C is a normalization constant and

$$\delta=(1-\beta)/\alpha. \tag{2·11}$$

$\delta=0.52$ for Landau's theory.

2·3. *Inelasticitiy*

It may be convenient to estimate the value of inelasticity in our model for understanding of the nucleonic cascade. The most probable energy of the pion that carries away the largest energy among secondary pions is in Landau's theory, estimated in the paper B, and is of the order of several tens of percent of the total energy. In our model we may regard it as the "apparent" elasticity and can estimate its value from eq. (2·10). The most probable energy of the pion having the largest energy is given by

$$\Delta E^*=\int_{E'^*}^{E^*_{\max}} E^*\frac{dn}{dE^*}dE^*,$$

where E'^* is determined from the requirement

$$1=\int_{E'^*}^{E^*_{\max}}\frac{dn}{dE^*}dE^*.$$

A simple calculation shows

$$\frac{\Delta E^*}{E_0^*}=1-\left(\frac{1-\delta}{a\delta+1-\delta}\right)^{(1-\delta)/\delta},$$

and we may define the "apparent" inelasticity in our model as

$$\eta=\left(\frac{1-\delta}{a\delta+1-\delta}\right)^{(1-\delta)/\delta}. \tag{2·12}$$

In Table 1, the values of η for several different values of δ and a are listed. In Landau's theory η is approximately 0.5. Even for $a=1$, ΔE^* is smaller than E_0^* as it must be. Hereafter we shall in most cases put $a=1$ for numerical calcula-

Table I. The values of η for several different values of δ and a

a \ δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1	0.39	0.41	0.44	0.46	0.50	0.54	0.60	0·67	0.77	1.00
1/2	0.61	0.63	0.64	0.65	0.67	0.69	0.72	0.76	0.85	1.00

ions. In this case $\delta=1/2$ gives us the apparent inelasticity η of 0.5. For-unately, our diffusion equation could be solved exactly for $a=1$ and the cascade unction becomes so simple that we could easily observe consequences of changing alues of the parameters. As the solution for $a\neq 1$ is somewhat complicated, we vill give an approximate treatment in Appendix I for the sake of practical use.

Attention must be paid to our definition of the apparent inelasticity. In our nodel there is some possibility that a neutral pion has the largest energy, since a econdary particle having the largest energy is not always a survival of the incident article. If nucleons play an essential role compared with pions in the nucleonic ascade process, that is, nucleons carry away most of the energy in nucleon-nucleon ollisions, the fraction ν should be taken slightly larger than 2/3 compared with he case that pions play an essential role. That is, by taking the parameter ν roperly the probability that a nucleon or a charged pion has the largest energy ecomes nearly equal to unity. Since, as far as the particle having the large energy s concerned, the number of such particles is small and the decay effect of such articles may be negligible, there is no difference between nucleons and charged ions. Thus we hope that even if nucleons play an important role for the inelas-icity, the proper choice of the parameter ν will result in a satisfactory approxima-ion. In this approximation we put the energy spectrum including dn^c/dE^* both nucleons and charged pions as

$$\frac{dn^c}{dE^*}=\nu\frac{dn}{dE^*}. \qquad (2\cdot 13)$$

If the inelasticity were small, however, our device of the inelasticity would be far rom the real situation. In eq. (2·13) we took the same spectrum for charged ions (including nucleons) and neutral pions. This is not necessary for the follow-ing calculation, but is used only for simplicity.

2·4. *The energy distribution in the laboratory system*

We use the notation in the laboratory system (L. S.) as follows,

$$E_0^*\to E_0, \qquad E^*\to E,$$
$$E^*_{\max}\ \text{(forward)}\to E_M, \qquad E^*_{\min}\to E_1,$$
$$E^*_{\max}\ \text{(backward)}\to E_2.$$

The relations between the quantities in the C. M. S. and those in the L. S. are listed in Table 2. These values in Landau's theory are approximately given by

$$E_M\sim E_0^{(2+\sqrt{3})/4}=E_0^{0.933}\sim E_0,$$
$$E_1\sim E_0^{7/12},$$
$$E_2\sim E_0^{(2-\sqrt{3})/4}=E_0^{0.067}\sim\text{constant}.$$

The energy distribution of secondary pions in the L. S. can be calculated from eqs. (2·2), (2·3), (2·8) and Table 2, and is approximately given by

Table 2. Relations between the energies in the C. M. S. and in the L. S. in nucleon rest mass unit

$$E_0=2E_0^{*2}$$

$$E=E_0^*E^*(1+\cos\theta^*)\sim\begin{cases}2E_0^*E^* & \text{for forward}\\ E_0^*E^*\dfrac{\theta'^{*2}}{2} & \text{for backward } (\theta'^*=\pi-\theta^*)\end{cases}$$

$$E_M=2E_0^*E^*_{\max}=aE_0$$

$$E_1=E_0^*E^*_{\min}=E_M\left(\frac{E_0}{E_M}\left(\frac{1-\delta}{\delta}\right)\frac{1}{n/2}\right)^{1/\delta},\quad \delta=\frac{1-\beta}{\alpha}$$

$$E_2=E_0^*E^*_{\max}\frac{\theta^{*2}_{\min}}{2}\sim E_M\frac{1}{4}\left(\frac{E_0}{E_M}\left(\frac{1-\delta}{\delta}\right)\frac{1}{n/2}\right)^{2/(1-\beta)}$$

$$dn=\begin{cases}(E_0/E_M)^{1-\delta}(1-\delta)/\delta\cdot d(E_0/E)^{\delta} & \text{for } E_M>E>E_1,\\ \tilde{\xi}(E_0/E_1)^{\kappa\delta}\cdot d(E/E_0)^{\kappa\delta} & \text{for } E_1>E>E_2.\end{cases}\qquad(2\cdot14)$$

The first one of eq. (2·14) corresponds to the distribution of pions emitted in the forward direction and the second corresponds to that in the backward direction in the C. M. S.. Here

$$\kappa=\frac{\alpha}{2-\alpha}=\begin{cases}0.677 & \text{for Landau's theory,}\\ 1 & \text{for } P_T=\text{const.,}\end{cases}\qquad(2\cdot15)$$

where we assume the symmetrical distribution between the forward and backward directions in a nucleon- (or pion-) nucleon collision, i. e., $\tilde{\xi}=1$. If it is a nucleon-nucleus collision, we use the quantities α' and δ' which are related to the power indices in the backward direction in eq. (2·14) as stated in § 2·2 and $\tilde{\xi}$ may be, in Landau's theory, proportional to the number of nucleons in a jet tunnel of the nucleus. The distribution in the forward direction in a nucleon-nucleus collision does not change in Landau's theory (see the paper B). Eq. (2·14) is the fundamental assumption, which we constructed from Landau's theory, for the calculation of nucleonic cascade processes. Any model which resembles or is well approximated by eq. (2·14) by changing parameters E_M, E_1, E_2, δ, κ and $\tilde{\xi}$ properly can be treated in a framework of our cascade theory.

§ 3. The diffusion equation and the recurrence formula

The diffusion equation of the nucleonic cascade is if the decay of charged pions is neglected given by

$$\frac{\partial N(E,x)}{\partial x}=-N(E,x)+\int dE'\frac{dn^c(E',E)}{dE}N(E',x),\qquad(3\cdot1)$$

where $N(E,x)\,dE$ is the differential number of the nucleonic component having an energy between E and $E+dE$ and dn^c/dE is given by eq. (2·13) with eq. (2·14).

x is the depth from the top of the atmosphere measured in the nuclear collision mean free path.

Expanding $N(E, x)$ as

$$N(E, x)=\sum_{l=0}^{\infty} e^{-x}\frac{x^l}{l!}f_l(E), \tag{3·2}$$

and inserting it into eq. (3·1), we can derive the recurrence formula for each generation as follows.

$$f_l(E)=\int dE'\frac{dn^c(E', E)}{dE}f_{l-1}(E'). \tag{3·3}$$

Eq. (3·2) gives the most probable number. Fluctuations in the depth, at which the collision of the nucleonic component of each generation occurs, are neglected. As there are many particles after the 2nd generation, fluctuations will be smeared out and be small. Only the important fluctuation is due to the depth at which the first collision occurs. This fluctuation, as pointed out by S. Miyake,[6] may change the aspect of EAS at the elevation of observation even if we observe EAS initiated by primary particles with the same energies. In this case, measuring the depth from the point at which the first collision occurs, $x'(x'=x-x_1$, x_1 is the depth of the first collision measured from the top of the atmosphere), the differential number of the nucleonic component is given by

$$N(E, x')=\sum_{l=0}^{\infty} e^{-x'}\frac{x'^l}{l!}f_{l+1}(E), \tag{3·4}$$

instead of eq. (3·2). The number of neutral pions at a depth between x and $x+dx$ and with an energy between E and $E+dE$, is given by

$$N_{\pi^0}(E, x)=\frac{1-\nu}{\nu}\sum_{l=0}^{\infty} e^{-x}\frac{x^l}{l!}f_{l+1}(E). \tag{3·5}$$

While the number of neutral pions as a function of $x'(=x-x_1)$ is

$$N_{\pi^0}(E, x')=\frac{1-\nu}{\nu}e^{-x'}\left\{\sum_{l=0}^{\infty}\frac{x'^l}{l!}f_{l+2}(E)+\delta(x')f_1(E)\right\}, \tag{3·6}$$

where $\delta(x')$ is the delta function. The number of muons at depth x is approximately given by

$$N_\mu(E, x)=\frac{B}{E}\int_0^x\frac{dx}{x}e^{-x}\sum_{l=0}^{\infty}\frac{x^{l+1}}{(l+1)!}f_{l+1}(E), \tag{3·7}$$

and the number of muons as a function of x' is given by

$$N_\mu(E, x)=\frac{B}{E}\int_0^{x'}\frac{dx}{x+x_1}e^{-x}\sum_{l=1}^{\infty}\frac{x^l}{(l+1)!}f_{l+1}(E). \tag{3·8}$$

where B is the decay constant of pions.

In order to solve eq. (3·3) using eqs. (2·1), (2·13) and (2·14), it is convenient to introduce a new energy unit. In this unit we represent the energy with the notation ε.

$$E=A\varepsilon, \quad A=\frac{1}{b}\left(\frac{1}{a^{1-\delta}}\left(\frac{1-\delta}{\delta}\right)\right)^{1/d}. \tag{3·9}$$

The factor A is nearly equal to unity, hence this unit is of the same order as the nucleon rest mass unit or GeV unit. On this scale the corresponding relations of E_M, E_1 and E_2 in Table 2 turn out to be

$$\begin{aligned} &E_M \to \varepsilon_M = a\varepsilon_0, \\ &E_1 \to \varepsilon_1 = \varepsilon_0^{1/k}, \\ &E_2 \to \varepsilon_2 = \text{const.} = 0 \quad \text{(for simplicity)}, \end{aligned} \tag{3·10}$$

where

$$1/k = 1 - d/\delta \simeq 1/2.$$

If we put

$$f_l(E) \equiv f_l(\varepsilon) = \left(\left(\frac{1}{a}\right)^{1-\delta} \nu(1-\delta)\right)^l \left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta} \frac{1}{\varepsilon_0} h_l(\varepsilon), \tag{3·11}$$

eq. (3·3) can be replaced by the relation between $h_l(\varepsilon)$ and $h_{l-1}(\varepsilon)$, i. e.,

$$h_l(\varepsilon) = \int_{\varepsilon/a}^{\varepsilon^k} \frac{d\varepsilon'}{\varepsilon'} h_{l-1}(\varepsilon') + \xi_K \int_{\varepsilon^k}^{\infty} \frac{d\varepsilon'}{\varepsilon'} \left(\frac{\varepsilon^k}{\varepsilon'}\right)^\gamma h_{l-1}(\varepsilon'), \tag{3·12}$$

where

$$\gamma = 2(\delta - d)/(2-\alpha).$$

The first term in eq. (3·12) is a contribution from forward secondary particles and the second term is a contribution from backward respectively in the l-th collision in its C. M. S..

§ 4. The solution of the diffusion equation for a single particle incident in the case of $a=1$

The boundary condition for the case in which a single particle with energy E_0 (or ε_0 in our unit) impinges at the top of the atmosphere is

$$f_0(\varepsilon) = \delta(\varepsilon - \varepsilon_0) \quad \text{or} \quad h_0(\varepsilon) = \varepsilon_0 \delta(\varepsilon - \varepsilon_0). \tag{4·1}$$

The solution of eq. (3·12) under this boundary condition can be divided into different types corresponding to different energy regions. Putting

$$h_l(\varepsilon)=\begin{cases} h_l^0(\varepsilon) & \text{for } \varepsilon_0>\varepsilon>\varepsilon_0' \quad \text{(Region I)},\\ p_l(\varepsilon) & \text{for } \varepsilon_0'>\varepsilon>\varepsilon_0'' \quad \text{(Region II)},\\ q_l(\varepsilon) & \text{for } \varepsilon_0''>\varepsilon>\varepsilon_0''' \quad \text{(Region III)}, \end{cases} \tag{4·2}$$

where

$$\varepsilon_0'=\varepsilon_0^{1/k}, \quad \varepsilon_0''=\varepsilon_0^{1/k^2}, \quad \varepsilon_0'''=\varepsilon_0^{1/k^3}.$$

we divide eq. (3·12) correspondingly into the following three equations:

$$h_l^0(\varepsilon)=\int_{\varepsilon}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}h_{l-1}^0(\varepsilon'),$$

$$p_l(\varepsilon)=\int_{\varepsilon}^{\varepsilon_0'}\frac{d\varepsilon'}{\varepsilon'}p_{l-1}(\varepsilon')+\int_{\varepsilon_0'}^{\varepsilon^k}\frac{d\varepsilon'}{\varepsilon'}h_{l-1}^0(\varepsilon')+\kappa\int_{\varepsilon^k}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)^{\tau}h_{l-1}^0(\varepsilon'),$$

$$q_l(\varepsilon)=\int_{\varepsilon}^{\varepsilon_0''}\frac{d\varepsilon'}{\varepsilon'}q_{l-1}(\varepsilon')+\int_{\varepsilon_0''}^{\varepsilon^k}\frac{d\varepsilon'}{\varepsilon'}p_{l-1}(\varepsilon')+\kappa\int_{\varepsilon^k}^{\varepsilon_0'}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)p_{l-1}(\varepsilon')$$

$$+\kappa\int_{\varepsilon_0'}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)^{\tau}h_{l-1}^0(\varepsilon'). \tag{4·3}$$

Our model is valid only for the energy region where the jet model is valid, i. e. $E\gtrsim 10^{12}$ eV for air nuclei and the main contributions to EAS come from such a high-energy region (it will be shown that this is actually so for the electron component). Thus it is sufficient only to give $h_l^0(\varepsilon)$, $p_l(\varepsilon)$ and $q_l(\varepsilon)$ in most cases.

The solutions for eq. (4·3) with the boundary condition (4·1) are

$$h_0^0(\varepsilon)=\varepsilon_0\delta(\varepsilon-\varepsilon_0), \quad p_0(\varepsilon)=q_0(\varepsilon)=0,$$

$$h_1^0(\varepsilon)=1, \qquad p_1(\varepsilon)=q_1(\varepsilon)=\kappa\left(\frac{\varepsilon^k}{\varepsilon_0}\right)^{\tau},$$

and for $l\geqq 2$

$$h_l^0(\varepsilon)=\frac{1}{(l-1)!}\left(\ln\frac{\varepsilon_0}{\varepsilon}\right)^{l-1},$$

$$p_l(\varepsilon)=h_l^0(\varepsilon)-\frac{1-\frac{1}{k'}}{1-\frac{1}{k}}\left\{\frac{\left(\ln\frac{\varepsilon_0}{\varepsilon^k}\right)^{l-1}}{(l-1)!}-\kappa F_{l-2}(\varepsilon_0,\ \varepsilon^k)\right\},$$

$$q_l(\mathcal{E}) = p_l(\mathcal{E}) + \frac{1-\frac{1}{k^l}}{1-\frac{1}{k}} \frac{\frac{1}{k}-\frac{1}{k^l}}{1-\frac{1}{k^2}} \left\{ \frac{1}{(l-1)!} \left(\ln \frac{\mathcal{E}_0}{\mathcal{E}^{k^2}} \right)^{l-1} \right.$$

$$\left. -k^{l-1} \left(\kappa + \kappa^2 \frac{1}{k-1} \right) F_{l-2}(\mathcal{E}_0', \mathcal{E}^k) - \left(\kappa - \kappa^2 \frac{k}{k-1} \right) F_{l-2}(\mathcal{E}_0, \mathcal{E}^{k^2}) \right\}, \quad (4\cdot4)$$

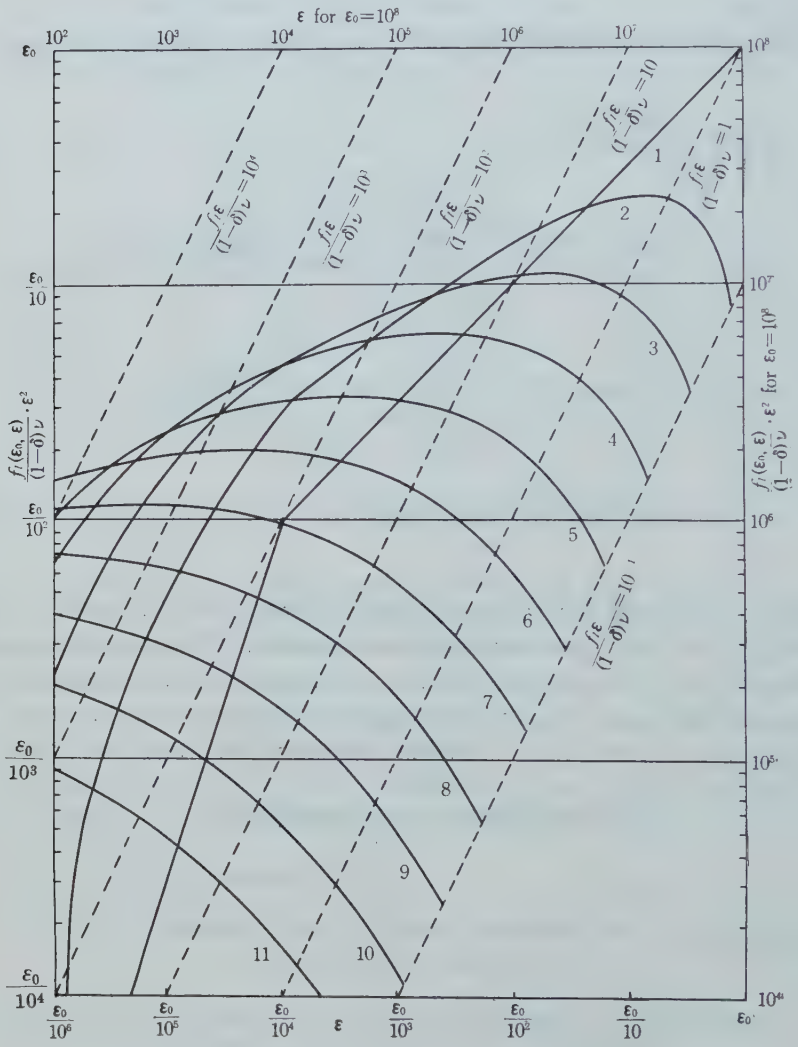

Fig. 2. The numerical curves of $f_l(\varepsilon)\varepsilon^2(\nu(1-\delta))^{-1}$ for various generations as a function of ε with use of eq. (4·4).

These curves have the same values of $f_l(\varepsilon)\varepsilon^2(\nu(1-\delta))^{-1}$, where $\varepsilon f_l(\varepsilon)$ is an energy carried by nuclear active particles of l-th generation, at intersecting points on them with the broken line representing $f_l(\varepsilon)\varepsilon(\nu(1-\delta))^{-1}$=const. The scales of the right-hand ordinate and the upper abscissa are represented for incident energy $\varepsilon_0=10^8$.

where

$$F_{l-2}(\varepsilon_0,\ \varepsilon^k)=\int_{\varepsilon^k}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)^{\gamma}\frac{\left(\ln\frac{\varepsilon_0}{\varepsilon'}\right)^{l-2}}{(l-2)!}. \tag{4·5}$$

The numerical values of $f_l(\varepsilon)\varepsilon^2(\nu(1-\delta))^{-1}$ as a function of ε with use of eq. (4·4) are given by a set of curves for various generations in Fig. 2. On these curves the loci of $f_l(\varepsilon)\varepsilon(\nu(1-\delta))^{-1}=$const. form a set of straight lines represented by the broken lines.

The solutions (4·4) are somewhat complicated but eq. (3·4), i. e. the solution of the case in which the nucleonic cascade process starts at the depth x_1 can be reduced to a rather simple form. Corresponding to eq. (4·4), we divide $N(E,\ x')$ in eq. (3·4) to three forms, i. e.

$$\begin{aligned}&N(E,\ x')\\ &\equiv N(\varepsilon,\ x)=\nu(1-\delta)\,e^{-x'}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta}\frac{1}{\varepsilon_0}\begin{cases}\mathcal{H}(\varepsilon,\ x') & \text{for } \varepsilon_0>\varepsilon>\varepsilon_0',\\ \mathcal{P}(\varepsilon,\ x') & \text{for } \varepsilon_0'>\varepsilon>\varepsilon_0'',\\ \mathcal{Q}(\varepsilon,\ x') & \text{for } \varepsilon_0''>\varepsilon>\varepsilon_0''',\end{cases}\end{aligned} \tag{4·6}$$

where

$$\begin{aligned}\mathcal{H}(\varepsilon,\ x')&=\sum_{l=0}^{\infty}\frac{x'^l}{l!}((1-\delta)\nu)^l h^0_{l+1}(\varepsilon),\\ \mathcal{P}(\varepsilon,\ x')&=\sum_{l=0}^{\infty}\frac{x'^l}{l!}((1-\delta)\nu)^l p_{l+1}(\varepsilon),\\ \mathcal{Q}(\varepsilon,\ x')&=\sum_{l=0}^{\infty}\frac{x'^l}{l!}((1-\delta)\nu)^l q_{l+1}(\varepsilon).\end{aligned} \tag{4·7}$$

Using the following notations,

$$z=4(1-\delta)\nu x',$$

$$I_0(\sqrt{z})=\sum_{l=0}^{\infty}\frac{1}{(l!)^2}\left(\frac{z}{4}\right)^l \quad (I_0:\ \text{Imaginary Bessel function of zeroth order})$$

$$J(\varepsilon,\ z)=\frac{1}{k-1}\left\{kI_0\left(\sqrt{z\ln\frac{\varepsilon_0}{\varepsilon^k}}\right)-I_0\left(\sqrt{z\ln\frac{\varepsilon_0'}{\varepsilon}}\right)\right\}, \tag{4·8}$$

$$\begin{aligned}G(\varepsilon,\ z)=&\frac{k}{k-1}\frac{1}{k^2-1}\left\{kI_0\left(\sqrt{z\ln\frac{\varepsilon_0}{\varepsilon^{k^2}}}\right)-(k+1)\,I_0\left(\sqrt{z\ln\frac{\varepsilon_0}{\varepsilon^k}}\right)\right.\\ &\left.+I_0\left(\sqrt{z\ln\frac{\varepsilon_0''}{\varepsilon}}\right)\right\},\end{aligned}$$

we get

$$\mathcal{H}(\varepsilon,\ z)=I_0\left(\sqrt{z\ln\frac{\varepsilon_0}{\varepsilon}}\right),$$

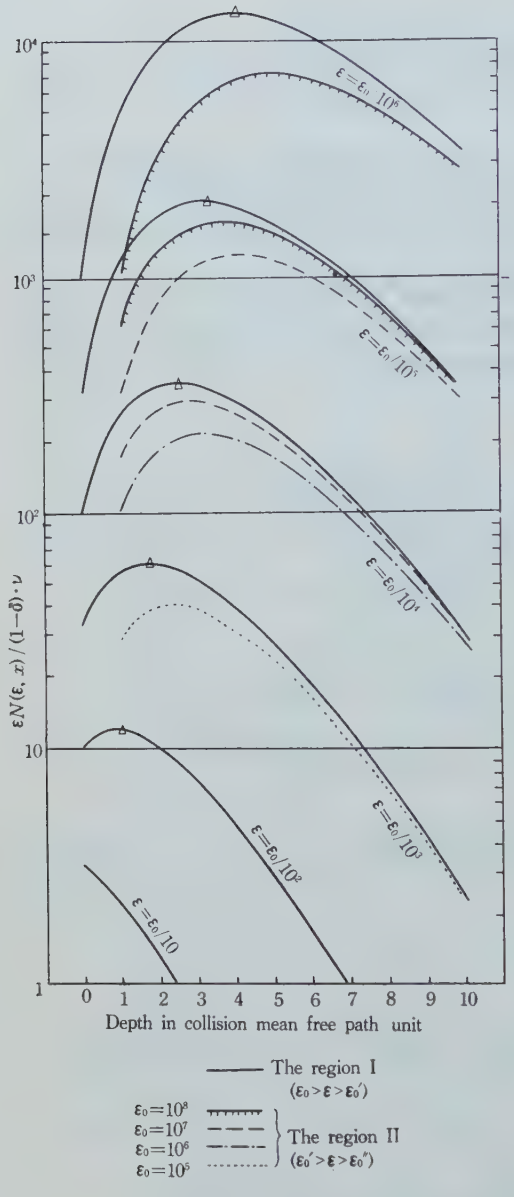

Fig. 3. Number of nuclear particles of energy ε as a function of depth.

The depth is measured from the depth at which the first collision occurs.

$\delta=1/2$, $k=2$, $\nu=2/3$.

Δ: the optimum point derived from eq. (5•5)

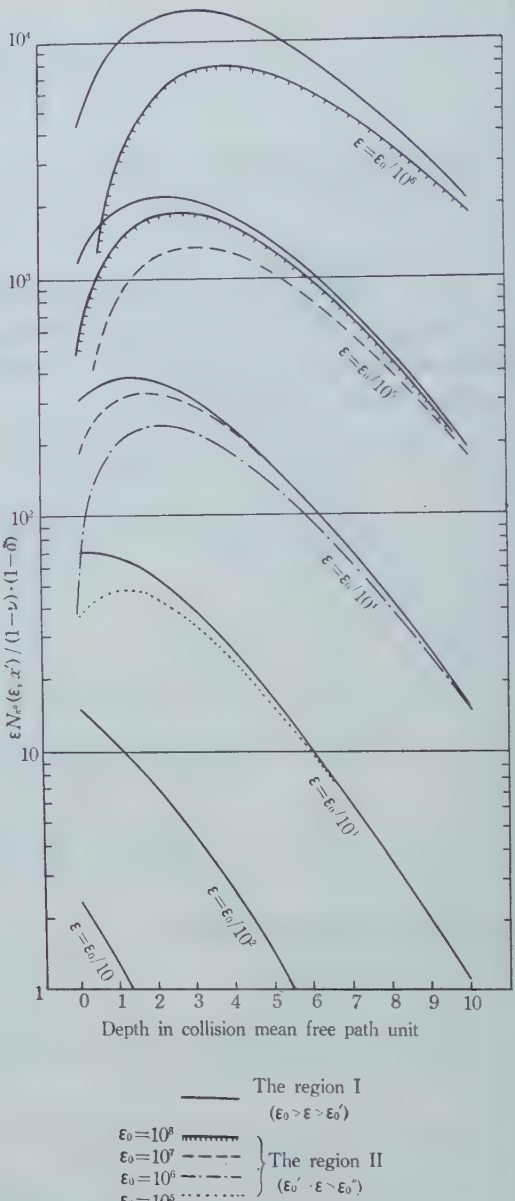

Fig. 4. Number of neutral pions ot energy ε as a function of depth. The ordinate represents only the first term of the right hand of eq. (3•6). The depth is measured from the depth at which the first collision occurs.

$\delta=1/2$, $k=2$, $\nu=2/3$.

$$\mathcal{P}(\varepsilon,\ z)=\mathcal{K}(\varepsilon,\ z)-J(\varepsilon,\ z)+\kappa\int_{\varepsilon^k}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)^\tau\varepsilon_0\frac{\partial}{\partial\varepsilon_0}J(\varepsilon',\ z)+\kappa\left(\frac{\varepsilon^k}{\varepsilon_0}\right)^\tau,$$

$$\mathcal{Q}(\varepsilon,\ z)=\mathcal{P}(\varepsilon,\ z)+G(\varepsilon,\ z)$$

$$-\kappa\left\{\int_{\varepsilon^{k^2}}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)^\tau\varepsilon_0\frac{\partial}{\partial\varepsilon_0}G(\varepsilon'^{1/k^2},\ z)\left(1-\frac{\kappa}{k-1}\right)+k\int_{\varepsilon^k}^{\varepsilon_0'}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)^\tau\right.$$

$$\left.\times\varepsilon_0\frac{\partial}{\partial\varepsilon_0}G(\varepsilon'^{1/k},\ z)\left(1+\frac{\kappa}{k-1}\right)\right\}. \qquad (4\cdot9)$$

Thus we have the solution of the diffusion equation, using the imaginary Bessel function and its single integration. The functions in eqs. (3·2) and (3·5) are given in Appendix III, for which we get solutions similar to eq. (4·9) only by changing the imaginary Bessel function of zeroth order to that of 1st order. Asymptotically they are quite the same as eq. (4·9) except simple factors for large values of the argument of the imaginary Bessel function.

The numerical values of eq. (4·6) with use of eqs. (4·7) and (4·9) are shown for various pion energies in Fig. 3. For $\varepsilon_0=10^7$, for example, the curves of $\varepsilon=10^6$, 10^5 and 10^4 belonging to the region I are represented by full curves and those of $\varepsilon=10^3$ and 10^2 belonging to the region II represented by broken curves. The similar curves for eq. (3·6) are also presented in Fig. 4 where the first term in the right-hand side of eq. (3·6) is taken as the ordinate. It will be interesting to note that in the region I the curves depend only on the ratio $\varepsilon_0/\varepsilon$ and do not depend separately on ε_0 and ε. Moreover, the effect of δ and ν appears only as a function of $(1-\delta)\nu$, which may be roughly interpreted as inelasticity in the sense stated in § 2·3. The boundary of the regions I and II or II and III depends on the value of k only. For the region II and III the contribution of backward emitted particles which contains the factor κ or κ^2 is of the order of several per cent to that of forward emitted particles and tends to zero with increasing depth where low-energy particles become dominant. Thus the forms of the solutions for the regions II and III are not so different in the first approximation from that of the region I. From Figs. 3 and 4, we can use approximately the solution I all over the region within errors of 10 per cent.

§ 5. Semiquantitative properties of the solution

5·1 *Semiquantitative properties of the nucleonic cascade process*

The nuclear cascade function given by eq. (4·6) can be well approximated with use of the asymptotic formula of the imaginary Bessel function

$$I_0(\tau)\fallingdotseq e^\tau/\sqrt{2\pi\tau} \qquad (5\cdot1)$$

with an error of $1/8\tau$. With this approximation the asymptotic formula of eq. (4·6) is given by

$$N(\varepsilon,\ x')\,d\varepsilon=\frac{(1-\delta)\nu}{\sqrt{4\pi\sqrt{1-\delta)\nu x'y}}}\exp\{-x'+\delta y+2\sqrt{(1-\delta)\nu x'y}\}\,dy, \qquad (5\cdot2)$$

where $y=\ln(\varepsilon_0/\varepsilon)$. When $x'=1$, $\delta=1/2$ and $\nu=2/3$, errors are of 7 per cent for $\varepsilon_0/\varepsilon=10$ and 1 per cent for $\varepsilon_0/\varepsilon=10^4$ respectively. Similar formulas for the regions II and III can be obtained, but since differences between them and eq. (5·2) are not significant for application eq. (5·2) will be used over all energy regions with some caution.

The coefficient of the exponential in eq. (5·2) is closely related to a generation which contributes mainly to the number of nuclear particles. The generation l contributed mainly by nuclear particles having energy $y(=\ln\varepsilon_0/\varepsilon)$ is from eq. (4·7) given by the condition

$$\frac{\partial}{\partial l}\left[\frac{x''}{l!}f_{l+1}(\varepsilon)\right]_{l=\bar{l}}=0 \qquad (5\cdot3)$$

and $\bar{l}$ is approximately given by

$$\bar{l}\sim\sqrt{(1-\delta)\nu x'y} \qquad (5\cdot4)$$

and the width of $\bar{l}$ is of the order of $\sqrt{\bar{l}}$.

From comparison with an exact numerical calculation with fixed parameters, it is shown that this approximation gives very nice agreement with true values. This circumstance could be interpreted as follows. In the upper atmosphere the cascade process is mainly affected by particles of energies in the region I, and towards the lower atmosphere the difference between the solutions in the regions I and II becomes small.

Using this approximation, we shall analyse the semiquantitative properties of the cascade process.

It will be interesting to know the depth x_m at which the number of pions of energy y becomes maximum. This can be easily estimated from eq. (5·2) as

$$x_m=(1-\delta)\nu y-\tfrac{1}{2}. \qquad (5\cdot5)$$

The half width of the cascade curve is

$$\Delta x_m=\tfrac{1}{2}+2\sqrt{(1-\delta)\nu y}=\tfrac{1}{2}+2\sqrt{x_m+\tfrac{1}{2}}. \qquad (5\cdot6)$$

The comparison of eq. (5·5) with the exact calculation is shown in Fig. 3. It gives nice agreement in the region I as well as the regions II and III. The number of nuclear particles of energy y at x_m is given by

$$N(\varepsilon,\ x_m)\,d\varepsilon\cong\sqrt{(1-\delta)\nu/4\pi y}\exp\{y-(1-\delta)(1-\nu)y\}\,dy. \qquad (5\cdot7)$$

The energy of those nuclear particles, ε_m, which contains most of the energy of a cascade in $y(=\ln\varepsilon_0/\varepsilon)$ unit at any fixed depth, x, is also estimated from eq. (5·2), and is given by

$$\ln(\varepsilon_0/\varepsilon_m)=\nu/(1-\delta) \quad \text{or} \quad \varepsilon_m=\varepsilon_0\exp\{-\nu/(1-\delta)\cdot x\} \qquad (5\cdot8)$$

5·2. *An approximate formula for the electron component*

Next we shall derive an approximate formula for the electron component. Assuming that two γ-rays produced by a decay of a neutral pion of energy E' have the same energies $E'/2$, we obtain as the total number of electrons at depth t in radiation unit in a shower initiated by a primary of energy E_0

$$N_e(E_0,\ 0,\ t)=\int_{2\varepsilon_c}^{E_0} dE' \int_0^t \frac{dt'}{m} N_{\pi^0}\left(E_0, E', \frac{t'}{m}\right)\cdot 2\Pi\left(\frac{E'}{2},\ 0,\ t-t'\right), \quad (5\cdot 9)$$

where m is the ratio of the collision mean free path to the radiation length.

With use of the approximate formulae for $\Pi(E'/2,\ 0,\ t-t')$ given in Appendix IV and for $N_{\pi^0}(E_0,\ E',\ t'/m)$ corresponding to eq. (5·2), eq. (5·9) turns out to be

$$N_e(E_0,\ 0,\ t)=\frac{1.4}{\sqrt{\pi}}\frac{(1-\delta)(1-\nu)}{m}\int_0^t dt' \int_{2\varepsilon_c}^{E_0} dE' \frac{1}{\left\{\frac{(1-\delta)}{m}\nu t' y'\right\}^{1/4}}$$

$$\times\sqrt{\frac{y'-\beta}{t-t'-\alpha}}\frac{1}{(t-t')^{1/2}}\exp\{f(E',\ t')\}, \quad (5\cdot 10)$$

where

$$f(E',\ t')=\delta y'+\left(1-\frac{1}{m}\right)t'+2\sqrt{\frac{(1-\delta)\nu}{m}t' y'}+2\sqrt{(t-t'-\alpha)(y'-\beta)}\ -t$$

and

$$y_0=\ln(E_0/2\varepsilon_c),\quad y'=\ln(E_0/E'),\quad \alpha=1.4,\quad \beta=0.56\,.$$

The integration of eq. (5·10) can be made by the double saddle point method, as long as the saddle point $(y_1,\ t_1)$ falls into the following region:

$$0<y_1<y_0,\quad 0<t_1<t, \quad (5\cdot 11)$$

where y_1, t_1 can be determined by the requirements:

$$\partial f/\partial y'|_{\substack{y'=y_1\\ t'=t_1}}=0,\quad \partial f/\partial t'|_{\substack{y'=y_1\\ t'=t_1}}=0. \quad (5\cdot 12)$$

We can find out two solutions of eq. (5·12) satisfying the condition (5·11). One solution situates in the region A, where a shower is in a growing up stage, and the other in the region C, where a shower is in a decreasing stage. In the intermediate region B between the regions A and C, where a shower develops near the optimum depth, there is no solution satisfying eq. (5·11). However, in the region B, $N_e(E_0,\ 0,\ t)$ may be well approximated by the approximate formula of the cascade function initiated by a single photon.

Thus, using the results derived in Appendix IV, we obtain

for region A

$$\frac{(y_0-\beta)m\delta^2}{(m-1)^2(1-\delta)\nu}\left(\frac{\lambda-m}{\lambda}\right)^2<t-\alpha<\frac{(y_0-\beta)}{(m-1)^2}\delta^2m^2\left(\frac{\lambda-1}{\lambda}\right)^2,$$

$$N_e(E_0,\ 0,\ t)=2.8\sqrt{\pi}\,e^{-\alpha}\frac{1-\nu}{\sqrt{\ }}\left(\frac{1-\delta}{\nu}\right)^{1/2}\frac{m^{1/2}}{m-1}\left(\frac{m-1}{\delta m}\right)^{1/2}\left(\frac{\lambda}{\lambda-1}\right)^{1/2}$$

$$\times\left(\frac{t-t_1-\alpha}{t-t_1}\right)^{1/2}\exp\left[\frac{\delta m}{m-1}\frac{\lambda-1}{\lambda}(y_0-\beta)+\left\{\frac{m-1}{\delta m}\frac{\lambda}{\lambda-1}-1\right\}(t-\alpha)\right],$$

for region B

$$\frac{(y_0-\beta)\delta^2m^2}{(m-1)^2}\left(\frac{\lambda-1}{\lambda}\right)^2<t-\alpha<(y_0-\beta)\left(\frac{\lambda}{\lambda-1}\right)^2,$$

$$N_e(E_0,\ 0,\ t)\simeq 2.8\sqrt{\pi}\,e^{-\alpha}\frac{1-\nu}{\sqrt{\ }}\left(\frac{1-\delta}{\nu}\right)^{1/2}\frac{m^{1/2}}{m-1}$$

$$\times\left(\frac{t-t_1-\alpha}{t-t_1}\right)^{1/2}\left(\frac{y_0-\beta}{t-\alpha}\right)^{1/4}\exp[-(t-\alpha)+2\sqrt{(t-\alpha)(y_0-\beta)}],\qquad(5\cdot13)$$

and for region C

$$(y_0-\beta)\left(\frac{\lambda}{\lambda-1}\right)^2<t-\alpha<(y_0-\beta)\frac{m}{(1-\delta)\nu}\left(\frac{\lambda m}{\lambda-m}\right)^2,$$

$$N_e(E_0,\ 0,\ t)=2.8\sqrt{\pi}\,e^{-\alpha}\frac{1-\nu}{\sqrt{\ }}\left(\frac{1-\delta}{\nu}\right)^{1/2}\frac{m^{1/2}}{m-1}\left(\frac{\lambda-1}{\lambda}\right)^{1/2}$$

$$\times\left(\frac{t-t_1-\alpha}{t-t_1}\right)^{1/2}\exp\left[\frac{\lambda}{\lambda-1}(y_0-\beta)-\frac{1}{\lambda}(t-\alpha)\right]$$

where t_1 is given by eq. (A IV. 4) and

$$\frac{1}{\lambda}=1-\frac{1}{2\delta m}\left\{m+(m-1)\delta-(1-\delta)\nu-\sqrt{\ }\right\},$$

$$\sqrt{\ }=\sqrt{\{m-(m-1)\delta-(1-\delta)\nu\}^2-4(m-1)(1-\delta)\delta\nu},\qquad(5\cdot14)$$

$$y_0=\ln E_0/2\varepsilon_c.$$

As shown in eq. (5·13), λ means the attenuation length at the tail of the electron component in EAS and is larger than the nuclear collision mean free path. Moreover, the changes of λ for changes of δ, ν and m are easily calculated.

$$\Delta\lambda/\Delta\nu=\lambda(1-\delta)/\sqrt{\ }\cdot(\lambda-1),$$

$$\Delta\lambda/\Delta\delta=\lambda/\delta\sqrt{\ }\cdot\{m-\nu-\lambda(1-\nu)\},$$

$$\Delta\lambda/\Delta m=\lambda/m\sqrt{\ }\cdot\{\delta+(1-\delta)\nu+(1-\delta)(1-\nu)\lambda\}.\qquad(5\cdot15)$$

From eqs. (5·15), the effect of the dynamical character of the high-energy collision on the electron component in EAS could be estimated.

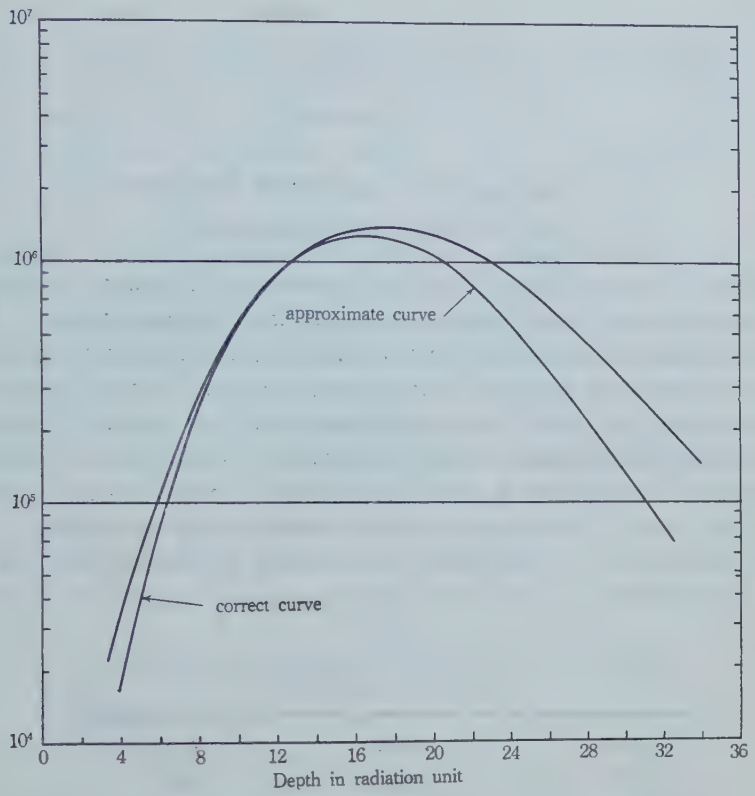

Fig. 5. Comparison of the approximate curve derived with eq. (5·13) with the correct altitude variation of electrons for $E_0=10^6\ mc^2$. The values of the parameters are given in Table 3.

As an example, if we take the values of the parameters given in Table 3, we obtain

$$\lambda=4.1 \text{ (rad. unit)} \qquad (5\cdot14')$$

and eqs. (5·13) and (5·15) turn out to be

for region A $\quad 0.16(y_0-0.56)<t-1.4<0.43(y_0-0.56),$

$$N_e(E_0,\ 0,\ t)=0.48\exp[0.66(y_0-0.56)-0.53(t-1.4)],$$

for region B $\quad 0.43(y_0-0.56)<t-1.4<1.7(y_0-0.56),$

$$N_e(E_0,\ 0,\ t)=0.39\left(\frac{y_0-0.56}{t-1.4}\right)^{1/4}\exp\left[-(t-1.4)+2\sqrt{(t-1.4)(y_0-0.56)}\right], \qquad (5\cdot13')$$

and for region C $\quad 1.7(y_0-0.56)<t-1.4<209(y_0-0.56),$

$$N_e(E_0,\ 0,\ t)=0.34\exp\left[1.3(y_0-0.56)-0.24(t-1.4)\right],$$

and

$$\Delta\lambda/\Delta\nu=7.1 \text{ (rad. unit)}, \quad \Delta\lambda/\Delta\delta=3.9 \text{ (rad. unit)}, \qquad (5\cdot15')$$
$$\Delta\lambda/\Delta m=2.8 \text{ (rad. unit)}.$$

§ 6. Analysis of extensive air showers

—Results of numerical calculations—

The values of the parameters used in numerical calculations are tabulated in Table 3. The choice of these values are due to the following reasons: First, the parameters are fixed so as to satisfy the constancy of the transverse momenta, that is, $\alpha=1$. a is put to be unity for mathematical simplicity. Secondly, the apparent inelasticity η is taken to be 0.5 for comparison with the results derived with the model I. Using these values of the parameters, $\beta=1/2$, $\delta=1/2$ are obtained. Further assuming the collision to be a nucleon- (or a pion-) nucleon collision, we obtain $\kappa=1$ and $\xi=1$. Taking the formula of multiplicity to be $n=2E_0^{1/4}$, i. e. $b=1$, $d=1/4$, we obtain $k=2$. The values of the other parameters are the same as those used in the paper A. The transverse momentum is taken to be 400 MeV/c.

Table 3. The values of the parameters used in the numerical calculation

$a=1$	$\nu=0.7$	$\delta=1/2$
$\alpha=1$	$\kappa=1$	$\xi=1$
$\beta=1/2$	$k=2$	$\gamma=1/2$
multiplicity $n=2E_0^{1/4}$		
collision mean free path	90 gcm^{-2}	
radiation unit	38 gcm^{-2}	
critical energy	85 MeV	
decay constant of charged pions, $B=148\ Mc^2$		

With use of these values of the parameters the energy spectrum of secondary pions at the high-energy collision in the L. S., eq. (2·14) turns out to be

$$dn=\begin{cases}\delta(E_0/E)^{1+\delta}\,dE/E_0=\frac{1}{2}(E_0/E)^{3/2}\,dE/E_0 & \text{for narrow cone,}\\ \kappa\delta(E/E_1)^{\kappa\delta}\,dE/E=\frac{1}{2}(E_0E_1)^{-1/2}\,dE & \text{for wide cone.}\end{cases} \qquad (6\cdot1)$$

It must be noted that the differential energy spectrum in the narrow cone is proportional to $E^{-3/2}dE$.

6·1. *The energy spectrum of high-energy nuclear particles*

There are evidences[15] that EAS always accompanies high-energy nuclear particles ($\gtrsim 10^{12}$ eV) very near a shower core. However, at this time, no data enable us to discuss their properties more quantitatively. Therefore, we present only the

rgy spectrum at mountain elevation and sea level for a primary energy $E_0=10^6$ ^{2} in Figs. 6 and 7, respectively. Comparing these figures with those of high- rgy muons, we see that the contribution of early generations is smaller for high- rgy nuclear particles than for high-energy muons. (see Fig. 8) Thus the study he energy spectrum of high-energy nuclear particles would be useful only for cking rough consistency of a model.

2. *The lateral distribution of high-energy muons*

Since muons have very weak interactions with matter and the long life-time, y come directly to the points of observation from the points of their creation. ides, we showed in the paper A that most of high energy muons ($\gtrsim 100$ GeV) created at the first stage of nucleonic cascade. Moreover, the Coulomb scatter- is quite negligible for such high energy muons. These situations make it easier derive information about the high-energy collisions from experimental data of ons. Actually we can estimate the energy spectrum of secondary particles produced

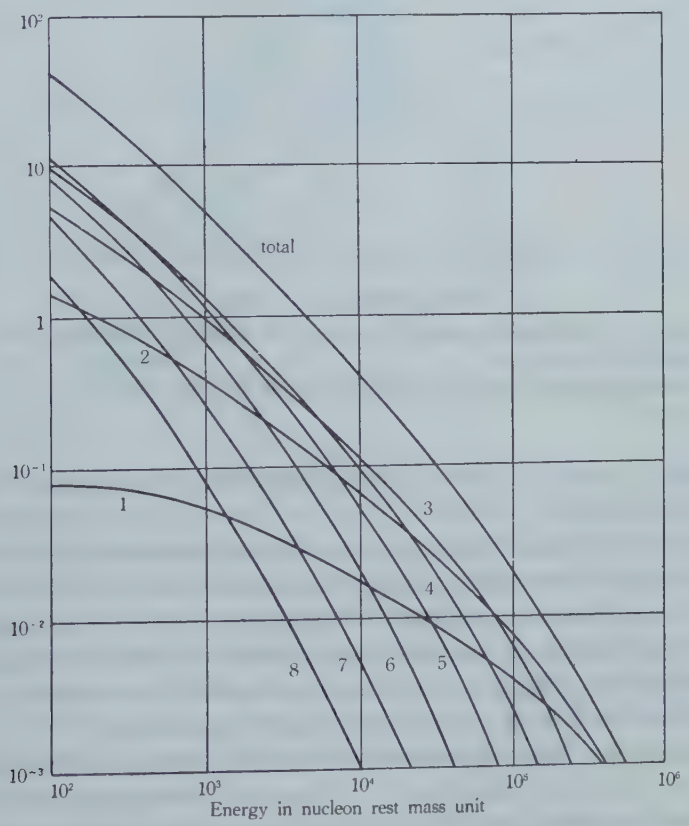

Fig. 6. Integral energy spectrum of nuclear particles at mountain elevation (720 gcm^{-2}) for $E_0=10^6\,Mc^2$.

Each numbered curve represents the contribution from each generation.

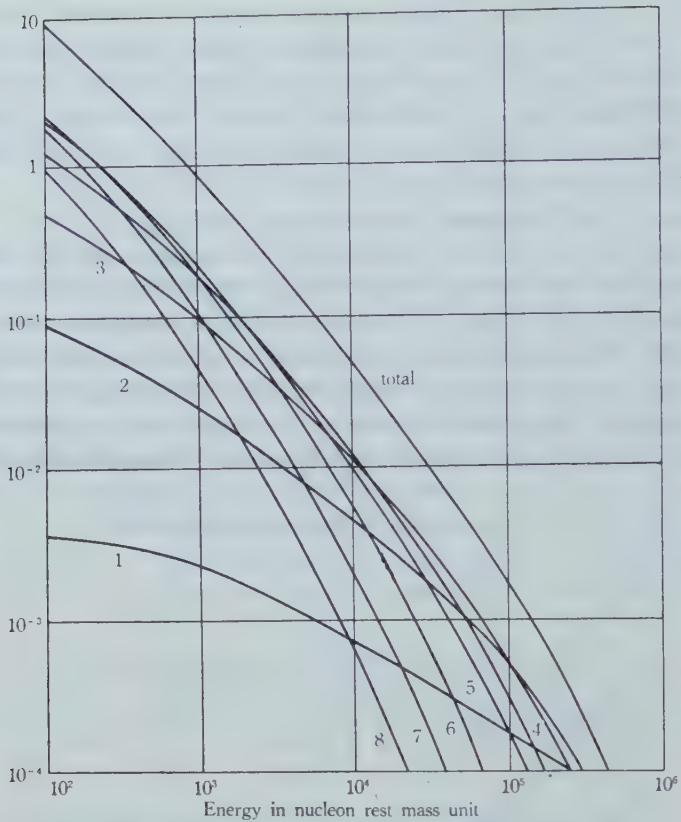

Fig. 7. The integral energy spectrum of nuclear particles at sea level for $E_0=10^6\,Mc^2$.

Each numbered curve represents the contribution from each generation.

in the high-energy collision from the lateral distribution of high-energy muons, usin the fact that $P_T \simeq$ const.

An experimental datum fitting the above stated situations is the one given b the Cornell group.[16] The experiment was performed far underground, its dept being 1600 m. w. e. which corresponds to about 500 GeV of the lowest energy muons. The experiment shows from the analysis of the decoherence curve tha the lateral distribution of such muons is nearly constant near a shower core withi a radius of 13 meters and falls off rapidly outside the radius.

If we assume the energy spectrum of secondary pions at high-energy collision in the L. S. to be proportional to $E^{-n}dE$ and the constancy of the transvers momenta of the secondary pions and also the pions to be produced at the averag height ~ 15 km, we can obtain the lateral distribution of high-energy muons, th result being $2\pi\rho(r)\,rdr \propto 2\pi r^{n-2}\,rdr$, within a radius of about 10 meters. In ou model we obtain $\rho(r) \propto r^{-0.5}$ from eq. (5·2), only taking muons into account

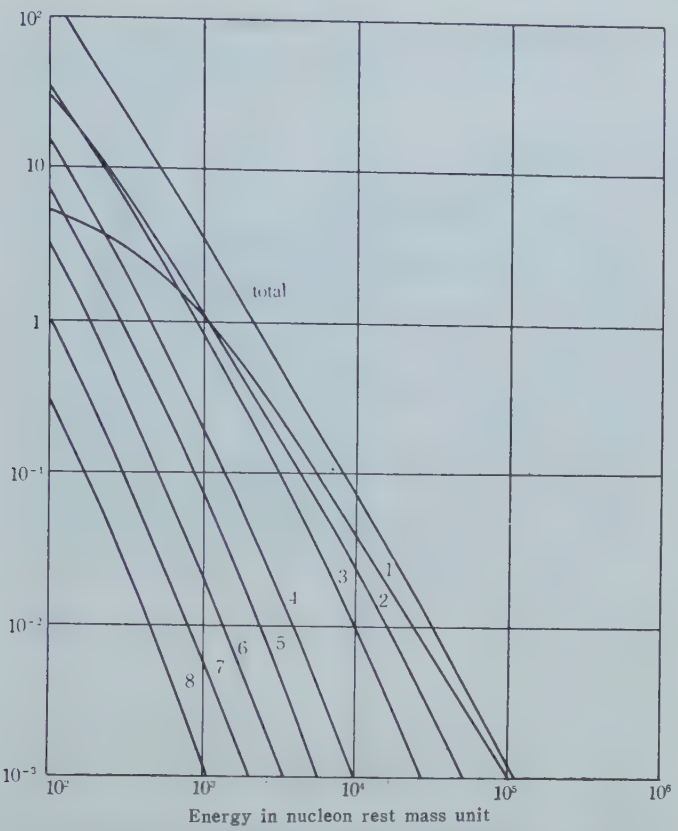

Fig. 8. Integral energy spectrum of high energy muons at sea level for $E_0=10^6\ Mc^2$.

Each numbered curve represents the contribution from each generation.

hich parental pions belong to the first generation.

It is interesting to check how much the nucleonic cascade process changes this tuation. The numerical curve of the lateral distribution for $E_0=10^6\ Mc^2$ is shown Fig. 9. In the calculation, assumptions are made that a muon moves along the me direction as its parental pion and the angular spread of pions is only due to at of the last collisions by which they are produced and also the production point muons is the average height of respective generations. The lowest energy of ons is taken to be 500 Mc^2. The effect of backward emitted pions in the C. M. is taken into account only for the first generation. Their contribution from other enerations is negligibly small.

As seen from the figure, $\rho(r)$ is nearly constant for $r \lesssim 10$ m. The mean dius is found to be a little too small, partly due to the neglect of the angular pread of pions belonging to earlier generations. A similar distribution was also btained for $E_0=10^4\ Mc^2$. Thus if the energy spectrum of secondary pions at the

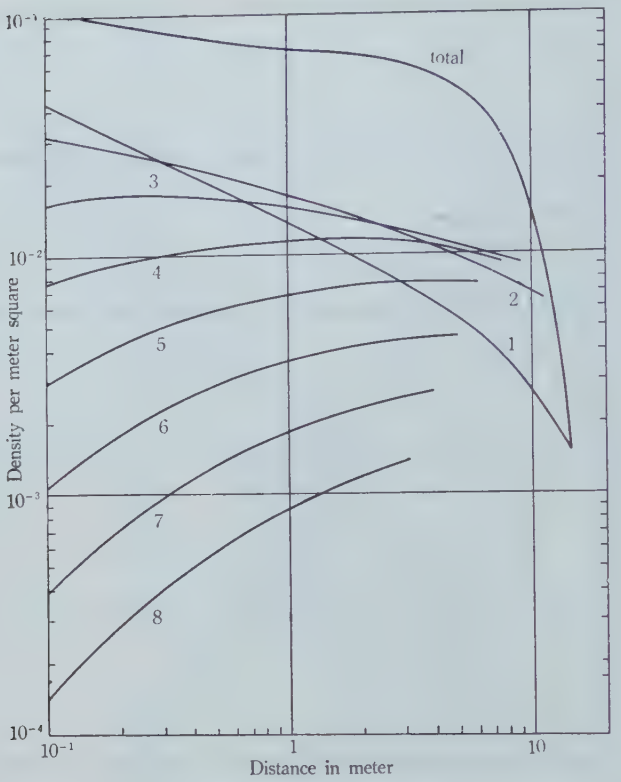

Fig. 9. The lateral distribution of high energy muons for $E_0=10^6\ Mc^2$.

Each numbered curve represents the contribution from each generation.

high-energy collision could be well approximated by a power law $E^{-n}dE$, n shou not be smaller than 1.5. On the other hand, $n\cong 2$ from the analysis of j showers.[2),3)] If $n\gtrsim 2$, the form of the lateral distribution of high-energy muo would become concave near a shower core. Therefore, it may be concluded $2\gtrsim n\gtrsim 1.5$, if the decoherence curve corresponds really to the flatness of the later distribution.*

The lateral distribution of high-energy muons is insensitive to a value of t nuclear collision mean free path; then it depends only on the energy spectrum secondary pions and the ratio of charged to neutral pions. If we considered th the contribution of the first generation is main (see Fig. 9), we expect that t lateral distribution of high-energy muons will give an important clue to the ener

* It is noticed by J. Nishimura that the lateral distribution becomes slightly steeper than our case near a shower core, when the fluctuations of transverse momenta are taken into accou

spectrum of the high-energy collision. Therefore, it is desirable to measure a lateral distribution of high-energy muons directly, not a decoherence curve.

6·3. *The altitude variation of the number of electrons*

The total number of electrons was calculated with use of the cascade function in Approximation B, by taking into account neutral pions produced by nuclear particles of energies greater than $10^2\ Mc^2$. It was confirmed numerically that the contribution from nuclear particles of energies lower than $10^2\ Mc^2$ is negligible. The results for the altitude variation of the number of electrons with the model II are shown in Figs. 10 and 11. These are very similar to those with the model I in their shape. But there is an essential difference in the optimum depths of the electron component as shown in figures. In the case of the model II the optimum depths are deeper than those in the case of the model I. Corresponding to this difference, the attenuation lengths at mountain elevation and at sea level in the former are somewhat longer than in the latter.

These differences result from the following fact: Neutral pions of high energies are produced much more in the model II than in the model I. Such high energy

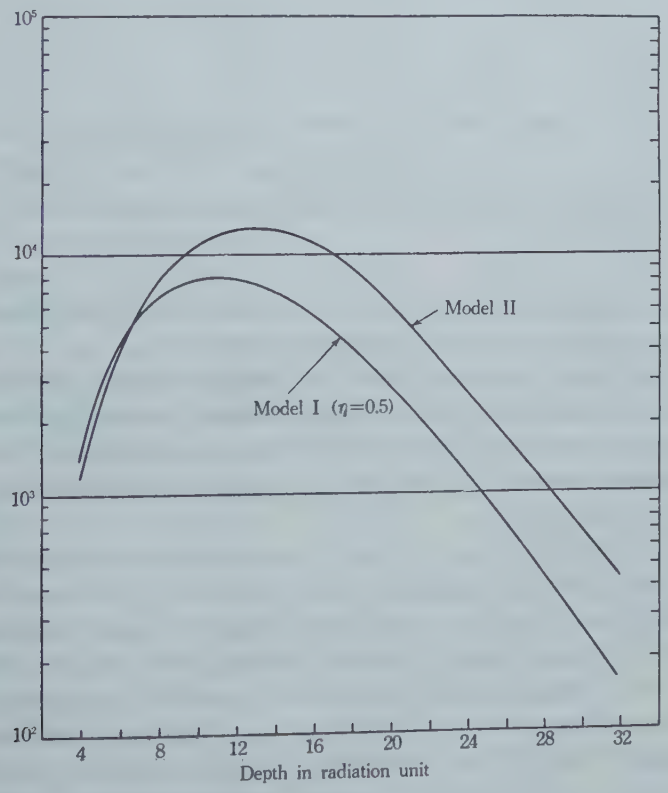

Fig. 10. The altitude variation of the electron component for $E_0=10^4\ Mc^2$.

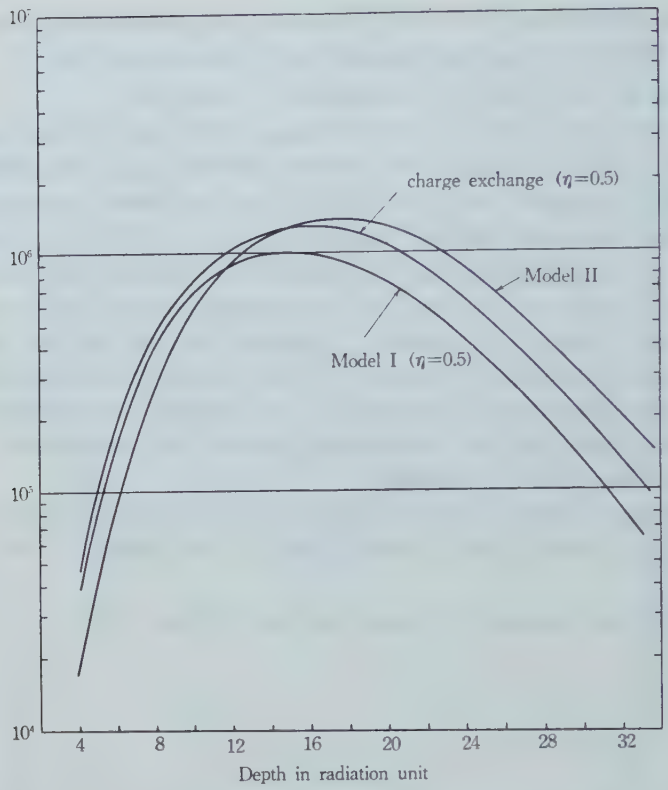

Fig. 11. The altitude variation of the electron component for $E_0=10^6\ Mc^2$.

neutral pions are produced in the earliest two or three generations and conduct dominantly the optimum depth of the altitude variation.

This fact is actually confirmed with the charge exchange model which is the modification of the model I so that colliding charged pions with air nuclei exchange their charges and escape as neutral pions with a probability of 1/3. With this model the optimum depth of the electron component situates between those of the models I and II (see Fig. 11).

We pointed out in the paper A with the model I that the attenuation length varies largely with primary energies. This fact is also unchanged in the case of model II as expected. Moreover, the optimum depth of the electron component depends on the primary energy strongly, though the optimum depth for EAS initiated by a fixed primary energy could be adjustable by changing the energy spectrum of neutral pions and the parameter ν. Therefore it is difficult to explain with our models I and II the experimental data available at this time, that the attenuation length of EAS is nearly constant over the region of primary energies $10^{14}\sim10^{17}$ eV. From the theoretical point of view the attenuation length should

increase with the primary energy, if the mechanism of the high-energy collision, i. e., the energy spectrum, the inelasticity, the ratio of charged to neutral pions, and so on, would be independent of the incident energy over a wide range $10^{12}\sim10^{17}$ eV. Before considering this discrepancy between theory and experiment, it is most desirable to obtain experimental altitude variation curves for different primary energies, taking into account the fluctuation in the depth at which the first collision of a primary particle occurs, as pointed out by S. Miyake. If the experimental fact as stated above were still correct, even if an interpretation was made by considering this fluctuation, it would become necessary to introduce some serious modification of the mechanism of the high-energy collision, as has been conceived to date.

6·4. *The lateral distribution of electrons*

With the model I we have shown[10] that the lateral distribution of electrons is insensitive to the value of inelasticity and also to the primary energy. We reexamined this conclusion with the model II. The calculation was made on an assumption that the lateral spread of electrons is only due to multiple Coulomb scattering of electrons, and the effects of the angular spreads of nuclear active particles and two-photon-decay of neutral pions are negligible. The procedure of calculation is given in Appendix V.

The calculated results are shown in Figs. 12 and 13. In the broken parts of curves there are some errors, because of the following reason. The shower age in the three dimensional cascade theory depends not only on the primary enery E but also on the distance from the shower axis as a function of $(Er/\varepsilon_c r_1)$, where ε_c is the critical energy in air and r_1 is the Molière unit. Accordingly, the smaller r/r_1 gets, the more showers initiated by individual neutral pions are found to have ages larger than two. The approximate formula is no longer applied to the case of such an age. For $r/r_1 \lesssim 5\cdot10^{-2}$ the number of such showers becomes considerable, but the contribution of such show-

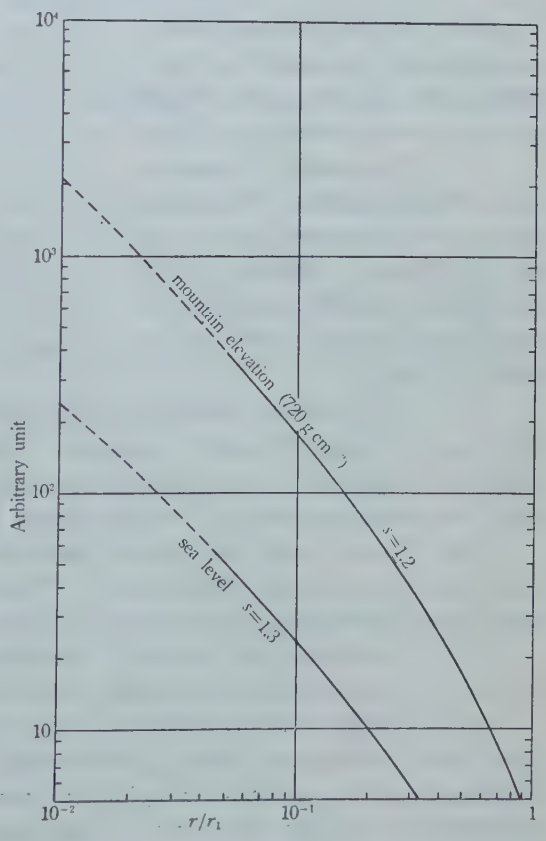

Fig. 12. The lateral distribution of the electron component for $E_0=10^4\,Mc^2$.

ers to the density of electrons is not large. For such a region, however, it becomes meaningless to evaluate the density more accurately, because of the neglect of the angular spreads as stated above.

We again find that the lateral distribution is insenistive both to the energy spectrum, as is found by comparison of the model II with the model I, and to the primary energy.

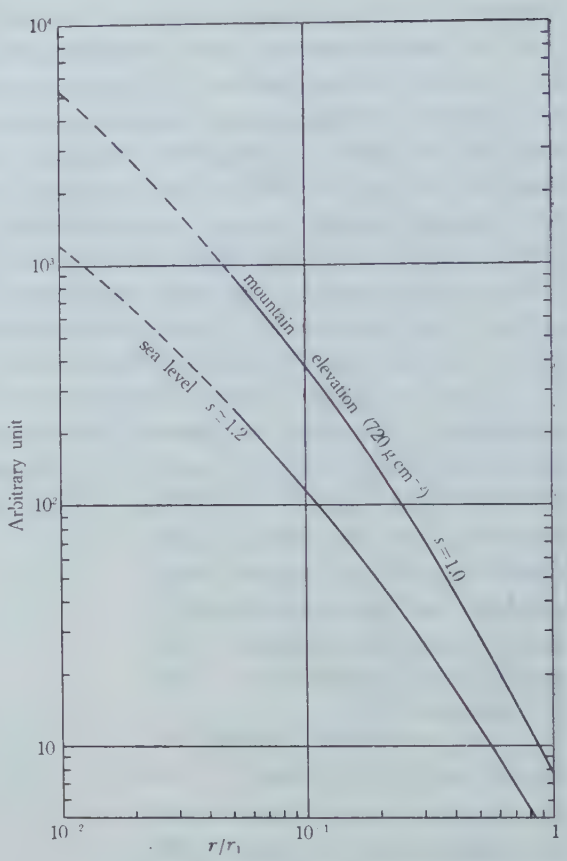

Fig. 13. The lateral distribution of the electtron component for $E_0=10^6\ Mc^2$.

§ 7. Summary and conclusion

First we shall summarize the advantages of our model and cascade theory and the main results obtained from the analysis of EAS.

(1) The model II contains some free parameters which characterize the dynamical property of the high-energy collision. The parameters are adjustable so as to satisfy the constancy of the transverse momenta.

(2) The diffusion equation can be solved analytically and rather simply if the decay of charged pions is neglected. This fact makes it easy to investigate unknown properties of the high-energy collision. The accuracy of the solution is good for nuclear particles of energies larger than $10^2\ Mc^2$.

(3) In our model the energy spectrum of secondary pions in the forward cone in the L. S. is proportional to $E^{-(1+\delta)}dE$. $2 \gtrsim 1+\delta \gtrsim 1.5$ is consistent with the lateral distribution of high-energy muons far underground.

(4) The lateral distribution of the electron component is insensitive to a model of the high-energy collision and also to the primary energy.

(5) For the altitude variation of the electron component, the numerical curves show that the optimum depth is rather deep. To shift it to the upper atmosphere could be made by properly changing the energy spectrum of neutral pions and the parameter ν and other parameters as far as the altitude variation for a fixed primary energy is concerned. However, it is doubtful that the altitude variation as a function of the primary energy can be explained with use of the energy spectrum of

neutral pions and ν and other parameters fitted for one fixed primary energy.

In this paper we have analysed EAS in connection with the dynamical characters of the high-energy collision taking into account the constancy of the transverse momenta. And we have seen that the constancy of the transverse momenta and the energy spectrum of secondary charged pions are consistent with the results derived from the analysis of jet showers. As stated in the introduction, EAS should furnish us information about inelasticity. However, we could not say anything about it at this time. This is partly due to the absence of reliable data concerning the altitude variation of the electron component over wide range of primary energies. Therefore we have only examined how much the energy spectrum of secondary neutral pions changes the altitude variation as compared with the case of the model I in which the spectrum is monochromatic.

Acknowledgement

We should like to express our sincere thanks to Professors S. Hayakawa, S. Miyake, J. Nishimura and M. Oda for valuable discussions and advices. Thanks are also due to Mr. I. Matsumoto and Miss H. Kunitaka for their assistance in carrying out the numerical calculations. One of the authors (A. U.) is much indebted to the Yukawa Memorial Foundation for a fellowship, which was extended to him in the last year when this work was initiated.

Appendix I. The approximate solution of the diffusion equation for $a<1$

For $a<1$, the solution corresponding to eq. (4·4) is somewhat complicated, as we cannot divide the energy region in a simple manner as in eq. (4·9). It depends on the generation l and must be divided into more energy sections than in eq. (4·9). We divide the energy region of the n-th generation as follows:

$$\begin{aligned}
&\text{Region I}: && a^{l}\mathcal{E}_0 > \mathcal{E} > a^{l-1}\mathcal{E}_0', \\
&\text{Region II}: && a^{l-1-j}a^{j}\mathcal{E}_0' > \mathcal{E} > a^{l-1-j-1}a^{j+1}\mathcal{E}_0', \quad j=0, 1, \cdots, l-1, \\
&\text{Region III}: && a^{l-1}\mathcal{E}_0' > \mathcal{E} > a^{l-2}\mathcal{E}_0'',
\end{aligned} \tag{AI·1}$$

where

$$\mathcal{E}_0' = \mathcal{E}_0^{1/k}, \quad \mathcal{E}_0'' = \mathcal{E}_0^{1/k^2}, \quad a' = a^{1/k}, \quad a'' = a^{1/k^2}.$$

We define the solution in each energy region as follows:

$$\begin{aligned}
&\text{Region I}: && h_l^{0}(\mathcal{E}), \\
&\text{Region II}: && h_l^{j}(\mathcal{E}) = h_l^{j-1}(\mathcal{E}) + \bar{h}_l^{j}(\mathcal{E}), \quad j=0, 1, \cdots, n-1, \\
&\text{Region III}: && p_l(\mathcal{E}),
\end{aligned} \tag{AI·2}$$

and

$$h_{l+1}^{0}(\varepsilon)=\frac{\left(\ln\left(\frac{\varepsilon_0}{\varepsilon}a^{l+1}\right)\right)^{l}}{l!},$$

$$\bar{h}_{j+1}^{j+1}(\varepsilon)=\frac{1}{k^j}\int_{\varepsilon^k}^{\varepsilon_0 a^{l+j(l-1)}}\frac{d\varepsilon'}{\varepsilon'}\left(-1+\kappa\left(\frac{\varepsilon^k}{\varepsilon'}\right)^{\tau}\right)\frac{\left(\ln\left(\frac{\varepsilon_0}{\varepsilon'}a^{l+j(k-1)}\right)\right)^{l-1}}{(l-1)!},$$

$$\bar{h}_{l+1}^{l+1}((\varepsilon)=p_{l+1}(\varepsilon)-h_{l+1}^{l}(\varepsilon). \qquad \text{(AI·3)}$$

As the energy region II is much smaller than the regions I and III, we omit the region II extending the solutions in the regions I and III to the region II and connect them properly; then we have two new regions,

Region I: $a^{l-1}\varepsilon_0'<\varepsilon<a^{l}\varepsilon_0$ solution $h_l^{0}(\varepsilon)$,

Region II: $a^{l-2}\varepsilon_0''<\varepsilon<a^{l-1}\varepsilon_0'$ solution $p_l(\varepsilon)$,

$$h_l^{0}(\varepsilon)=\frac{1}{(l-1)!}\left(\ln\left(\frac{a^l\varepsilon_0}{\varepsilon}\right)\right)^{l-1},$$

$$p_l(\varepsilon)=\frac{1}{(l-1)!}\left\{\left(\ln\left(\frac{a^l\varepsilon_0}{\varepsilon}\right)\right)^{l-1}-\frac{1-\frac{1}{k^l}}{1-\frac{1}{k}}\left(\ln\left(\frac{a^{l-1}\varepsilon_0}{\varepsilon^k}\right)\right)^{l-1}\right\}$$

$$+\kappa\frac{1-\frac{1}{k^l}}{1-\frac{1}{k}}\int_{(\varepsilon_0/a^{l-1})^k}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\left(\frac{\varepsilon}{a^{l-1}}\right)^k}{\varepsilon'}\right)^{\tau}\frac{\left(\ln\left(\frac{\varepsilon_0}{\varepsilon'}\right)\right)^{l-2}}{(l-2)!}. \qquad \text{(AI·4)}$$

This solution coincides exactly with the solution (4·11) if we put $a=1$, and as far as $h_l^{0}(\varepsilon)$ is concerned, it is exact. As we mentioned already that the main term in $p_l(\varepsilon)$ is the same as $h_l^{0}(\varepsilon)$, and we can use the solution I all over the region without introducing a large error (10%). The solution of eq. (4·2) in the region I is given by

$$N(\varepsilon,\ x)=\frac{1}{\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta}\left(\frac{1}{a}\right)^{1-\delta}\nu(1-\delta)e^{-x}\sum_{l=0}^{\leqq l_0}\frac{\left(\left(\frac{1}{a}\right)^{1-\delta}\nu(1-\delta)\ln\left(\frac{a^l\varepsilon_0}{\varepsilon}\right)\right)^{l}}{(l!)^2}. \qquad \text{(AI, 5)}$$

The upper limit l_0 for the summation is given by

$$\frac{a^{l_0}\varepsilon_0}{\varepsilon}=1 \quad \text{or} \quad l_0=\frac{\ln\varepsilon_0/\varepsilon}{\ln(1/a)}. \qquad \text{(AI·6)}$$

The generation giving the most contribution to the summation of eq. (AI·5) is given by

$$l'=\sqrt{\left(\frac{1}{a}\right)^{1-\delta}\nu(1-\delta)\ln\left(\frac{a'\varepsilon_0}{\varepsilon}\right)}. \qquad \text{(AI·6')}$$

It is noted l' is smaller than l_0.

We could approximate eq. (AI·5) as

$$N(E,\ x)=\frac{1}{\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta}\left(\frac{1}{a}\right)^{1-\delta}\nu(1-\delta)\,e^{-t}\,I_0\left(2\sqrt{\left(\frac{1}{a}\right)^{1-\delta}\nu(1-\delta)\,x\ln\left(\frac{a''\varepsilon_0}{\varepsilon}\right)}\right),$$

where

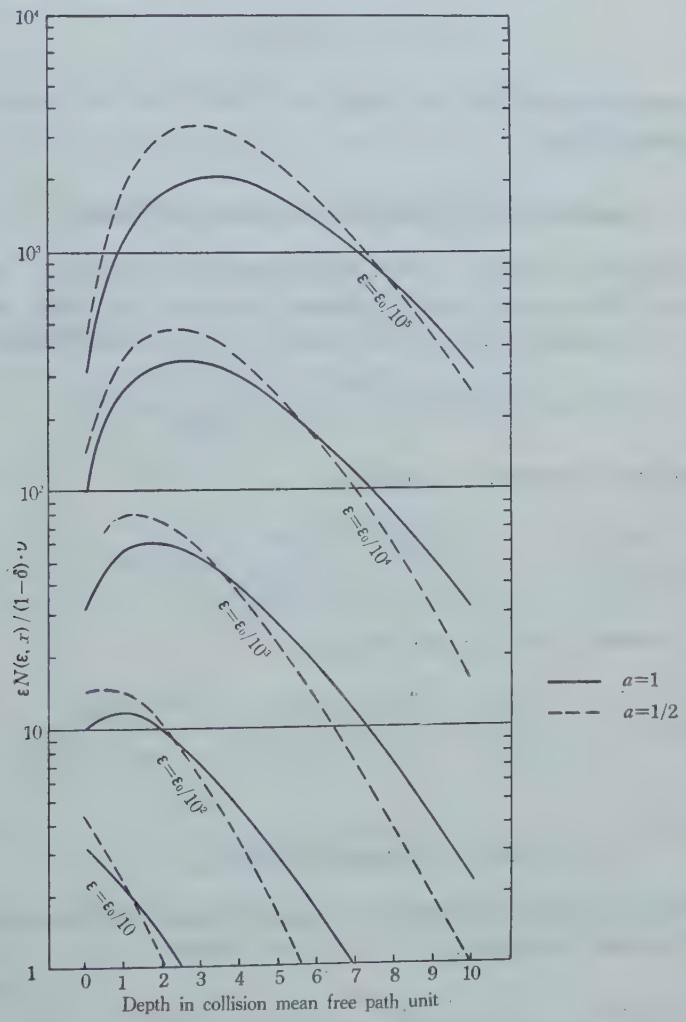

Fig. 14. Number of nuclear particles of energy ε as a function of depth.
Comparison of the case $a=1/2$ with $a=1$.
$\delta=1/2$, $k=2$, $\nu=2/3$.

$$l'^2=\left(\frac{1}{a}\right)^{1-\delta}\nu(1-\delta)\,x\left(\ln\left(\frac{\varepsilon_0}{\varepsilon}\right)-l'\ln\left(\frac{1}{a}\right)\right).$$

We introduce a new parameter

$$\nu\left(\frac{1}{a}\right)^{1-\delta}=\nu',$$

$$N(\varepsilon,\ x)=\frac{1}{\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta}\nu'(1-\delta)\,e^{-x}I_0\left(2\sqrt{\nu'(1-\delta)\,x\ln\left(\frac{a^{l'}\varepsilon_0}{\varepsilon}\right)}\right), \qquad \text{(AI·7)}$$

where

$$l'\fallingdotseq\sqrt{\nu'(1-\delta)\,x\ln(\varepsilon_0/\varepsilon)}.$$

Using these approximations we could apply eq. (AI·7) to the analysis of EAS similar to the case of $a=1$.

Appendix II. The effect of decay of pions

Here we only give the decay effect of charged pions which is important for low-energy pions. We assume that all charged particles are pions, and put $a=1$ for simplicity. Corresponding to eqs. (3·11) and (3·12), a simple calculation shows

$$f_l^*(\varepsilon)=\nu^l(1-\delta)^l\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta}\frac{1}{\varepsilon_0}\,\frac{1}{1+(B/l\varepsilon)}\,h_l^*(\varepsilon), \qquad \text{(AII·1)}$$

where B is the decay constant of pions and

$$h_l^*(\varepsilon)=\int_{\varepsilon}^{\varepsilon_k}\frac{d\varepsilon'}{\varepsilon'+B/(l-1)}\,h_{l-1}^*(\varepsilon')+\kappa\int_{\varepsilon_k}^{\infty}\frac{d\varepsilon'}{\varepsilon'+B/(l-1)}\,h_{l-1}^*(\varepsilon')\left(\frac{\varepsilon_k}{\varepsilon'}\right)^{\gamma}. \qquad \text{(AII·2)}$$

In the region I where $h_l^{0*}(\varepsilon)$ is the solution,

$$h_l^{*0}(\varepsilon)=\int_{\varepsilon}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'+B/(l-1)}\int_{\varepsilon'}^{\varepsilon_0}\frac{d\varepsilon''}{\varepsilon''+B/(l-2)}\cdots\cdots\int_{\varepsilon^{l-2}}^{\varepsilon_0}\frac{d\varepsilon^{l-1}}{\varepsilon^{l-1}+B}, \qquad \text{(AII·3)}$$

that is,

$$h_l^{*0}(\varepsilon)=h_l^0(\varepsilon)+B(\partial h_l^{*0}(\varepsilon)/\partial B)\,|_{B=0}+\cdots. \qquad \text{(AII·4)}$$

We only take into account the first and second terms. From eq. (AII·3) the second term is derived as follows:

$$-\frac{\partial h_l^{*0}(\varepsilon)}{\partial B}\Big|_{B=0}=\int_0^1 d\lambda\int_{\varepsilon}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'^2}\,\frac{1}{(l-2)!}\left(\ln\frac{\varepsilon'}{\varepsilon}+\lambda\ln\frac{\varepsilon_0}{\varepsilon'}\right)^{l-2}. \qquad \text{(AII·5)}$$

A simple integration of (AII·5) gives

$$-\frac{\partial h_l^{*0}}{\partial B}\Big|_{B=0}=\frac{(\ln \varepsilon_0/\varepsilon)^{l-2}}{(l-1)!}A_l(\varepsilon_0,\ \varepsilon), \tag{AII·6}$$

where

$$A_l(\varepsilon_0,\ \varepsilon)=\frac{1}{\varepsilon_0}\left(1+\frac{1}{2}+\cdots+\frac{1}{l-1}+\sum_{i=1}^{\infty}\frac{(\ln \varepsilon_0/\varepsilon)^i}{i}\left(\frac{1}{i!}-\frac{(l-1)!}{(l-1+i)!}\right)\right). \tag{AII·7}$$

Since the depth and the generation to which pions of energy ε mainly contribute are from eqs. (5·5) and (5·4)

$$x_m\sim(1-\delta)\nu\ln\varepsilon_0/\varepsilon, \tag{AII·8}$$

$$\bar{l}\sim\sqrt{(1-\delta)\nu x\ln\varepsilon_0/\varepsilon}\sim(1-\delta)\nu\ln\varepsilon_0/\varepsilon,$$

the order of magnitude of eq. (AII·3) is estimated by using these relations;

$$A(\varepsilon_0,\ \varepsilon)\sim\frac{1}{\varepsilon_0}\left\{\ln(2(1-\delta)\nu y+1)+\frac{e^y-1}{y}-\frac{2.86}{\sqrt{y}}e^{0.3y}\right\}\sim\frac{1}{\varepsilon}\frac{1}{y}, \tag{AII·9}$$

where $$y=\ln\varepsilon_0/\varepsilon.$$

Then eq. (AII·6) becomes

$$-\frac{\partial h_l^{*0}(\varepsilon)}{\partial B}\Big|_{B=0}\simeq\frac{(\ln\varepsilon_0/\varepsilon)^{l-2}}{(l-1)!}\frac{1}{\varepsilon}\frac{1}{\ln\varepsilon_0/\varepsilon}. \tag{AII·10}$$

Assuming the form of $h_l^{*0}(\varepsilon)$ to be

$$h_l^{*0}(\varepsilon)=\frac{1}{(l-1)!}\left(\ln\frac{\varepsilon_0}{\varepsilon+\mu B}\right)^{l-1},$$

and comparing it with eq. (AII·4) using eq. (AII·10), we obtain $\mu=(l-1)^{-1}$, i. e.

$$h_l^{*0}(\varepsilon)=\frac{1}{(l-1)!}\left(\ln\frac{\varepsilon_0}{\varepsilon+B/(l-1)}\right)^{l-1}, \tag{AII·11}$$

and from eq. (AII·1),

$$f_l^*(\varepsilon)=\nu^l(l-\delta)^l\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta}\frac{1}{\varepsilon_0}\frac{1}{1+(B/l\varepsilon)}\left(\ln\left(\frac{\varepsilon_0}{\varepsilon+B/(l-1)}\right)\right)^{l-1}\Big/(l-1)!. \tag{AII·12}$$

Thus if we approximate l contained in the factor $(\varepsilon+B/l)$ of eq. (AII·12) by $\bar{l}$ given in eq. (AII·8), the cascade function taking account of the decay could be well approximated by

$$N(\varepsilon,\ x')=(1-\delta)\nu\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta}\frac{1}{\varepsilon_0}\frac{e^{-x'}}{1+(B/l\varepsilon)}I_0\left(2\sqrt{\nu(1-\delta)x'\ln\left(\frac{\varepsilon_0}{\varepsilon+(B/\bar{l})}\right)}\right). \tag{AII·13}$$

This solution may be useful even at a rather low energy of about 10 GeV with an error of 10 per cent.

Next we shall estimate the muon component. The number of muons having energy ε is given by

$$N_\mu(\varepsilon,\ x)=\frac{B}{\varepsilon}\int_0^x\frac{dx'}{x'}e^{-x'}\sum_{l=0}^{\infty}\frac{x'^{l+1}}{(l+1)!}f_{l+1}^{*}(\varepsilon). \tag{AII·14}$$

The main contribution in the summation for fixed x' comes from the generation $\bar{l}$ determined by

$$\bar{l}^2\sim(1-\delta)\nu x'\ln\left(\frac{\varepsilon_0}{\varepsilon+B/(l-1)}\right), \tag{AII·15}$$

thus for $\varepsilon_0=10^8$ and $\varepsilon=10^2$, for example,

$$\bar{l}^2\sim 4.6\,x'. \tag{AII·16}$$

On the other hand, for fixed l, the maximum of $e^{-x'}x'^l$ in eq. (AII·14) occurs at

$$x'_{\max}\sim l.$$

Thus, if the depth of observation x is much greater than $\bar{l}$, i.e. $x\gg\bar{l}$ or $x\gg 4.6$, the difference between $\int_0^x e^{-x'}x'^{\bar{l}}dx'$ and $\int_0^\infty e^{-x'}x'^{\bar{l}}dx'$ becomes negligible. Then at a deep enough depth, say at sea level where $x\sim 12>4.6$, eq. (AII·14) could be approximated as

$$N_\mu(\varepsilon,\ x)\cong N_\mu(\varepsilon,\ \infty)=\sum\frac{B}{\varepsilon}\frac{1}{l+1}f_{l+1}^{*}(\varepsilon)\cong\frac{B}{\varepsilon}\frac{1}{\bar{l}+1}f_{\bar{l}+1}^{*}(\varepsilon). \tag{AII·17}$$

Using eqs. (AII·12) and (AII·15), we finally obtain

$$N_\mu(\varepsilon,\ x)\cong\frac{B}{\varepsilon}\frac{1}{\varepsilon+B/((1-\delta)\nu\ln\varepsilon_0/\varepsilon)}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{\delta+\nu(1-\delta)}\cdot\frac{1}{\ln\varepsilon_0/\varepsilon}. \tag{AII·18}$$

The approximate formulae in this chapter were derived by using only $h_l^0(\varepsilon)$ which is correct in the region I, i.e., $\varepsilon_0>\varepsilon>\varepsilon_0^{1/k}$. The approximate formulae in the region II and III could be obtained in a similar manner, though the derivation is rather complicated. However, as stated in § 5, the differences between the solutions in the region I and the regions II and III become small at the lower atmosphere. Thus the formula will be useful to analyse the quantitative characters of EAS.

Appendix III. The differential number spectra of nuclear particles and neutral pions without taking the fluctuation into account

In this appendix the full expression of the differential number spectra of nuclear particles and neutral pions, eqs. (3·2) and (3·5), is given without taking into

account fluctuations in the depth at which the first collision occurs.

The differential number spectrum of nuclear particles is given by

$$N(\varepsilon,\ x)=(1-\delta)\nu e^{-x}\Big(\frac{\varepsilon_0}{\varepsilon}\Big)^{1+\delta}\frac{1}{\varepsilon_0}\begin{cases}\mathcal{H}_1(\varepsilon,\ x) & \text{for } \varepsilon_0>\varepsilon>\varepsilon_0',\\ \mathcal{P}_1(\varepsilon,\ x) & \text{for } \varepsilon_0'>\varepsilon>\varepsilon_0'',\\ \mathcal{Q}_1(\varepsilon,\ x) & \text{for } \varepsilon_0''>\varepsilon>\varepsilon_0''',\end{cases} \tag{AIII·1}$$

where

$$\mathcal{H}_1(\varepsilon,\ x)=\sum_{l=0}^{\infty}\frac{x^l}{l!}((1-\delta)\nu)^l h_l^{\,0}(\varepsilon),$$

$$\mathcal{P}_1(\varepsilon,\ x)=\sum_{l=0}^{\infty}\frac{x^l}{l!}((1-\delta)\nu)^l p_l(\varepsilon),$$

$$\mathcal{Q}_1(\varepsilon,\ x)=\sum_{l=0}^{\infty}\frac{x^l}{l!}((1-\delta)\nu)^l q_l(\varepsilon). \tag{AIII·2}$$

Using the notations

$$z=4(1-\delta)\nu x,$$

$$I_1(\sqrt{z})=\sum_{l=0}^{\infty}\frac{1}{l!(l+1)!}\Big(\frac{z}{2}\Big)^{l+1},\quad (I_1:\ \text{The imaginary Bessel function of 1st order})$$

$$J_1(\varepsilon,\ z)=\sqrt{\frac{x}{(1-\delta)\nu\ln\varepsilon_0/\varepsilon^k}}\,\frac{1}{k-1}\Big\{kI_1(\sqrt{z\ln\varepsilon_0/\varepsilon^k})-\sqrt{k}\,I_1(\sqrt{z\ln\varepsilon_0'/\varepsilon}\Big\}, \tag{AIII·3}$$

$$G_1(\varepsilon,\ z)=\sqrt{\frac{x}{(1-\delta)\nu\ln\varepsilon_0/\varepsilon^{k^2}}}\,\frac{1}{k-1}\,\frac{1}{k^2-1}$$

$$\times\Big\{kI_1(\sqrt{z\ln\varepsilon_0/\varepsilon^{k^2}})-(k+1)\sqrt{k}\,I_1(\sqrt{z\ln\varepsilon_0'/\varepsilon^k}+kI_1(\sqrt{z\ln\varepsilon_0''/\varepsilon})\Big\},$$

we obtain

$$\mathcal{H}_1(\varepsilon,\ x)=\sqrt{\frac{x}{(1-\delta)\nu\ln\varepsilon_0/\varepsilon}}\,I_1(\sqrt{z\ln\varepsilon_0/\varepsilon}),$$

$$\mathcal{P}_1(\varepsilon,\ x)=\mathcal{H}_1(\varepsilon,\ x)-J_1(\varepsilon,\ z)+\kappa\int_{\varepsilon^{k^2}}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\Big(\frac{\varepsilon^k}{\varepsilon'}\Big)^{\tau}\varepsilon_0\frac{\partial}{\partial\varepsilon_0}J_1(\varepsilon'^{1/k},\ z), \tag{AIII·4}$$

$$\mathcal{Q}_1(\varepsilon,\ x)=\mathcal{P}_1(\varepsilon,\ x)+G_1(\varepsilon,\ z)-\kappa\Big\{\int_{\varepsilon^k}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\Big(\frac{\varepsilon^{k^2}}{\varepsilon'}\Big)^{\tau}\varepsilon_0\frac{\partial}{\partial\varepsilon_0}G_1(\varepsilon^{1/k^2},\ z)\Big(1+\frac{\kappa}{k-1}\Big)$$

$$+k\int_{\varepsilon^k}^{\varepsilon_0'}\frac{d\varepsilon'}{\varepsilon'}\Big(\frac{\varepsilon^k}{\varepsilon'}\Big)^{\tau}\varepsilon_0\frac{\partial}{\partial\varepsilon_0}G_1(\varepsilon'^{1/k},\ z)\Big(1-\frac{\kappa k}{k-1}\Big)\Big\}.$$

In a similar manner the differential number of neutral pions is given by

$$N_{\pi^0}(\varepsilon,\ x)=\frac{(1-\delta)(1-\nu)}{\nu}\,e^{-x}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1+\delta}\frac{1}{\varepsilon_0}\begin{cases}\mathcal{H}_2(\varepsilon,\ x) & \text{for } \varepsilon_0>\varepsilon>\varepsilon_0',\\ \mathcal{P}_2(\varepsilon,\ x) & \text{for } \varepsilon_0'>\varepsilon>\varepsilon_0'',\\ \mathcal{Q}_2(\varepsilon,\ x) & \text{for } \varepsilon_0''>\varepsilon>\varepsilon_0''',\end{cases}\quad\text{(AIII·5)}$$

where

$$\mathcal{H}_2(\varepsilon,\ x)=\sqrt{\frac{(1-\delta)\nu\ln\varepsilon_0/\varepsilon}{x}}\,I_1(\sqrt{z\ln\varepsilon_0/\varepsilon})+\delta(x),$$

$$\mathcal{P}_2(\varepsilon,\ x)=\mathcal{H}_2(\varepsilon,\ x)-J_2(\varepsilon,\ x)+\kappa\int_{\varepsilon^k}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)^{\tau}\varepsilon_0\frac{\partial}{\partial\varepsilon_0}J_2(\varepsilon'^{1/k},\ z),\quad\text{(AIII·6)}$$

$$\mathcal{Q}_2(\varepsilon,\ x)=\mathcal{P}_2(\varepsilon,\ x)+G_2(\varepsilon,\ z)-\kappa\left\{\int_{\varepsilon^{k^2}}^{\varepsilon_0}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^{k^2}}{\varepsilon'}\right)^{\tau}\varepsilon_0\frac{\partial}{\partial\varepsilon_0}G_2(\varepsilon'^{1/k^2},\ z)\left(1+\frac{\kappa}{k-1}\right)\right.$$

$$\left.+k\int_{\varepsilon^k}^{\varepsilon_0'}\frac{d\varepsilon'}{\varepsilon'}\left(\frac{\varepsilon^k}{\varepsilon'}\right)^{\tau}\varepsilon_0\frac{\partial}{\partial\varepsilon_0}G_2(\varepsilon'^{1/k},\ z)\left(1-\frac{\kappa k}{k-1}\right)\right\},$$

and

$$z=4(1-\delta)\nu x$$

$$J_2(\varepsilon,\ z)=\sqrt{\frac{(1-\delta)\nu\ln\varepsilon_0/\varepsilon^k}{x}}\,\frac{1}{k-1}\left\{kI_1(\sqrt{z\ln\varepsilon_0/\varepsilon^k})-\sqrt{1/k}\,I_1(\sqrt{z\ln\varepsilon_0'/\varepsilon})\right\},\quad\text{(AIII·7)}$$

$$G_2(\varepsilon,\ z)=\sqrt{\frac{(1-\delta)\nu\ln\varepsilon_0/\varepsilon^{k^2}}{x}}\,\frac{1}{k-1}\,\frac{1}{k^2-1}\left\{kI_1(\sqrt{z\ln\varepsilon_0/\varepsilon^{k^2}})\right.$$

$$\left.-\sqrt{1/k}\,(k+1)I_1(\sqrt{z\ln\varepsilon_0'/\varepsilon^k})+\frac{1}{k}I_1(\sqrt{z\ln\varepsilon_0''/\varepsilon})\right\}.$$

Appendix IV. The derivation of an approximate formula for the electron component

a. *The cascade function used in* § 5

Heisenberg,[18] following the method of Landau and Rumer,[19] derived the following approximately analytic expression for the number of electrons of energies greater than ε at the depth t in a shower initiated by a photon of an energy W_0:

$$\Pi_H(W_0,\ \varepsilon,\ t)=\left(\frac{y-\beta}{t-\alpha}\right)^{1/2}t^{-1/2}\exp\left\{-t+2[(t-\alpha)(y-\beta)]^{1/2}\right\},\quad\text{(AIV·1)}$$

where $y=\ln(W_0/\varepsilon)$, $\alpha=1.4$ and $\beta=0.56$. Comparing $\Pi_H(W_0,\ \varepsilon_c,\ t)$, where ε_c

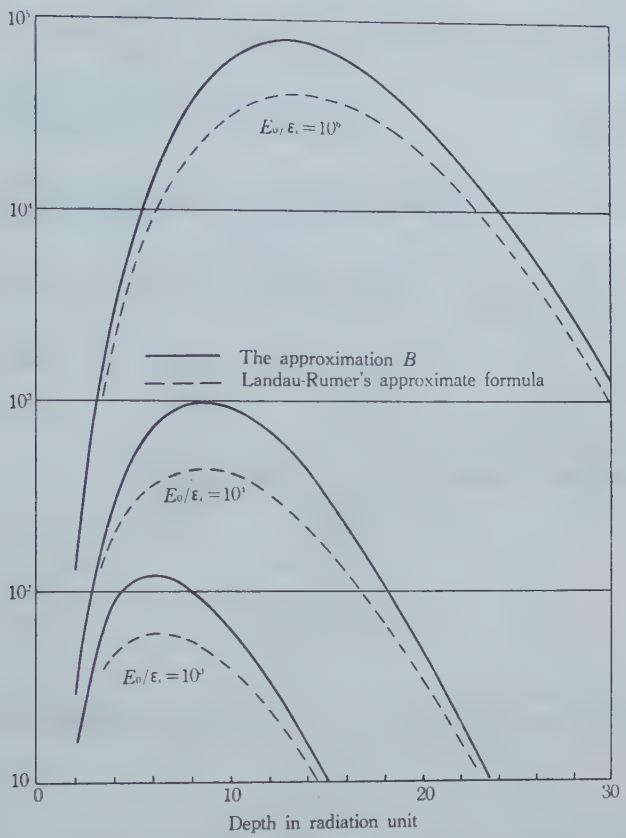

Fig. 15. Comparison of the Landau-Rumer's approximate formula with the approximation B for electron cascade initiated by a single photon of energy E_0. The curves with the approximation B were reproduced from the paper by Rossi and Greisen (Rev. Mod. Phys. **13** (1941), 240)

is the critical energy, with the cascade function in the approximation B, we find that the error is small concerning both the optimum depth and its longitudinal spread as shown in Fig. 16. The total number of electrons at the optimum depth in Heisenberg's one is smaller by a factor of 1.4 than that in the approximation B, independently of the initial photon energy. Thus we use as a cascade function Heisenberg's one multiplied by a factor of 1.4. The main differences between this modified cascade function and that in the approximation B appear at the beginning stage and the tail of the cascade process. But these differences will be small enough to be ignored, so far as we are concerned with the electron component arising from the nucleonic cascade process.

b. *The details of the saddle point method used in* § 5

First we shall find the saddle point (y_1, t_1) defined in eq. (5·12). From the next expression of eq. (5·10), we obtain

$$\left.\frac{\partial f}{\partial y'}\right|_{\substack{y'=y_1\\ t'=t_1}}=\delta+\sqrt{\frac{(1-\delta)\nu}{m}}\sqrt{\frac{t_1}{y_1}}-\sqrt{\frac{t-t_1-\alpha}{y_0-y_1-\beta}}=0,$$

$$\left.\frac{\partial f}{\partial t'}\right|_{\substack{y'=y_1\\ t'=t_1}}=1-\frac{1}{m}+\sqrt{\frac{(1-\delta)\nu}{m}}\sqrt{\frac{y_1}{t_1}}-\sqrt{\frac{y_0-y_1-\beta}{t-t_1-\alpha}}=0. \quad \text{(AIV·2)}$$

These equations can be solved with respect to $\frac{t_1}{y_1}$ and $\frac{t-t_1-\alpha}{y_0-y_1-\beta}$ as follows:

$$\frac{t_1}{y_1}=\frac{m}{4(m-1)^2(1-\delta)\nu}\{m-(m-1)\delta-(1-\delta)\nu\pm\sqrt{\quad}\}^2\equiv A_\pm,$$

$$\frac{t-t_1-\alpha}{y_0-y_1-\beta}=\frac{1}{4(m-1)^2}\{m+(m-1)\delta-(1-\delta)\nu\pm\sqrt{\quad}\}^2\equiv B_\pm, \quad \text{(AIV·3)}$$

where $\sqrt{\quad}$ is given by eq. (5·14).

Therefore the saddle point (y_1, t_1) is represented by

$$y_1=\frac{t_0'-B_\pm y_0'}{A_\pm-B_\pm},$$

$$t_1=\frac{A_\pm(t_0'-B_\pm y_0')}{A_\pm-B_\pm}, \quad \text{(AIV·4)}$$

where $y_0'=y_0-\beta$ and $t_0'=t-\alpha$.

With use of the conditions $m>1$, $1>\nu>0$ and $1>\delta>0$, it is easily shown that

$$A_+>B_+>B_->A_->0. \quad \text{(AIV·5)}$$

Instead of the condition (5·11), we use the condition that

$$0<t_1<t_0',\ 0<y_1<y_0'. \quad \text{(AIV·6)}$$

The difference between these conditions is not essential. Substituting eq. (AIV·4) into eq. (AIV·6) under the inequality (AIV·5), we finally obtain

$$B_-y_0'>t_0'>A_-y_0', \quad \text{(AIV·7)}$$

and

$$A_+y_0'>t_0'>B_+y_0'. \quad \text{(AIV·8)}$$

As long as the initial energy E_0 and the depth t satisfy the inequalities (AIV·7) and (AIV·8), the saddle point method can be used for the evaluation of eq. (5·10). The regions A and C called in § 5 are defined by the conditions (AIV·7) and (AIV·8) respectively.

Next we shall evaluate eq. (5·10). Eq. (5·10) turns out, by means of the saddle point method, to be

$$N_e(E_0,\ 0,\ t)=\frac{1.4}{\sqrt{\pi}}\frac{(1-\delta)(1-\nu)}{m}\left\{\frac{(1-\delta)\nu}{m}t_1y_1\right\}^{-1/4}\left\{\frac{y_0'-y_1}{t_0'-t_1}\right\}^{1/2}(t-t')^{-1/2}$$

$$\times \exp\{f(y_1,\ t_1)\} \cdot \int dt' \int dy' \exp\left[\left(\frac{\partial^2 f}{\partial t'^2}\right)_{\substack{y'=y_1\\ t'=t_1}} \frac{(t'-t_1)^2}{2!} + \left(\frac{\partial^2 f}{\partial y'^2}\right)_{\substack{y'=y_1\\ t'=t_1}} \frac{(y'-y_1)^2}{2!}\right.$$

$$\left. + \left(\frac{\partial^2 f}{\partial t' \partial y'}\right)_{\substack{y'=y_1\\ t'=t_1}} (y'-y_1)(t'-t_1)\right]$$

$$= \frac{1.4}{\sqrt{\pi}} \frac{(1-\delta)(1-\nu)}{m} \left\{\frac{(1-\delta)\nu}{m} t_1 y_1\right\}^{-1/4} \left\{\frac{y_0'-y_1}{t_0'-t_1}\right\}^{1/2} (t-t')^{-1/2}$$

$$\times \exp\{f(y_1,\ t_1)\} \cdot 2\pi \left[\frac{\partial^2 f}{\partial y'^2} \frac{\partial^2 f}{\partial t'^2} - \left(\frac{\partial^2 f}{\partial y' \partial t'}\right)^2\right]^{-1/2}_{\substack{y'=y_1\\ t'=t_1}} \qquad \text{(AIV·9)}$$

where

$$f(y_1,\ t_1) = -\alpha + \frac{y_0-\beta}{2(m-1)} \{m+(m-1)\delta-(1-\delta)\nu \pm \sqrt{\quad}\}$$

$$-(t-\alpha)\left\{1-\frac{1}{2\delta m}[m+(m-1)\delta-(1-\delta)\nu \mp \sqrt{\quad}]\right\}, \qquad \text{(AIV·10)}$$

and

$$\left[\frac{\partial^2 f}{\partial t'^2} \frac{\partial^2 f}{\partial y'^2} - \left(\frac{\partial^2 f}{\partial t' \partial y'}\right)^2\right]_{\substack{y'=y_1\\ t'=t_1}}$$

$$= \frac{(\sqrt{\quad})^2 \{m+(m-1)\delta-(1-\delta)\nu \mp \sqrt{\quad}\} \{m-(m-1)\delta-(1-\delta)\nu \mp \sqrt{\quad}\}}{16\delta^2 m^3 (1-\delta)\nu y_1 (y_0-y_1-\beta)}.$$

$$\text{(AIV·11)}$$

Substituting eqs. (AIV·10), (AIV·11) into eq. (AIV·9), we finally obtain

$$N_e(E_0,\ 0,\ t) = 2.8\sqrt{2\pi}\, e^{-\alpha} \frac{1-\nu}{\sqrt{\quad}} \left(\frac{1-\delta}{\nu}\right)^{1/2} \left(\frac{m}{m-1}\right)^{1/2}$$

$$\times \frac{1}{\sqrt{m+(m-1)\delta-(1-\delta)\nu \pm \sqrt{\quad}}} \left(\frac{t-t_1-\alpha}{t-t_1}\right)^{1/2}$$

$$\times \exp\left[\frac{m+(m-1)\delta-(1-\delta)\nu \pm \sqrt{\quad}}{2(m-1)} (y_0-\beta)\right.$$

$$\left. + \left\{\frac{1}{2\delta m}[m+(m-1)\delta-(1-\delta)\nu \mp \sqrt{\quad}]-1\right\}(t-\alpha)\right]. \qquad \text{(AIV·12)}$$

Putting

$$\frac{1}{\lambda} = 1 - \frac{1}{2\delta m} \{m+(m-1)\delta-(1-\delta)\nu - \sqrt{\quad}\}, \qquad \text{(AIV·13)}$$

we obtain

$$A_+ = \frac{m}{(1-\delta)\nu} \left(\frac{\lambda m}{\lambda - m}\right)^2, \qquad B_+ = \left(\frac{\lambda}{\lambda - 1}\right)^2,$$

$$A_-=\frac{m\delta^2}{(1-\delta)\nu(m-1)^2}\left(\frac{\lambda m}{\lambda-m}\right)^2, \quad B_-=\frac{\delta^2 m^2}{(m-1)^2}\left(\frac{\lambda-1}{\lambda}\right)^2.$$

From eqs. (AIV·7), (AIV·8), (AIV·12), (AIV·13) and (AIV·14), we get eqs. (5·13) and (5·14).

Appendix V. Calculation of the lateral distribution of the electron component

It is assumed that the lateral spread of total electrons are only due to multiple Coulomb scattering, whereas the lateral spreads due to the angular spreads of nuclear active particles and of two-photons-decays of neutral pions are negligible.

As the structure function an approximate formula of the Nishimura-Kamata function under Approximation B

$$f\left(\frac{r}{r_1}, s\right)=\left(\frac{r}{r_1}\right)^{s-2}C(s)\left[1+\frac{4}{s}\frac{r}{r_1}\right]\exp\{-a(s)(r/r_1)^{b(s)}\}$$
$$\equiv(r/r_1)^{s-2}\chi(r/r_1, s) \qquad \text{(AV·1)}$$

is used which was derived by Nishimura and Kamata.[19] This formula coincides with numerically evaluated values of the $N-K$ function for $s=0.6$, 1, 1.4 and 2 within errors of several per cent.

Let $\pi_l^0(E, x)\,dE$ be the differential number of neutral pions produced by nuclear active particles of the $(l-1)$th generation, at the depth x measured in the nuclear collision mean free path. The number of electrons at depth t (in the radiation unit) and at the distance r/r_1 (r_1: Molière unit) from the core in the showers initiated by the neutral pions is given by

$$\Pi_l(E, 0, t, r/r_1)\,dE\cdot 2\pi(r/r_1)\,d(r/r_1)$$
$$=dE\cdot 2\pi(r/r_1)\,d(r/r_1)\int_0^t \pi_l^0\left(E, \frac{t'}{m}\right)\frac{dt'}{m}\Pi(E, 0, t-t', r/r_1), \qquad \text{(AV·2)}$$

where

$$\Pi(E, 0, t-t', r/r_1)2\pi(r/r_1)\,d(r/r_1)=2\pi(r/r_1)\,d(r/r_1)$$
$$\times\frac{1}{\pi i}\int_{\delta-i\infty}^{\delta+i\infty} ds\,\frac{M(s)K_1(s_1-s)}{s+1}\left(\frac{E}{\varepsilon_c}\right)^s\exp[\lambda_1(s)(t-t')-\tfrac{1}{2}\log s]f(r/r_1, s)$$
$$\text{(AV·3)}$$

is the lateral distribution of an electron shower started by a single neutral pion having energy E at depth t'. Inserting eq. (AV·3) into eq. (AV·2) and using the saddle point method, we get

$$\Pi_l(E, 0, t, r/r_1)\,dE\cdot 2\pi(r/r_1)\,d(r/r_1)=\pi_l^0(E)\,dE\cdot 2\pi(r/r_1)\,d(r/r_1)$$
$$\times\sqrt{\frac{2}{\pi}}\,\frac{M(\bar{s})K_1(\bar{s}_1-\bar{s})}{(\bar{s}+1)\sqrt{\bar{s}\vartheta_l''(\bar{s})}}\left(\frac{Er}{\varepsilon_c r_1}\right)^{\bar{s}}\exp[\lambda_1(\bar{s})t](m\bar{\xi})^{-l}\cdot f_{l-1}(\bar{\xi}t)\,(r/r_1)^{-2}\chi(r/r_1, \bar{s}),$$
$$\text{(AV·5)}$$

where

$$\mathfrak{g}_l(s)=\ln\left(\frac{Er}{\varepsilon_0 r_1}\right)\cdot s+\lambda_1(s)t-\tfrac{1}{2}\ln s-\ln(s+1)+\ln M(s)+\ln\{f_{l-1}(\bar{\xi}t)/\bar{\xi}^l\},$$

$$f_l(\bar{\xi}t)=\int_0^{\bar{\xi}t}\frac{e^{-y}y^l}{l!}dy,\quad \bar{\xi}=\frac{1}{m}+\lambda_1(s) \qquad \text{(AV·5′)}$$

and $\bar{s}$ is determined by $\mathfrak{g}_l'(\bar{s})=0$. The other notations except for m are the same as those used in the paper A.

It is worth noting that in the right-hand side of eq. (AV·4) the factor

$$\sqrt{\frac{2}{\pi}}\cdots\cdots f_{l-1}(\bar{\xi}t)$$

coincides with the total number of electrons in a shower initiated by a neutral pion of energy $(Er/\varepsilon_c r_1)$ (see eq. (13) in the paper A). Consequently, to evaluate eq. (AV·4) the same procedure can be used as was done in the paper A, if we evaluate the factor $(r/r_1)^{-2}\chi(r/r_1, s)$ for several values of r/r_1 and s.

A shower age s in the three dimensional cascade theory depends not only on the primary energy and the depth but also on the distance from the shower axis

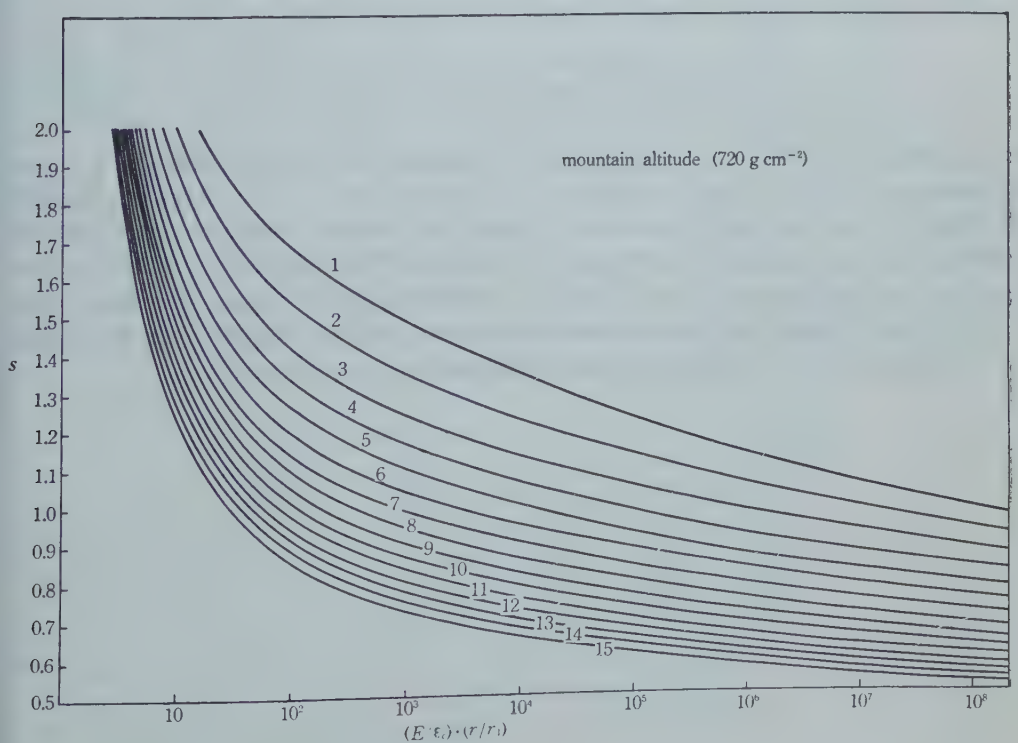

Fig. 16

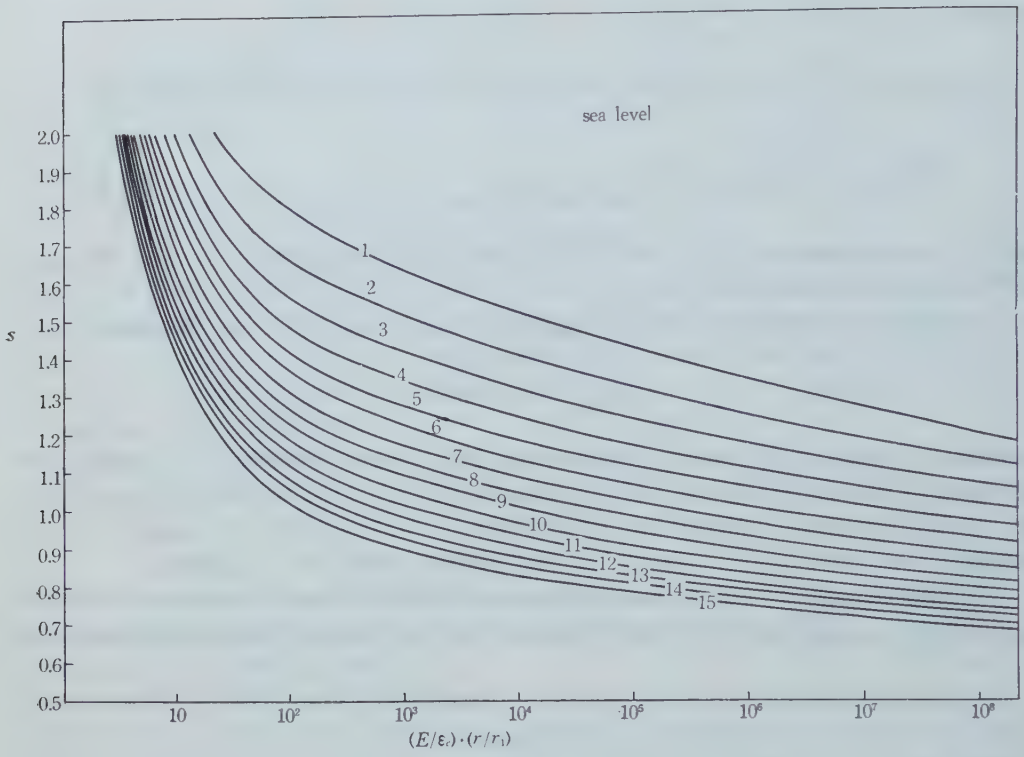

Fig. 17

as a function of $(Er/\varepsilon_c r_1)$. In our case s depends also on the generation l. s as a function of $(Er/\varepsilon_c r_1)$ and l is shown in Figs. 16 and 17 at mountain altitude (720 gcm^{-2}) and sea level respectively, where the collision mean free path and the radiation unit are taken to be 90 gcm^{-2} and 38 gcm^{-2} respectively. Since shower ages in the figures coincide with those of one-dimensional cascade showers when $r/r_1=1$, these figures will give a rough measure of ages of cascade showers initiated by neutral pions of various energies produced at various depths.

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Octupole Type Deformation of Nuclear Surface

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The shell structure dependence of the surface rigidity to nuclear octupole type deformation is examined. In order to calculate the energy change due to octupole type deformation, a coordinate transformation which transforms a deformed surface into a spherical one is introduced. The results of our calculations are in good agreement with the trend of experimental data suggested by low lying odd parity states of even-even nuclei and display a characteristic shell structure dependence as well as quadrupole type deformation. According to the deformed potenital model, it is shown that octupole type deformations are closely connected with quadrupole type deformations. This fact may play an important role in determining the spin values of low lying odd parity states of even-even nuclei and in understanding a mechanism of asymmetric fission.

§ 1. Introduction

In certain regions of elements, even-even nuclei possess rather low lying states of odd parity. These states were observed first in α-decay spectrum of the heavy even-even nuclei having $Z\sim 90$ and $N\sim 136$.[1] The states of this type are found also in the even-even nuclei in the rare earth region with $Z\sim 60$ and $N\sim 90$.[2] Analyses of gamma-transitions from these states suggest that these states may represent the existence of the octupole type oscillations of nuclear surface and systematic occurrence of these states may suggest a close relation between octupole type surface oscillations and shell structure. An asymmetric fission of heavy nuclei is also one of the evidence of existence of this type deformation of nuclear surface.[3]

The E 3-transitions of the j-forbidden type occur in odd-A nuclei having neutron or proton numbers 43, 45 and 47. This may suggest that the surface rigidity for octupole type deformation is very small in the subshell of nucleon number 40.

Some of the qualitative properties of octupole type oscillations of nuclear surface have already been discussed by Copenhagen group.[4] Bohr and Mottelson have estimated the electric-dipole transition probabilities and attempted to obtain information about octupole type distortion.[5] Fröman has studied odd parity α-transitions of even-even nuclei and estimated the order of magnitude of octupole type distortion.[6] Recently, Lee and Inglis have calculated the energy change associated with pear-shaped deformation using the nucleon wave functions of a spheroidal harmonic oscillator potential.[7] They found that this energy change was always positive but they could not examine the shell structure dependence of pear-shaped deformation

because of their neglect of spin-orbit coupling term.

In this paper, according to the theory of MSY[8] on the surface rigidity, we calculate the surface rigidity C_3 of closed shell (core) to octupole type deformation (§ 2), discuss the shell structure dependence of C_3 (§ 3), and examine qualitatively the properties of octupole type oscillations (§ 4). The theory of MSY is not a complete theory of the quantum mechanical description of collective oscillations, but it can reflect the behavior of surface rigidity[8),9] correctly. In § 2, we introduce a coordinate transformation which transforms an arbitrarily deformed surface into a spherical one. As the results of this transformation, the kinetic energy part of a single particle Hamiltonian becomes to depend on the deformation parameters, and in the case of small deformation we can expand this part as power series into the deformation parameters and treat these terms as perturbations. This coordinate transformation may be applicable to many problems as a general method for a boundary shape perturbation. C_2 (the surface rigidity for quadrupole type deformation) by our coordinate transformation coincides with that by Moszkowski's transformation.[10] In order to examine the shell structure dependence of surface rigidity, we take into account a spin-orbit coupling, and for simplicity neglect the Coulomb interactions between protons. Therefore, C_3 of this paper is the surface rigidity of neutron core.

According to our calculations, the surface rigidities to octupole type deformation display characteristic shell structure dependence, which agrees well with the general trend suggested by empirical data. According to the deformed potential model, octupole type deformations are closely connected with quadrupole type deformations. This fact may play a very important role in determining the spins of low lying odd parity states of even-even nuclei and in understanding the relation between growth of nuclear deformations and the process of asymmetric fission. Simple and qualitative discussions on the behavior of octupole type oscillation are given in § 4.

§ 2. Deformed potential model and surface rigidity

Using the theory of MSY[8] on the quantum mechanical description of nuclear collective motion, we calculate the surface rigidity to nuclear octupole type deformation. This theory is not complete in order to account for the moment of inertia of nucleus,[9] but reproduced the qualitative behavior of the surface rigidity to quadrupole type deformation in good agreement with empirical data.[8),11] We suppose that the behavior of the surface rigidity to octupole type deformation is also well reproduced by this theory, hence we use this theory as a basis of the calculation.

We assume that a nucleus is composed of a core (closed shell) and several extra-particles in an unfilled shell, and use the adiabatic approximation to the surface motion of the core and the motion of the extra-particles compared with the individual particle motions of core-particles. On these assumptions, the motions of core-particles are described approximately by a deformed infinite square well potential. If we define the deformed surface of core as

$$R(\theta, \varphi)=R_0[1+\sum_{\lambda,\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta, \varphi)], \tag{1}$$

then the surface rigidity $C_{\lambda\mu}$ is defined as the expansion coefficient of the total energy $E_{\text{core}}(\alpha)$ of the individual particle motion of core-particles in this deformed potential[8]:

$$E_{\text{core}}(\alpha)=\sum_{p=1}^{N}E_p(\alpha)=E_0+\sum_{\lambda,\mu}\alpha_{\lambda\mu}E^{(1)}_{\lambda\mu}+\sum_{\lambda,\mu}|\alpha_{\lambda\mu}|^2E^{(2)}_{\lambda\mu}, \tag{2}$$

and

$$C_{\lambda\mu}=2E^{(2)}_{\lambda\mu}, \tag{3}$$

where N is the number of the core-particles and R_0 is the mean radius of the core and we assume that the deviation from the equilibrium spherical shape is small. $E_p(\alpha)$ is a single particle energy in the deformed infinite square well potential. This potential is expressed as follows:

$$U(\boldsymbol{r};\alpha)=\begin{cases}0: & \text{inside the deformed core,}\\ \infty: & \text{outside the deformed core.}\end{cases} \tag{4}$$

The Hamiltonian that describes the single particle motion is

$$H=\boldsymbol{p}^2/2M+k\boldsymbol{l}\cdot\boldsymbol{s}+U(\boldsymbol{r};\alpha). \tag{5}$$

In the following calculations, we neglect the μ-dependence of $C_{\lambda\mu}$. This simplification is equivalent with the assumption of axially symmetric deformation. We describe the particle motion in the coordinate system fixed with the core and select the z-axis so as to coincide with the symmetry axis of the core. The surface of the core is expressed as

$$R(\theta)=R_0f(\theta)=R_0[1+\sum_{l}a_lY_{l0}(\theta, \varphi)], \tag{1'}*$$

in this coordinate system.

In order to calculate the energy eigenvalue $E_p(\alpha)$ to the Hamiltonian (5), we introduce a coordinate transformation

$$\begin{aligned}&r=r'f(\theta'), \quad \theta=\theta' \quad \text{and} \quad \varphi=\varphi',\\ &f(\theta')=1+\sum_{l}a_lY_{l0}(\theta', \varphi'),\end{aligned} \tag{6}$$

where (r, θ, φ) and (r', θ', φ') are polar-coordinates in the body fixed frame before

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* In general, the principal axis for the deformations of different l are independent except for the conditions of constancy of volume and center of mass. If a deformed surface is expressed as eq. (1'), then the relative phase of each order (l) of deformation are strongly restricted. We must pay attention to this fact in the case of the quantum mechanical treatment of collective oscillation of nuclear surface.

nd after this coordinate transformation, respectively.* We consider only $l=2$ and , in the following calculations.** In the expression (1′), a_0 and a_1 are eliminated y the conditions of constancy of volume and center of mass:

$$a_0+\frac{1}{\sqrt{4\pi}}\sum_{n=0}^{\infty}a_n{}^2+\frac{1}{3\sqrt{4\pi}}\sum_{n,n',n''=0}^{\infty}\langle nn'n''\rangle a_n a_{n'}a_{n''}=0,$$

$$a_0+\frac{3}{2}\sum_{n,n'=0}^{\infty}\langle 1nn'\rangle a_n a_{n'}+\sum_{n,n',n''=0}^{\infty}\langle 1nn'n''\rangle a_n a_{n'}a_{n''}$$

$$+\frac{1}{4}\sum_{n,n',n'',n'''=0}^{\infty}\langle 1nn'n''n'''\rangle a_n a_{n'}a_{n''}a_{n'''}=0.$$

here

$$\langle n\cdots n'\rangle=\iint Y_{n0}\cdots Y_{n'0}\sin\theta\,d\theta\,d\varphi.$$

olving these equations and picking up the first several terms, we have

$$a_0=-\frac{1}{\sqrt{4\pi}}\left[a_2{}^2+a_3{}^2+\sqrt{\frac{4}{5\pi}}a_2a_3{}^2\right],$$

nd

$$a_1=-\frac{9}{\sqrt{4\pi}}\sqrt{\frac{3}{35}}\,a_2a_3\,.$$

As the results of the coordinate transformation (6), the deformed potential $U(\boldsymbol{r}\,;a)$ is transformed into a spherically symmetric potential $U(r')$:

---

* The Moszkowski's transformation[10]

$$X=D^{-1/2}X',\ Y=D^{-1/2}Y' \text{ and } Z=DZ',$$

$$X'^2+Y'^2+Z'^2=R_0{}^2$$

s written as

$$r=r'\sqrt{D^{-1}\sin^2\theta'+D^2\cos^2\theta'},\ \tan\theta=D^{-3/2}\tan\theta' \text{ and } \varphi=\varphi',$$

n the polar-coordinate. Here, we put $D=1+\sqrt{5/4\pi}\,\alpha_2$ and expand it up to $\alpha_2{}^2$, then this transformation is approximately

$$r=r'f_M(\theta'),\quad \theta=g_M(\theta') \text{ and } \varphi=\varphi',$$

$$f_M(\theta)=1+\frac{2}{4\pi}\alpha_2{}^2+\alpha_2\left(1-\frac{1}{7}\sqrt{\frac{5}{4\pi}}\alpha_2\right)Y_{20}(\theta,\varphi)-\frac{3}{7\sqrt{4\pi}}\alpha_2{}^2Y_{40}(\theta,\varphi),$$

$$g_M(\theta)=\theta+\frac{\alpha_2}{2}\left(1-\frac{11}{28}\sqrt{\frac{5}{4\pi}}\alpha_2\right)\frac{\partial Y_{20}}{\partial\theta}-\frac{13}{14\sqrt{4\pi}}\alpha_2{}^2\frac{\partial Y_{20}}{\partial\theta}.$$

In this approximation, the surface of core is

$$R_M(\theta)=R_0\left[1-\frac{\alpha_2{}^2}{4\pi}+\alpha_2\left(1-\frac{4}{7}\sqrt{\frac{5}{4\pi}}\alpha_2\right)Y_{20}(\theta,\varphi)+\frac{9}{7\sqrt{4\pi}}\alpha_2{}^2Y_{40}(\theta,\varphi)\right].$$

** The deformation of $l=4$ were examined by Suekane and Yamaguchi.[12]

$$U(r')=\begin{cases} 0: & r'\leq R_0, \\ \infty: & r'>R_0, \end{cases} \tag{4'}$$

and the Laplacian operator of the kinetic energy term is expressed as

$$\Delta=\Delta'+\frac{1}{f^3}\Big[(f-f^3)\Delta'+\frac{1}{f}\Big(\frac{\partial f}{\partial\theta'}\Big)^2\Big(\Delta'+\frac{\boldsymbol{L}^2}{r'^2}\Big)+\frac{(\boldsymbol{L}^2 f)}{r'}\frac{\partial}{\partial r'}-\Big(\frac{\partial f}{\partial\theta'}\Big)\frac{2}{r'}\frac{\partial^2}{\partial r'\partial\theta'}\Big], \tag{7}$$

where

$$\boldsymbol{L}^2=-\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\Big(\sin\theta'\frac{\partial}{\partial\theta'}\Big)-\frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\varphi'^2}.$$

Then, Hamiltonian (5) is expressed as follows:*

$$H=H_0+H' \tag{8}$$

$$H_0=\boldsymbol{p}'^2/2M+U(\boldsymbol{r}')+k\boldsymbol{l}'\cdot\boldsymbol{s}', \tag{9}$$

$$H'=a_2H_2+a_3H_3+a_2^2H_{22}+a_3^2H_{33}+a_2a_3H_{23}+a_2a_3^2H_{233}, \tag{10}$$

where

$$H_2=2Y_{20}\Delta'-Y_{20}\frac{6}{r'}\frac{\partial}{\partial r'}+\frac{\partial Y_{20}}{\partial\theta'}\frac{2}{r'}\frac{\partial^2}{\partial r'\partial\theta'}, \tag{10·1}$$

$$H_3=2Y_{30}\Delta'-Y_{30}\frac{12}{r'}\frac{\partial}{\partial r'}+\frac{\partial Y_{30}}{\partial\theta'}\frac{2}{r'}\frac{\partial^2}{\partial r'\partial\theta'}, \tag{10·2}$$

$$H_{22}=(3Y_{20}Y_{20}-2/4\pi)\Delta'-\frac{\partial Y_{20}}{\partial\theta'}\frac{\partial Y_{20}}{\partial\theta'}\Big(\Delta'+\frac{\boldsymbol{L}^2}{r'^2}\Big), \tag{10·3}$$

$$H_{33}=(3Y_{30}Y_{30}-2/4\pi)\Delta'-\frac{\partial Y_{30}}{\partial\theta'}\frac{\partial Y_{30}}{\partial\theta'}\Big(\Delta'+\frac{\boldsymbol{L}^2}{r'^2}\Big), \tag{10·4}$$

$$H_{23}=\{6Y_{20}Y_{30}-9(3/35\pi)^{1/2}Y_{10}\}\Delta'-2\frac{\partial Y_{20}}{\partial\theta'}\frac{\partial Y_{30}}{\partial\theta'}\Big(\Delta'+\frac{\boldsymbol{L}^2}{r'^2}\Big)$$
$$+9(3/35\pi)^{1/2}\Big(\frac{Y_{10}}{r'}\frac{\partial}{\partial r'}-\frac{\partial Y_{10}}{\partial\theta'}\frac{1}{r'}\frac{\partial^2}{\partial r'\partial\theta'}\Big), \tag{10·5}$$

$$H_{233}=\{3Y_{20}Y_{30}Y_{30}-27(3/35\pi)^{1/2}Y_{10}Y_{30}-3Y_{20}/2\pi-1/\sqrt{5\pi}\}\Delta'$$
$$+\Big\{9(3/35\pi)^{1/2}\frac{\partial Y_{10}}{\partial\theta'}\frac{\partial Y_{30}}{\partial\theta'}+Y_{20}\frac{\partial Y_{30}}{\partial\theta'}\frac{\partial Y_{30}}{\partial\theta'}+2\frac{\partial Y_{20}}{\partial\theta'}\frac{\partial Y_{30}}{\partial\theta'}Y_{30}\Big\}\Big(\Delta'+\frac{\boldsymbol{L}^2}{r'^2}\Big). \tag{10·6}$$

* In this treatment, we assumed that $\boldsymbol{l}\cdot\boldsymbol{s}$-coupling term is not affected by this coordinate transformation. It may depend on the origin of the $\boldsymbol{l}\cdot\boldsymbol{s}$-coupling term whether such assumption is valid or not. Since the $\boldsymbol{l}\cdot\boldsymbol{s}$-coupling term used in the shell model is set up under the assumption of the spherically symmetric averaged potential, the adoption of $\boldsymbol{l}\cdot\boldsymbol{s}$-coupling term in such a transformed coordinate system would not always be nonsense.

We treat H' as a perturbation. The eigen-function of H_0 is a single particle wave function $\psi^0_{nljm}(\boldsymbol{r})$ of the jj-coupling shell model and its corresponding eigen-value is E^0_{nlj}. In the expressions (10), we neglected the terms of higher order than a_2^2 and a_3^2, and reserved the terms of a_2a_3 and $a_2a_3^2$ in order to investigate the coupling between the deformations of order 2 and order 3. As the results of the coordinate transformation (6), the volume element is transformed as follows:

$$r^2\sin\theta\, dr d\theta\, d\varphi = f^3(\theta')\, r'^2 \sin\theta' dr' d\theta' d\varphi', \tag{11}$$

so that we must pay attention to that the orthogonality of the wave function ψ^0_{nljm} does not hold now and we must use the non-orthogonal perturbation method.

The energy of a single particle in a deformed potential is expressed as

$$E_p(a) = E_{nljm}(a) = E^0_{nlj} + a_2E^2_{nljm} + a_2^2E^{22}_{nljm} + a_3^2E^{33}_{nljm} + a_2a_3^2E^{233}_{nljm}. \tag{12}$$

The total energy $E_{\text{core}}(a)$ of the system is given by summing up $E_{nljm}(a)$ according to a suitable order of shell filling of the core-particles* and is written as follows:

$$E_{\text{core}}(a) = \sum E_{nljm}(a) = E_0 + Aa_2 + Ba_2^2 + Ca_3^2 + Da_2a_3^2. \tag{2'}$$

Any odd power term of a_3 never appears in the expression (2'). Accordingly, the equilibrium value of a_3 is exactly zero. This fact means that, if C_3 is positive, the octupole type surface oscillations oscillate around a spherical equilibrium shape, even if the surface rigidity C_3 is very small, i.e. even if the surface is very deformable octupole type deformation. We can rewrite the expression (2') as

$$E_{\text{core}}(a) = E_0 - B(a_2^0)^2 + Ba_2'^2 + (C + Da_2^0)a_3^2 + Da_2'a_3^2, \tag{2''}$$

where a_2^0 is an equilibrium value of a_2 defined by

$$a_2^0 = -A/2B, \tag{13}$$

and

$$a_2' = a_2 - a^0_2.$$

Therefore, the surface rigidities to quadrupole and octupole type deformations are given by

$$C_2 = 2B, \tag{14}$$

and

$$C_3 = 2(C + Da_2^0), \tag{15}$$

respectively. The expression (15) shows that the surface rigidity (or deformability) to the octupole type deformation is closely related with the quadrupole type deformation.

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* In the calculations of $E_{\text{core}}(a)$, the order of shell filling of the core particles is determined so as to make the absolute value of A in eq. (2') as large as possible. And the sign of the deformation a_2 is determined so as to make the product a_2A negative. The order of shell filling and the value of A determined in this way agree with the results of Moszkowski and Townes[13].

§ 3. Surface rigidity to octupole type deformation

For the "spherical" square well potential, the effect of the "finite" wall height can be given approximately by[14]

$$E_{V_0,R_0} \simeq E_{\infty,R_0}(1+\lambda_0/R_0)^{-2},$$
$$\lambda_0=[\hbar^2/2MV_0]^{1/2}. \tag{16}$$

Therefore, we can take into account the effect of the "finite" wall height by replacing the nuclear radius R_0 by

$$R_0'=R_0+\lambda_0.$$

In the case of the spheroidal potential, we adopt this procedure as the first approximation, since our calculations are performed in particular under the condition that the deformation of the potential is small. We take the same values of R_0 and V_0 as MSY;

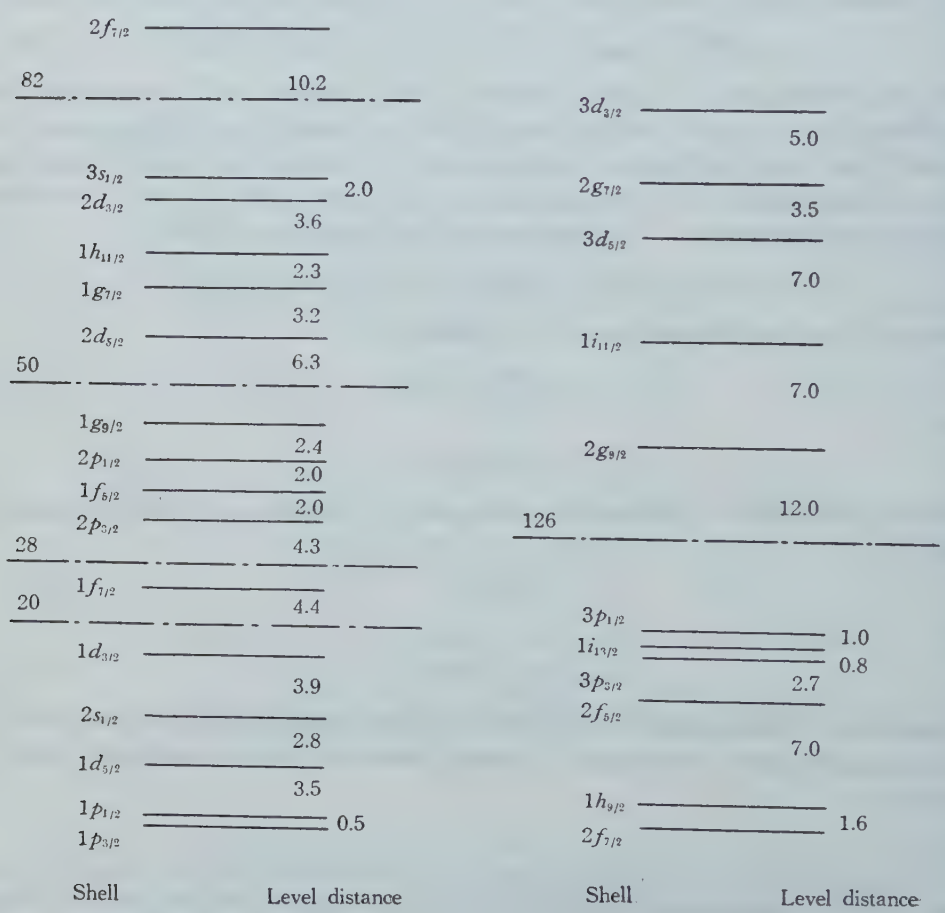

Fig. 1. Shell scheme and level distance. The level distances in this figure are written in unite of $\hbar^2/2MR_0'^2$.

$$R_0=1.4\times A^{1/3}\times 10^{-13}\,\text{cm}, \quad V_0=36\ \text{Mev}. \tag{17}$$

The energy denominator $E^0_{nlj}-E^0_{n'l'j'}$ in the perturbation calculations are determined from the empirical values of binding energies in a similar way to MSY. These values are listed in Fig. 1. The distance of two succesive levels (nlj) and $(n'l'j')$ is calculated from the difference between the binding energy of the last neutron of $[(nlj)$-closed shell+one $(n'l'j')$-neutron$]_N$-nucleus and that of $[(nlj)$-closed shell−one (nlj)-neutron$]_N$-nucleus. We picked up only a set of nuclei with proton closed shell. When we cannot find such a set of nuclei, we use a set of nuclei with neighboring even Z.*

Results of our calculations on C_3^N of neutron shells are shown in Fig. 2.** Fig. 2 shows that C_3^N also have similar strong shell structure dependence as C_2.[8] C_3^N corresponding to the shells of neutron numbers 16, 20, 32, 40, 56, 64, 82 and 90 are very small. Therefore, we may expect that the effects of octupole type deformations are easily observed in nuclei in the neighborhood of these shells. As pointed out in reference 8, C_2 are very large at the

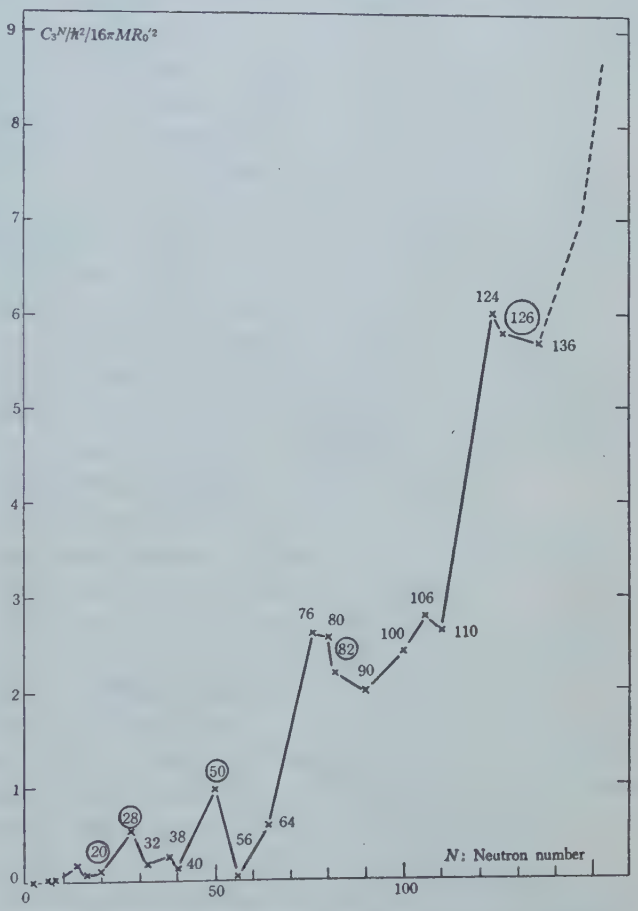

Fig. 2. The surface rigidity of neutron core, C_3^N, in unit of $\hbar^2/16\pi MR_0'^2$. In this figure, the rigidity of the region between two successive closed shells may be reduced very much because of the coupling of the core and extra-particles.

* These values are in general smaller than those of MSY. We calculated them from a single set of proton-closed shell nuclei but in MSY the level distances of several sets of proton configurations around a closed shell were averaged. This is a reason for the discrepancy. In general, the level distance determined from a set of nuclei with odd proton number is considerably larger than that of nuclei with even proton number.

** In order to calculate the surface rigidity C_3^P of proton shell, we must use the level distnces of proton shells.

the closed shells of magic numbers, while C_3 are not always large there, e.g. 20 and 82. In nuclei in the neighborhood of these closed shells, the low lying levels due to octupole type collective oscillation may be observed.

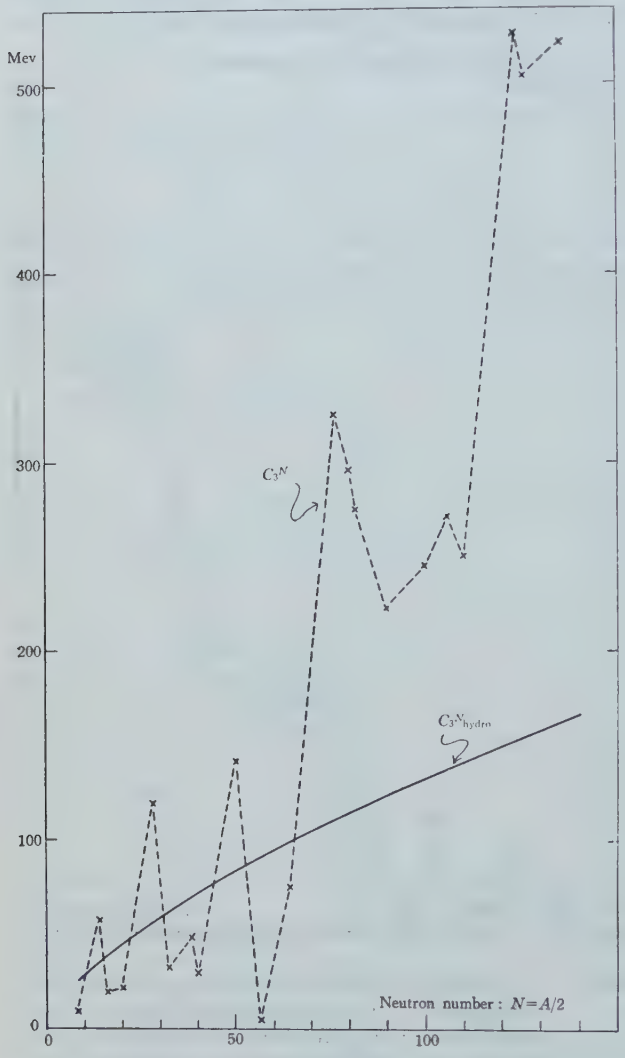

Fig. 3. The surface rigidity of the neutron core with $N=Z$. C_3^N is calculated from our model, $C_3^{N}{}_{\text{hydro}}$ is calculated from hydrodynamical aproximation. We assumed that $N=Z$ for simplicity.

C_3^N calculated from our model are compared with $C_3^{N}{}_{\text{hydro}}$ due to hydrodynamical approximation[15] in Fig. 3. In this figure, we assumed $N=Z$, for simplicity. The qualitative behavior of C_3 agrees well with the empirical evidences described in § 1. For nucleon numbers greater than 100, the calculated values are not very reliable and we signified this situation by the dotted line. This unreliableness is due to lack of the exact knowledge about the order and distances of the levels of shells in this region. In our calculations, we used Klinkenberg's level scheme[16] up to the $3d_{3/2}$-level. If we take into account the contributions from the $1j$-level (and higher odd parity levels), then C_3 of $N=90$ and 136 are reduced very much compared with the values of Fig. 2. The low lying odd parity states found in α-decay spectrum may be related with smallness of C_3 of the core of $Z=90$ and and $N=136$. Those of the nuclei in the rare earth region are related with $Z=56$, 64 and $N=82$. C_3 of the shell of nucleon number 40 is very small as expected. This may be closely related with the j-forbidden type $E3$ transitions. In these comparatively light nuclei, however, the configuration mixing also has an important effect and we can not draw a simple conclusion.[17] The final conclusion may be

given by a detailed study of the structure of levels in this transition.

§ 4. Discussion and conclusion

According to the results of our calculations, if a_2^0 (the equilibrium deformation to the quadrupole type deformation) vanishes, the energy change of a single particle level associated with the octupole type deformation is always positive. This situation occurs near a closed shell. But, in nuclei far from closed shells, a_2^0 does not vanish and the stability of particle levels against octupole type deformation depends on the magnitude and the sign of Da_2^0 in general, as we can see from the equations (2″) and (15) : If the sign of D is different from a_2^0 and the absolute value of Da_2^0 is larger than that of C, then C_3 is negative. Such a nucleus is unstable against the octupole type deformation. These circumstances may play a very important role in the process of nuclear asymmetric fission, that is, as the nuclear deformation becomes large, single particle level scheme undergoes a notable change and crosses over each other,[18] and a_2^0 becomes large. Now, the nucleus whose C_3 becomes negative for large a_2^0 may undergo an asymmetric fission.* In this process, the superfluous energy which arises as the results of change of a particle configuration due to nuclear deformation excites the octupole type surface oscillations and as the results of this excitation the deformation of the nucleus grows up.

The equation (15) shows that deformability C_3 to octupole type deformation is closely correlated with the quadrupole moment through a_2^0. As pointed out in § 3, the equilibrium value of the octupole type deformation is always zero. Therefore, it is sufficient to treat this type of surface oscillations according to a weak coupling scheme. This suggest that the levels due to this type surface oscillations will be observed as the levels due to surfon jump and the spin and parity of the lowest excited state of this type of excitation is 3^-. In general, the 3^- state due to the octupole type oscillations is higher than the 2^+ state due to the quadrupole type oscillations. In the nuclei with small C_3, we can observe the low lying odd parity levels. The level scheme of these nuclei is determined by the values of a_2^0 and D in equation (2″). The value of a_2^0 may divide the nucleus into two categories so called spherical and non-spherical nucleus, and the values of D is related with the coupling between quarupole type oscillations and octupole type oscillations. It may closely depend on a_2^0 and D which is the lowest odd parity state 1^- state or 3^- state. Therefore, in order to study the effects of the octupole type deformation, it is very important to know the behavior of D.

According to our calculations, the j-forbidden type $E3$-transitions cannot be considered as a simple surfon transition between a ground state and an excited state due to the octupole type surface oscillations. Our rough estimate of the

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* There may also be a case in which C becomes negative for large deformation. In this case, the stability of a nucleus to the octupolt type deformation is determined from the sign of C_3 due to the net effect of C and Da_2^0.

transition probabilities based on this simple picture are too large. The experimental evidences show that the nuclei which undergo this type of $E3$-transition possess a very complicated and delicate level scheme and, on the other hand, the surface rigidities of these nuclei are extremely small so that the interactions between collective motions and motions of extra-particles may be essentially important in these nuclei.

We have studied behavior of the surface rigidity C_3 to octupole type deformations. The results of our calculations show that C_3 also displays the characteristic shell structure dependence as well as C_2 and agrees well qualitatively with experimental facts though the experimental data we have at present are not enough. We must calculate the coupling strength D between quadrupole type deformations and octupole type deformations in order to study these experimental data in detail. This calculation will be very complicated and laborious.

Simple considerations indicate that the octupole type surface oscillations play a very important role in the properties of low lying states and the mechanism of the asymmetric fission. This type of collective oscillations may contribute also to the phenomena of nuclear reactions.

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Interpretation of the Optical Model in Nuclear Reactions

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The theoretical basis of the nuclear optical model is investigated for the elastic scattering of neutrons at low energies. It is argued in connection with statistical mechanics of irreversible processes that the characteristic feature of the optical model can be revealed by means of wave packets whose energy spread is wide enough to cover many compound levels. The giant resonance of the Lorentzian shape is fonnd to result owing to fluctuations in the interaction potential which is experienced by an incident nucleon in a target nucleus. The imaginary part of the optical potential or the half width of a giant resonance is expressed by the product of the second moment and the correlation time. The latter is closely related to the concept of *portée* introduced by Bloch, with whom we share the same opinion in many respects. The correlation time is shown by our method to reveal a feature of the fluctuations and can be evaluated, if the fourth moment is known.

§ 1. Introduction

Following the discovery of the gross structure of neutron cross sections,[1] a great number of experiments have provided evidence for the validity of the optical model in low energy nuclear reactions.[2] The theoretical basis of the optical model has also been investigated by many authors, since the pioneer work by Feshbach, Porter and Weisskopf.[3] In the present paper we wish to add a contribution to the theoretical interpretation of the optical model with a somewhat different point of view from previous ones.

The basic points on which we stand are the following. (i) The characteristic feature of the nuclear optical model can be revealed by such a wave packet of a bombarding particle that has a coherent energy spread enough to cover many levels of the compound nucleus. (ii) The wave packet decays owing to fluctuations in its interaction potential while it passes through the target nucleus. (iii) The gross structure of the cross section is determined not only by the second moment of the compound level distribution but also by the average period of the fluctuations ; the imaginary part of the optical potential is found to be proportional to the product of these two quantities.

Friedman and Weisskopf[4] have emphasized that the concept of the wave packet is of great importance to interpreting the nuclear reactions. In order to get information about individual compound levels, we have to know what is happening during the period of at least one oscillation of the compound nucleus. This requires

the wave packet to be longer in time than the oscillation period. In other words, the energy spread of the wave packet must be smaller than the level distance. If the wave packet is shorter than the oscillation period, or the energy spread is larger than the level distance, on the other hand, there is room for introducing the irreversible decay of the incident wave, as has been shown by Van Hove[9] for macroscopic systems. For nuclear systems the oscillation period corresponds to the period of the Poincarè cycle, during which an initial state recurs. The irreversible process is meaningful only for time shorter than the period of the Poincarè cycle.

When such a short wave packet strikes a nucleus, it interacts with individual nucleons in the nucleus through a potential $\mathcal{V}$. This results in the excitation of compound states as well as the phase change of the incident wave. If one observes the potential $\mathcal{V}$ sitting upon the incident particle, $\mathcal{V}$ fluctuates rapidly in time with the period of τ. The fluctuations may be regarded as random, because of the random distribution of the target nucleons; hence this gives rise to a sort of noise which causes the random dissipation of the incident wave.* The dissipation eventually appears as the formation of compound states, if one waits for long enough time, so that the outgoing wave consists of two parts, the one similar to the incident packet and the other being a tail coming through the compound states. The optical character of the nucleus is revealed, if one is concerned with the first part alone.** The separation of the outgoing wave into two parts is meaningful only with the initial wave of the short packet length; otherwise the separation is merely formal, because the dissipation of the initial wave cannot be defined but there appear periodic oscillations representing the compound states.

The above considerations allow us to infer how the amplitude of the incident wave declines. The incident wave may be represented by the superposition of waves corresponding to respective compound states with such phase changes of the order of $\mathcal{V}\tau$ that possess randomly distributed values. The simple average of these phase gives us a constant phase shift that can be interpreted as due to a real optical potential. The noise arises from the mean square of fluctuations, so that the amplitude decreases according to the square of time in the initial stage. Since the phase correlation does not extend beyond the correlation time τ_c, the amplitude decrease turns out to be directly proportional to time, as is seen in the problem of random walk. Hence the rate of change in the amplitude may be given by***

$$\langle(\mathcal{V}-\langle\mathcal{V}\rangle)^2\rangle\tau_c \simeq W, \tag{1·1}$$

the second moment times the correlation time. Here we assumed that τ_c is of the order of the internucleon distance divided by the velocity of the bombarding particle in the nucleus, and is practically equal to τ.

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* The interpretation of nuclear reactions in terms of noise theory was attempted by C. E. Porter, Nuclear Physics, **4** (1957), 472.

** A similar consideration was also made by C. De Dominicis, Jour. de Phys, **19** (1958), 1.

*** In this paper we take $\hbar$ as unity. Therefore, energy has the dimension of reciprocal time.

The quantity W given by (1·1) is nothing but the imaginary part of the optical potential. This interpretation is in sharp contrast with that proposed by Lane, Thomas and Wigner,[6] who identified W with the square root of the second moment. Our interpretation is quite analogous to that by Bloch,[7] if τ_c is related to his "*portée*" $\vec{\mathcal{G}}$ through

$$\tau_c \simeq 1/\vec{\mathcal{G}}. \tag{1·2}$$

The existence of the correlation time or the portée results in the Lorentzian gross structure which corresponds to the one described by the optical potential.

The qualitative considerations described above are formulated in a mathematical way in § 2, taking the elastic scattering of neutrons as an example. Our mathematical formulation is made as elementary as possible, emphasizing physical implications. Hence important, but involved parts, for example, the investigation of detailed behaviours of wave packets in nuclear reactions, are left for a separate paper.[8] In § 3 various ways of the interpretation of the optical model are compared in the light of our interpretation.

§ 2. Mathematical formulation

The Hamiltonian of the total system consists of the target or the core Hamiltonian, H_c, the kinetic energy of the incident nucleon, K_n, and its interaction with the target nucleus, $\mathcal{V}(n, c)$:

$$H = H_c + K_n + \mathcal{V}(n, c). \tag{2·1}$$

The interaction energy is decomposed into an average potential, $U(n)$, and the residual part, $V(n, c)$, which is regarded as a fluctuating part of the potential:

$$\mathcal{V} = U(n) + V(n, c). \tag{2·2}$$

We choose the unperturbed Hamiltonian as

$$H_0 = H_c + K_n + U(n), \tag{2·3}$$

and denote its eigenstate and the eigenenergy by $|a\rangle$ and E_a respectively, so that

$$(H_0 - E_a)|a\rangle = 0. \tag{2·4}$$

The eigenstates $|a\rangle$'s describe the scattering by the fixed potential U; hence E_a's form a continuous spectrum. Since we are concerned with the elastic scattering, the core state is left in its ground state when we speak of $|a\rangle$. The average potential $U(n)$ is so chosen as to satisfy

$$\langle a'|U|a\rangle = 0. \tag{2·5}$$

It is convenient for our discussion to introduce the interaction representation, in which the fluctuating potential is expressed as

$$V(t) = e^{iH_0 t} V e^{-iH_0 t}. \tag{2·6}$$

Then the fluctuations in space due to the distribution of target nucleons appear as the fluctuations in time which the incident nucleon sees as it goes through the force fields of the individual target nucleons. Of course, the motion of the latter also contributes to the fluctuations but its period is so long that we may neglect the time correlation of the target nucleons for simplicity.

The wave function of the total system in the interaction representation varies in time as

$$\Psi(t)=U(t,\ t_0)\,\Psi(t_0), \tag{2·7}$$

where $U(t,\ t_0)$ is the transformation function expressed as

$$U(t,\ t_0)=e^{iH_0 t}e^{-iHt}e^{iHt_0}e^{-iH_0 t_0}. \tag{2·8}$$

Suppose we have such an unperturbed wave Φ at time t_0 that is obtained from $\Psi(t_0)$ by switching off the interaction with the target nucleons adiabatically, and ask how large the amplitude of the unperturbed wave at time t is. The amplitude is given by

$$\langle\Phi|\Psi(t)\rangle=\langle\Phi|U(t,\ t_0)|\Phi\rangle. \tag{2·9}$$

In our case Φ is expressed by the superposition of the unperturbed eigenstates

$$\Phi=\int|a\rangle\,dE_a\rho_s(E_a)\langle a|\Phi\rangle, \tag{2·10}$$

where $\rho_s(E_a)$ is the unperturbed state density. The integrand is practically constant over an energy interval δ with its center at E_{a_0}.

For calculating (2·9) it is convenient to express the transformation function, by employing the integral equations

$$U(t,\ t_0)=1-i\int_{t_0}^{t}dt'\,V(t')U(t,\ t_0)=1-i\int_{t_0}^{t}dt'\,U(t,\ t')\,V(t'),$$

as

$$U(t,\ t_0)=1-i\int_{t_0}^{t}V(t')\,dt'-\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\,V(t')U(t',\ t'')\,V(t''). \tag{2·11}$$

Substituting this in (2·9), the second term in the right-hand side vanishes by virtue of (2·5), since the core is left in the ground state in elastic scattering. The contribution from the last term has to be investigated carefully. This gives us

$$\langle\Phi|U(t,\ t_0)|\Phi\rangle_{II}=-\int dE_{a'}\int dE_a\rho_s(E_{a'})\rho_s(E_a)\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''$$

$$\times e^{i(E_{a'}t'-E_at'')}\sum_s e^{-iE_s(t'-t'')}\cdot\langle\tilde{a}'|V|s\rangle\langle|\tilde{s}V|a\rangle\langle\Phi|a'\rangle\langle a|\Phi\rangle, \tag{2·12}$$

where $|s\rangle$ means a compound nucleus state and $|\tilde{s}\rangle$ its time reversed state, as we use the convention adopted by Bloch.[9]

First, we notice that the non-diagonal elements with respect to the unperturbed states can be dropped owing to the randomness in the phases of $\langle|\tilde{s}V|a\rangle$.[10]* If there is any phase correlation, there does not occur the dissipation but the scattering which results in a fine structure in the cross section. The latter may be the case, if the energy width of the wave packet covers only two or three resonance levels. The randomness in phase ensures that the wave packet propagates without changing its shape, consequently without delay. In this approximation we may write

$$\langle\Phi|U(t,\ t_0)|\Phi\rangle_{II}=-\langle\Phi|U(t_0,\ t_0)|\Phi\rangle\cdot\sigma_a^{\,2}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\,f_a(t'-t''),\qquad(2\cdot12')$$

where we used the abbreviations

$$\sigma_a^{\,2}\equiv\sum_{\Delta}\sigma_a^{-2}(\Delta)\equiv\sum_{\Delta}\frac{\int dE_a\rho_s(E_a)\langle\Phi|\tilde{a}\rangle\langle a|\Phi\rangle\sigma_a^{\,2}(\Delta)}{\int dE_a\rho_s(E_a)\langle\Phi|\tilde{a}\rangle\langle a|\Phi\rangle},\qquad(2\cdot13)$$

with

$$\sigma_a^{\,2}(\Delta)\equiv\langle\tilde{a}|V|s\rangle\langle\tilde{s}|V|a\rangle\quad\text{for }\Delta\equiv E_s-E_a,$$

and

$$f_a(\tau)=\frac{\sum_{\Delta}\sigma_a^{-2}(\Delta)\,e^{-i\Delta\tau}}{\sum_{\Delta}\sigma_a^{-2}(\Delta)}.\qquad(2\cdot14)$$

For a fixed state $|a\rangle$ the quantity $\sigma_a^{\,2}(\Delta)$ has a discrete spectrum as a function of Δ with an average separation D of the levels of the compound nucleus. We note that $\langle\Phi|U(t,\ t_0)|\Phi\rangle_{II}$ declines to a vanishing value after a time interval σ_a^{-1} even if there appears no appreciable fluctuations due to $f_a(\tau)$. At this stage we may discriminate several different situations according to the relative magnitude of various parameters defined above.

I. $\delta\ll D$

This corresponds to the use of a sharp coherent beam or a long packet, and the result is essentially the same as the case without E_a integration in (2·13). Due to the dicrete spectrum of $\sigma_a^{\,2}(\Delta)$ an appreciable contribution appears only when E_a is equal to one of the compound levels E_s, and therefore the result is a collection of many sharp resonances, if the incident beam of a wide energy width is a mixture of such long packets. Let us confine ourselves to a special case in which this collection reveals a giant resonance behaviour corresponding to the intermediate coupling in Wigner's sense.[6] For the case in which the envelope of a single giant

* In the so-called direct interaction the assumption of randomness no longer holds. As far as we are concerned with the over-all property of nuclear reactions, however, this assumption seems to be allowed. It may be worth remarking that the same assumption is adopted for explaining the distribution of level spacing; see N. Rosenzweig, Phys. Rev. Letters, 1 (1958), 24.

collection has an approximately Gaussian shape the width is given by the root second moment, σ_a.

II. $\sigma_a \gg \delta \gg D$

This corresponds to the use of a medium spread of the wave packet which is enough to wash out the structure of the individual sharp resonances but still retains the gross structure which has been described successfully by the cloudy crystal ball model. In contrast with the case I, $\sigma_a^2(\Delta)$ is now a slowly varying, continuous function of Δ; therefore, we may replace the Δ summation in (2·14) by an integration.

$$f_a(\tau) = \frac{\int d\Delta \chi(\Delta)\, e^{-i\Delta\tau}}{\int d\Delta \chi(\Delta)}; \qquad \chi(\Delta) \equiv \rho_s(\Delta)\sigma_{\bar{a}}^2(\Delta). \tag{2·15}$$

In this case $f_a(\tau)$ may be considered as a continuous function of time which declines to nought in a characteristic time τ_c. The inverse of τ_c is approximately given by the overall energy spread $\bar{\mathfrak{G}}$ of the function $\chi(\Delta)$, which is closely related to the concept of *portée* proposed by Bloch. The second moment which was defined by (2·13) is independent of $f_a(\tau)$, and thus insensitive to the fluctuations, whereas the fourth moment $\langle (\Delta E)^4 \rangle = \langle a | V^4 | a \rangle$ involves the first indication of fluctuations. Explicitly, we have

$$\langle (\Delta E)^4 \rangle = 3\sigma_a^4 - \sigma_a^2 \left(\frac{d^2 f_a}{d\tau^2} \right)_{\tau=0}, \tag{2·16}$$

where the second term stands for the effect of fluctuations. For the case in which the fluctuations are represented by a Gaussian random function, it is easy to determine the function $f_a(\tau)$, which is

$$f_a(\tau) = \exp\left(-\frac{\tau^2}{2} \bar{\mathfrak{G}}^2 \right), \tag{2·17}$$

where $\bar{\mathfrak{G}}$ may be given by

$$\bar{\mathfrak{G}}^2 \equiv \frac{\sum_\Delta \sigma_{\bar{a}}^2(\Delta)\Delta^2}{\sum_\Delta \sigma_{\bar{a}}^2(\Delta)}. \tag{2·18}$$

The result shows clearly that $\bar{\mathfrak{G}}$ is the measure of the energy spread of $\sigma_{\bar{a}}^2(\Delta)$. At this stage we may again discriminate two cases according to the relative magnitude of σ_a and $\bar{\mathfrak{G}}$.

II. A. $\bar{\mathfrak{G}} \gg \sigma_a$

In this case we have the asymptotic behaviour

$$\sigma_a^2 \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' f_a(t'-t'') \simeq \frac{1}{2}(t-t_0) W, \tag{2·19}$$

where

$$W=\sqrt{\pi}\left(\frac{\sigma_a^2}{\mathcal{E}}\right). \qquad (2\cdot20)$$

Physically, the period of fluctuations $\mathcal{E}^{-1}$ is much shorter than the time scale in which we are interested, so that it is possible to choose a time interval which includes many fluctuations and is at the same time much shorter than the relaxation period for a giant resonance. Under these assumptions we have

$$\begin{aligned} F(t)\equiv\langle\Phi|U(t,\ t_0)|\Phi\rangle &=1+\langle\Phi|U(t,\ t_0)|\Phi\rangle_{II} \\ &=1+F'(t_0)\,(t-t_0)+\cdots\cdots \\ &=1-\frac{W}{2}\cdot(t-t_0)+\cdots\cdots. \end{aligned}$$

In conformity with expressing $f_a(\tau)$ as (2·17) and (2·19), which hold only approximately for higher orders in τ, we may express $F(t)$ as

$$F(t)=\exp\left[-\frac{W}{2}\cdot(t-t_0)\right]F(t_0),$$

for $|t-t_0|\ll\mathcal{E}^{-1}$. Then the differential equation for $\langle\Phi|U(t,\ t_0)|\Phi\rangle$ is given by

$$\frac{d}{dt}\langle\Phi|U(t,\ t_0)|\Phi\rangle=-\frac{W}{2}\langle\Phi|U(t,\ t_0)|\Phi\rangle. \qquad (2\cdot21)$$

The solution of this equation (2·21), when transformed back to the original representation, is given by

$$\langle\Phi|U(t,\ t_0)|\Phi\rangle=\langle\Phi|U(t_0,\ t_0)|\Phi\rangle\cdot\exp\{-W/2(t-t_0)+iEa_0(t-t_0)\} \qquad (2\cdot22)$$

This means that we have the Lorenzian shape of the giant resonance, of which the centre is located at E_{a_0} and the width is given by W. Clearly, this special case corresponds to that which was described by the cloudy crystal ball model. Under the assumption of Gaussian random modulation the fluctuation frequency $\mathcal{E}^{-1}$ is given by

$$\mathcal{E}^{-1}=\left[\frac{\langle(\Delta E)^4\rangle-3\langle(\Delta E)^2\rangle^2}{\langle(\Delta E)^2\rangle}\right]^{-1/2}. \qquad (2\cdot23)$$

II. B. $\mathcal{E}\ll\sigma$

In this case the effect of fluctuations does not show up in the observed result and we have

$$\langle\Phi|U(t,\ t_0)|\Phi\rangle=-\langle\Phi|U(t_0,\ t_0)|\Phi\rangle\exp\{-(t-t_0)^2\sigma_a^2/2\} \qquad (2\cdot24)$$

In general, the validity of this formula is limited only to the region of small $(t'-t_0)$; however, we may infer the long time behaviour from this knowledge for a special case in which the envelope of the collection of compound resonances as approximately Gaussian. This corresponds to the case treated by Lane, Thomas and

Wigner[6] and the width of the giant resonance is given by the root second moment. The situation is evidently different from that described by a complex potential and it is only natural that we have quite a different value as the width of the giant resonance.

III. $\delta \gg \sigma$

This corresponds to the use of a wave packet which is so broad in energy compared with the root second moment that even the gross structure of the resonance is washed out. Therefore, we expect a uniform, monotonic behaviour of the cross section as a function of the incident energy. For high energy scattering we observe such a behaviour of cross sections, but this seems to be due chiefly to the overlap of giant resonances, namely W being larger than the distance between two adjacent giant resonances.

§ 3. Comparison with other theories

Our interpretation may become clearer, if it is compared with other theories.

Feshbach-Porter-Weisskopf's theory.[3] In their original theory the scattering amplitude, proportional to η below, is separated into two parts as

$$\eta = \langle \eta \rangle + (\eta - \langle \eta \rangle), \qquad (3\cdot1)$$

where $\langle\ \rangle$ means an average over the energy spread of an incident beam, and the first part is interpreted as to contribute to the shape elastic scattering, while the second part to the compound elastic scattering. The physical meaning of the separation is, however, dependent on the length of the wave packet of a bombarding particle relative to the oscillation period of the compound nucleus, as has been discussed in § 1. For a long wave packet the first part represents the potential scattering due to the contributions from distant levels and the second part the resonance scattering. Since the outgoing wave packets from these two sources are approximately of the same length, they interfere with one another, so that the cross section reveals an envelope of ordinary resonance cross sections observed with sharp beams, if the beam consists of a mixture of such long packets. For a short wave packet, on the contrary, the first and the second parts are distinguishable, so that they do not practically interfere with one another. Thus one can get a distinct separation, one being interpreted as shape elastic and the other as compound elastic.*

Friedman-Weisskopf's theory.[4] They correctly emphasized the importance of introducing the wave packet. Our theory is an extension of theirs.

Lane-Thomas-Wigner's theory.[6] They identified the root second moment with the imaginary potential, and this was calculated to give too large a value, about 20 MeV, compared with the observed one. Several authors have then tried

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* It is almost needless to mention that the distinction is practically possible, only if inelastic channels are open.

to reduce the value of the second moment, for example, by taking into account the effect of the core polarization.[11] However, this is found in § 2 to correspond to the case II. B, in which the current description of scattering in terms of the optical model is not possible.

Lane, Thomas and Wigner assumed the strength function of a Lorentzian shape and correlated it with the observed gross structure through the second moment. It is well known, however, that the second moment diverges for the Lorentzian strength function. In order to obtain convergent moments, we must have a distribution dropping rapidly towards wings, such as a Gaussian distribution. In fact, the distribution of reduced widths in a certain narrow energy interval is found to be Gaussian,[12] and the superposition of many Gaussian distributions is likely to give a Gaussian one. Hence we expect that the envelope of the strengths of individual compound levels shows a Gaussian gross structure which gives convergent moments. However, such a gross structure is not always observed. It has been familiar in the magnetic resonance absorption in solids that the presence of modulation may reduce the absorption line shape to Lorentzian, the absorption width being given by (1·1).[13] Such a modulational narrowing effect seems to take place in nuclear reactions, too, due to the fluctuation in potential, as has been discussed before. Once this is taken for granted, we can derive the Lorentzian gross structure as well as a correct order of magnitude of the imaginary potential.*

Bloch's theory.[7] Our theory is a time-dependent version of Bloch's one. Bloch has shown that the relation between the second moment, M_2, and the imaginary potential is given by

$$W \simeq \pi M_2 / \tilde{\varepsilon}^*, \tag{3·2}$$

where $\tilde{\varepsilon}^*$ is the *portée* introduced by him, slightly different by a numerical factor from our $\tilde{\varepsilon}$ defined in (2·18). The *portée* is defined in such a way that the matrix element of the interaction V, $|\langle a|V|b\rangle|$, is practically constant for $|E_a - E_b| < \tilde{\varepsilon}^*$, while it practically vanishes otherwise, where E_a and E_b are the eigen-energies of respective single particle states, $|a\rangle$ and $|b\rangle$. The introduction of the *portée* allows him to calculate the second moment as

$$M_2 \simeq 2|\langle a|V|b\rangle|^2 \tilde{\varepsilon}^* / D_s, \tag{3·3}$$

where D_s is the single particle level spacing. He has further argued that the contribution to the imaginary potential comes mainly from the repeated second order processes that come back to the original single particle state in every two steps in the sense of perturbation theory. This procedure is nothing but the method adopted by Van Hove[5] in his theory of irreversible processes, although this was not mentioned by Bloch.

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* Such an argument has been made by N. Yajima and S. Takagi, Soryushiron Kenkyu, **15** (1957), 245 and by S. Takagi and M. Yasuno, ibid, 287. These authors reached a conclusion very close to ours.

Furthermore, Bloch investigated the structure of the Green function in nuclear reactions based on Wigner's theory.[10] He compared the distance between poles of the Green function in the energy plane with the energy spread of the irradiating beam. Since the integral over energy passing through the poles can be replaced by an integral along a path, parallel to the real axis, shifted by a positive imaginary value equal to the energy spread, the distribution of the poles appear to be continuous, thus giving rise to a dissipative process. A corresponding procedure is made in our calculation in (2·12′) and thereafter, in which the rapid oscillation caused by the discrete spectrum of $\sigma_a(\Delta)$ is smoothed out by taking the Δ-summation. It has been important in this procedure that we have a coherent energy width large enough to cover many levels. Although Bloch did not notice that the beam width should be that of a wave packet, this fact was implicitly taken into account in his theory. Indeed, Sunakawa[14] has shown that the introduction of the imaginary part in energy, which makes it possible to shift the integral path, is equivalent to the use of the wave packet.

From the above discussions one can see the essential equivalence of our theory to Bloch's. However, our version may appeal to intuition, by introducing the wave packet, serves to give a clear meaning to the *portée* in relation to the fourth moment or to the fluctuation in potential, and is successful to demonstrate the energy independence of W which was merely assumed by Bloch.

Multiple scattering theory. Theories other than those mentioned above seem to have little connection with the present discussions, and one has only to consult with a review article by Lane.[15] However, we shall add a remark on the multiple scattering theory.

The substitution of (3·3) in (3·2) gives one

$$W \cong 2\pi |\langle a|V|b\rangle|^2 / D_s . \qquad (3\cdot4)$$

Since the right-hand side corresponds to the transition probability, this may express a simple interpretation of the imaginary potential in terms of the absorption of an incident nucleon by individual nucleon-nucleon collisions,[16] namely

$$W \cong \langle v\sigma \rangle \cdot n , \qquad (3\cdot5)$$

where v is the velocity of the incident nucleon inside the target nucleus, σ the nucleon-nucleon collision cross section and n the density of target nucleons; the average of $v\sigma$ should be taken by taking into account the Fermi distribution of target nucleons and the Pauli principle for the final states. The evaluation of W along this line is refined with the aid of the multiple scattering theory.[17]

In such theories an implicit assumption is always made that the transition amplitude due to the collision with a target nucleon can be defined unambiguously. This is not always the case, however, as we have seen in the case II. B in § 2, where the amplitude changes quadratic in time, but not linear in time. The linear variation of the amplitude begins only after τ_c, owing to the effect of fluctuations;

before τ_c no irreversibility comes in, so that one cannot speak of the transition. Thus the multiple scattering theory has the basis in the occurrance of the irreversible process and its validity conditions should be examined in a way described in the present paper.*

§ 4. Remarks and discussions

Since we have restricted ourselves in the foregoing discussions only to the essential points, in order to elucidate the physical nature of the optical model in nuclear reactions, a number of remarks may be useful to help understanding.

In our formulation the concept of the wave packet has been employed only in the Δ-summation in (2·13) as the coherent energy spread and the spatial spread of the packet is referred only qualitatively. It will be more interesting to study how the shape of a packet changes in the course of a nuclear reaction. This has been carried out by Nagata and Yasuno[18] for resonance scattering. A more extensive analysis suitable for our purpose has been made by Sasakawa[8] and has, indeed, confirmed our qualitative considerations. Since mathematics of this analysis is involved, this work will be published as separated papers.

Our way of approach will be extended in several respects. For inelastic scattering, for example, the motion of target nucleons may become so important that we shall have to take the space-time correlation into account, although the space correlation alone is considered in the present paper. The effect of the space-time correlation will be treated in an analogous way to the theory of neutron scattering by Van Hove,[19] and such a theory will shed light on the transport phenomena in nuclear matter.

Leaving out such extension of our theory to future problems, we add some comments on the validity of the optical model. Since the exponential decay of the single particle amplitude holds after τ_c, the correlation time, the Lorentzian shape of the giant resonance does not extend to wings of a resonance peak. Practically, however, the wings of two resonance peaks overlap with one another, so that the shape at wings is of no practical interest. The shape at a resonance peak is determined by the behaviour at the tail in time of the amplitude. The detailed shape of a resonance line depends or how good the approximation adopted in (2·19) is, or how good the convergence of our expansion with respect to moments is. Although the expansion in powers of $V\tau_c$ is expected to converge rapidly enough, in contrast with perturbation theory, because τ_c or $\overline{\mathcal{E}}^{-1}$ is very small, the convergence will have to be examined by calculating higher moments.

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* In general, the transition probability should be interpreted in such a way that it is expressed by the product of the second moment and the modulation time. This remark will clarify the quantum mechanical formulation of the decaying state attempted, for example, by A. Salam. His paper read at the High Energy Conference at Genève in July, 1958, was kindly informed by Professor G. Takeda.

Although our aim in the present paper is not the detailed comparison with experiments, it may be worth while to mention a feasible way to test our theory. We have remarked that the Lorentzian shape of the giant resonance is not identical with the distribution of reduced widths, the latter being expected to be Gaussian on the basis of indirect evidence. If one observes elastic cross sections by means two beams of different energy resolution, both the giant resonance and the individual resonances due to compound levels may be observed for the same target nucleus in the same energy region; the former may be replaced by the mass number dependence of cross sections for nearby nuclei. This will make the comparison between the above two distributions possible. However, one has to be cautious about the coherent energy width, because this is usually different from the energy width of a beam and is difficult to know how large it is.

Finally, we give a rough value of the imaginary potential based on eq. (2·20). Since the correlation time is approximately given by the internucleon distance, about 1.2×10^{-13} cm, divided by the velocity of the incident nucleon, about 1×10^{10} cm·sec^{-1}, we estimate

$$\overline{\mathcal{G}}=\tau_c^{-1}\cong 0.8\times10^{23}\ \text{sec}^{-1}\simeq 80\ \text{MeV}.$$

Adopting the value of the second moment calculated by Lane, Thomas and Wigner,[6]

$$\sigma_a\cong 20\ \text{MeV},$$

we obtain

$$W\cong 9\ \text{MeV}.$$

This is considerably larger than the observed value, about 3 MeV, but the corrections due to the polarization and statistical effects[11] may reduce the second moment so as to give a right value of W. A more quantitative result will be obtained, if the fourth moment is evaluated.

In conclusion the authors would like to express their thanks to Professor M. Namiki for his helpful discussions.

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Radiative K-μ Decay and Quantum Electrodynamics at Small Distances

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The radiative K-μ decay is analysed as a means of obtaining information on the properties of the muon and Q. E. D. at small distances and (or) on the type of the weak Fermi interaction. To express modifications of the properties of the muon and of Q. E. D. at small distances cut-off factors are used. If we use, for example, Feynman's cut-off factor with cut-off radius $(1/\Lambda) \gtrsim 0.3\times10^{-13}$ *cm*, it is shown that the effects of these modifications may be measurable with the present experimental techniques, provided that the values of the axial vector and vector coupling constants are greater than or equal to that of the tensor coupling constant: $|C_A| \sim |C_V| \geq |C_T|$. If $|C_T| > |C_A| \sim |C_V|$ and $(1/\Lambda) < 0.3\times10^{-13}$ *cm*, on the contrary, the effects of the tensor interaction are measurable. When $(1/\Lambda) \gtrsim 0.3\times10^{-13}$ *cm* and $|C_T| > |C_A| \sim |C_V|$, it is hard to distinguish these two effects.

§ 1. Introduction

The present quantum field theory with point interactions has the divergence difficulties. In order to overcome these difficulties and to get a consistent finite theory, quantum field theory would have to be improved in some small regions. We are thus interested in testing experimentally the limit of applicability of the present quantum field theory at small distances. To make sure the existence of these small regions and know their properties experimentally, it is necessary to compare some theoretical predictions based on the present theory with the experimental results, which may reflect the properties of these small regions. The electron and the muon which have no strong interactions, would be suitable for this purpose. In the case of strongly interacting particles, on the other hand, it is difficult to calculate the theoretical predictions with required accuracy.

Recently the value of the anomalous magnetic moment (a.m.m.) of the muon has been reported by Coffin, Garwin, Penman, Lederman and Sachs.[1] The value obtained by them is

$$\Delta\mu = (2.17 \pm 0.75)\frac{\alpha}{2\pi}\mu_0, \tag{1·1}$$

where $\Delta\mu$ and μ_0 are anomalous and normal magnetic moments of the muon respectively. If we assume that the muon is just a heavy electron, then the theoretical value is, on the other hand.

$$\Delta\mu=\left\{\frac{\alpha}{2\pi}+0.75\,\frac{\alpha^2}{\pi^2}+\cdots\right\}\mu_0{}^{2)}. \qquad (1\cdot2)$$

Many authors have pointed out that the difference between the theoretical value and the experimental one of the a.m.m. of the muon may reflect the limit of applicability of Q.E.D. or the structure and interaction properties of the muon. Though the standard deviation of $\Delta\mu$ in (1·1) is too large to conclude that the theoretical value of $\Delta\mu$ deviates from the experimental one, we may infer the limit of applicability of Q.E.D. as[3)]

$$\frac{\Lambda_\mu{}^2}{m^2}\gtrsim 1, \qquad (1\cdot3)^*$$

where Λ_μ is the cut-off momentum for a muon and m the muon mass.

Next, let us consider the high energy electron-proton scattering observed at Stanford.[4)] From this experiment we may put the lower limit of Λ_e, the cut-off momentum for an electron, as

$$\frac{\Lambda_e{}^2}{M^2}\gtrsim 3/4, \qquad (1\cdot4)^{**}$$

where M means the nucleon mass. Thus there may be a possibility that the muon has some electromagnetic structure different from that of the electron.[5)]*** In this paper we shall propose an experiment by which this possibility may be tested.****

For this purpose we shall consider $K^+\to\mu^++\nu+\gamma$ decay process. The reasons why we would like to select this decay process are as follows.

I. For the radiative K-μ decay we have the three Feynman diagrams shown in Fig. 1 (a), (b) and (c). In Fig. 1(c) there is a virtual muon line from which a photon is emitted. If the virtual muon line and the muon-photon vertex are

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* We shall use the unit, $\hbar=c=1$, throughout this paper.

** To get this value we assume that the charge distribution of the electron is of an exponential type and its r. m. s. radius is smaller than 0.8×10^{-13} *cm*.

*** The electron and the muon have very similar properties except for their masses. If one assumes that the masses of elementary particles consist of their self energies alone, the difference between the electron and muon masses seems to mean that both particles have different electromagnetic structures.

**** After completing this manuscript a new value of the a. m. m. of the muon is reported by Lundy, Sens, Swanson, Telegdi and Yovanovitch (*Phys. Rev. Letter*, **1** (1958), 38). The value obtained by them is

$$\Delta\mu=\ (1.3\pm0.5)\ \ (\alpha/2\pi)\ \ \mu_0,$$

which agrees with theoretical value (1·2) within the experimental error.

As will be seen from Table 2 and Fig. 3 the radiative K-μ decay may be used for testing the validity of Q. E. D. in the domain $(1/\Lambda)\gtrsim0.3\times10^{-13}$ *cm*. If Q. E· D. is modified in this domain the a. m. m. of the muon may differ from the theoretical value (1·2) by $\delta\mu$, where $|\delta\mu|\gtrsim0.03(\alpha/2\pi)\ \mu_0$. This corrected value, $\Delta\mu=$ theoretical value $(1\cdot2)+\delta\mu$, also agrees with the above experimental value.

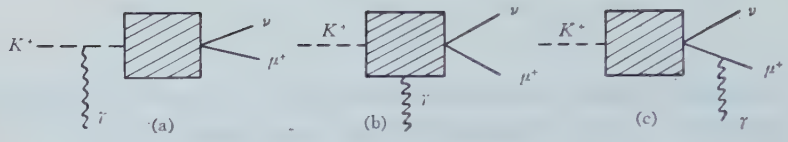

Fig. 1. Feynman diagrams for $K^+ \to \mu^+ + \nu + \gamma$ decay. The black box represents virtual strong interactions such as the creation and annihilation of a baryon pair and other more complicated processes.

modified at small distances, then the distribution of the angles of emitted photons to the muon direction is modified at large angles by measurable extent.

II. In the case of the electron-proton elastic scattering[4] and the large angle electron pair production on hydrogen,[3)6)] the momenta of virtual photons and electrons in these processes are space-like vectors. On the other hand, our process contains a virtual muon with time-like momentum. This process may, therefore, exploit further indication on the modification of Q.E.D. at small distances.

III. The mass of a K-meson m_K is greater than that of a pion m_π. The modification of Q.E.D. at small distances would, therefore, be more effective for the radiative K-μ decay than for the radiative π-μ decay. The ratio of the frequency of radiative decays to non-radiative ones for K-mesons is predicted to be larger than the same ratio for pions:

$$\frac{K^+ \to \mu^+ + \nu + \gamma}{K^+ \to \mu^+ + \nu} \approx 72 \frac{\pi^+ \to \mu^+ + \nu + \gamma}{\pi^+ \to \mu^+ + \nu}. \qquad (1\cdot5)$$

These are the reasons why we would like to select the radiative K-μ decay and not the radiative π-μ decay.

IV. The production ratio of K^+-mesons to K^--mesons is greater than unity and the positive charged muons produced in the decay processes is not captured by nuclei. For this reason we have selected the process, $K^+ \to \mu^+ + \nu + \gamma$, and not the process $K^- \to \mu^- + \nu + \gamma$.

Although we have shown that the radiative K^+ decay is suitable for our purpose, one might think that the influence due to the types of interactions among K, μ and ν would be so great as to mask the effects of the electromagnetic interactions discussed above. This is not the case, however, if only the vector and the axial vector interactions are dominant in weak Fermi interactions,

$$C_P \approx C_S \approx 0 \text{ and } |C_A| \approx |C_V| \gtrsim |C_T|, \qquad (1\cdot6)$$

as has been deduced from pion decays[7] and nuclear β-decays[8]. If, on the contrary, the tensor interaction were dominant in K-meson decays, as has been argued by Hiroshima group[9] through the analysis of the K^c_{e3} decay process,* the radiative K^+

* There seems to be a possibility that further experiments on K decay may exclude the tensor coupling, just as in the case of the experiment on H_e[6 8]

decay would provide another means to verify the strong tensor interaction, and the effects of the electromagnetic interactions would be masked to great extent. Anyhow the radiative K^+ decay may be used in either way.

To express the modification of the virtual muon line as well as that of the muon-photon vertex to which one virtual muon line is connected, we shall use two typical form factors F_1 and F_2 given by (5·2) and (5·3). If F_1 function with $(1/\Lambda_\mu) \gtrsim 0.3\times10^{-13}cm$ is used and there holds (1·6), it will be shown in §§ 4 and 5 that the effects of the tensor interaction on the radiative K^+ decay are smaller than those of the modification at small distances. If $|C_T| > |C_A| \approx |C_V|$ and $(1/\Lambda_\mu) < 0.3\times10^{-13}cm$, on the contrary, the effects of the former becomes larger than those of the latter. If $(1/\Lambda_\mu) \gtrsim 0.3\times10^{-13}cm$ and $|C_T| > |C_A| \approx |C_V|$, it is hard to distinguish these two effects. If F_2 function (with $(1/\Lambda_\mu) \gtrsim 0.9\times10^{-13}cm$ or $(1/\Lambda_\mu) < 0.9\times10^{-13}cm$) is used instead of F_1 function, then the same relations hold.

The fraction of the radiative $K^+\to\mu^+$ decay, in which the energy of the emitted photon is greater than 1 *MeV*, amounts about 0.6% compared with all other K^+ decay modes. When K^+-mesons are stopped in a large liquid bubble chamber, one may identify some K^+-decays with the radiative $K^+\to\mu^+$ decays. To know the existence of the above mentioned effects, it is necessary to observe the distribution of angles of the emitted photons which produce electron pairs with respect to the muon direction. If a number of large angle decays are observed, then we may conclude that there exists either the break down of Q.E.D. at small distances or the tensor interaction in the radiative K-μ decay, or both.

§ 2. Transition matrix elements

Before entering into the radiative K-μ decay, we shall recall previous analyses of the pion decay.[7)10)~14)] As is well known, a pion decays into a muon or possibly into an electron, with or without the emission of a photon. Experimental relative frequencies of such decay modes are[15)],

$$\frac{\pi\to e+\nu}{\pi\to\mu+\nu} < 10^{-5},$$

and

$$\frac{\pi\to e+\nu+\gamma}{\pi\to\mu+\nu} < 10^{-5}. \qquad (2\cdot1)$$

It has been pointed out by Huang and Low[7)] that, if one estimates the magnitude of transition matrix elements using a cut-off perturbation calculation with a cut-off momentum equal to the nucleon mass, the following set of magnitudes of coupling constants of weak Fermi interactions may explain the experimental relative frequencies given by (2·1):

$$|C_A| \approx |C_V| \gtrsim |C_T| \text{ and } C_P \approx 10^{-3}\, C_A. \qquad (2\cdot2)$$

The scalar coupling constant C_S is not determined from the pion decay alone.

Next we shall enter into the radiative K-meson decay. We do not know at present whether the parity of a K-meson is even or odd and its parity is definite or not. For definiteness we shall, for the time being, discuss the transition matrix elements assuming that K-mesons are pseudoscalar particles. We shall further assume that weak Fermi interactions take place at a point and Q.E.D. are modified at small distances. Then the requirements of gauge invariance when $F=1$ and selection rules lead us to the following form for the matrix elements:

$$\mathfrak{M}=(2\pi)^4 u(p_1)\frac{(1+\gamma_5)}{2}\Gamma_\mu v(-p_2)\ \phi(p)\,A_\mu(k)\,\delta(p_1+p_2+k-p),$$

where

$$\begin{aligned}\Gamma_\mu=&(MC_F f_F-mC_A f_A)\frac{(i\gamma p+m)}{(p-p_1)^2+m^2}\gamma_\mu F[(p-p_1)^2]\\&+C_A f_A\Big\{F[(p-p_1)^2]-1\Big\}\gamma_\mu\\&+\Big\{C_A G_A(p\cdot k)+C_V G_V(p\cdot k)\Big\}\frac{(p\cdot k)}{M^2}\gamma_\mu\\&+C_T G_T(p\cdot k)\frac{(i\gamma k)}{M}\gamma_\mu,\end{aligned}\qquad(2\cdot3)$$

where u and v are Dirac spinors for the neutrino and the muon emitted respectively, ϕ and A are, respectively, the amplitudes of the K-meson and the photon, and p, p_1, p_2 and k are energy-momentum four-vectors of the K-meson, neutrino, muon and photon respectively.

The function F in (2·3) may depend on the momentum of the virtual muon line in Fig. 1 (c), and exhibits the modification of Q.E.D. at small regions. This function can express not only the modification of the virtual muon line but also that of the muon-photon vertex at small distances. If Q.E.D. is correct in any small domain, then this function reduces to unity and the second term in (2·3) vanishes.

If weak Fermi interactions take place at a point, then f_F and f_A are constants. G_A, G_V and G_T are functions of $(p\cdot k)$. Estimates of f_F, f_A, G_A, C_V and G_T based on a cut-off perturbation calculation of black box parts in Fig. 1 (b) and (c), with a cut-off momentum equal to nucleon mass, give[7)12)]

$$|f_F|\approx|f_A|\approx|G_A|\approx|G_V|\approx|G_T|.\qquad(2\cdot4)$$

When the K-meson is a scalar particle and not a pseudoscalar one, following relations hold instead of (2·3) and (2·4)

$$\Gamma_\mu'=(MC_S f_S-mC_V f_V)\frac{(i\gamma p+m)}{(p-p_1)^2+m^2}\gamma_\mu F[(p-p_1)^2]$$

$$+C_V f_V\left\{F[(p-p_1)^2]-1\right\}\gamma_\mu$$

$$+\left\{C_V G_V'(p\cdot k)+C_A G_A'(p\cdot k)\right\}\frac{(p\cdot k)}{M^2}\gamma_\mu$$

$$+C_T G_T'(p\cdot k)\frac{i\gamma k}{M}\gamma_\mu \qquad (2\cdot5)$$

and $$|f_S|\approx|f_V|\approx|G_V'|\approx|G_A'|\approx|G_T'|\approx|f_A|. \qquad (2\cdot6)$$

If the parity of the K-meson is not definite, the following quantity must be used instead of $\Gamma_\mu\phi(p)$ or $\Gamma_\mu'\phi(p)$,

$$\Gamma_\mu\phi_p(s)+\Gamma_\mu'\phi_s(p), \qquad (2\cdot7)$$

where ϕ_p and ϕ_s are the pseudoscalar and scalar parts of $\phi(p)$ respectively.

Now we may use (2·2) and

$$C_s\approx0 \qquad (2\cdot8)$$

in the process concerned. In fact the experiments on K-meson decays show

$$\frac{K\to e+\nu}{K\to\mu+\nu}\ll1,$$

which means $$C_P\approx C_S\approx0.$$

In (2·3) and (2·5), we have chosen the special gauge, $p_\mu\cdot A_\mu=0$, which means $\boldsymbol{p}=0$. The δ-function in (2·3) leads us to

$$k_0\leq\frac{(m_K^2-m^2)}{2m_K} \qquad \text{(See, § 3)}$$

We get, therefore, $$-(p\cdot k)/M^2\leq(m_K^2-m^2)/2M^2\approx1/8. \qquad (2\cdot9)$$

(2·9) means that the third terms in (2·3) and (2·5) are negligible compared with $C_A f_A$ or $C_V f_V$ term. In fact, it will be shown in §§ 4 and 5 that not only the third terms but also the fourth terms in (2·3) and (2·5) are negligible provided that $|C_T|\leq|C_V|\approx|C_A|$.

In the lowest order perturbation calculation G_A, G_V, G_A' and G_V' are finite but f_A, f_V, G_T and G_T' are logarithmically divergent. With a large variation of the cut-off parameter, therefore, the latter constants and functions of $(p\cdot k)$ might vary by many orders of magnitude. G_T/f_A and G_T'/f_V, however, remain unchanged approximately[14] and

$$|G_A/f_A|\approx|G_V/f_V|\approx|G_A'/f_V|\approx|G_V'/f_V|\lesssim1.$$

These properties as to the magnitudes of the transition matrix elements would hold for higher order radiative corrections* of the black box parts in Fig. 1.

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* The diagrams which belong to Fig. 1 (b) are divided into two classes according to whether

According to the reasons mentioned above, one finally gets for pseudoscalar K-mesons

$$\Gamma_\mu \fallingdotseq -mC_A f_A \frac{(i\gamma p+m)}{(p-p_1)^2+m^2}\gamma_\mu F[(p-p_1)^2]$$

$$+C_A f_A\{F[(p-p_1)^2]-1\}\gamma_\mu \qquad (2\cdot10)$$

$$+C_T G_T(p\cdot k)\frac{i\gamma k}{M}\gamma_\mu,$$

and

$$|f_A|\approx|G_T|. \qquad (2\cdot11)$$

(2·11) will remain unchanged, as the cut-off parameter is varied in a large interval. Analogous relations hold for scalar and parity mixed K-mesons.

§ 3. Transition probability

In this section we shall calculate the transition probability of the radiative K-μ decay, and use the following notations,

$$\begin{cases} C=m_K/m, \quad d=M/m, \quad P_2=(m\boldsymbol{R}, mE), \quad E=\sqrt{R^2+1}, \\ R=|\boldsymbol{R}|, \quad k=(m\boldsymbol{l}, ml), \quad l=|\boldsymbol{l}|, \\ 2m_e=m\varepsilon \text{ and } h=\dfrac{(C^2-1)}{2C}-\varepsilon\dfrac{(C^2+1)}{C^2}, \end{cases} \qquad (3\cdot1)$$

where m_e is the electron mass.

The transition probability w is given by

$$w=(2\pi)\sum_{\boldsymbol{F}}\sum_{S}|\mathfrak{M}|^2\rho_{\boldsymbol{F}}/\int dV\int dt, \qquad (3\cdot2)$$

where $\rho_{\boldsymbol{F}}$ is the final state density, $\int dV\int dt$ the total volume of space-time, $\sum_{\boldsymbol{F}}$ and $\sum_{S}$ respectively, the summations over the final states and the spin states concerned. After the integration over $\boldsymbol{p}_1$ in (3·2), the following type of integrations appears:

$$X=\int_0^\infty dR\int_{-1}^{1}d(\cos\theta)\int_\varepsilon^\infty dl\ f(R, l, \cos\theta)$$

$$\times\delta(C-E-l-\sqrt{R^2+l^2+2Rl\cos\theta}), \qquad (3\cdot3)$$

where θ is the angle between $\boldsymbol{l}$ and $\boldsymbol{R}$. This integration can be rewritten as

$$X=\int_0^h dR\int_{-1}^{1}d(\cos\theta)\int_{-\infty}^{\infty}dl\frac{(C-E-l)}{(C-E+R\cos\theta)}$$

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a photon is emitted by a virtual meson line or by a virtual baryon line. When a virtual meson emits a photon, this diagram has no contribution to the matrix element, (2·3) or (2·5).

$$\times f(R, l, \cos\theta)\delta\left(l-\frac{C^2+1-2CE}{2[C-E+R\cos\theta]}\right). \tag{3·4}$$

If the energy of the emitted photon is smaller than $2m_e$, the photon can not create an electron pair and the radiative K^+-μ^+ decay would be difficult to distinguish from $K^+\to\mu^++\nu$ decay.

For this reason we have cut the l-integration in (3·3) at ε. This corresponds to cutting the R-integration at h and not $(C^2-1)/2C$ as in (3·4). In this procedure $0(\varepsilon^2)$ terms have been neglected.

In this section we shall treat K-mesons as pseudoscalar particles. The results obtained would may be translated at once to the other cases.

Case 1. $F=1$ and $C_T=0$.

This is the case which was resently proposed by Feynman and Gell-Mann.[16] Inserting (2·10) into (3·2), one obtains

$$\begin{aligned}A_1&=\omega|_{F=1 \text{ and } C_T=0}\\&=\frac{\alpha}{2\pi}m^4|C_A f_A|^2\int_0^h dR\int_0^\pi d\theta\int_{-\infty}^{\infty}dl\ \Delta(R, l, \cos\theta)\\&\times\frac{1}{[E-R\cos\theta]^2}\Big\{2C(C-E)(CE-1)\\&+(C^2-1)(E^2-CE-R^2\cos^2\theta)\\&+[2C+(C^2-1)(E-R\cos\theta)-2C^2E]l\Big\},\end{aligned} \tag{3·5}$$

where

$$\Delta(R, l, \cos\theta)=\frac{R^2\sin\theta}{E(C^2+1-2CE)}\delta\left[l-\frac{C^2+1-2CE}{2(C-E+R\cos\theta)}\right]. \tag{3·6}$$

Further integrations with respect to l and θ of (3·5) lead us to

$$\begin{aligned}A_1=\frac{2\alpha}{\pi}|C_A f_A|^2m^4\int_0^h dR\ \frac{R}{E}\Big\{&\frac{C^2+1-2CE}{8C}\log\frac{(E+R)^2(C^2+1-2CE)}{(C-E-R)^2}\\&+\frac{(C^2-1)}{(C^2+1-2CE)}\langle E\log(E+R)-R\rangle\Big\}.\end{aligned} \tag{3·7}$$

(3·7) agrees exactly with (7) of the paper of Flalho and Tiomno.[11]

The transition probability for the non-radiative K-μ decay is, on the other hand, given by

$$w'=\tfrac{1}{4}|C_A f_A|^2m^4\frac{(C^2-1)^2}{C^2}. \tag{3·8}$$

The numerical calculation of the ratio of A_1 to w' gives us

$$A_1/w'\cong 1.08\times10^{-2}. \tag{3·9}$$

This ratio is very large compared with the corresponding ratio for the pion, which is about 1.50×10^{-4}*. The latter value is smaller than the experimental value, $(3.3\pm1.3)\times10^{-4}$.[17] This difference need not be considered too seriously because the number of the radiative π-μ decays observed is very few.

Using (3·5) we can obtain the angular dependence of the emitted photon

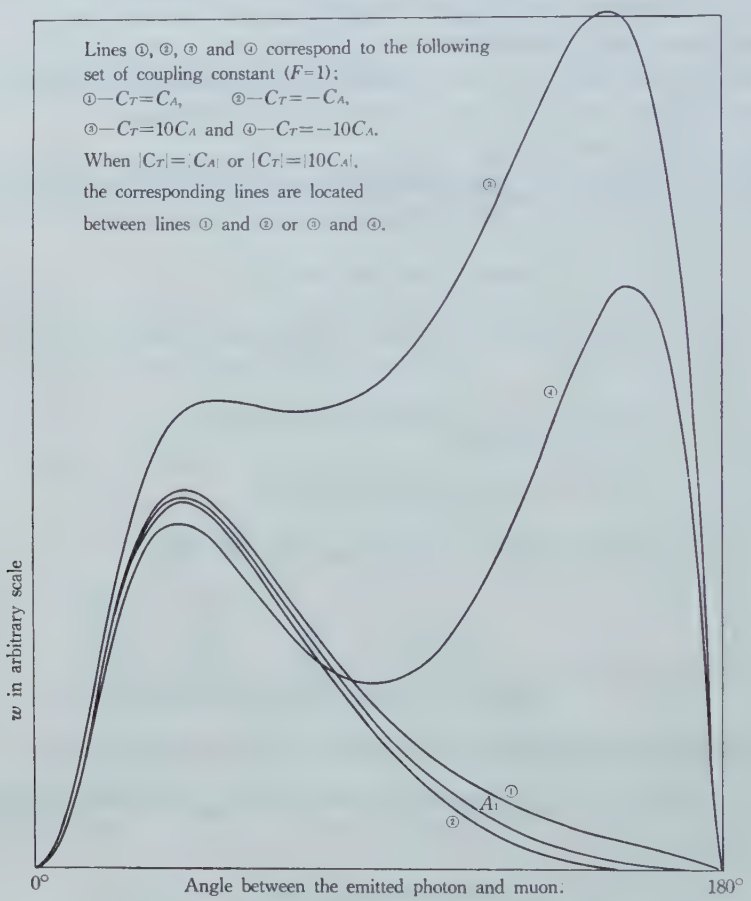
Lines ①, ②, ③ and ④ correspond to the following set of coupling constant ($F=1$):
①—$C_T=C_A$, ③—$C_T=-C_A$,
②—$C_T=10C_A$ and ④—$C_T=-10C_A$.
When $|C_T|=|C_A|$ or $|C_T|=|10C_A|$, the corresponding lines are located between lines ① and ② or ③ and ④.
③
④
①
②
A_1
w in arbitrary scale
0°
Angle between the emitted photon and muon.
180°

Fig. 2. Angular dependence of the photon to the muon.

with respect to the muon. It will be seen from Fig. 2 and Table 1 that the large angle decay of the photon is very small.

Case 2. $F=1$ and $C\neq0$.

Since the $(p\cdot k)$ dependence of $G_T(p\cdot k)$ is believed to be negligible, we obtain

$$\omega|_{F=1}=A_1+A_2+A_3, \tag{3·10}$$

* This value is small by $\frac{1}{2}$ compared with the value given by Huang and Low[7].

$$A_2=\frac{\alpha}{2\pi}m^4\Big\{(C_A f_A)^{Re}(C_T G_T(0))^{Re}+(C_A f_A)^{Im}(C_T G_T(0))^{Im}\Big\}$$
$$\times\frac{4}{d}\int_0^h dR\int_0^\pi d\theta\int_{-\infty}^{\infty}dl\,\Delta(R,l,\cos\theta)\frac{1}{[E-R\cos\theta]}$$
$$\times\Big\{(C^2+1-2CE)(E-R\cos\theta)+C(R^2\sin^2\theta)\Big\}l^2 \qquad (3\cdot11)$$

and

$$A_3=\frac{\alpha}{2\pi}m^4|C_T G_T(0)|^2\frac{8}{d^2}\int_0^h dR\int_0^\pi d\theta\int_{-\infty}^{\infty}dl\,\Delta(R,l,\cos\theta) \qquad (3\cdot12)$$
$$\times(C-E+R\cos\theta)(E-R\cos\theta)l^4,$$

where *Re* and *Im* mean real and imaginary parts respectively. The effects of the tensor interaction will be estimated in § 4.

Case 3. $F\neq1$ and $C_T=0$.

In this case we get

$$\omega|_{C_T=0}=A_1+A_4+A_5, \qquad (3\cdot13)$$

$$A_4=\frac{\alpha}{2\pi}m^4|C_A f_A|^2\int_0^h dR\int_0^\pi d\theta\int_{-\infty}^{\infty}dl\,\Delta(R,l,\cos\theta)$$
$$\times\frac{4l}{[E-R\cos\theta]}\Big\{|F^2|-ReF\Big\}$$
$$\times\Big\{-(C-E)^2+R^2\cos^2\theta+(C-E+R\cos\theta)l\Big\}, \qquad (3\cdot14)$$

$$A_5=\frac{\alpha}{2\pi}m^4|C_A f_A|^2\int_0^h dR\int_0^\pi d\theta\int_{-\infty}^{\infty}dl\,\Delta(R,l,\cos\theta)$$
$$\times\Big\{|F|^2+1-2ReF\Big\} \qquad (3\cdot15)$$
$$\times4l^2\Big\{E(C-E)+R^2\cos\theta-(E-R\cos\theta)l\Big\}$$

and

$$F=F[-2m^2l(E-R\cos\theta)-m^2]. \qquad (3\cdot16)$$

The effects of the break down of Q.E.D. at small distances will be evaluated in § 5.

§ 4. Effects of the tensor interaction

As was mentioned before, the following set of Fermi coupling constants must be adopted to explain the relative frequency of the pion decay given by (2·1) and the rareness of *K-e* decays :

$$C_S \approx C_P \approx 0 \text{ and } |C_A| \approx |C_V| \gtrsim |C_T|^{*}. \tag{1·6}$$

This set of coupling constants seems to be "*universal*." In fact (1·6) is consistent with all existing experiments in nuclear β-decays[8]. The only exception is the K^{e}_{e3} decay, which was analysed by Hiroshima group.[9] Their analysis about the energy spectrum of secondary electrons seems to show that there are scalar and tensor interactions and probably $|C_T| > |C_V|$. From the view point of universal Fermi interaction, however, one may hope that further experiments on K^{e}_{e3} decay exclude scalar and tensor interactions.

In this section we shall estimate the effects of the tensor interaction keeping in mind the above mentioned situation. As is well known, A_1 has the infra-red catastrophe. This means that the main contribution to A_1 comes from the process emitting low energy photons. As will be seen from the δ-function in (3·6) low energy photons are, in general, emitted at small angles. The angular dependence expected from A_1 alone, which is shown in Fig. 2 by curve A_1 shows this property. On the other hand, effects of the modification of Q.E.D. at small distances and of the tensor interaction appear at large angles. (See, Fig. 2) As a measure of these effects, we shall introduce the following quantity:

$$R_\theta = \frac{\int_0^\theta d\theta' \; w_\theta'}{\int_\theta^\pi d\theta' \; w_\theta'}, \tag{4·1}$$

where $$w = \int_0^\pi d\theta' \; w_\theta'.$$

The values of R_θ against some values of θ are tabulated in Table 1. From this table it seems difficult to know the existence of the tensor interactions if $|C_T| \lesssim |C_A|$. Fig. 2 and Table 1 remain approximately unchanged for scalar and parity mixing K-mesons if $|C_A| \approx |C_V|$, $|f_A| \approx |f_V|$ and $|G_T| \approx |G_T'|$.

line \ R_θ	R_{160°	R_{150°	R_{135°	R_{120°
A_1 line ($F=1$ and $C_T=0$)	532.	201.	65.5	26.2
line 1 $F=1$ and $(C_T=C_A)$	106.	50.2	23.3	12.6
line 2 $F=1$ and $(C_T=-C_A)$	2460.	938.	178.	47.5

Table 1. Small angle-large angle ratio of transition probabilities.

* If K-mesons are pseudoscalar particles, then $C_S \neq 0$ does not contradict with the rareness of K-e decay.

§ 5. Effects of modification of Q.E.D. at small distances

Unfortunately we have not a consistent finite theory yet. If we introduce the form factor F in the present theory as in the case of (2·10), therefore, the result becomes inconsistent with the framework of the present theory. For example, at least one of the condition of the unitarity of the S matrix, microscopic causality, gauge invariance and Lorentz invariance, etc., is not satisfied. However, we may hope that this technique reflects some properties of consistent finite theory which will be found in future.

Though this technique violates the framework of the present theory, we may impose the following conditions on the form factor F.

a) F has a covariant form.

b) F plays the role of a convergence factor.

c) When all momenta involved in F satisfy the energy-momentum relation which is satisfied by free particles, F reduces to unity.*

We are interested in the form factor which involves only one virtual muon momentum. In this case F has, in general, the form:

$$F(q^2)=\theta(q^2)F_s(q^2)+\theta(-q^2)F_t(q^2), \tag{5·1}$$

where

$$\theta(q^2)=\begin{cases}1 & \text{for} \quad q^2>0\,,\\ 0 & \text{for} \quad q^2<0\,,\end{cases}$$

and q is the virtual muon momentum. Even if we knew the character of one of these two parts, F_s and F_t, in principle, we could have no information on the other part. In the case of large angle electron pair production on hydrogen[3)6)], one may infer the feature of F_s. On the other hand, our process concerns F_t. We may, therefore, obtain further information of Q.E.D. at small distances.

In order to estimate the effect of the modification of Q.E.D. at small distances, we shall use the following two form factors:

$$F_t(q^2)=\frac{\Lambda_\mu^2}{(q^2+m^2)+\Lambda_\mu^2}\equiv F_1(q^2,\Lambda_\mu^2) \tag{5·2}$$

and

$$F_t(q^2)=\frac{\Lambda_\mu^4}{(q^2+m^2)^2+\Lambda_\mu^4}\equiv F_2(q^2,\Lambda_\mu^2)\,. \tag{5·3}$$

F_1 or F_2 may express the modification due to the virtual muon line as well as the muon-photon vertex part. Using these form factors one obtains Fig. 3 and

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* For example we may adopt the following form factor for the muon-photon vertex part,

$$\Lambda_\mu^2/\{(q_1^2+m^2)+(q_2^2+m^2)+q_3^2+\Lambda_\mu^2\},$$

where q_1 and q_2 are four vectors of muons and q_3 is a four vector of the photon. This form satisfies the conditions a), b) and c).

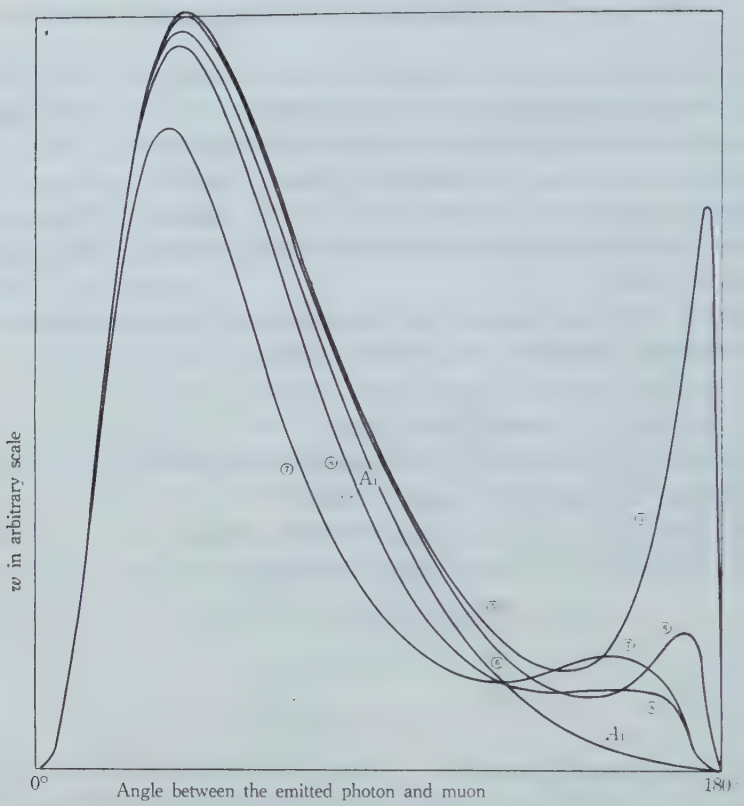

Fig. 3. Angalar dependence of the photon to the muon.

Table 2. Lines 5, 6, 7 and 8 corresponed to the following form factors and cut-off parameters:

$$\text{line 5} \quad — \quad F_t = F_1 \text{ and } \frac{\Lambda_\mu^2}{m^2} = C^2 - \frac{1}{2} \simeq 21.3,$$

$$\text{line 6} \quad — \quad F_t = F_1 \text{ and } \frac{\Lambda_\mu^2}{m^2} = 25.0,$$

$$\text{line 7} \quad — \quad F_t = F_2 \text{ and } \frac{\Lambda_\mu^2}{m^2} = 1.00$$

and

$$\text{line 8} \quad — \quad F_t = F_2 \text{ and } \frac{\Lambda_\mu^2}{m^2} = 4.00.$$

If $(\Lambda^2_\mu/m^2) \leq C^2 - 1$, there may appear a pole in the F_1 function. In this case F_1 function needs to be modified and the corresponding line would be changed a great deal from A_1 line.

As will be seen from Tables 1 and 2, the effect of the modification of Q.E.D.

at small distances seems to be measurable, if $(\Lambda_\mu^2/m^2) \lesssim 36$ and is larger than the effect of the tensor interaction provided that $|C_T| \lesssim |C_A|\ (\approx |C_V|)$. Our analysis has been based on the assumption that the muon and neutrino are created at a point. If weak Fermi interactions are smoothed out over some regions, then there is a possibility that this effect also appears at large angles. However, this would be negligible in comparison with the effect of the modification of Q.E.D. at small distances if the domain over which Fermi interactions are smoothed out is about $1/M$ or smaller. The feature of F_s appearing in (5·1) would be determined by large angle muon pair production on Hydrogen.

line \ R_θ	R_{160^0}	R_{150^0}	R_{135^0}	R_{120^0}
A_1 line	532.	201.	65.5	26.2
line 5	7.63	6.15	5.11	4.23
line 6	25.2	17.8	12.7	8.94
line 7	39.4	18.6	10.2	7.16
line 8	55.1	28.3	16.3	11.0

Tabl 2. Small angle-large angle ratio of transition probabilities.

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Exciton and Plasmon in Insulating Crystals

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The relation between an exciton and a plasmon in insulators is investigated from a view-point of an electron-hole pair approximation. It is shown that the usual exciton model is valid in the insulator having appreciably localized wave functions and that if the wave functions are not appreciably localized but rather extended, a plasmonlike excitation becomes possible and appears above the continuum pair-band besides the exciton level below it. These behaviors are discussed in comparison with experimental facts.

§ 1. Introduction

The several peaks in optical absorption spectra of alkali halides have been analysed with the aid of the so-called exciton model by many authors.[1)–11)] The electron excited by photon and the positive hole left behind in valence band are coupled together through an effectively attractive Coulomb force so that a bound state will be formed. The bound electron-hole pair is known as an exciton. The exciton model has also been applied to interpreting not only the optical absorption peaks found in other ionic crystals like Cu_2O,[12)] which constitute the hydrogen-like series, but also the structures of X-ray absorption and emission spectra in alkali halides.[13)]

On the other hand, it has been well-known that the charged particles with high speed suffer characteristic energy losses of about ten to twenty electron volts, when they pass through a crystal.[14)] Some peaks in the energy loss spectra in insulators have been considered to be due to excitation of plasma oscillations,[15)] as in metals. The plasma oscillation, whose quantum is called "plasmon" according to Pines, is an organized motion of many electrons coupled together by a long-range Coulomb force. Since, in insulators, the valence band is filled up by electrons and the conduction band is empty at zero temperatures, the organized plasma oscillation will be facilitated only by exciting electron-hole pairs. Thus, the plasmon in insulators appears to be related, to some extent, to the exciton model.

It is the purpose of the present paper to make clear the relation between the models of an exciton and of a plasmon in insulators. This is made easy by the use of an electron-hole pair approximation. It is shown that in the insulator having appreciably localized wave functions the usual exciton model is valid as a possible excitation. If the wave functions are not appreciably localized but rather extended,

so that collective couplings of electrons and holes become possible, then a plasmon-like excitation can appear above the continuum pair-band besides the exciton level below it. These behaviors will be discussed in § 5.

The fundamental formulation of eigenvalue problem for a possible excitation state is given in § 2. § 3 and § 4 are devoted to the reviews of behaviors of a plasmon and an exciton in insulators from the view-point of an electron-hole pair approximation. The plasmon model in insulators is similar to that in metals except the effect of existence of a finite energy gap between conduction and valence bands. Both models are somewhat modified if we proceed to the second approximation in § 5, where the relation between an exciton and a plasmon is discussed. In § 6, the concluding remarks are given and experimental data observed in various insulators are discussed in connection with the present results. The important correction to an electron-hole pair approximation is the effect of scatterings of electrons to electrons and holes to holes, and the coupling between pairs having different momentum transfers. Even though they may serve screening effects to the effective potentials for an exciton and a plasmon, the essential results of the present approximation will not qualitatively be changed by them.

§ 2. General formulation

The Coulomb interaction between electrons in crystal is expressed by

$$H_c=\frac{1}{2}\iint d\boldsymbol{x}d\boldsymbol{x}'\Psi^*(\boldsymbol{x})\Psi^*(\boldsymbol{x}')\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\Psi(\boldsymbol{x}')\Psi(\boldsymbol{x}). \qquad (2\cdot1)$$

Annihilation and creation operators of electrons are introduced by expanding the electron field $\Psi(\boldsymbol{x})$ with respect to Bloch functions, which constitute an orthonormal complete set:

$$\Psi(\boldsymbol{x})=\sum_{\mu}\sum_{\boldsymbol{k}}a_{\mu,\boldsymbol{k}}\psi_{\mu,\boldsymbol{k}}(\boldsymbol{x}), \qquad \psi_{\mu,\boldsymbol{k}}(\boldsymbol{x})=\Omega^{-1/2}\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]u_{\mu,\boldsymbol{k}}(\boldsymbol{x}), \qquad (2\cdot2)$$

where μ is the index of band, $\boldsymbol{k}$ is the reduced wave vector which runs over values within the first Brillouin zone, and the Bloch function $\psi_{\mu,\boldsymbol{k}}(\boldsymbol{x})$ is normalized in the crystal having the volume Ω.

In insulators, the valence bands are completely filled up by electrons and the conduction bands are empty at zero temperature. If an electron is excited from a valence band into a conduction band, there is left a hole in the valence band. As we are interested in the excitation state in insulators, it is more convenient to introduce creation and annihilation operators of a hole besides that of an electron. These operators are introduced by transforming $a_{\mu,\boldsymbol{k}}$ and $a_{\mu,\boldsymbol{k}}^*$ into $b_{\mu,\boldsymbol{k}}^*$ and $b_{\mu,\boldsymbol{k}}$, respectively, if μ belongs to the valence band. New operators $a_{\mu,\boldsymbol{k}}$, $a_{\mu,\boldsymbol{k}}^*$, $b_{\mu,\boldsymbol{k}}^*$, and $b_{\mu,\boldsymbol{k}}$ correspond to the annihilation and creation of an electron in $\boldsymbol{k}$-state of μ-conduction band and the creation and annihilation of a hole in the $\boldsymbol{k}$-state of μ-valence band, respectively. They satisfy the following commutator relations:

$$\left.\begin{aligned}
&[a_{\mu,k},\ a_{\mu',k'}]_+=0, \qquad [a_{\mu,k},\ a^*_{\mu',k'}]_+=\delta_{\mu,\mu'}\,\delta_{k,k'},\\
&[b_{\mu,k},\ b_{\mu',k'}]_+=0, \qquad [b_{\mu,k},\ b^*_{\mu',k'}]_+=\delta_{\mu,\mu'}\delta_{k,k'},\\
&[a_{\mu,k},\ b_{\mu',k'}]_+=0, \qquad [a_{\mu,k},\ b^*_{\mu',k'}]_+=0.
\end{aligned}\right\} \tag{2·3}$$

The Hamiltonian of the system is given by

$$H=H^{(1)}+H_c, \tag{2·4}$$

where $H^{(1)}$ consists of kinetic terms of electrons and of interactions of electrons with nuclei at rest*, and H_c is the Coulomb interaction between electrons given by (2·1). As we use the Bloch scheme, we must be careful in the choice of the unperturbed Hamiltonian. Let the Bloch function $\psi_{\mu,k}(\boldsymbol{x})$ be the eigenfunction of Hartree-Fock self-consistent field Hamiltonian H_0 and $\varepsilon_\mu(\boldsymbol{k})$ be the corresponding eigenvalue:

$$H_0\psi_{\mu,k}(\boldsymbol{x})=\varepsilon_\mu(\boldsymbol{k})\psi_{\mu,k}(\boldsymbol{x}). \tag{2·5}$$

Then the total Hamiltonian can be written as

$$H=H_0+H_c-\Gamma, \tag{2·6}$$

where Γ is determined in such a way that an eigenvalue of H_0 is equal to an eigenvalue of H in the Hartree-Fock approximation. Determination of Γ has been studied by many authors.[16] Thus, the terms on the right-hand side of (2·6) are given, by the use of new operators, as follows.

$$H_0=\sum_c\sum_k \varepsilon_c(\boldsymbol{k})\,a^*_{c,k}a_{c,k}-\sum_v\sum_k \varepsilon_v(\boldsymbol{k})\,b^*_{v,k}b_{v,k}+\sum_v\sum_k \varepsilon_v(\boldsymbol{k}), \tag{2·7}$$

$$\begin{aligned}
H_c=&\tfrac{1}{2}\sum_{\mu\nu\lambda\sigma}\sum_q\sum_k\sum_{k'} V_{k+q,k,k'-q,k'}^{\mu\ \nu\ \lambda\ \sigma}(\boldsymbol{q})\,(a^*_{\mu,k+q}b^*_{\nu,k}+b_{\mu,k+q}a_{\nu,k}+a^*_{\mu,k+q}a_{\nu,k}+b_{\mu,k+q}b^*_{\nu,k})\\
&\times(a_{\lambda,k'-q}b^*_{\sigma,k'}+b_{\lambda,k'-q}a_{\sigma,k'}+a^*_{\lambda,k'-q}a_{\sigma,k'}+b_{\lambda,k'-q}b^*_{\sigma,k'})\\
&-\tfrac{1}{2}\sum_{\mu\nu\lambda}\sum_q\sum_k V_{k,k-q,k-q,k}^{\mu\ \lambda\ \lambda\ \nu}(\boldsymbol{q})\,(a^*_{\mu,k}b^*_{\nu,k}+b_{\mu,k}a_{\nu,k}+a^*_{\mu,k}a_{\nu,k}+b_{\mu,k}b^*_{\nu,k}),
\end{aligned} \tag{2·8}$$

$$\begin{aligned}
\Gamma=&\sum_{\mu\nu}\sum_\lambda^{(b)}\sum_k\sum_{k'}\{V_{k',k',k,k}^{\lambda\ \lambda\ \mu\ \nu}(0)-V_{k+k',k,k,k+k'}^{\lambda\ \nu\ \mu\ \lambda}(\boldsymbol{k})\}\\
&\times(a^*_{\mu,k}b^*_{\nu,k}+b_{\mu,k}a_{\nu,k}+a^*_{\mu,k}a_{\nu,k}+b_{\mu,k}b^*_{\nu,k}),
\end{aligned} \tag{2·9}**$$

where

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* Displacements of nuclei and thermal vibrations of lattice are not taken into account in the present paper.

** A half of the diagonal term of the Hartree-Fock potential Γ may still be subtracted from (2·9), as has been discussed by Tobocman (see reference 16)). Instead, the terms concerning infinite Coulomb energies of an electron and a hole will be dropped after calculations, ensuring the neutrality of the crystal.

$$V_{k+q,k,k'-q,k'}^{\mu\ \nu\ \lambda\ \sigma}(\boldsymbol{q})=\iint d\boldsymbol{x}d\boldsymbol{x}'\psi_{\mu,k+q}^{*}(\boldsymbol{x})\psi_{\nu,k}(\boldsymbol{x})\psi_{\lambda,k'-q}^{*}(\boldsymbol{x}')\psi_{\sigma,k'}(\boldsymbol{x}')\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{2·10}$$

$\sum_{\lambda}^{(v)}$ in (2·9) means that the summation over λ must be restricted only to valence bands.

Since we are interested in low-lying excitation states, it will be sufficient to restrict ourselves to the one-pair approximation in which the excited state is assumed as

$$\Psi_{\text{excited}}=A^{*}\Psi_{0}, \tag{2·11}$$

$$A^{*}=\sum_{q}\sum_{\mu,\nu}\sum_{p}[\alpha_{p}^{\mu\nu}(\boldsymbol{q})a_{\mu,p+q}^{*}b_{\nu,p}^{*}+\beta_{p}^{\mu\nu}(\boldsymbol{q})b_{\mu,p+q}a_{\nu,p}+\gamma_{p}^{\mu\nu}(\boldsymbol{q})a_{\mu,p+q}^{*}a_{\nu,p}+\delta_{p}^{\mu\nu}(\boldsymbol{q})b_{\mu,p+q}b_{\nu,p}^{*}]. \tag{2·12}$$

Ψ_0 is the ground state of the system, where there are no real electrons in conduction bands at zero temperatures. The first term in (2·12) represents the state in which holes and electrons are created in valence and conduction bands, respectively. If the electron and the hole are coupled to form a bound state, so that the coefficient has a localized character when it is transformed into coordinate space, then the corresponding state will be the usual exciton state. Otherwise, the corresponding state will represent the entities of electrons and holes moving freely to each other, in which we have no interest. The second term in (2·12) represents destructions of the electron-hole pairs. If it couples completely with the first term, resulting a stationary wave of charge fluctuations, then the state expressed by them will correspond to the plasmon. The third and fourth terms represent scatterings of electrons to electrons and holes to holes, respectively. If we take these terms as virtual states for the real electron-hole pair state represented by the first two terms of (2·12), it will be seen that they serve some screening effects of higher order* to the effective potential for the electron-hole pair state. We ignore, therefore, the last two terms in the followings, because they do not seem essential in order to see the relation between an exciton and a plasmon.

Eigenvalue problem is

$$H\Psi_{\text{excited}}=E\Psi_{\text{excited}}, \tag{2·13}$$

or

$$(E-H)A^{*}\Psi_{0}=[A^{*},\ H]_{-}\Psi_{0}+(E-E_{0})A^{*}\Psi_{0}=0, \tag{2·14}$$

where E_0 is the ground state energy of the crystal:

$$H\Psi_{0}=E_{0}\Psi_{0}. \tag{2·15}$$

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* The effect of the last two terms of (2·12) is partly taken into account in the unperturbed Hamiltonian H_0, in the lowest order, that is, in the meaning of Hartree-Fock approximation.

Multiplying (2·14) by $\langle \Psi_0 | b_{v,\boldsymbol{p}} a_{c,\boldsymbol{p}+\boldsymbol{q}}$ and $\langle \Psi_0 | a_{c,\boldsymbol{p}}^* b_{v,\boldsymbol{p}+\boldsymbol{q}}^*$, respectively, from the left, we obtain the equations for $\alpha_{\boldsymbol{p}}^{cv}(\boldsymbol{q})$ and $\beta_{\boldsymbol{p}}^{cv}(\boldsymbol{q})$.

$$[\hbar\omega - \{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_v(\boldsymbol{p})\}]\alpha_{\boldsymbol{p}}^{cv}(\boldsymbol{q}) + \sum_{c'v'} \sum_{\boldsymbol{p}'} U_{cvc'v'}^{cvc'v'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q})\alpha_{\boldsymbol{p}'}^{c'v'}(\boldsymbol{q}) - \sum_{c'v'} \sum_{\boldsymbol{p}'} U_{cvc'v'}^{cvc'v'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q})\beta_{\boldsymbol{p}'}^{c'v'}(\boldsymbol{q}) = 0, \tag{2·16}$$

$$[\hbar\omega + \{\varepsilon_c(\boldsymbol{p}) - \varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}]\beta_{\boldsymbol{p}}^{cv}(\boldsymbol{q}) - \sum_{c'v'} \sum_{\boldsymbol{p}'} U_{vcvc'}^{vc'vc}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q})\beta_{\boldsymbol{p}'}^{c'v'}(\boldsymbol{q}) + \sum_{c'v'} \sum_{\boldsymbol{p}'} U_{vcvc'}^{vc'vc}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q})\alpha_{\boldsymbol{p}'}^{c'v'}(\boldsymbol{q}) = 0, \tag{2·17}$$

where c and c' are the indices of conduction bands, and v and v' that of valence bands. The following notations were used.

$$\hbar\omega = E - E_0, \tag{2·18}$$

$$U_{\mu'\nu'\lambda'\sigma'}^{\mu\nu\lambda\sigma}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}) = V_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}'+\boldsymbol{q},\boldsymbol{p}',\boldsymbol{p}}^{\mu\nu\lambda\sigma}(\boldsymbol{p}-\boldsymbol{p}') - V_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p},\boldsymbol{p}',\boldsymbol{p}'+\boldsymbol{q}}^{\mu'\nu'\lambda'\sigma'}(\boldsymbol{q}). \tag{2·19}$$

In the reduction, we have neglected $\langle \Psi_0 | a_{c,\boldsymbol{p}}^* a_{c,\boldsymbol{p}} | \Psi_0 \rangle$ and $\langle \Psi_0 | b_{v,\boldsymbol{p}}^* b_{v,\boldsymbol{p}} | \Psi_0 \rangle$ in comparision to unity. This is reasonable in the present one-pair approximation, because the number of excited electrons and holes is assumed to be a fraction of the total number of electrons in the crystal. It should be noted here that the coefficients with different $\boldsymbol{q}$ are not coupled with each other, as is seen from (2·16) and (2·17). This is the consequence of the one-pair approximation.

In order to solve (2·16) and (2·17), we shall approximate U's by using a reasonable condition realizable in real insulators.

§ 3. Plasmon

In this section, it will be shown that the plasmon can be a possible solution of a set of equations, (2·16) and (2·17), for an ideal case in which wave functions are not appreciably localized but are rather plane wave-like.

By the use of the Fourier transform of $e^2/|\boldsymbol{x}-\boldsymbol{x}'|$ and of the periodicity of Bloch functions, (2·19) can be written as

$$U_{\mu'\nu'\lambda'\sigma'}^{\mu\nu\lambda\sigma}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}) = \frac{4\pi e^2}{\Omega|\boldsymbol{p}-\boldsymbol{p}'|^2} C_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}'+\boldsymbol{q}}^{\mu\nu} C_{\boldsymbol{p}',\boldsymbol{p}}^{\lambda\sigma} - \frac{4\pi e^2}{\Omega q^2} C_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}}^{\mu'\nu'} C_{\boldsymbol{p}',\boldsymbol{p}'+\boldsymbol{q}}^{\lambda'\sigma'}, \tag{3·1}$$

where

$$C_{\boldsymbol{p},\boldsymbol{p}'}^{\mu\nu} = \int d\boldsymbol{x}\, \psi_{\mu,\boldsymbol{p}}^*(\boldsymbol{x})\psi_{\nu,\boldsymbol{p}'}(\boldsymbol{x}) \exp[i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}]$$

$$= (N/\Omega)\int_{\text{cell}} d\boldsymbol{x}\, u_{\mu,\boldsymbol{p}}^*(\boldsymbol{x}) u_{\nu,\boldsymbol{p}'}(\boldsymbol{x}). \tag{3·2}$$

Ω/N is the volume of unit cell and $\int_{\text{cell}} d\boldsymbol{x}$ means the integral over the unit cell.

Condsider the ideal case where wave functions $\psi_{\mu,\boldsymbol{p}}(\boldsymbol{x})$'s are nearly plane

wave-like, and assume that values of $C_{p,p'}^{\mu\nu}$'s do not appreciably change from one pair of bands, μ and ν, to the other. This may be a possible assumption for the semiconductor having a nearly metallic character. Then, the value of $C_{p+q,p'+q}^{\mu\nu}\cdot C_{p',p}^{\lambda\sigma}$ will be comparable to that of $C_{p+q,p}^{\mu'\nu'}\cdot C_{p',p'+q}^{\lambda'\sigma'}$. If this is the case, the ratio of the first term to the second one on the right-hand side of (3·1) is roughly estimated by replacing them by their averages over values of $\boldsymbol{p}$ and $\boldsymbol{p}'$ within the first Brillouin zone, and is found to be of the order of $(q^2/p_{\max}^2)$, where $p_{\max}$ is the radius of the sphere having the same volume as the first Brillouin zone.* Therefore, if we restrict ourselves to sufficiently small value of $\boldsymbol{q}$, the first term is negligible compared with the second one, and (3·1) is approximated by

$$U_{\mu'\nu'\lambda'\sigma'}^{\mu\nu\lambda\sigma}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q})\simeq-\frac{4\pi e^2}{\Omega q^2}C_{p+q,p}^{\mu'\nu'}C_{p',p'+q}^{\lambda'\sigma'},\quad q\ll p_{\max}. \tag{3·3}$$

Then, multiplying (2·16) and (2·17) by $C_{p,p+q}^{v\,c}/[\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}]$ and $C_{p,p'+q}^{c\,v}/[\hbar\omega+\{\varepsilon_c(\boldsymbol{p})+\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}]$, respectively, and summing them over c, v and $\boldsymbol{p}$, we obtain

$$\left(1-\frac{4\pi e^2}{\Omega q^2}\sum_{c,v}\sum_{p}\frac{|C_{p+q,p}^{c\,v}|^2}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}\right)\sum_{c'v'}\sum_{p'}C_{p',p'+q}^{v'c'}\alpha_{p'}^{c'v'}(\boldsymbol{q})$$

$$+\frac{4\pi e^2}{\Omega q^2}\sum_{c,v}\sum_{p}\frac{|C_{p+q,p}^{c\,v}|^2}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}\sum_{c'v'}\sum_{p'}C_{p',p'-q}^{c'v'}\beta_{p'}^{c'v'}(\boldsymbol{q})=0, \tag{3·4}$$

$$\left(1+\frac{4\pi e^2}{\Omega q^2}\sum_{c,v}\sum_{p}\frac{|C_{p,p+q}^{c\,v}|^2}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}\right)\sum_{c'v'}\sum_{p'}C_{p',p'+q}^{c'v'}\beta_{p'}^{c'v'}(\boldsymbol{q})$$

$$-\frac{4\pi e^2}{\Omega q^2}\sum_{c,v}\sum_{p}\frac{|C_{p,p+q}^{c\,v}|^2}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}\sum_{c'v'}\sum_{p'}C_{p',p'+q}^{v'c'}\alpha_{p'}^{c'v'}(\boldsymbol{q})=0. \tag{3·5}$$

These have a solution only if the determinant of the coefficients vanishes, or

$$1-\frac{4\pi e^2}{\Omega q^2}\sum_{c,v}\sum_{p}\left\{\frac{|C_{p+q,p}^{c\,v}|^2}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}-\frac{|C_{p,p+q}^{c\,v}|^2}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}\right\}=0. \tag{3·6}$$

The corresponding wave function is given by

$$\Psi_{pl}(\boldsymbol{q})=N_q\sum_{c,v}\sum_{p}\left\{\frac{C_{p+q,p}^{c\,v}}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}a_{c,p+q}^{*}b_{v,p}^{*}\right.$$

$$\left.+\frac{C_{p,p+q}^{c\,v}}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}b_{v,p+q}a_{c,p}\right\}|\Psi_0\rangle, \tag{3·7}$$

where

* When $\boldsymbol{p}=\boldsymbol{p}'$, $\mu=\nu$, and $\lambda=\sigma$, the first term of (3·1) represents an infinite Coulomb energy, which should be dropped as noted before in the footnote.

$$|N_q|^{-2}=\sum_{c,v}\sum_{\boldsymbol{p}}\left\{\frac{|C^{c\ v}_{\boldsymbol{p}-\boldsymbol{q},\boldsymbol{p}}|^2}{[\hbar\omega-\{\mathcal{E}_c(\boldsymbol{p}+\boldsymbol{q})-\mathcal{E}_v(\boldsymbol{p})\}]^2}-\frac{|C^{c\ v}_{\boldsymbol{p},\boldsymbol{p}+\boldsymbol{q}}|^2}{[\hbar\omega+\{\mathcal{E}_c(\boldsymbol{p})-\mathcal{E}_v(\boldsymbol{p}+\boldsymbol{q})\}]^2}\right\}. \tag{3·8}$$

It is seen that (3·6) and (3·7) are formally identical to the dispersion relation and the wave function for the plasmon in metals,[17] respectively.

We have assumed above that the energy denominator is not zero, because we are interested in the bound state. Another solution with the energy

$$\hbar\omega^{cv}=\mathcal{E}_c(\boldsymbol{p}+\boldsymbol{q})-\mathcal{E}_v(\boldsymbol{p})+O(1/\Omega) \tag{3·9}*$$

corresponds to the scattering solution, in which an electron and a hole move freely to each other, and (3·9) constitutes the corresponding band spectrum, which is called " pair-band ".

It is evident that the bound solution of (3·6) lies above the lowest continuum pair-band, say $\hbar\omega^{cv}$, because the left-hand side of (3·6) cannot vanish for $\hbar\omega$ less than $\hbar\omega^{cv}$. Assume, for simplicity, that the lowest pair-band $\hbar\omega^{cv}$ is separated by an energy gap from the nextly low-lying pair-band. If we restrict ourselves to the bound solution lying just above $\hbar\omega^{cv}$ and neglect contributions of other terms concerning higher pair-bands in (3·6), the dispersion relation (3·6) is reformulated, in the limit of small $\boldsymbol{q}$, as follows.

$$\sum_{\boldsymbol{p}}\frac{(\hbar\omega_{el})^2 f^{cv}}{(\hbar\omega)^2-\{\mathcal{E}_c(\boldsymbol{p})-\mathcal{E}_v(\boldsymbol{p})\}^2}=n, \tag{3·10}$$

where

$$f^{cv}=\left(\frac{2\hbar^2}{m}\right)\frac{N}{\Omega}\int_{\text{cell}}d\boldsymbol{x}\,u^*_{c,\boldsymbol{p}}(\boldsymbol{x})\left(\frac{1}{i}\nabla\right)u_{v,\boldsymbol{p}}(\boldsymbol{x})\Big|^2/\{\mathcal{E}_c(\boldsymbol{p})-\mathcal{E}_v(\boldsymbol{p})\}, \tag{3·11}$$

$$(\hbar\omega_{el})^2=4\pi e^2\hbar^2 n/m\Omega, \tag{3·12}$$

and n is the total number of electrons occupying a v-band in the ground state. f^{cv} is the optical oscillator strength for interband transition and satisfies the relation

$$\sum_{\boldsymbol{p}}f^{cv}=n. \tag{3·13}$$

(3·10) is the well-known formula for a plasmon in the limit of small $\boldsymbol{q}$, for which a detailed discussion has been given by Nozières and Pines.[18] The plasmon energy is easily estimated from (3·10) and is given roughly by

$$\hbar\omega\simeq\hbar\omega_{el}\{1+(\Delta_{cv}/\hbar\omega_{el})^2\}^{1/2}+O(d_{cv}/\hbar\omega_{el}), \tag{3·14}$$

where Δ_{cv} is the energy gap between c- and v-bands and d_{cv} is a quantity of the order of the sum of their band-widths. Thus, the presence of a finite energy gap Δ_{cv} is found to give a shift of the plasmon energy from the free electron value.

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* It should be noted that $\hbar\omega+\{\mathcal{E}_c(\boldsymbol{p})-\mathcal{E}_v(\boldsymbol{p}+\boldsymbol{q})\}\neq 0$, because $\mathcal{E}_c-\mathcal{E}_v$, involving a finite energy gap, is always positive and $\hbar\omega$ must not be negative.

If the higher pair-bands lie nearer the lowest one, so that terms concerning them cannot be neglected in (3·6), the dispersion relation (3·6) cannot be simplified to (3·10) and is difficult to be solved. It is also noted that if the plasmon level born out from $\hbar\omega^{cv}$ overlaps the other pair-band, the plasmon will discipate its energy quickly to the continuum, because then the wave function (3·7) is not normalizable. These situations together with the shift of the level due to the presence of an energy gap might disturb the plasmon to be identified in the energy loss spectra in insulators. Nevertheless, the same plasmon model as in metals seems applicable to interpreting energy losses found in the metallic semiconductors having a small energy gap.

§ 4. Exciton in ionic crystals

In order to review the exciton model in insulators, we shall take alkali halides as an example of typical ionic crystal and assume that the wave functions are highly localized to respective ionic sites, contrary to the case discussed in § 3. It is convenient to use the Wannier wave function, which is related to the Bloch wave function by the relation

$$\varphi'_{\mu,k}(\boldsymbol{x}) = N^{-1/2}\sum_{\boldsymbol{R}} \exp[i\boldsymbol{k}\cdot\boldsymbol{R}]\varphi_{\mu}(\boldsymbol{x}-\boldsymbol{R}), \tag{4·1}$$

where N is the number of unit cells and $\boldsymbol{R}$ is taken as the coordinate of the halogen lattice point of the $\boldsymbol{R}$-th cell. The Wannier function $\varphi_{\mu}(\boldsymbol{x}-\boldsymbol{R})$ is assumed to be appreciably localized around the $\boldsymbol{R}$-th cell, so that it may be considered to behave like a linear combination of atomic orbitals from alkali and halogen ions.

Let c and v specify the lowest conduction band and the highest valence band, respectively. The bands are assumed to be well-separated from the rest. Consider a pair consisted of an electron and a hole excited in c- and v-bands, respectively. We shall begin with the estimation of the last coefficient $U^{cv'c'v}_{cvc'v'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q})$ in (2·16), which can be expressed with the Wannier functions as follows.

$$\begin{aligned} U^{cv'c'v}_{cvc'v'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}) = N^{-1}\sum_{\boldsymbol{R},\boldsymbol{R}',\boldsymbol{R}''} \exp[-i\boldsymbol{q}\cdot(\boldsymbol{R}-\boldsymbol{R}') - i\boldsymbol{p}\cdot\boldsymbol{R} + i\boldsymbol{p}'\cdot(\boldsymbol{R}'-\boldsymbol{R}'')] \\ \times\Big\{\iint d\boldsymbol{x}d\boldsymbol{x}'\,\varphi^*_c(\boldsymbol{x}-\boldsymbol{R})\varphi_{v'}(\boldsymbol{x}-\boldsymbol{R}')\varphi^*_{c'}(\boldsymbol{x}'-\boldsymbol{R}'')\varphi_v(\boldsymbol{x}')\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|} \\ -\iint d\boldsymbol{x}d\boldsymbol{x}'\,\varphi^*_c(\boldsymbol{x}-\boldsymbol{R})\varphi_v(\boldsymbol{x})\varphi^*_{c'}(\boldsymbol{x}'-\boldsymbol{R}'')\varphi_{v'}(\boldsymbol{x}'-\boldsymbol{R}')\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\Big\}. \end{aligned} \tag{4·2}$$

We shall discuss separately about cases which are expected to give a dominant contribution to integrals in (4·2), as c' and v' run over conduction and valence bands, respectively.

(i) When $v'=v$. For $\boldsymbol{R}'\neq 0$, the overlap between $\varphi_{v'}(\boldsymbol{x}-\boldsymbol{R}')$ and $\varphi_v(\boldsymbol{x}')$ is considered to be sufficiently small so that values of the integrals will be much

smaller than that for $\boldsymbol{R}'=0$, because $\varphi_v(\boldsymbol{x}-\boldsymbol{R}')$ will have a predominant contribution from localized p-orbitals of a halogen located at $\boldsymbol{R}'$. For $\boldsymbol{R}'=0$, however, the first integral cancels out the second one, so that (4·2) vanishes.

(ii) When $c'=c$. For $\boldsymbol{R}''\neq\boldsymbol{R}$, values of integrals will again be negligibly small on account of a localized character of $\varphi_c(\boldsymbol{x}-\boldsymbol{R})$. For $\boldsymbol{R}''=\boldsymbol{R}$, the first integral cancels out the second one, so that (4·2) vanishes.

(iii) When $c'\neq c$ and $v'\neq v$. The main contribution to the integrals will come when c' and v' specify the nearest bands to c- and v-bands, respectively. In this case, $\varphi_{c'}(\boldsymbol{x}-\boldsymbol{R})$, which may be contributed dominantly from the orbital of excited state of a halogen, is expected to have the largest amplitude on $\boldsymbol{R}$-th halogen site, even though its amplitude will be less than that of $\varphi_v(\boldsymbol{x}-\boldsymbol{R})$. Similarly, $\varphi_{v'}(\boldsymbol{x}-\boldsymbol{R})$ is expected to have a large amplitude on alkali sites adjacent to the $\boldsymbol{R}$-th halogen site. If this is the case, we can approximate (4·2) by taking $\boldsymbol{R}''=0$ and $\boldsymbol{R}'=\boldsymbol{R}$, and obtain

$$\begin{aligned}U^{cvc'v'}_{cvc'v'}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q})\simeq N^{-1}\sum_{\boldsymbol{R}}\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]\\ \times\Big\{\iint d\boldsymbol{x}d\boldsymbol{x}'\,\varphi_c^*(\boldsymbol{x}-\boldsymbol{R})\varphi_{v'}(\boldsymbol{x}-\boldsymbol{R})\varphi_{c'}^*(\boldsymbol{x}')\varphi_v(\boldsymbol{x}')\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\\ -\iint d\boldsymbol{x}d\boldsymbol{x}'\,\varphi_c^*(\boldsymbol{x}-\boldsymbol{R})\varphi_v(\boldsymbol{x})\varphi_{c'}^*(\boldsymbol{x}')\varphi_{v'}(\boldsymbol{x}'-\boldsymbol{R})\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\Big\}.\end{aligned}\tag{4·3}$$

However, the values of integrals in (4·3) are considered to be much less than that of the Coulomb and exchange integrals between c-electron and v-hole, which constitute the main part of the coefficient $U^{cc'vv'}_{cvv'c'}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q})$, as is seen below. Further, it is found that the last term of (2·16) with the coefficient (4·3) is less than the second term of the same equation by a factor of the order of $U^{cvc'v'}_{cvc'v'}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q})/[\hbar\omega+\{\varepsilon_{c'}(\boldsymbol{p})-\varepsilon_{v'}(\boldsymbol{p}+\boldsymbol{q})\}]$. Therefore, we can neglect the last term in (2·16), in the first approximation. We also neglect the last term in (2·17) by the similar consideration. In the same approximation as above, the second terms of (2·16) and (2·17) are replaced by the terms having the largest coefficients with $c'=c$ and $v'=v$. The coefficients are given by

$$U^{ccvv}_{cvvc}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q})=[U^{vvcc}_{vccv}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q})]^*\simeq N^{-1}\sum_{\boldsymbol{R}}\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]v(\boldsymbol{R}),\tag{4·4}$$

where

$$\begin{aligned}v(\boldsymbol{R})=v^*(\boldsymbol{R})\\ =\iint d\boldsymbol{x}d\boldsymbol{x}'\,\varphi_c^*(\boldsymbol{x}-\boldsymbol{R})\varphi_c(\boldsymbol{x}-\boldsymbol{R})\varphi_v^*(\boldsymbol{x}')\varphi_v(\boldsymbol{x}')\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\\ -\iint d\boldsymbol{x}d\boldsymbol{x}'\,\varphi_c^*(\boldsymbol{x}-\boldsymbol{R})\varphi_v(\boldsymbol{x})\varphi_v^*(\boldsymbol{x}')\varphi_c(\boldsymbol{x}'-\boldsymbol{R})\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}.\end{aligned}\tag{4·5}$$

The first term is the Coulomb interaction and is usually larger than the second term, which is the exchange interaction, so that $v(\boldsymbol{R})$ is positive.

(2·16) and (2·17) are, now, reduced to

$$[\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}]\alpha_{\mu}^{cv}(\boldsymbol{q})+N^{-1}\sum_{R}\sum_{p'}\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]v(\boldsymbol{R})\alpha_{\mu'}^{cv}(\boldsymbol{q})=0, \tag{4·6}$$

$$[\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}]\beta_{\mu}^{cv}(\boldsymbol{q})-N^{-1}\sum_{R}\sum_{p'}\exp[i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]v(\boldsymbol{R})\beta_{\mu'}^{cv}(\boldsymbol{q})=0. \tag{4·7}$$

Thus, it is found that the equations for α and β are independent of each other in the first approximation.

Multiplying both sides of (4·6) by $\exp[i\boldsymbol{p}\cdot\boldsymbol{R}_0]/[\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}]$ and summing over $\boldsymbol{p}$ within the first Brillouin zone, we obtain

$$\sum_{p}\exp[i\boldsymbol{p}\cdot\boldsymbol{R}_0]\alpha_{\mu}^{cv}(\boldsymbol{q})+\frac{1}{N}\sum_{R}\sum_{p}v(\boldsymbol{R})\frac{\exp[i\boldsymbol{p}\cdot(\boldsymbol{R}_0-\boldsymbol{R})]}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}\times\sum_{p'}\exp[i\boldsymbol{p}'\cdot\boldsymbol{R}]\alpha_{\mu'}^{cv}(\boldsymbol{q})=0, \tag{4·8}$$

where it is assumed that the energy denominator does not vanish because we are not interested in scattering solution. We can take, in a good approximation, only the term with $\boldsymbol{R}=\boldsymbol{R}_0$ in the second term of (4·8). To see this, we assume the usual spectra for $\varepsilon_c(\boldsymbol{p})$ and $\varepsilon_v(\boldsymbol{p})$, i.e.,

$$\left.\begin{aligned}\varepsilon_c(\boldsymbol{p})&=\Delta_c+\alpha p^2,\\ \varepsilon_v(\boldsymbol{p})&=\Delta_v+\beta p^2,\end{aligned}\right\} \tag{4·9}$$

where α is positive and β is negative, so that c- and v-bands are concaved upwards and downwards, respectively, and $\Delta_{cv}\equiv\Delta_c-\Delta_v$ specifies the positive energy gap between the bands. If we replace the summation with respect to $\boldsymbol{p}$ by the integration over the sphere having the same volume as the first Brillouin zone, we can see for small $\boldsymbol{q}$ that

$$I(\boldsymbol{R}-\boldsymbol{R}_0)\equiv\sum_{p}\frac{\exp[i\boldsymbol{p}\cdot(\boldsymbol{R}_0-\boldsymbol{R})]}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}$$

$$\simeq-\frac{\Omega}{(2\pi)^3}\frac{2\pi^2}{(\alpha-\beta)}\frac{1}{|\boldsymbol{R}_0-\boldsymbol{R}|}\exp\left[-\sqrt{\frac{\Delta_{cv}-\hbar\omega}{(\alpha-\beta)p_{\max}^2}}\cdot p_{\max}\cdot|\boldsymbol{R}_0-\boldsymbol{R}|\right],\quad \text{for}\quad \Delta_{cv}>\hbar\omega, \tag{4·10}$$

where $p_{\max}$ is the radius of the sphere replacing the first Brillouin zone. As the magnitude of $\Delta_{cv}-\hbar\omega$ for an exciton in alkali halides may be considered as comparable with $(\alpha-\beta)p_{\max}^2$, which gives roughly the sum of widths of c- and v-bands, the value of $I(\boldsymbol{R}-\boldsymbol{R}_0)$ will sharply fall off for $|\boldsymbol{R}_0-\boldsymbol{R}|>1/p_{\max}$. In fact, it has been calculated by Takeuti[19] that even when $|\boldsymbol{R}_0-\boldsymbol{R}|$ equals to a primitive translation, the value of $I(\boldsymbol{R}-\boldsymbol{R}_0)/I(0)$ comes out to be about one tenth or less for $(\Delta_{cv}-\hbar\omega)/[(\alpha-\beta)p_{\max}^2]\simeq 1/4$.

Thus, leaving only the term with $\boldsymbol{R}=\boldsymbol{R}_0$ in the summation with respect to $\boldsymbol{R}$, (4·8) becomes

$$\left(1+\frac{1}{N}v(\boldsymbol{R}_0)\sum_{\nu}\frac{1}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}\right)\sum_{p'}\exp[i\boldsymbol{p}'\cdot\boldsymbol{R}_0]\alpha_{p'}^{cv}(\boldsymbol{q})=0. \quad (4\cdot11)$$

In order that $\sum_{p'}\exp[i\boldsymbol{p}'\cdot\boldsymbol{R}_0]\alpha_{p'}^{cv}(\boldsymbol{q})\neq0$, it must hold that

$$1+\frac{1}{N}v(\boldsymbol{R}_0)\sum_{\nu}\frac{1}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p}\}}=0. \quad (4\cdot12)$$

This is equivalent to the equation for exciton level, which has been derived by Takeuti[19] by the use of the Wannier scheme. The wave function $\sum_{p}\exp[i\boldsymbol{p}\cdot\boldsymbol{R}_0]\cdot\alpha_{p}^{cv}(\boldsymbol{q})a_{c,p+q}^{*}b_{v,p}^{*}|F_0$ represents the internal behavior of the exciton. It consists of a bound electron-hole pair in which an electron is at a distance $\boldsymbol{R}_0$ from a hole, as is seen from that the wave function falls off exponentially as $|\boldsymbol{R}_0-\boldsymbol{R}|$ increases because $\alpha_{p}^{cv}(\boldsymbol{q})\propto\sum_{R}\exp[-i\boldsymbol{p}\cdot\boldsymbol{R}]/[\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}]$ and that $\boldsymbol{R}_0$ in $v(\boldsymbol{R}_0)$ specifies the distance between cells to which an electron and a hole are localized respectively. It should be noted that the exciton state is represented by a linear combination of various pairs with a momentum transfer $\boldsymbol{q}$, which represents the translation vector of the exciton as is seen below.

We shall, now, discuss under which condition (4·12) has a bound solution. As has been given by Takeuti, the simplified calculation, assuming the usual spectra (4·9) for $\varepsilon_c(\boldsymbol{p})$ and $\varepsilon_v(\boldsymbol{p})$ and replacing the summation by the integration, shows that (4·12) has a positive solution $\hbar\omega$ only when the condition

$$(\gamma/2\pi^2)v(\boldsymbol{R}_0)/(\alpha-\beta)p_{\max}^2\geq1 \quad (4\cdot13)$$

is satisfied, where $\gamma\equiv(\Omega/N)p_{\max}^3$, which equals $6\pi^2$ for f.c.c.. This condition has a natural physical meaning that if an electron is remoted so far from a hole that the effective potential $v(\boldsymbol{R}_0)$ becomes too small to satisfy the condition, the pair cannot form a bound state. When the condition (4·13) is satisfied, the energy level for deep-lying exciton is given by

$$\hbar\omega\simeq\Delta_{cv}-(\gamma/6\pi^2)v(\boldsymbol{R}_0)-(\alpha\beta/(\alpha-\beta))q^2. \quad (4\cdot14)$$

The condition that $\hbar\omega>0$ requires a condition

$$(\gamma/6\pi^2)v(0)/\Delta_{cv}<1. \quad (4\cdot15)$$

This condition, however, seems too stringent. As was mentioned in the introduction, the potential $v(0)$ will be reduced by effects of virtual processes which are not taken into account in the present first approximation of an electron-hole pair model, so that $v(0)$ in (4·15) should be replaced by the effective potential thus reduced.

It is expected from (4·14) that for crystals with a large energy gap Δ_{cv} the separation between the lowest and next low-lying exciton levels will be given by $(\gamma/6\pi^2)[v(0)-v(\tau)]$ where τ is a primitive translation. Therefore, the separation will be smaller for a crystal with less localized wave functions.

We shall discuss the equation (4·7). By a similar consideration as before, (4·7) can be transformed into

$$\left(1-\frac{1}{N}v(\boldsymbol{R}_0)\sum_{p}\frac{1}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}\right)\sum_{p'}\exp[-i\boldsymbol{p}'\cdot\boldsymbol{R}_0]\beta_{p'}^{cv}(\boldsymbol{q})=0. \quad (4\cdot16)$$

When the conditions (4·13) and (4·15) are satisfied, the expression in the bracket in (4·16) cannot vanish for positive $\hbar\omega$. Therefore, $\sum_{p'}\exp[-i\boldsymbol{p}'\cdot\boldsymbol{R}_0]\beta_{p'}^{cv}(\boldsymbol{q})$ must identically be zero for $\hbar\omega$ given by (4·12). This allows us to normalize the wave function of exciton, $\sum_{p}\exp[i\boldsymbol{p}\cdot\boldsymbol{R}_0]\alpha_{p}^{cv}(\boldsymbol{q})\,a_{c,p+q}^{*}b_{v,p}^{*}\Psi_0$.

Lastly, we shall show that our pair model allows us to derive an effective mass equation for a moving exciton, which affords also a better understanding of the exciton model. Assume that $v(\boldsymbol{R})$ can be regarded as a continuous function of $\boldsymbol{R}$ and behaves like $\sim(e^2/R)\exp[-\kappa R]$ for a large separation R between an electron and a hole. Then it is found that

$$\sum_{R}\exp[i\boldsymbol{p}\cdot\boldsymbol{R}]v(\boldsymbol{R})\propto 4\pi e^2/(\kappa^2+p^2). \quad (4\cdot17)$$

This suggests us that the summation over $\boldsymbol{p}$ within the first Brillouin zone in the second term of (4·6) may be extended to the whole momentum space, if $1/\kappa$ is of the order of or larger than a primitive translation. This approximation will be permitted for the crystal where the wave functions are not appreciably localized. Thus, if we assume, for simplicity, the spectra (4·9) for $\varepsilon_c(\boldsymbol{p})$ and $\varepsilon_v(\boldsymbol{p})$, (4·6) can be written as follows.

$$\left\{\hbar\omega-\Delta_{cv}+\frac{\alpha\beta}{\alpha-\beta}q^2-(\alpha-\beta)p^2\right\}\alpha_{p-(\alpha/\alpha-\beta)q}^{cv}(\boldsymbol{q})$$
$$+N^{-1}\sum_{R}\sum_{p'}^{\text{all}}\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]v(\boldsymbol{R})\alpha_{p'-(\alpha/\alpha-\beta)q}^{cv}(\boldsymbol{q})=0, \quad (4\cdot18)$$

where variables $\boldsymbol{p}$ and $\boldsymbol{p}'$ were replaced by $\boldsymbol{p}-\frac{\alpha}{\alpha-\beta}\boldsymbol{q}$ and $\boldsymbol{p}'-\frac{\alpha}{\alpha-\beta}\boldsymbol{q}$, respectively. Define $\psi_q(\boldsymbol{r},\ \boldsymbol{R}_G)$ and $\phi_q(\boldsymbol{r})$ by

$$\psi_q(\boldsymbol{r},\ \boldsymbol{R}_G)=\exp[i\boldsymbol{q}\cdot\boldsymbol{R}_G]\phi_q(\boldsymbol{r}), \quad (4\cdot19)$$

$$\phi_q(\boldsymbol{r})=\Omega^{-1/2}\sum_{p}^{\text{all}}\exp[i\boldsymbol{p}\cdot\boldsymbol{r}]\alpha_{p-(\alpha/\alpha-\beta)q}^{cv}(\boldsymbol{q}). \quad (4\cdot20)$$

Multiplying from the left of (4·18) by $\Omega^{-1/2}\exp[i\boldsymbol{q}\cdot\boldsymbol{R}_G]\cdot\exp[i\boldsymbol{p}\cdot\boldsymbol{r}]$ and summing over all $\boldsymbol{p}$, we obtain

$$\left\{\hbar\omega-\Delta_{cv}-\frac{\alpha\beta}{\alpha-\beta}\Gamma_{R_G}^2-(\alpha-\beta)\Gamma_{r}^2\right\}\psi_q(\boldsymbol{r},\ \boldsymbol{R}_G)$$
$$+N^{-1}\Omega^{-1/2}\exp[i\boldsymbol{q}\cdot\boldsymbol{R}_G]\sum_{R}\sum_{p}^{\text{all}}\sum_{p'}^{\text{all}}\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot(\boldsymbol{R}-\boldsymbol{r})]$$
$$\times v(\boldsymbol{R})\exp[i\boldsymbol{p}'\cdot\boldsymbol{r}]\alpha_{p'-(\alpha/\alpha-\beta)q}^{cv}(\boldsymbol{q})=0. \quad (4\cdot21)$$

By replacing $\sum_{p}^{\text{all}} \sum_{p'}^{\text{all}}$ by $\sum_{(p-p')}^{\text{all}} \sum_{p'}^{\text{all}}$ and using the relation that

$$N^{-1} \sum_{R} \sum_{(p-p')}^{\text{all}} \exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot(\boldsymbol{R}-\boldsymbol{r})]v(\boldsymbol{R}) = v(\boldsymbol{r}), \tag{4·22}$$

(4·21) is rewritten as

$$\{(\alpha\beta/\alpha-\beta)\nabla_{R_G}^2 - (\alpha-\beta)\nabla_r^2 - v(\boldsymbol{r}) - E\}\psi_q(\boldsymbol{r}, \boldsymbol{R}_G) = 0, \tag{4·23}$$

where

$$E = \hbar\omega - \Delta_{cv}. \tag{4·24}$$

(4·23) is the well-known effective mass equation for an exciton moving with a translation wave vector $\boldsymbol{q}$, and E is the energy measured from the bottom of conduction band. $\boldsymbol{R}_G$ specifies the coordinate of the center of mass, and $\boldsymbol{r}$ the radius of the orbit. If we separate the center of mass motion, we have an effective mass equation for the internal motion of the exciton.

$$[-(\alpha-\beta)\nabla_r^2 - v(\boldsymbol{r}) - \varepsilon_q]\phi_q(\boldsymbol{r}) = 0, \tag{4·25}$$

where

$$\varepsilon_q = E + (\alpha\beta/\alpha-\beta)q^2. \tag{4·26}$$

It is found again that the effectively attractive force is essential to form a bound state.

§ 5. Relation between exciton and plasmon

In the preceding two sections, it has been shown that an electron-hole pair model can easily lead us to either plasmon or exciton model in the extreme cases. However, if we advance to the second approximation, both plasmon and exciton may be possible excitations, in which they will be coupled and modified by one another so that the situation will become more complex. In order to make understanding of the situation easy, it will be convenient to consider a monatomic valence crystal like Ge and Si. This is because their wave functions will be less localized than that in low-lying bands in ionic crystals.

Before entering into the study of possible excitations in such a valence crystal, we remind how either picture of a plasmon or of an exciton has been obtained in the first approximation, according to different behaviors of wave functions. Two points are instructive. First, when the wave functions are extended so that there arises an enough coupling between creations and annihilations of pairs, there becomes possible an organized motion of charge fluctuation with a long wavelength, which is a plasmon. On the other hand, when the wave functions are appreciably localized, then the created-pair state is stable without being disturbed by other virtual states and can form a bound state, i.e., an exciton. The second point is the difference of the sign of potential U. It is effectively negative for the plasmon, as is seen

from (3·3), and is positive for the exciton, as is seen from (4·4) and (4·5). These general situations will not be essentially altered even for valence crystals.

$U^{cvc'v'}_{cvc'v'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q})$ is given again by (4·2), where $\varphi_\mu(\boldsymbol{x}-\boldsymbol{R})$ is assumed to be localized around the $\boldsymbol{R}$-th cell, so that it may be considered as a hybridized μ-orbital concentrated around the $\boldsymbol{R}$-th cell. Neglecting three- and four-centered integrals, a dominant contribution to the first integral in (4·2) is considered to be given when $\boldsymbol{R}'=\boldsymbol{R}$ and $\boldsymbol{R}''=0$, and that to the second one in (4·2) is given when $\boldsymbol{R}''=\boldsymbol{R}'$ and $\boldsymbol{R}=0$. Therefore, (4·2) can be approximated by

$$U^{cvc'v'}_{cvc'v'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}) = N^{-1}\sum_{\boldsymbol{R}} \{\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]u_{cvc'v'}(\boldsymbol{R}) - \exp[i\boldsymbol{q}\cdot\boldsymbol{R}]u_{cvc'v'}(\boldsymbol{R})\}, \tag{5·1}$$

where

$$u_{\mu\nu\lambda\sigma}(\boldsymbol{R}) = \iint d\boldsymbol{x}d\boldsymbol{x}'\,\varphi^*_\mu(\boldsymbol{x}-\boldsymbol{R})\varphi_\nu(\boldsymbol{x}-\boldsymbol{R})\varphi^*_\lambda(\boldsymbol{x}')\varphi_\sigma(\boldsymbol{x}')\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{5·2}$$

By the similar consideration, other U's are approximated by

$$\left.\begin{aligned} U^{vcv'c'}_{vcv'c'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}) &= N^{-1}\sum_{\boldsymbol{R}} \{\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]u_{vcv'c'}(\boldsymbol{R}) - \exp[i\boldsymbol{q}\cdot\boldsymbol{R}]u_{vcv'c'}(\boldsymbol{R})\},\\ U^{cc'v'v}_{cvc'v'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}) &= N^{-1}\sum_{\boldsymbol{R}} \{\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]u_{cc'v'v}(\boldsymbol{R}) - \exp[i\boldsymbol{q}\cdot\boldsymbol{R}]u_{cvc'v'}(\boldsymbol{R})\},\\ U^{vv'c'c}_{vcv'c'}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}) &= N^{-1}\sum_{\boldsymbol{R}} \{\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]u_{vv'c'c}(\boldsymbol{R}) - \exp[i\boldsymbol{q}\cdot\boldsymbol{R}]u_{vcv'c'}(\boldsymbol{R})\}. \end{aligned}\right\} \tag{5·3}$$

Inserting these expressions into (2·16), we are given

$$\begin{aligned} &[\hbar\omega - \{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_v(\boldsymbol{p})\}]\alpha^{cv}_{\boldsymbol{p}}(\boldsymbol{q})\\ &+N^{-1}\sum_{c'v'}\sum_{\boldsymbol{p}'}\sum_{\boldsymbol{R}} \{\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]u_{cc'v'v}(\boldsymbol{R}) - \exp[i\boldsymbol{q}\cdot\boldsymbol{R}]u_{cv'c'v}(\boldsymbol{R})\}\alpha^{c'v'}_{\boldsymbol{p}'}(\boldsymbol{q})\\ &-N^{-1}\sum_{c'v'}\sum_{\boldsymbol{p}'}\sum_{\boldsymbol{R}} \{\exp[-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{R}]u_{cv'c'v}(\boldsymbol{R}) - \exp[i\boldsymbol{q}\cdot\boldsymbol{R}]u_{cvc'v'}(\boldsymbol{R})\}\beta^{c'v'}_{\boldsymbol{p}'}(\boldsymbol{q}) = 0. \end{aligned} \tag{5·4}$$

We shall further take the simplification that in the terms including the factor $\exp[-i\boldsymbol{p}\cdot\boldsymbol{R}]$ only the term with $\boldsymbol{R}=0$ is left in the summation with respect to $\boldsymbol{R}$, because the value of $\sum_{\boldsymbol{p}}\{\exp[-i\boldsymbol{p}\cdot\boldsymbol{R}]/[\hbar\omega - \{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_v(\boldsymbol{p})\}]\}$ will sharply fall off for $\boldsymbol{R}\neq 0$ as discussed before. Instead of the approximation that only the term with $c'=c$ and $v'=v$ is left in the summation over c' and v', which corresponds to neglecting the coupling between (c, v)-pair and (c', v')-pair, we shall replace $u_{cc'v'v}(\boldsymbol{R})$'s by the averages over various combinations of c' and v' when c and v are fixed, and assume, for simplicity, $\langle u_{cc'v'v}(\boldsymbol{R})\rangle$'s thus averaged are real.

Multiplying (5·4) by $[\hbar\omega - \{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_v(\boldsymbol{p})\}]^{-1}$, and summing over c, v, and $\boldsymbol{p}$ within the first Brillouin zone, we obtain

$$\left(1+\frac{1}{N}\sum_{c,v}\sum_{\boldsymbol{p}}\frac{V_{cv}(\boldsymbol{q})}{\hbar\omega - \{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_v(\boldsymbol{p})\}}\right)\sum_{c'v'}\sum_{\boldsymbol{p}'}\alpha^{c'v'}_{\boldsymbol{p}'}(\boldsymbol{q})$$

$$-\frac{1}{N}\sum_{c,v}\sum_{\boldsymbol{p}}\frac{W_{cv}(\boldsymbol{q})}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}\sum_{c'v'}\sum_{\boldsymbol{p}'}\beta_{\boldsymbol{p}'}^{c'v'}(\boldsymbol{q})=0, \tag{5·5}$$

where

$$V_{cv}(\boldsymbol{q})=\langle u_{cc'vv'}(0)\rangle-\sum_{\boldsymbol{R}}\exp[i\boldsymbol{q}\cdot\boldsymbol{R}]\langle u_{cvvc'}(\boldsymbol{R})\rangle, \tag{5·6}$$

$$W_{cv}(\boldsymbol{q})=\langle u_{cv'c'v}(0)\rangle-\sum_{\boldsymbol{R}}\exp[i\boldsymbol{q}\cdot\boldsymbol{R}]\langle u_{cvc'v'}(\boldsymbol{R})\rangle. \tag{5·7}$$

$\langle\ \rangle$ means the averaging procedure mentioned above. In the same approximation as above, we obtain from (2·17)

$$\left(1-\frac{1}{N}\sum_{c,v}\sum_{\boldsymbol{p}}\frac{V_{cv}(\boldsymbol{q})}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}\right)\sum_{c'v'}\sum_{\boldsymbol{p}'}\beta_{\boldsymbol{p}'}^{c'v'}(\boldsymbol{q})$$
$$+\frac{1}{N}\sum_{c,v}\sum_{\boldsymbol{p}}\frac{W_{cv}(\boldsymbol{q})}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}\sum_{c'v'}\sum_{\boldsymbol{p}'}\alpha_{\boldsymbol{p}'}^{c'v'}(\boldsymbol{q})=0. \tag{5·8}$$

The equations, (5·5) and (5·8), have a solution only if the following equation is satisfied.

$$\left(1+\frac{1}{N}\sum_{c,v}\sum_{\boldsymbol{p}}\frac{V_{cv}(\boldsymbol{q})}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}\right)\left(1-\frac{1}{N}\sum_{c,v}\sum_{\boldsymbol{p}}\frac{V_{cv}(\boldsymbol{q})}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}\right)$$
$$+\left(\frac{1}{N}\sum_{c,v}\sum_{\boldsymbol{p}}\frac{W_{cv}(\boldsymbol{q})}{\hbar\omega-\{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})\}}\right)\left(\frac{1}{N}\sum_{c,v}\sum_{\boldsymbol{p}}\frac{W_{cv}(\boldsymbol{q})}{\hbar\omega+\{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})\}}\right)=0. \tag{5·9}$$

This is the general dispersion relation for a possible excitation. We shall show that both plasmon and exciton can be possible solutions of (5·9). Let the lowest conduction band be c_0 and the highest valence band be v_0. Assume, for simplicity, that the spectra of $\varepsilon_{c_0}(\boldsymbol{p})$ and $\varepsilon_{v_0}(\boldsymbol{p})$ are expressed by

$$\left.\begin{aligned}\varepsilon_{c_0}(\boldsymbol{p})&=\Delta_{c_0}+\alpha_{c_0}p^2, \quad \alpha_{c_0}>0,\\ \varepsilon_{v_0}(\boldsymbol{p})&=\Delta_{v_0}+\beta_{v_0}p^2, \quad \beta_{v_0}<0,\end{aligned}\right\} \tag{5·10}$$

and are well-separated from the rest.

If we restrict our consideration to the (c_0, v_0)-pair and separate the terms concerning it from the others, then (5·9) can be written as follows.

$$1+A^{-1}\{(1-f)V_{c_0v_0}(\boldsymbol{q})+gW_{c_0v_0}(\boldsymbol{q})\}N^{-1}$$
$$\times\sum_{\boldsymbol{p}}\left\{\frac{1}{\hbar\omega-\{\varepsilon_{c_0}(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_{v_0}(\boldsymbol{p})\}}-\frac{1}{\hbar\omega+\{\varepsilon_{c_0}(\boldsymbol{p})-\varepsilon_{v_0}(\boldsymbol{p}+\boldsymbol{q})\}}\right\}$$
$$-A^{-1}(V_{c_0v_0}^2-W_{c_0v_0}^2)\left(\frac{1}{N}\sum_{\boldsymbol{p}}\frac{1}{\hbar\omega-\{\varepsilon_{c_0}(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_{v_0}(\boldsymbol{p})\}}\right)$$
$$\times\left(\frac{1}{N}\sum_{\boldsymbol{p}}\frac{1}{\hbar\omega+\{\varepsilon_{c_0}(\boldsymbol{p})-\varepsilon_{v_0}(\boldsymbol{p}+\boldsymbol{q})\}}\right)=0, \tag{5·11}$$

where

$$\left.\begin{aligned} f &\equiv \frac{1}{N}\sum_{\substack{c\neq c_0\\ v\neq v_0}}\sum_{\boldsymbol{p}}\frac{V_{cv}(\boldsymbol{q})}{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})} \simeq \frac{1}{N}\sum_{\substack{c\neq c_0\\ v\neq v_0}}\sum_{\boldsymbol{p}}\frac{V_{cv}(\boldsymbol{q})}{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})}, \\ g &\equiv \frac{1}{N}\sum_{\substack{c\neq c_0\\ v\neq v_0}}\sum_{\boldsymbol{p}}\frac{W_{cv}(\boldsymbol{q})}{\varepsilon_c(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_v(\boldsymbol{p})} \simeq \frac{1}{N}\sum_{\substack{c\neq c_0\\ v\neq v_0}}\sum_{\boldsymbol{p}}\frac{W_{cv}(\boldsymbol{q})}{\varepsilon_c(\boldsymbol{p})-\varepsilon_v(\boldsymbol{p}+\boldsymbol{q})}, \\ A &\equiv (1-f)^2-g^2. \end{aligned}\right\} \qquad (5\cdot12)$$

In the expressions of f and g, $\hbar\omega$ in the denominators were neglected since it was assumed that $\hbar\omega \ll \Delta_c-\Delta_v$ for $c\neq c_0$ and $v\neq v_0$, and the second equalities come from the assumption that $\Delta_c-\Delta_v \gg \alpha_c q^2 \simeq \beta_v q^2$ for $c\neq c_0$ and $v\neq v_0$.

To see the qualitative behavior of bound solutions of (5·11), we take the simplification that $\Delta_{c_0}-\Delta_{v_0} \gg (\alpha_{c_0}-\beta_{v_0})\,p_{\max}^2$. If the bound solution lies below the scattering solution

$$\hbar\omega_s \simeq \varepsilon_{c_0}(\boldsymbol{p}+\boldsymbol{q})-\varepsilon_{v_0}(\boldsymbol{p}), \qquad (5\cdot13)$$

it will be called the "exciton-like" level, and, on the other hand, if it lies above $\hbar\omega_s$, it will be called the "plasmon-like" level. Performing integrations instead of the summation $\sum_{\boldsymbol{p}}$, it is easily found that we can obtain the following alternate solution according to the character of $\tilde{V}(\boldsymbol{q})$ described below.

(A) *Exciton-like solution.* The exciton-like solution is possible only if the following condition is satisfied.

$$(\gamma/6\pi^2)\,3\tilde{V}(\boldsymbol{q})/(\alpha_{c_0}-\beta_{v_0})\,p_{\max}^2 > 1 > (\gamma/6\pi^2)\,2\tilde{V}(\boldsymbol{q})/\Delta_q > 0, \qquad (5\cdot14)$$

where

$$\left.\begin{aligned} \Delta_q &= \Delta_{c_0}-\Delta_{v_0}-(\alpha_{c_0}\beta_{v_0}/(\alpha_{c_0}-\beta_{v_0}))\,q^2, \\ \tilde{V}(\boldsymbol{q}) &= A^{-1}[(1-f)\,V_{c_0v_0}+gW_{c_0v_0}]-(\gamma/6\pi^2)\,(1/2A\Delta_q)[V_{c_0v_0}^2-W_{c_0v_0}^2], \\ \gamma &= (\Omega/N)\,p_{\max}^3. \end{aligned}\right\} \qquad (5\cdot15)$$

Then, the shallow level is given roughly by

$$\hbar\omega_{\mathrm{ex}} \simeq \Delta_q-(2/\pi)^2(\alpha_{c_0}-\beta_{v_0})\,p_{\max}^2[1-(6\pi^2/\gamma)\,(\alpha_{c_0}-\beta_{v_0})\,p_{\max}^2/(3\tilde{V})]^2, \qquad (5\cdot16)$$

and the deep lying level is given by

$$\hbar\omega_{\mathrm{ex}} \simeq \Delta_q-(\gamma/6\pi^2)\,\tilde{V}. \qquad (5\cdot17)$$

From (5·12) and (5·14), we can see that $f\simeq g \ll V_{c_0v_0}(\boldsymbol{q})/(\Delta_{c_0}-\Delta_{v_0})<1$. Therefore, f and g can be neglected and A can be taken as unity. Thus, $\tilde{V}$ can be written as

$$\tilde{V} = V_{c_0v_0}(\boldsymbol{q})/\kappa_{\mathrm{inter}}, \qquad (5\cdot18)$$

where

$$\kappa_{\mathrm{inter}}^{-1} = 1-(\gamma/6\pi^2)\,(V_{c_0v_0}(\boldsymbol{q})/2\Delta_q)[1-(W_{c_0v_0}/V_{c_0v_0})^2]. \qquad (5\cdot19)$$

Comparing (5·17) with (4·14), we find that the exciton-like solution can be considered as an exciton, even though it is modified somewhat from its simplest model. The modification arises through the term of destruction of pairs, which may be considered to play a role of screening the effective potential for the exciton by $1/\kappa_{\text{inter}}$. The positive value of $\tilde{V}$ is contributed predominantly by the Coulomb interaction between c_0-electron and v_0-hole, as in § 4.

(B) *Plasmon-like solution.* In order that (5·11) has a stable plasmon-like solution, it is necessary that the energy level lies above the continuum (5·13), that is,

$$\hbar\omega_{pl} > \text{Maximum of } \{\varepsilon_{c_0}(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_{v_0}(\boldsymbol{p})\}. \tag{5·20}$$

Then, the plasmon-like level is given approximately by

$$\hbar\omega_{pl} \simeq \Delta_q [1 - (\gamma/6\pi^2)\,(2\tilde{V}(\boldsymbol{q})/\Delta_q)]^{1/2}. \tag{5·21}$$

Therefore, it is required that $\tilde{V}(\boldsymbol{q})$ is negative. This will be realized if the second terms on the right-hand sides of (5·6) and (5·7) overwhelm their first terms for sufficiently small $\boldsymbol{q}$. If this should be the case for a sufficiently small $\boldsymbol{q}$ and if we can take that

$$V_{cvv_0}(\boldsymbol{q}) \simeq W_{cvv_0}(\boldsymbol{q}) \equiv -w(\boldsymbol{q}), \tag{5·22}$$

then the dispersion relation (5·11) turns out approximately to be

$$1 - w(\boldsymbol{q}) N^{-1} \sum_{\boldsymbol{p}} \left\{ \frac{1}{\hbar\omega - \{\varepsilon_{c_0}(\boldsymbol{p}+\boldsymbol{q}) - \varepsilon_{v_0}(\boldsymbol{p})\}} - \frac{1}{\hbar\omega + \{\varepsilon_{c_0}(\boldsymbol{p}) - \varepsilon_{v_0}(\boldsymbol{p}+\boldsymbol{q})\}} \right\} = 0. \tag{5·23}$$

Comparing (5·23) with (3·6), we find that (5·23) corresponds to the dispersion relation for the plasmon discussed in § 3. That the effective potential for the plasmon is $-w(\boldsymbol{q})$ with a small $\boldsymbol{q}$ means, from (5·6), (5·7), and (5·22), that the long-range correlation between electrons and holes is essential for the plasmon. The deviation of $V_{cvv_0}(\boldsymbol{q})$ from $W_{cvv_0}(\boldsymbol{q})$ in real crystal acts to modify the dispersion relation given by (5·23) and to bring unharmonicities into the wave function given by (3·7), resulting a shorter lifetime of the plasmon.

Thus, the different character of the solutions mentioned above seems to be decisively dependent upon the $\boldsymbol{q}$-dependence of $\tilde{V}(\boldsymbol{q})$. To see it qualitatively, we assume, from the localized character of wave functions, that $\langle u_{cvv'c'}(\boldsymbol{R}) \rangle$ falls off like $\langle u_{cvv'c'}(0) \rangle \cdot \exp[-QR]$* as R increases, where $1/Q$ may be considered as a parameter giving a rough estimate of linear extension of the wave functions. Taking $f=g=0$, $A=1$, and $\gamma=6\pi^2$, and replacing the summation over $\boldsymbol{R}$ by the integration, we obtain from (5·6), (5·7), and (5·15) the following expression

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* Preliminary calculation, taking exponential s-wave functions, shows that the behavior of $\langle u_{c_0v_0v'c'}(R) \rangle$ for not so large R can be approximated by the exponential function. The deviation from the exponential decay for large R seems to bring qualitatively no essential change in the results.

for $\tilde{V}(\boldsymbol{q})$.

$$\tilde{V}(\boldsymbol{q})=u_c^*[1-(4/3\pi)(u_{ex}^*/u_c^*)(\Omega_Q/\Omega_0)\{1+(\Omega_Q/\Omega_0)^{2/3}(q/p_{max})^2\}^{-2}], \quad (5\cdot24)$$

where

$$\left.\begin{aligned} u_c^* &= u_c[1-(u_c/2\Delta_q)\{1-(u_{ex}/u_c)^2\}], \\ u_{ex}^* &= u_{ex}[1-(u_c/\Delta_q)\{1-(u_{ex}/u_c)\}], \\ u_c &= \langle u_{c_0c_0v_0v_0}(0)\rangle, \\ u_{ex} &= \langle u_{c_0v_0v_0c_0}(0)\rangle, \\ \Omega_Q &= 6\pi^2/Q^3, \end{aligned}\right\} \quad (5\cdot25)$$

and Ω_0 is the volume of unit cell. The values of u_c and u_{ex} may be determined predominantly by the Coulomb and exchange interactions, respectively, between c_0-electron and v_0-hole, and Ω_Q may be considered to represent a measure of extension of their wave functions. It is expected from (5·24) that $\tilde{V}(\boldsymbol{q})$ becomes negative for a sufficiently small $\boldsymbol{q}$, if the wave functions are appreciably extended so that $\Omega_Q \gg \Omega_0$. Fig. 1 shows an example of the $\boldsymbol{q}$-dependence of $\tilde{V}(\boldsymbol{q})$ for various values of Ω_Q/Ω_0.

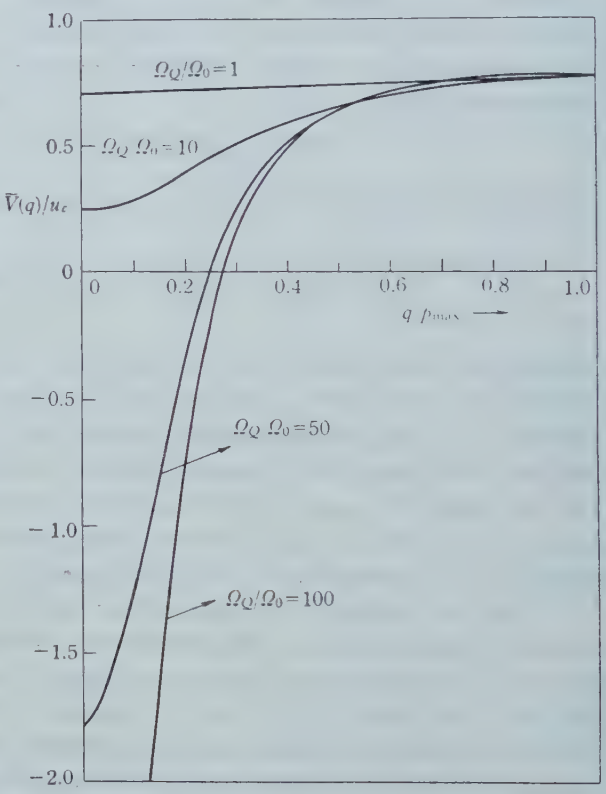

Fig. 1. Typical q-dependencies of $\tilde{V}(\boldsymbol{q})$ for $\Omega_Q/\Omega_0=1$, 10, 50, and 100, which were calculated from (5·24) by taking $u_{ex}/u_c=1/5$, $u_c/\Delta_{c_0v_0}=1/2$, $\alpha_{c_0}p_{max}^2/\Delta_{c_0v_0}=1/4$, and $(-\beta_{v_0})=\alpha_{c_0}$.

It will be suggested from (5·24) that it will be possible to classify insulators into the following three groups.

Group I. $u_{ex}/u_c \ll 1$ and $\Omega_Q/\Omega_0 \gtrsim 1$, so that $\tilde{V}(\boldsymbol{q})$ is always positive for all $\boldsymbol{q}$. In this case, the condition (5·20) cannot be satisfied and the plasmon-like solution is impossible for the (c_0, v_0)-pair. Therefore a possible low-lying bound state in the insulator belonging to this group will be only the exciton-like solution. Ionic crystals, which have appreciably localized wave functions, will belong to this group.

Group II. $u_{ex}/u_c \lesssim 1$ and $\Omega_Q/\Omega_0 \gg 1$, so that $\tilde{V}(\boldsymbol{q})$ becomes negative for small $\boldsymbol{q}$, and, moreover, when $\tilde{V}(\boldsymbol{q})$ is positive its magnitude is too small to satisfy the the condition (5·14). A possible bound state in the insulator belonging to this group will be only the plasmon-like solution. The conditions that $u_{ex}/u_c \lesssim 1$ and $\Omega_Q/\Omega_0 \gg 1$ allow us to take $V_{c_0v_0}(\boldsymbol{q}) \simeq W_{c_0v_0}(\boldsymbol{q})$. Therefore, the dispersion relation (5·23) will be valid in this case, so that the usual plasmon picture is obtained except that the level is shifted by the presence of an energy gap, as discussed in § 3.

Group III. $(u_{ex}/u_c)(\Omega_Q/\Omega_0) \gtrsim 1$, so that $\tilde{V}(\boldsymbol{q})$ is negative for sufficiently small $\boldsymbol{q}$ and positive for remaining values of $\boldsymbol{q}$. If $\tilde{V}(\boldsymbol{q})$ has a sufficient magnitude to satisfy the conditions (5·20) and (5·14) for appropriate values of $\boldsymbol{q}$, both types of excitation, the plasmon-like and the exciton-like, will be obtained. This is the intermediate case between two groups mentioned above. Some of valence crystals or intermetallic compounds may belong to this group. The plasmon-like excitation is responsible for a motion of the pair with a long wavelength and the exciton-like excitation is concerned with the remaining degrees of freedom of a motion of the pair. Fig. 2 shows schematically the appearance of those levels, depending upon q. Using the value of $\tilde{V}(\boldsymbol{q})$ for $\Omega_Q/\Omega_0=50$, given in Fig. 1, the plasmon-like level was calculated from (5·21), and the exciton-like level from (5·16) and (5·17). Broken lines indicate unstable solutions which violate the conditions (5·14) and (5·20).

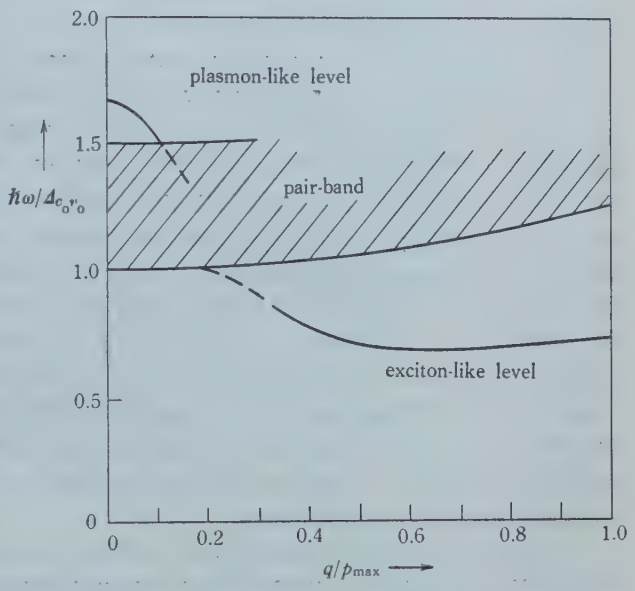

Fig. 2. Energy level scheme of possible excitations in the case of Group III. Both plasmon-like and exciton-like levels were calculated for $V(\boldsymbol{q})$ with $\Omega_Q/\Omega_0=50$, given in Fig. 1.

We have hitherto assumed that (c_0, v_0) pair-band is well-separated from the rest and that $\Delta_{c_0v_0} \gg (\alpha_{c_0}-\beta_{v_0})p_{max}^2$. These assumptions are evidently not satisfied in Ge and Si, so that the results obtained above should be considered only as qualitative for them.

§ 6. Concluding remarks

We have discussed in the previous section the relation between a plasmon-like and an exciton-like solution, and have found that they are more or less deformed

from the simplest models of usual plasmon and exciton, respectively. Although the discussions have been given under rather simplified assumptions, the conclusion derived there appears to supply us with a qualitatively right direction to understand the relation between two models of a plasmon and an exciton in insulators.

In reality, if an electron-hole pair is optically excited in alkali halides, an exciton-like bound state will be formed by the attractive potential, $-\tilde{V}(\boldsymbol{q})$ between the electron and the hole. As alkali halides are considered to belong to Group I, $\tilde{V}(\boldsymbol{q})$ is nearly independent of $\boldsymbol{q}$, as is seen from Fig. 1, and is given by

$$\tilde{V} \simeq u_c/\kappa_{\text{inter}} \simeq v(0)/\kappa_{\text{inter}}, \qquad (6\cdot1)$$

where $v(0)$ is given by $(4\cdot5)$, $\boldsymbol{R}$ being zero. Thus, the usual exciton model with a due modification of the effective potential seems applicable to alkali halides. On the other hand, if an electron ejected from v_0-band is excited to higher conduction band, say c_1-band, by charged particles with a high speed, relevant values of u_{ex}/u_c and Ω_Q/Ω_0 may be somewhat increased, because the wave functions belonging to the higher conduction band will be less localized. Thus, there arises a possibility that $\tilde{V}(\boldsymbol{q})$ becomes negative for sufficiently small $\boldsymbol{q}$ and a plasmon-like level appears above the continuum pair-band (c_1, v_0). The plasmon-like level will be given by the similar formula as $(5\cdot21)$, and, therefore, the separation from the continuum may be small on account of a large gap $\Delta_{c_1v_0}$. Further, the plasmon-like excitation will have a shorter lifetime than that expected from the usual plasmon model. Therefore, the energy loss due to excitation of the plasmon-like solution may hardly be dissolved from the loss due to excitation of scattering solution. Those situations appear to make the plasmon-like level difficult to be identified in the energy loss spectra found in alkali halides.[20]

In the case of X-ray absorption, both exciton-like and plasmon-like levels will be shifted from that discussed above, because the hole left behind remains at the X-ray level instead of the v_0-band. Characteristic structures near the K-absorption edge of KCl have recently been discussed by Muto and Okuno,[21] and Parratt and Jossem[22] on the basis of an exciton model. On the other hand, Leder, Mendlowitz, and Marton[23] have investigated the correlation between the characteristic energy losses and the fine structure near the X-ray absorption edge and have found that there is a relatively good correlation between them for NaCl, KCl, and KBr. However, it seems difficult to explain the energy losses in those crystals by the plasmon model.

Valence crystal like Ge and Si is characterized by that the wave function is not appreciably localized and the energy gap between c_0- and v_0-bands is small. It is supposed from $(5\cdot19)$ that the effective potential $\tilde{V}$ is strongly reduced on account of the small gap, so that the exciton level will lie close to the bottom of the conduction band. The experimental evidence of the exciton has been obtained in Ge by Macfarlane et al.,[24] and recently by Zwerdling, Roth, and Lax,[25] using a magneto-optical technique. If we consider that such a valence crystal belongs to Group III, we can expect that the plasmon-like excitation concerning the pair with

a long wavelength will also be obtained. If the plasmon-like excitation is brought up by the pair in which an electron is excited to higher conduction band, then it may be approximated by the usual plasmon, because the wave function will be extended. In fact, the second peaks of about 17eV in the energy loss spectra found in Ge and Si are matched with the theoretical value estimated by Pines[26] on the basis of the plasmon model. However, the first peaks of about 5eV still remain to be explained.

Leder and Marton[27] have reported that the energy loss peaks found in several elements, Si, Te, Pb, Sb, and Mg, are shifted towards higher energies when they form compounds with O and S. If a plasmon model is applied to this phenomenon, it might be explained by the shift due to the larger energy gap in compounds, as discussed in § 3. It has also been known that ionic crystals like BaO, MgO, PbS,[28] etc., have several peaks in the energy loss spectra. However, it seems necessary to have a further study to identify the plasmon-like excitation in those crystals.

Shallow levels of exciton in Cu_2O,[29] CdS,[30] PbI_2,[31] etc., have been observed to form a hydrogen-like series. An analysis by means of an effective mass equation shows that the orbit has a large radius. This suggests that the wave functions of conduction and valence bands are not appreciably concentrated around ionic sites. If this is the case, the plasmon-like excitation may also be observed in those crystals, for which the experimental study of the energy loss is desired.

So far as an electron-hole pair model is concerned, the plasmon-like excitation is responsible for a degree of freedom of motion of the pair having a long wavelength and the exciton is a possible bound state within the remaining degrees of freedom. The electron-hole pair model is, of course, only an approximation to many-particles problem. However, it seems that the relation between two models of a plasmon and an exciton has been qualitatively clarified by the pair model. We have not taken into account of the effect of thermal vibrations and displacements of lattices, which will be reported in future.

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On the Validity of Dispersion Relations in Perturbation Theory

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The dispersion relations with the ordinary threshold energies are proved in every order of perturbation expansion for $\pi-\pi$, $\pi-K$, $\pi-N$, $\pi-\Lambda$, $\pi-\Xi$, $K-K$, $N-N$, $\Lambda-\Lambda$, $N-\Xi$, $K-N$, $K-\Lambda$ forward scatterings. A counter-example is presented for $N-\Lambda$, $N-\Sigma$, $\Sigma-\Sigma$, $\Lambda-\Xi$, $\Sigma-\Xi$ scatterings. Non-forward scatterings also are treated.

§ 1. Introduction

The dispersion relation was first introduced to quantum field theory by Gell-Mann et al.,[1] and its general proof was given by Goldberger.[2] But recently Bogoliubov et al.[3] have pointed out that his proof was not mathematically correct in the treatment of the unphysical region, and using analytic continuation, they rigorously proved the dispersion relation for $\pi-N$ scattering. Improving their technique, Bremermann et al.[4] showed that using the usual axioms alone one cannot prove the dispersion relations for $N-N$ scattering and $NN\pi$ vertex function unless $m_\pi > (\sqrt{2}-1)m_N$. Oehme,[5] Karplus et al.[6] and Nambu[7] independently investigated the lowest order term of the perturbation expansion of vertex function, and showed that the dispersion relation for $\Sigma\Sigma\gamma$ vertex function actually breaks down. Nambu[7] also proved, using his parametric representation, that the dispersion relation for $NN\gamma$ (or $NN\pi$) vertex function is valid in every order of perturbation expansion if baryon number is conserved and if nucleon is the lowest-mass baryon. Very recently, Symanzik[8] has independently proved, using the same formula, that, in addition to this, the dispersion relation for $\pi-N$ scattering is valid in every order. He also treated $N-N$ scattering, but could not give the dispersion relation with the ordinary threshold energy.

In the present paper, we investigate the ordinary dispersion relations for various cases in perturbation theory, making use of the general integral formula of perturbation term presented previously by the present author.[9] We treat only the elastic scatterings by strong interactions. Our fundamental postulates are baryon number conservation and strangeness conservation. Iso-spin conservation and parity conservation are not taken into account. No assumptions are made for interaction types.

In § 2, we summarize the main results of the previous paper and give a fundamental theorem for the present investigation. § 3 and § 4 are devoted to prove the ordinary dispersion relations for forward scatterings, which hold in every

order of perturbation expansion for $\pi-\pi$, $\pi-K$, $\pi-N$, $\pi-\Lambda$, $\pi-\Xi$, $K-K$, $N-N$, $\Lambda-\Lambda$, $N-\Xi$, $K-N$, $K-\Lambda$ scatterings, but break down for $N-\Lambda$, $\Lambda-\Xi$ and $\Sigma-$ baryon scatterings. Finally, in § 5 we treat the non-forward scatterings, and prove the ordinary dispersion relation for $N-N$ scattering with half momentum transfer $\Delta \leqq m_\pi$, which cannot be furthermore improved by the present method.

§ 2. Preliminary

As a preliminary, we summarise the main results obtained in the previous paper.[9] Without loss of generality, we may assume that all particles are scalar and all couplings are direct,* since we are interested only in mass spectra. However, of course, selection rules are taken into account. Being unimportant, renormalizations are not considered.

Consider an arbitrary Feynman diagram G. Each line of G corresponds to a propagator

$$1/\{(p_i+q_i)^2+m_i^2-i\varepsilon\}, \quad (i=1, 2, \cdots, \alpha)$$

where p_i, q_i, and m_i stand for an integration momentum, a constant momentum, and a mass, respectively. The propagators which do not contain any integration momentum are separately considered for the convenience of the present purpose. α is the total number of the propagators which contain an integration momentum. All of the integration momenta p_i are, of course, not independent. We denote by n the number of independent ones. Then the following general integral formula is derived by means of the generalized Feynman identity.

Theorem 1. The absorptive part of the matrix element corresponding to G (apart from unintegrated propagators) is given by the following parametric integral. The integration region is $x_i \geqq 0$ $(i=1, 2, \cdots, \alpha)$.

$$A_G=\text{const.}\int \frac{1}{U^2(x)}\,\delta^{(\alpha-2n-1)}(-V(q, x))\cdot\delta\Big(\sum_{i=1}^{\alpha} x_i-1\Big)\prod_{i=1}^{\alpha} dx_i, \qquad (2\cdot1)$$

where $\delta^{(s-1)}$-function for $s \leqq 0$** is defined by

$$\delta^{(s-1)}(u) \equiv (u^{|s|}/|s|!)\,\theta(u),$$
$$\theta(u)=1 \quad \text{for} \quad u>0,$$
$$=0 \quad \text{for} \quad u<0.$$

The notations used in (2·1) are as follows.

$$U(x) \equiv \sum x_{\nu_1} x_{\nu_2} \cdots x_{\nu_n}. \qquad (2\cdot2)$$

---

* Otherwise we have only to operate $f(D)$ on (2·1). That does not bring any changes for the following arguments.

** This case often appears for the second kind interactions. For $s \leqq 0$, the usual expression $\mathrm{Im}\, 1/(V-i\varepsilon)^s$ loses its meaning.

The summation covers all possible sets $(\nu_1, \nu_2, \cdots, \nu_n)$ such that $p_{\nu_1}, p_{\nu_2}, \cdots, p_{\nu_n}$ are independent.

$$V(q, x) \equiv \sum_{i=1}^{\alpha} x_i(m_i^2+q_i^2) - Q(q, x)/U(x), \tag{2·3}$$

$$Q(q, x) \equiv \sum_{C} U_C(x)\,(\pm x_{C_1} q_{C_1} \pm x_{C_2} q_{C_2} \pm \cdots \pm x_{C_l} q_{C_l})^2. \tag{2·4}$$

C is a closed circuit belonging to the graph G, and $C_1, C_2, \cdots, C_l$ are the lines belonging to C. The double signs correspond to the relative directions of $q_{C_1}, q_{C_2}, \cdots, q_{C_l}$ on C.

$$U_C(x) \equiv \sum x_{\nu_1} x_{\nu_2} \cdots x_{\nu_{n-1}}. \tag{2·5}*$$

The summation covers all possible sets $(\nu_1, \nu_2, \cdots, \nu_{n-1})$ such that none of $\nu_1, \nu_2, \cdots, \nu_{n-1}$ coincides with any one of $C_1, C_2, \cdots, C_l$ and $p_{\nu_1}, p_{\nu_2}, \cdots, p_{\nu_{n-1}}$ are independent. The summation $\sum_C$ in (2·4) covers all possible closed circuits belonging to G.

The proof of this theorem is found in the previous paper.

Now, it is obvious from (2·2) and (2·5) that $U(x) \geqq 0$ and $U_C(x) \geqq 0$. Hence if all q_i's are Euclidean vectors,** i. e.

$$-(\sum_i a_i q_i)^2 \geqq 0$$

for arbitrary real numbers a_i, then we have an inequality

$$V(q, x) \geqq \sum_{i=1}^{\alpha} x_i(m_i^2+q_i^2), \tag{2·6}$$

since the second term of (2·3) is non-negative definite. The constant momenta q_i are, of course, not all independent. When we denote the external momenta by k_j, q_i is expressed by

$$q_i = \sum_j a_{ij} k_j$$

where the coefficients a_{ij} are real numbers. It is of importance that the choices of a_{ij} are not unique. If there is a special choice for which inequalities

$$m_i^2+q_i^2 \geqq 0 \quad (\text{for } i=1, 2, \cdots, \alpha) \tag{2·7}$$

hold, then we obtain

$$V(q, x) \geqq 0 \tag{2·8}$$

from (2·6), hence $A_G=0$ from (2·1). Thus we get the following important theorem.

Theorem 2. When all external momenta k_j are Euclidean, we have $A_G=0$

---

* For $n=1$, $U_c=1$.

** Euclidean vectors can appear in the analytic continuation to unphysical region.

if we can find such a set of q_i that momentum is conserved at every vertex and the inequalities (2·7) hold.

§ 3. Forward scatterings—General considerations

In this and next sections we treat the dispersion relations for elastic, forward scatterings. We denote the momentum and the mass of the target particle by p and m, respectively, and those of the projectile by k and $\mu(\leqq m)$. For two momenta p and k, there are three independent scalar products: $p^2=-m^2$, $k^2=-\mu^2$ and $(p+k)^2=-\kappa^2$. κ represents the total rest energy of the system.

Fig. 1

The dispersion relation is generally written in the following form* :

$$D_G(\sigma^2)=\frac{1}{\pi}\mathrm{P}\int\frac{A_G(\kappa^2)}{\kappa^2-\sigma^2}d\kappa^2+\text{a polynomial of }\sigma^2 \qquad (3\cdot1)$$

where D_G is the dispersive part of G. In the ordinary dispersion relations the spectrum of κ is nothing else but the energy spectrum of the possible intermediate states. But the ordinary dispersion relations do not always hold, as has been pointed out recently. Our problem is to clarify in what cases they hold.

Outside the unphysical region

$$(m-\mu)^2\leqq\kappa^2\leqq(m+\mu)^2, \qquad (3\cdot2)$$

$A_G(\kappa^2)$ is generally non-vanishing in the ordinary dispersion relations. Hence we have only to consider the mass spectrum in the region (3·2), in which p and k become Euclidean, and therefore theorem 2 is applicable.

For convenience, we introduce the following notion. When a diagram G is obtained from a simpler diagram S through the following operations, we call S a skeleton of G.

1) to add simply any radiative correction.
2) to increase the mass of any propagator.
3) to add the self-energy part to any stable particle.

It is obvious that if for S there is a choice of q_i satisfying the conditions of theorem 2, then also for G such a choice is always possible. Thus G is reduced to S, but of course S is not uniquely determined. It will be appropriately chosen case by case.

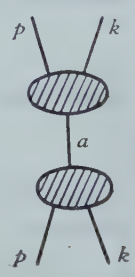


Fig. 2

Now, we investigate the spectrum of $A_G(\kappa^2)$ for various diagrams G. First, consider a diagram that contains one-particle intermediate state (see Fig. 2). Denoting the one-particle line by a, we have

$$A_G(\kappa^2)=\frac{1}{\pi}\mathrm{Im}\left[\frac{1}{-\kappa^2+m_a^2-i\varepsilon}\int\frac{A_{G'}(\sigma^2)}{-\kappa^2+\sigma^2-i\varepsilon}d\sigma^2\right], \qquad (3\cdot3)$$

* For the location of the singularities of a matrix element, see, for example, the Appendix of Matsuyama and Miyazawa.[10]

where $A_{G'}(\kappa^2)$ is the absorptive part for the graph without the line a. (3·3) is rewritten as

$$A_G(\kappa^2)=\int \frac{A_{G'}(\sigma^2)}{\sigma^2-m_a^2}d\sigma^2\cdot\delta(\kappa^2-m_a^2)-\frac{A_{G'}(\kappa^2)}{\kappa^2-m_a^2}. \qquad (3\cdot4)$$

Namely, the continuous spectrum of $A_G(\kappa^2)$ is same with that of $A_{G'}(\kappa^2)$. Thus such a diagram is reduced to vertex functions. The same is true for the one-particle-exchanging diagrams, Fig. 3 (a) and (b). Fig. 3 (a) is especially simple, which is reduced to two self-energy graphs, and (b) is equivalent to Fig. 2.

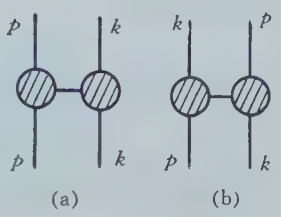


Fig. 3

Next, we consider the general diagrams other than Fig. 2 and Fig. 3. Apart from self-energy parts, these cannot be divided into two parts by cutting any one line. Any intermediate state contains at least two particles. We denote the lowest energy of intermediate states by $M(\leqq m+\mu)$.

The non-exothermic case ($M=m+\mu$) is especially simple. In this case the region in which A_G should vanish is the whole of the unphysical region. Since V is a linear function of κ^2, it must take its minimum at one of the end points of the region. Hence in order to prove $A_G(\kappa^2)=0$ for the region (3·2), it is sufficient to show $V\geqq 0$ for $p=(0,\ m)$, $k=(0,\ \pm\mu)$. It has been conjectured but not yet proved that the ordinary dispersion relations always hold in the non-exothermic case, provided that all concerning particles are stable.* For actual examples, we can easily choose q_i so as to satisfy the conditions of theorem 2 for the above p, k.

The exothermic case ($M<m+\mu$) is the central problem of the present investigation. The general treatment is quite difficult. Hence we treat only the diagrams that can be reduced to the skeletons, Fig. 4 (a) and (b).

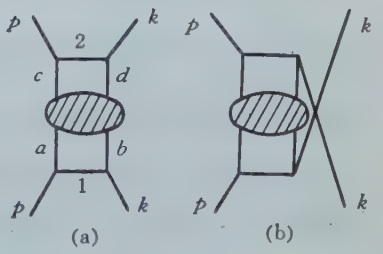


Fig. 4

(*a*) The region in which A_G should vanish is

$$(m-\mu)^2\leqq\kappa^2\leqq M^2. \qquad (3\cdot5)$$

To verify $A_G=0$ in this region, it is sufficient to prove $V\geqq 0$ for $\kappa^2=M^2$ and $\kappa^2=(m-\mu)^2$. For the latter no new technique is necessary. For the former we present a choice of q_i, which is the best one for these skeletons. We choose the momenta of the lines a, b, c and d as follows.

$$\left.\begin{aligned} q_a&=m_a(p+k)/(m_a+m_b), \quad q_b=m_b(p+k)/(m_a+m_b),\\ q_c&=m_c(p+k)/(m_c+m_d), \quad q_d=m_d(p+k)/(m_c+m_d).\end{aligned}\right\} \qquad (3\cdot6)$$

* This problem is very interesting as a purely mathematical game.

Then, of course, we have

$$m_j^2+q_j^2 \geqq 0 \quad (j=a, b, c, d) \tag{3·7}$$

because of $m_a+m_b \geqq M$ and $m_c+m_d \geqq M$. Since q_a, q_b, q_c and q_d in (3·6) are all parallel, the choice of q_i in the hatched part is generally easy.

q_1 and q_2 are uniquely determined by (3·6), namely

$$q_1=p-q_a=(m_b p-m_a k)/(m_a+m_b). \tag{3·8}$$

Hence we have

$$q_1^2=-(m_b^2 m^2+m_a^2 \mu^2+2m_a m_b pk)/(m_a+m_b)^2. \tag{3·9}$$

As $2pk=m^2+\mu^2-M^2$ when $\kappa^2=M^2$, the condition $m_1^2+q_1^2 \geqq 0$ is written as

$$m_1^2 \geqq \{m_b m^2+m_a \mu^2-m_a m_b M^2/(m_a+m_b)\}/(m_a+m_b). \tag{3·10}$$

In particular, when $m_a+m_b=M$, we obtain

$$m_1^2 \geqq (m_b m^2+m_a \mu^2)/(m_a+m_b)-m_a m_b. \tag{3·11}$$

For the special case $m=\mu$ and $m_a=m_b$, we have

$$m_1^2 \geqq m^2-m_a^2. \tag{3·12}$$

Quite analogously, we also obtain the condition

$$m_2^2 \geqq (m_a m^2+m_c \mu^2)/(m_c+m_d)-m_c m_d \tag{3·13}$$

when $m_c+m_d=M$.

(3·11) and (3·13) are our main results. These are also the *necessary* conditions for the validity of the ordinary dispersion relations. To see this, we consider the lowest order diagram, Fig. 5. From (2·3) and (2·4) with $q_3=p$, $q_4=k$, and $q_1=q_2=0$, we have

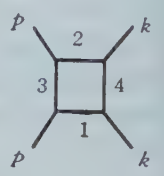


Fig. 5

$$V=\sum_{i=1}^{4} x_i m_i^2-x_3 m^2-x_4 \mu^2+x_3^2 m^2+x_4^2 \mu^2+2x_3 x_4 pk$$

on account of $U=\sum_{i=1}^{4} x_i=1$. Putting $-(p+k)^2=(m_3+m_4)^2$ (the expected threshold energy), we can rewrite this, using $\sum_{i=1}^{4} x_i=1$, as

$$V=x_1 m_1^2+x_2 m_2^2-(x_1+x_2)(x_3 m^2+x_4 \mu^2-x_3 m_3^2-x_4 m_4^2)+(x_3 m_3-x_4 m_4)^2.$$

Putting $x_3 : x_4=m_4 : m_3$, $x_3+x_4=1-\delta$, $x_1=\delta$, $x_2=0$, we have

$$V=\delta \cdot [m_1^2-(1-\delta)\{(m_4 m^2+m_3 \mu^2)/(m_3+m_4)-m_3 m_4\}].$$

Because of $m_3 \equiv m_a$, $m_4 \equiv m_b$, V becomes negative for very small δ if the condition (3·11) is violated. Thus (3·11) is necessary for $V \geqq 0$. (3·13) is likewise obtained.

In the Appendix B, we shall discuss the relation between (3·11) and the necessary and sufficient condition (in the axiomatic formulation) obtained by Bremermann et al.[4]

(*b*) Fig. 4 (b) is usually referred to as the crossing diagram. This is immediately reduced to Fig. 4 (a) if we replace k by $-k$. Hence it is unnecessary to consider these diagrams separately. For Fig. 4 (b) the region in which $A_G(\kappa^2)$ should vanish is

$$(m-\mu)^2 \leqq -(p-k)^2 \leqq M^2, \tag{3·14}$$

that is to say,

$$2(m^2+\mu^2)-M^2 \leqq \kappa^2 \leqq (m+\mu)^2. \tag{3·15}$$

Finally, we investigate the dispersion relation for vertex function. This is obtained immediately from that for the corresponding forward scattering by joining the two incident particles into a virtual one. For example, when the skeleton is represented by Fig. 6, the condition for m_1 is given by (3·11). In the lowest order (3·11) and (3·12) were presented by Karplus et al.[6] and Nambu[7] respectively.

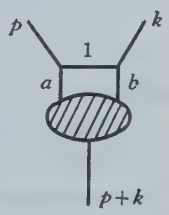


Fig. 6

When we write

$$V=\sum_i x_i m_i^2+\eta_1(x)\,p^2+\eta_2(x)\,k^2+\eta_3(x)\,(p+k)^2, \tag{3·16}$$

it will be shown in the Appendix A that $\eta_j(x) \geqq 0$ $(j=1, 2, 3)$ for vertex function. Therefore V is a monotonously decreasing function of κ^2. Thus if $V(\kappa^2) \geqq 0$ when $\kappa^2=M^2$, then we obtain

$$A_G(\kappa^2)=0 \quad \text{for } \kappa^2 \leqq M^2.$$

The vertex function thus is simpler than the corresponding forward scattering. Hence the cases of Fig. 2 and Fig. 3, which were reduced to vertex functions, may not be considered separately from the other general diagrams. In the next section therefore only the latters are considered.

§ 4. Forward scatterings—Actual examples

I. $\pi-N$ scattering

$\pi-N$ scattering is a non-exothermic process. Hence we have only to show the existence of q_i satisfying the conditions of theorem 2 for $p=(0, m_N)$, $k=(0, \pm m_\pi)$ about the non-crossing diagrams. On account of baryon number conservation, the initial and final nucleons are connected by baryon lines only. We call this set of lines "N-open-polygon", and any line which do not belong to the N-open-polygon "π-line".

i) $p=(0, m_N)$, $k=(0, m_\pi)$. First we let p flow along the N-open-polygon. Next we let k flow as follows. If the initial and final pion lines are connected by π-lines only, we choose them as the path along which k flows. If not so, starting from the initial pion line, we go along the π-lines which reach the vertex nearest to the final nucleon line on the N-open-polygon among the vertices which can be reached through π-lines alone, and proceed along the N-open-polygon toward the

initial nucleon line until the vertex which is determined in the following way. Starting from it, if we can reach the final pion line through π-lines only, we choose it. If there is not such a vertex, we choose the vertex starting from which we can reach the possible vertex nearest to the final nucleon line through π-lines. And we repeat the above process until we can reach the final pion line. As is easily verified, the path chosen above never passes the same vertex twice, provided that any intermediate state contains at least two particles. Hence it surely reaches the final pion line.

When p and k flow along the paths stated above, we have $q_i^2=-m_N^2$ or $-(m_N-m_\pi)^2$ on the N-open-polygon and $q_i^2=-m_\pi^2$ or 0 on π-lines. Thus the conditions of theorem 2 are satisfied.

ii) $p=(0, m_N), k=(0, -m_\pi)$. In this case, as the skeleton we have only to consider Fig. 7 alone. The intermediate line is a nucleon line and therefore $p+k$ can flow along it.

Fig. 7

Thus the ordinary dispersion relation for $\pi-N$ forward scattering holds. The same is true for $\pi-\pi$, $\pi-K$ scatterings. For $\pi-\Lambda$ scattering, in addition to the N-open-polygon the "strangeness-open-polygon" should be considered, along which an energy $m_\Lambda-m_N$ flows. $\pi-\Xi$ scattering is similar.

II. $N-N$ **scattering**

(*a*) *Pure scattering*

We have only to let p and k flow along the respective N-open-polygons.

(*b*) *Annihilation and creation**

This is an exothermic process. The skeletons are shown in Fig. 8 (b).

i) $-(p-k)^2=(2m_\pi)^2$. (3·12) is $m_N^2 \geqq m_N^2-m_\pi^2$, which is of course valid. The hatched part is nothing else but $\pi-\pi$ scattering, which is already proved.

ii) $p=k$. This case is trivial likewise to the case (*a*).

Thus the ordinary dispersion relation for $N-N$ forward scattering holds. $K-K$ scattering is quite similar.

$\Lambda-\Lambda$ scattering

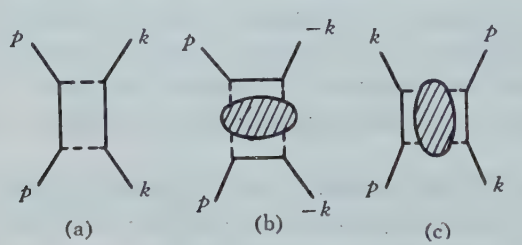


Fig. 8

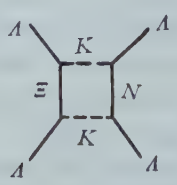


Fig. 9

* Originally, we should consider the exchange scattering Fig. 8 (c). But this can be converted equivalently to Fig. 8 (b), which is more intuitive.

(*a*) *Pure scattering*

The intermediate state $N+\Xi$ is non-exothermic. Hence an appropriate choice of q_i is easy.

(*b*) *Annihilation and creation*

In this case we have three kinds of (irreducible) exothermic processes: 2π, $K+\bar{K}$, $N+\bar{N}$. Therefore new skeletons other than Fig. 4 must be considered, namely the two parts other than the hatched part, which we call the "initial" and "final stages", respectively, are modified as Fig. 10, which are summarized by Fig. 11, where $m_a=m_b=m_\pi$ and $m_{a'}=m_{b'}$.

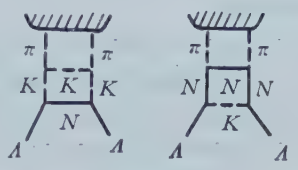


Fig. 10

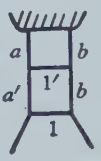


Fig. 11

i) When $-(p-k)^2=(2m_\pi)^2$, we must first have

$$q_a=q_b=(p-k)/2.$$

Next, we put

$$q_1=m_1(p-q_a)/(m_1+m_{1'})=m_1(p+k)/2(m_1+m_{1'}),$$

$$q_{1'}=m_{1'}(p-q_a)/(m_1+m_{1'})=m_{1'}(p+k)/2(m_1+m_{1'}).$$

Then, $m_1^2+q_1^2>0$ and $m_{1'}^2+q_{1'}^2>0$ are obvious from $m_1+m_{1'}>m_\Lambda$. Finally, we obtain

$$q_{a'}=p-q_1=\{(m_1+2m_{1'})p-m_1k\}/2(m_1+m_{1'}),$$

$$q_{a'}^2=\{-(m_1+2m_{1'})^2m_\Lambda^2-m_1^2m_\Lambda^2+(m_1+2m_{1'})m_1(2m_\Lambda^2-4m_\pi^2)\}/4(m_1+m_{1'})^2$$

$$=-\{(m_{1'}^2m_\Lambda^2+m_1(m_1+2m_{1'})m_\pi^2\}/(m_1+m_{1'})^2.$$

Therefore the condition $m_{a'}^2+q_{a'}^2\geqq 0$ becomes

$$m_K^2(m_N+m_K)^2\geqq m_K^2m_\Lambda^2+m_N(m_N+2m_K)m_\pi^2,$$

and

$$m_N^2(m_N+m_K)^2\geqq m_N^2m_\Lambda^2+m_K(m_K+2m_N)m_\pi^2.$$

Fortunately, the experimental values satisfy these inequalities. $q_{b'}$ is similar.

ii) $p=k$. This is quite easy.

Thus the ordinary dispersion relation for $\Lambda-\Lambda$ forward scattering holds.*

The same is valid also for $N-\Xi$ scattering because of $m_\Lambda^2+m_K^2\geqq(m_\Xi^2+m_N^2)/2$.

Contrary to these, $N-\Lambda$, $\Lambda-\Xi$ and any $\Sigma-$baryon scatterings do not allow the ordinary dispersion relations, because the condition (3·11) is violated for these cases. For example, we have

---

* But it should be noticed that the higher threshold for $K+\bar{K}$ and $N+\bar{N}$ are not ordinary ones, on account of $m_N^2+m_K^2<m_\Lambda^2$.

$$m_N^2 < (m_\pi m_\Lambda^2 + m_K m_N^2)/(m_\pi + m_K) - m_\pi m_K$$

for $N-\Lambda$ scattering. But it must be noticed that we have no definite conclusion for $\Lambda-\Sigma$ scattering if we take into account the isospin conservation.

III. $K-N$ **scattering**

For definiteness, we consider $K^- - N$ scattering. $K^+ - N$ scattering is its crossing counterpart.

The pure scattering is trivial. Hence we consider only the exothermic processes, the initial (or final) stages of which are shown in Fig. 12.

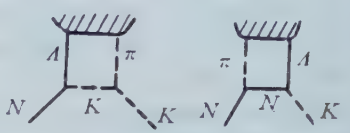


Fig. 12

i) $-(p+k)^2 = (m_\Lambda + m_N)^2$. From (3·11) we must have

$$m_K^2 \geqq (m_\pi m_N^2 + m_\Lambda m_K^2)/(m_\Lambda + m_\pi) - m_\Lambda m_\pi,$$

$$m_N^2 \geqq (m_\Lambda m_N^2 + m_\pi m_K^2)/(m_\Lambda + m_\pi) - m_\pi m_\Lambda,$$

which are indeed both valid. The hatched part is $\pi-\Lambda$ scattering.

ii) $p=(0,\ m_N)$, $k=(0,\ -m_K)$. We have only to put $q_1=q_2=(0,\ m_K)$ and $q_i=(0,\ m_N-m_K)$ for the other lines on the N-open-polygon.

Thus the ordinary dispersion relation for $K-N$ forward scattering holds. That for $K-\Lambda$ scattering likewise holds. But that for $K-\Xi$ scattering breaks down if parity is not conserved.

§ 5. Non-forward scatterings

In this section we discuss the dispersion relations for non-forward scatterings. We denote the initial and final momenta by p, k, and p', k', respectively. We choose the system

$$p=(\Delta,\ \sqrt{m^2+\Delta^2}),\quad p'=(-\Delta,\ \sqrt{m^2+\Delta^2}) \tag{5·1}$$

as usual. We then have*

$$\left.\begin{aligned} &p^2=p'^2=-m^2,\quad k^2=k'^2=-\mu^2,\\ &(k-k')^2=4\Delta^2,\\ &(p+k)^2=-(m^2+\mu^2+2\Delta^2+2\omega\sqrt{m^2+\Delta^2}),\end{aligned}\right\} \tag{5·2}$$

Fig. 13

where ω stands for the energy component of k. Unfortunately, these momenta are not Euclidean even in the unphysical region. Hence a modification is necessary.

In the present case V can be written as

$$V=\sum x_i m_i^2+\eta_1(x)\,p^2+\eta_2(x)\,p'^2+\eta_3(x)\,k^2+\eta_4(x)\,k'^2 + \eta_5(x)\,(k-k')^2+\eta_6(x)\,(p+k)^2. \tag{5·3}$$

---

* The last equation of (5·2) should be replaced by

$$(p-k')^2=-(m^2+\mu^2+2\Delta^2-2\omega\sqrt{m^2+\Delta^2})$$

for the crossing (or exchange) graphs.

Though η_5 and η_6 generally have no definite sign, it will be proved in the Appendix A that

$$\eta_1+\eta_2+\eta_3+\eta_4+4\eta_5 \geqq 0. \tag{5·4}$$

By V' we denote what is obtained by substituting

$$\left.\begin{aligned} &p^2=p'^2=-(m^2+\Delta^2), \quad k^2=k'^2=-(\mu^2+\Delta^2),\\ &(k-k')^2=0,\\ &(p+k)^2=-(m^2+\mu^2+2\Delta^2+2\omega\sqrt{m^2+\Delta^2}) \end{aligned}\right\} \tag{5·5}$$

instead of (5·2) in (5·3). Then from (5·4) we have

$$V \geqq V'. \tag{5·6}$$

Hence $V \geqq 0$ follows from $V' \geqq 0$. The momenta in (5·5) are Euclidean for $\omega^2 \leqq \mu^2+\Delta^2$. Non-forward scattering is thus reduced to the forward scattering of the particles with masses $\sqrt{m^2+\Delta^2}$ and $\sqrt{\mu^2+\Delta^2}$ (exothermic process).* Therefore it may seem that we could treat this quite similarly as we have done in the previous sections. But in this case a new difficulty appears. Since the external particles have masses larger than the ordinary ones, we must take into account infinitely various types of skeletons in the initial (or final) stage. Thus proofs become cumbersome. Hence we consider only $N-N$ scattering which is an important but simple case. Because of (3·12), we assume $\Delta^2 \leqq m_\pi^2$.

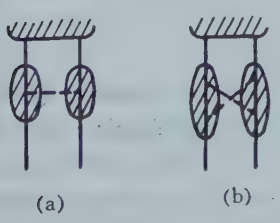
(*a*) *Pure scattering*

The skeletons in the initial stage are shown in Fig. 14. The vertex parts in Fig. 14 (a) are such graphs as Fig. 15,** and the hatched parts in Fig. 14 (b) are similar.

i) $-(p+k)^2=(2m_N)^2$. (i. e. $\omega=(m_N^2-\Delta^2)/\sqrt{m^2+\Delta^2}$).

An arbitrary internal momentum q is written as

$$q=\alpha\cdot\frac{p+k}{2}+\beta\cdot\frac{p-k}{2}, \tag{5·7}$$



(a) (b)

Fig. 14

Fig. 15

* This replacement is owing to Symanzik.[8]

** Self-energy parts should be removed by renormalizations.

where α and β are real numbers. Then we easily obtain

$$-q^2=\alpha^2 m_N^2+\beta^2 \Delta^2 \leqq (|\alpha| m_N+|\beta| m_\pi)^2. \tag{5·8}$$

Therefore it suffices if we can choose α and β so that

$$\begin{aligned} &|\alpha| m_N+|\beta| m_\pi \leqq m_N \quad \text{for } N\text{-line}, \\ &|\alpha| m_N+|\beta| m_\pi \leqq m_\pi \quad \text{for } \pi\text{-line}. \end{aligned} \tag{5·9}$$

For simplicity we abbreviate (5·7) as $q=\{\alpha, \beta\}$. We then have

$$p=\{1, 1\}, \quad k=\{1, -1\},$$

$$q_a=q_b=(p+k)/2=\{1, 0\}.$$

We prove (5·9) for the following two typical graphs among Fig. 14. Other cases can be treated quite analogously.

example 1

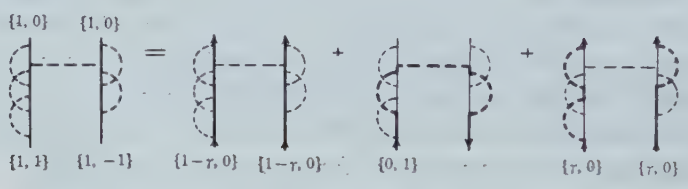


Fig. 16

$\{\alpha, \beta\}$'s in the left-hand graph of Fig. 16 are given by the superposition of those in the right-hand graphs, where the bold lines denote the paths along which the indicated $\{\alpha, \beta\}$'s flow, and where $\gamma=m_\pi/m_N$.

example 2

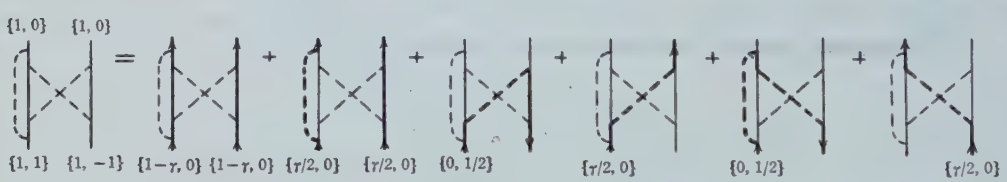


Fig. 17

Thus we can prove to be able to choose $\{\alpha, \beta\}$ so as to satisfy (5·9).

ii) $p=-k$ (i. e. $\omega=-\sqrt{m_N^2+\Delta^2}$). This case is very simple, since p and k are parallel. For example, we have only to replace $\{1-\gamma, 0\}$, $\{0, 1\}$ and $\{\gamma, 0\}$ by $\pm(m_N-m_\pi)p/(m_N+m_\pi)$, $m_\pi p/(m_N+m_\pi)$ and $\pm m_\pi p/(m_N+m_\pi)$, respectively, for the right-hand graphs of Fig. 16.

(*b*) *Annihilation and creation*

i) $-(p-k)^2=(2m_\pi)^2$, (i. e. $\omega=(m_N^2+\Delta^2-2m_\pi^2)/\sqrt{m_N^2+\Delta^2}$). Also in this case, the inequality (5·8) holds for q given by (5·7). Hence the decomposition

of $\{\alpha,\beta\}$ in the present case is quite same* as Fig. 16 or Fig. 17, except for the first graph of the right-hand side which is simply replaced by the N-lines in the present case.

ii) $-(p-k)^2=(2\Delta)^2$, (i. e. $\omega=(m_N^2-\Delta^2)/\sqrt{m_N^2+\Delta^2}$). This case is quite analogous to the above.

Finally, we must discuss the cases in which the lowest energy intermediate state does not appear, for these cases are not quite trivial for the non-forward scatterings. But we can easily treat these cases by applying the above-mentioned method, and so we omit the details.

Thus we can prove the ordinary dispersion relation for $N-N$ non-forward scattering with half momentum transfer $\Delta\leqq m_\pi$. When $\Delta=m_\pi$, the region of $A_G=0$ vanishes. Therefore we cannot further improve $\Delta_{\max}$ without investigating the signs of η_6.

The author is indebted to the financial aids of Yukawa Fellowship of Osaka University and Kumura Scholarship of Teikoku Rayon Co., Ltd.

Appendix A

For a self-energy diagram Fig. 18, V can be written as

$$\left.\begin{aligned} V(k,\,x)&=\sum x_i m_i^2+\eta(x)k^2\\ \text{with}\qquad \eta(x)&\equiv W(x)/U(x).\end{aligned}\right\} \qquad (A\cdot1)$$

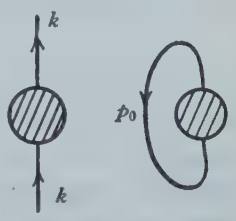


Fig. 18 Fig. 19

The explicit expression of $W(x)$ was presented in the previous paper,[9] namely

$$W(x)\equiv\sum x_{\nu_1}x_{\nu_2}\cdots x_{\nu_{n+1}}. \qquad (A\cdot2)$$

where the summation covers all possible sets $(\nu_1,\ \nu_2,\ \cdots,\ \nu_{n+1})$ such that p_{ν_1}, p_{ν_2}, $\cdots$, $p_{\nu_{n+1}}$ are independent in Fig. 19, which is obtained by connecting the external lines of Fig. 18 (of course the supposed integration momentum p_0 in Fig. 19 has no corresponding parameter x).

Now, let us consider the general diagram Fig. 20, in which the external lines are named as A, B, C, etc. When all external momenta except A and B vanish, Fig. 20 is reduced to a self-energy diagram like Fig. 18. We denote the η in this reduced graph by $\eta^{(AB)}$. Then

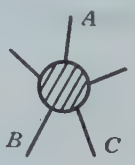


Fig. 20

Theorem 3. A triangle inequality

$$\eta^{(AC)}+\eta^{(BC)}\geqq\eta^{(AB)} \qquad (A\cdot3)$$

holds.

proof. (A·3) is, of course, equivalent to

$$W^{(AC)}+W^{(BC)}\geqq W^{(AB)}. \qquad (A\cdot4)$$

---

* But α and β exchange their roles on account of $q_a=q_b=\{0,1\}$ in this case (e. g. $\{0,1\}\longleftrightarrow\{\tau,0\}$ in Fig. 16).

Since each term of W is non-negative from (A·2), we have only to show that any term of $W^{(AB)}$ is contained also in either $W^{(AC)}$ or $W^{(BC)}$. We denote an arbitrary term of $W^{(AB)}$ by $x_1 x_2 \cdots x_{n+1}$, and the supposed momentum between A and B by $p_0^{(AB)}$. As the term $x_1 x_2 \cdots x_{n+1}$ belongs to $W^{(AB)}$, $p_0^{(AB)}$ must be expressible by p_1, p_2, $\cdots$, p_{n+1}, namely

$$p_0^{(AB)} = \sum_{i=1}^{n+1} a_i p_i, \tag{A·5}$$

where a_i's are some real numbers. Now, if we also have

$$p_0^{(AC)} = \sum_{i=1}^{n+1} b_i p_i, \tag{A·6}$$

where b_i's are real numbers, then p_1, p_2, $\cdots$, p_{n+1} are independent* in the vacuum polarization graph obtained by connecting A and C. Hence $x_1 x_2 \cdots x_{n+1}$ belongs to $W^{(AC)}$, which is what should be proved. We therefore may assume that (A·6) does not hold. Consider the graph in which $p_0^{(AB)}$ flows along AB and at the same time $p_0^{(AC)}$ flows along AC. In this graph there are $n+2$ independent momenta, and so $n+3$ momenta p_1, p_2, $\cdots$, p_{n+1}, $p_0^{(AB)}$, $p_0^{(AC)}$ cannot be independent, i. e.

$$\sum_{i=1}^{n+1} a_i' p_i + \alpha p_0^{(AB)} + \beta p_0^{(AC)} = 0, \tag{A·7}$$

where a_i''s, α and β are real numbers. From the assumption, $p_0^{(AC)}$ is not expressible by p_1, p_2, $\cdots$, p_{n+1} when $p_0^{(AB)}=0$. Hence it must be $\beta=0$, namely (A·7) is independent of $p_0^{(AC)}$, and so we can arbitrarily choose the value of $p_0^{(AC)}$. When we put $p_0^{(AC)}=p_0^{(AB)}$ in particular, the graph under consideration becomes equivalent to the graph obtained by connecting B and C. Thus (A·7) (which is to be identical with (A·5)) assures that $x_1 x_2 \cdots x_{n+1}$ belongs to $W^{(BC)}$. (q. e. d.)

In the following we state the applications of this theorem.

(i) *Vertex function*

When we denote the external lines of momenta p, k and $p+k$ by A, B and C, respectively, η_1, η_2 and η_3 in (3·16) are expressible by $\eta^{(AB)}$, $\eta^{(AC)}$ and $\eta^{(BC)}$ as follows.**

$$\left.\begin{aligned} \eta_1 &= (\eta^{(AB)} + \eta^{(AC)} - \eta^{(BC)})/2, \\ \eta_2 &= (\eta^{(AB)} + \eta^{(BC)} - \eta^{(AC)})/2, \\ \eta_3 &= (\eta^{(AC)} + \eta^{(BC)} - \eta^{(AB)})/2. \end{aligned}\right\} \tag{A·8}$$

Thus $\eta_j \geqq 0$ $(j=1, 2, 3)$ is obvious from theorem 3.

(ii) *Non-forward scattering*

When we denote the external lines of momenta p, k, p' and k' by A, B, C

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* Because all the other p_j's $(j=n+2, \cdots, \alpha)$ are, of course, expressible by $p_1, \cdots, p_{n+1}$.

** These are obtained by solving the simultaneous equations $\eta^{(AB)}=\eta_1+\eta_2$, $\eta^{(AC)}=\eta_1+\eta_3$ and $\eta^{(BC)}=\eta_2+\eta_3$, which correspond to $p+k=0$, $k=0$ and $p=0$, respectively.

and D, respectively, $\eta_j (j=1, 2, \cdots, 6)$ in (5·3) are expressible by $\eta^{(AB)}$, $\eta^{(AC)}$, etc., as follows.

$$\left.\begin{aligned}
&\eta_1+\eta_2=\eta^{(AC)}-\eta_6,\\
&\eta_3+\eta_4=\eta^{(BD)}-\eta_6,\\
&\eta_5=(\eta^{(AD)}+\eta^{(BC)}-\eta^{(AC)}-\eta^{(BD)})/2,\\
&\eta_6=(\eta^{(AD)}+\eta^{(BC)}-\eta^{(AB)}-\eta^{(CD)})/2.
\end{aligned}\right\} \qquad (A\cdot 9)$$

Hence

$$\begin{aligned}
\eta_1+\eta_2+\eta_3+\eta_4+4\eta_5&=\eta^{(AB)}+\eta^{(CD)}+\eta^{(AD)}+\eta^{(BC)}-\eta^{(AC)}-\eta^{(BD)}\\
&=(1/2)\{(\eta^{(AB)}+\eta^{(AD)}-\eta^{(BD)})+(\eta^{(AD)}+\eta^{(CD)}-\eta^{(AC)})\\
&\quad+(\eta^{(CD)}+\eta^{(BC)}-\eta^{(BD)})+(\eta^{(BC)}+\eta^{(AB)}-\eta^{(AC)})\}\geqq 0. \qquad (A\cdot 10)
\end{aligned}$$

The expression of $\eta_1+\eta_2+\eta_3+\eta_4+4\eta_5$ for the crossing (or exchange) graphs is identical with (A·10).

Appendix B

Bremermann et al. have presented the expression of the maximum momentum transfer below which the dispersion relation can be proved by the usual axioms alone. It is

$$\Delta^2_{\max}=\min G(t) \quad \text{for } 2t\geqq M \qquad (B\cdot 1)$$

with

$$G(t)\equiv\left[\left(t^2+\frac{ab}{2}-\frac{m^2+\mu^2}{4}\right)^2-\left(\frac{a+b}{2}t-\frac{a-b}{2}\frac{m^2-\mu^2}{4t}\right)^2\right]\left[t^2-\left(\frac{a-b}{2}\right)^2\right]^{-1}, \qquad (B\cdot 2)$$

where m, μ and M are same as in the text, and a and b are the thresholds of the mass spectra of the target and of the projectile, respectively, $(a\geqq b)$. Therefore the condition under which the dispersion relation for the forward scattering holds is

$$G(t)\geqq 0 \quad \text{for } 2t\geqq M. \qquad (B\cdot 3)$$

Now, we have obtained (3·11) as the condition for its validity. *If there were no restriction for the masses of the virtual particles*, we could put

$$\left.\begin{aligned}
&a=m_a+m_1,\\
&b=m_b+m_1,\\
&2t=m_a+m_b\geqq M,
\end{aligned}\right\} \qquad (B\cdot 4)$$

where we required even for the higher threshold $2t>M$ to be ordinary. Then (3·11) would be rewritten as

$$t^2+\frac{ab}{2}-\frac{m^2+\mu^2}{4}\geqq\frac{a+b}{2}t-\frac{a-b}{2}\,\frac{m^2-\mu^2}{4t}. \qquad \text{(B·5)}$$

When $2t\geqq a-b$, (B·5) is equivalent to (B·3).* In particular, for $N-N$ scattering (B·5) becomes $m_\pi\geqq(\sqrt{2}-1)m_N$ when $2t=m_N+m_\pi$.

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* It is easily verified that the sum of the both sides of (B·5) is positive.

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Low Energy Kaon-Nucleon Reactions

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The low energy KN and $\bar{K}N$ reactions are investigated in the case in which the kaon-nucleon-hyperon interaction is of the scalar type, with the nonrelativistic approximation. Direct interactions between kaons and pions are not considered. The higher order effects of kaons and pions are examined in detail and their importance is emphasized. It is shown that, as far as the parameters are adjusted to fit the experimental data on the KN scattering, those of the $\bar{K}N$ scattering can be hardly reproduced.

§ 1. Introduction

In making the theoretical analysis of the kaon-baryon interactions in the language of the current field theory, the points of arguments will be as follows: (i) The types of interactions. (Scalar, pseudoscalar, or in other words, relative parities of the baryons. Existence of direct interactions between kaons and pions.) (ii) Existence of symmetries other than the charge independence and the strangeness conservation. Moreover, as to the dynamical features of the interactions, the points are (iii) the relativistic effects and (iv) the strengths of the interactions and higher order effects of kaons and pions.

To any of these points we have yet no clear answer. The dispersion relations were applied to the determination of the types and the strengths of the kaon interactions.[1] With the present rough experimental data, it was shown that the pseudoscalar (NKY)*-interaction is more likely than the scalar one, and that the kaon coupling constants ${g_Y}^2$ are of the order of unity both for the pseudoscalar and the scalar cases. As for the point (ii), Gell-Mann[2] suggested an attractive idea of "global symmetry" which implies that the pions are coupled with Σ and Λ in a symmetrical way. This is, however, yet a mere conjecture. Pais[3] examined the case where not only pions but also kaons are symmetrically coupled with Σ and Λ. He showed, as far as one may neglect the $(\Sigma\Lambda)$ mass difference in dynamical

* Y means Σ and Λ.

calculations, that the assumption is apparently incompatible with the present experimental data. The possible existence and some favorable features of direct interactions between kaons and pions were suggested[4], but we do not touch upon such a kind of interaction in this paper.

In order to push a step further and construct a detailed theory we must explore the dynamical features as well as the kinematical ones. A number of such attempts have been made so far. The covariant lowest order perturbation calculation with the scalar and the pseudoscalar (NKY)-interaction was applied to the KN scattering[5], but this is not reliable since higher order effects are important as we shall see. As for the higher order effects of kaons on the KN scattering, the Chew-Low approach[6] and the lowest order Tamm-Dancoff approximation[7] in the static theory were performed. However, the effects of virtual $\bar{K}$ and pions on the KN scattering which were neglected in these works are not necessarily small. In particular, the role played by pions is essential in the $\bar{K}N$ reactions*. The dynamics of the KN reactions has not been properly investigated so far. Although the KN and the $\bar{K}N$ reactions are far more complicated than the πN reactions, the pion physics will serve as a helpful lesson. For instance, it would be possible to find a shortcut in such a meandering experienced in the dynamics of the πN reactions before Chew and Low's success.

In this paper we are mainly concerned with the above points (iii) and (iv), and shall investigate the KN and the $\bar{K}N$ reactions at low energies using the Chew-Low approach[8], deliberately considering the higher order effects of kaons and pions. We assume that the spins of kaons and hyperons are 0 and 1/2, respectively, and the parities of Σ and Λ are the same. The interaction Hamiltonian densities which satisfy the charge independence and the strangeness conservation are

$$H_\pi = (4\pi)^{1/2} \cdot i f_N^{(0)} \bar{N} \gamma_5 \boldsymbol{\tau} N \boldsymbol{\pi} + (4\pi)^{1/2} \cdot \{ i (f_\Lambda^{(0)} \bar{\Lambda} \gamma_5 \boldsymbol{\Sigma} \boldsymbol{\pi} + f_\Sigma^{(0)} (\bar{\boldsymbol{\Sigma}} \times \gamma_5 \boldsymbol{\Sigma}) \boldsymbol{\pi}) + \text{h.c.} \}, \tag{1·1}$$

$$H_K = (4\pi)^{1/2} \cdot (g_\Lambda^{(0)} \Lambda \Gamma N \bar{K} + g_\Sigma^{(0)} \bar{\boldsymbol{\Sigma}} \Gamma \boldsymbol{\tau} N \bar{K}) + \text{h.c.} \tag{1·2}$$

Here a particle symbol denotes the corresponding annihilation operator. A bold face letter denotes an isovector. $\Gamma = 1$ or $i\gamma_5$. $f^{(0)}$'s and $g^{(0)}$'s are the unrationalized and unrenormalized coupling constants. The natural unit $(c = \hbar = 1)$ is used. The terms containing Ξ are not written, as effects of these terms on the low energy KN and $\bar{K}N$ reactions turn to be negligible. Direct interactions between kaons and pions are not considered.

In the actual calculations we replace $i\gamma_5$ by $\sigma \boldsymbol{\nabla} / m_\pi$ in the pion interaction H_π. In the kaon interaction H_K, such a replacement would not necessarily be legitimate. Since we have at present no reliable means to treat the virtual pair creation and annihilation processes caused by the operator γ_5, we shall not consider the case of

* We use the following terminology. The process $\bar{K} + N \to \bar{K} + N$: $\bar{K}N$ scattering. The process $\bar{K} + N \to \pi + Y$: $\bar{K}N$ capture. The $\bar{K}N$ reaction means these both types of processes in a lump.

$\Gamma=i\gamma_5$ in this paper. On the other hand, the case of $\Gamma=\sigma\nabla/m_K$ will probably be excluded by the experimental information that the K^+p scattering at low energies occurs mainly through the S-wave kaons[9]. Hence we are eventually concerned with the case of $\Gamma=1$, the scalar interaction.

The scalar interaction is incompatible with the experimental fact that the K^+p scattering potential is likely repulsive[10]. Nevertheless it is we think worthwhile to investigate this case now because the experimental data still does not seem sufficiently accurate to exclude this case completely. Moreover, we have in this case a clear and reliable way to disclose some characteristic features of the kaon dynamics, then this will provide important clues to the kaon dynamics. At the same time the detailed investigation of this case will offer another argument *against* or *for* the scalar interaction. Hereafter it should be understood that the scalar interaction, $\Gamma=1$, is always assumed unless otherwise stated. For the order of magnitude of the kaon coupling constants, rather stringent restrictions were shown by the authors[11] in the case of the scalar interaction. This consideration provides a useful guide in the following.

In the following Sec. 2, we estimate the recoil effects in the KN and the $\bar{K}N$ scatterings using the Tamm-Dancoff approximation or the Lee model. Here the effects of pions and $\bar{K}(K)$ on the $KN(\bar{K}N)$ scattering are neglected for simplicity. Despite the relatively large mass of the kaon, the recoil effects are found to be fairly small at low energies and the static approximation can safely be applied. Furthermore it is pointed out that the higher order effects are so large that the perturbation calculation is inadequate. In Sec. 3, we write down the system of the Low equations for the KN and the $\bar{K}N$ reactions including virtual effects of K, $\bar{K}$ and pions. Then preliminary discussions about an approximate treatment of the equations are given. It is pointed out that somewhat detailed knowledge on the pion-hyperon scattering is needed especially for the estimate of the amplitudes of the $\bar{K}N$ reactions. Sec. 4 is devoted to the study of the pion-hyperon scattering. A significant result obtained in this section is that the "resonance" does not appear in the state with the total angular momentum $J=1/2$, provided that the ratio $(f_\Sigma/f_\Lambda)^2$ does not much differ from unity. In Sec. 5, the KN and the $\bar{K}N$ reactions are analysed on the basis of the results of the preceding sections. Characteristic features of the kaon dynamics are revealed. Comparison with the experimental data is also given and some difficulties are pointed out. Finally the results are summarized in Sec. 6. Details of calculations are presented in Appendix.

§ 2. Recoil effects on the KN and the $\bar{K}N$ scatterings

In the πN reactions at low energies the recoil effects due to pions are not very appreciable and the static approximation has been successfully applied. While in the KN and the $\bar{K}N$ reactions it is naturally supposed that the recoil effects will be important because of the relatively large mass of the kaon. In this section we shall estimate the recoil effects in the KN and the $\bar{K}N$ scatterings using the Lee model,

ignoring the effects of pions and $\bar{K}(K)$ on the $KN(\bar{K}N)$ scattering, for simplicity.

Starting from the relativistic Hamiltonian and neglecting the contributions from negative energy states*, we have the following equation for the KN scattering amplitude $f_\alpha(\boldsymbol{p})$ in the center-of-mass system.

$$(\varepsilon_p+\omega_p-\varepsilon_q-\omega_q)\,S(p)\,f_\alpha(\boldsymbol{p})=(\lambda_\alpha/4\pi^2)\int d^3k K(\boldsymbol{p},\boldsymbol{k})\,f_\alpha(\boldsymbol{k}) \tag{2·1}$$

with

$$K(\boldsymbol{p},\boldsymbol{k})=1/(\omega_p\omega_k)^{1/2}(E_{p+k}+\omega_k+\omega_p-\varepsilon_q-\omega_q) \tag{2·2}$$

and

$$S(p)=1-(\lambda_0+3\lambda_1)/8\pi^2(\varepsilon_p+\omega_p-\varepsilon_q-\omega_q)\times\int d^3k/\omega_k(E_{p-k}+\omega_k-\varepsilon_p)^2(E_{p-k}+\omega_p+\omega_k-\varepsilon_q-\omega_q). \tag{2·3}$$

Here $\boldsymbol{q}$ and $\boldsymbol{p}$ are the initial and the final momentum of the kaon, respectively. The energies of kaon, nucleon and hyperon are

$$\omega_p=(m_K^2+p^2)^{1/2},\quad \varepsilon_p=(m_N^2+p^2)\quad\text{and}\quad E_p=(m_Y^2+p^2)^{1/2}.$$

The masses are all observed ones and the $(\Sigma\Lambda)$ mass difference is neglected. The index α discriminates the two isospin states with $I=0$ and 1. The coefficient λ_α is related to the renormalized coupling constants** as follows: (for derivation, see Appendix A)

$$\lambda_\alpha=\begin{cases}-g_\Lambda^2+3g_\Sigma^2\\ g_\Lambda^2+g_\Sigma^2\end{cases}\quad\text{for}\quad\alpha=\begin{cases}0\\1\end{cases}. \tag{2·4}$$

The above equations differ from the corresponding ones in the static approximation[7] only in that the baryon masses are replaced by the corresponding total energies.

Since eq. (2·1) cannot be solved exactly we use the first approximation of the Fredholm method. Then we have

$$f_\alpha(\boldsymbol{p})=\delta(\boldsymbol{p}-\boldsymbol{q})+\frac{\lambda_\alpha K(\boldsymbol{p},\boldsymbol{q})}{4\pi^2S(p)[1-\lambda_\alpha D(q)]}\cdot\frac{p^2}{\varepsilon_p+\omega_p-\varepsilon_q-\omega_q} \tag{2·5}$$

with

$$D(q)=\mathrm{P}\int d^3k K(\boldsymbol{k},\boldsymbol{k})/4\pi^2S(k)\,(\varepsilon_k+\omega_k-\varepsilon_q-\omega_q). \tag{2·6}$$

The symbol P stands for principal value.

Now we examine the angular dependence of the amplitude. As is seen from eqs. (2·6) and (2·3), $D(q)$ and $S(p)$ are both independent on the angle $\widehat{\boldsymbol{qp}}$.

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* Contributions from negative energy states appear as the P-wave scattering, which is quite unappreciable at low energies.

** The renormalized masses and coupling constants weakly depend on the kaon momentum. We replace them by constants.

Consequently the angular dependence of the amplitude $f_\alpha(\boldsymbol{p})$ comes only from $K(\boldsymbol{p}, \boldsymbol{q})$. Then decomposing $K(\boldsymbol{p}, \boldsymbol{q})$ into partial waves, we get the phase shifts for the S-, P-, $\cdots$ waves.

$$\tan \delta_{\alpha, I(J)} = \frac{(-1)^I \lambda_\alpha R_{I(J)}(p)}{(2I+1)[1-\lambda_\alpha D(p)]} \tag{2·7}$$

with

$$\begin{aligned} R_{S(1/2)}(p) &= pQ \equiv p \cdot (\sqrt{m_Y^2+2p^2}+\omega_p-\varepsilon_p)^{-1}, \\ R_{P(1/2)}(p) &= R_{P(3/2)}(p) = p^3Q^2/2\sqrt{m_Y^2+2p^2} \end{aligned} \tag{2·8}$$

where only the terms with the lowest power of p are retained.

Fig. 1 shows $R_{I(J)}/(2I+1)$ in eq. (2·7) together with the corresponding static limit. We see that the recoil effects in the Born terms are fairly small. As for the higher order effects, since they are, roughly speaking, repeated accumulations of the Born terms, the recoil effects via the higher order effects will not be very appreciable also. Indeed, the calculation with a cut off energy 2.5 $\sim 3m_K$ shows that recoil effects including the higher order terms are at most of the order of 20%.

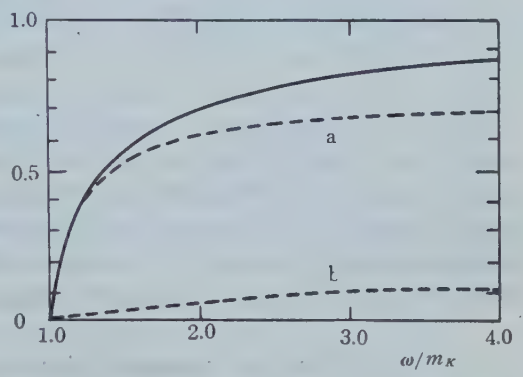

Fig. 1. The solid line: S-wave in the static limit.
The broken line a: S-wave.
b: P-wave. $(R_{P(1/2)}+R_{P(3/2)})/3$.
The abscissa indicates the kaon energy in m_K.

In case of the pseudovector interaction, though the scattering can occur through the S- and D-waves due to the recoil effects, the P-wave is dominant at low energies. Then in view of the experimental fact that the angular distribution of the K^+p scattering cross section is nearly isotropic, the case of the pseudovector interaction will probably be ruled out.

The scattering amplitudes for the $\bar{K}N$ scattering are obtained in a similar manner. We write only the results.

$$\tan \tilde{\delta}_\alpha = \frac{\tilde{\lambda}_\alpha p}{3(\varepsilon_p+\omega_p-m_Y)[1+\tilde{\lambda}_\alpha \tilde{D}(p)]} \tag{2·9}$$

with

$$\tilde{D}(p) = (\varepsilon_p+\omega_p-m_Y) \cdot \mathrm{P}\int d\omega_k \frac{k}{\pi} \frac{1}{(\varepsilon_k+\omega_k-m_Y)^2 (\varepsilon_k+\omega_k-\varepsilon_p-\omega_p)} \tag{2·10}$$

and

$$\tilde{\lambda}_\alpha = \begin{cases} -2g_\Lambda^2 \\ -2g_\Sigma^2 \end{cases} \quad \text{for} \quad \alpha = \begin{cases} 0 \\ 1 \end{cases}. \tag{2·11}$$

Contrary to the KN scattering, the integral equation for the $\bar{K}N$ scattering amplitude is solved exactly and, moreover, the waves other than the S-wave do not appear. These circumstances are due to the fact that only Λ or Σ appears in the intermediate state in the present approximation. The recoil effects in the $\bar{K}N$ scattering are of the same order with that in the KN scattering.

In effect, the quantitative estimate shows that the recoil effects in both cases of the KN and the $\bar{K}N$ scatterings are less than 20% up to about $1.5m_K$ of the kaon energy ω in the center-of-mass system. This estimate is little sensitive to the choice of the cutoff energy. Therefore the static approximation may be safely applied within the range of energy just mentioned.

Finally we remark on the higher order effects. The denominators $(1-\lambda_\alpha D(p))$ in eq. (2·7) and $(1+\tilde{\lambda}_\alpha \tilde{D}(p))$ in eq. (2·9) come from the higher order effects. $D(0)$, $\tilde{D}(0)\neq 0$, and $D(p)$, $\tilde{D}(p)$ are both slowly decreasing with increasing p. With a cutoff energy $\omega_{\max}\approx 2.5m_K$ ($\approx$hyperon mass), we have $D(0)\approx 0.7$. Then if $\lambda_\alpha \approx 1$, $(1-\lambda_\alpha D(0))\approx 0.3$; the higher order effects enhance the amplitude to about three or four times of the result of the Born approximation. As the energy becomes higher, $D(p)$ decreases and the higher order effects become less important. Such a tendency is in contrast with the case of the pseudovector interaction where the corresponding function D is increasing with increasing energy; the higher order effects are more important at higher energies.

§ 3. The Low equations for the KN and the $\bar{K}N$ reactions

In the preceding section we have seen that the recoil effects are not very appreciable while the higher order effects are important. In order to investigate the higher order effects it will be most appropriate to use the Chew-Low approach[8]. In this section we write down the system of the Low equations for the KN and the $\bar{K}N$ reactions in the static approximation and make preparation for an approximate treatment of the equations.

Noticing that the kaons are in the S-state and the pions are in the P-state, we write the T-matrix elements for various processes as follows.

$$\begin{aligned} &K+N\to K+N: \quad T_\alpha = -v_q v_p \cdot 2\pi(\omega_q\omega_p)^{-1/2} h_\alpha(\omega_p), \\ &\bar{K}+N\to \bar{K}+N: \quad \tilde{T}_\alpha^{sc} = -v_q v_p \cdot 2\pi(\omega_q\omega_p)^{-1/2} \tilde{h}_\alpha(\omega_p), \\ &\bar{K}+N\to \pi+Y: \quad \tilde{T}_\alpha^{Y} = -v_q v_p^\pi \cdot 2\pi(p_\pi/m_\pi)(\omega_q\omega_p^\pi)^{-1/2} \tilde{K}_\alpha^{Y}(\omega_p^\pi). \end{aligned} \tag{3·1}$$

Here the index $\alpha=0$ or 1 denotes the isopin as before; the angular momentum $J=1/2$ always. $\boldsymbol{q}$ and $\boldsymbol{p}$ are the initial and the final kaon momentum respectively, and $\boldsymbol{p}_\pi$ is the final pion momentum. v_p and v_p^π are the form factors in the kaon and the pion interactions. The pion energy is $\omega_p^\pi=(m_\pi^2+p_\pi^2)^{1/2}$.

Using the one meson approximation, or more precisely, taking into account only the states with one kaon or one pion in addition to the physical baryon states, we have the Low equations for h_α and $\tilde{h}_\alpha$:

$$h_\alpha(z)=\frac{\lambda_\alpha}{z+\Delta m}+\frac{1}{\pi}\int_{m_K}^{\infty}d\omega_p\cdot pv_p^2\left\{\frac{|h_\alpha(\omega_p)|^2}{\omega_p-z}+\sum_\beta A_{\alpha\beta}\frac{|\tilde{h}_\beta(\omega_p)|^2}{\omega_p+z}\right\}$$
$$+\frac{1}{\pi}\int_{m_\pi}^{\infty}d\omega_p^\pi\cdot\frac{p_\pi^3}{m_\pi^2}v_p^{\pi 2}\sum_{\beta,Y}A_{\alpha\beta}\frac{|\tilde{K}_\beta^Y(\omega_p^\pi)|^2}{\omega_p^\pi+\Delta m+z}, \qquad (3\cdot2)$$

$$\tilde{h}_\alpha(z)=\frac{\tilde{\lambda}_\alpha}{z-\Delta m}+\frac{1}{\pi}\int_{m_K}^{\infty}d\omega_p\cdot pv_p^2\left\{\frac{|\tilde{h}_\alpha(\omega_p)|^2}{\omega_p-z}+\sum_\beta A_{\alpha\beta}\frac{|h_\beta(\omega_p)|^2}{\omega_p+z}\right\}$$
$$+\frac{1}{\pi}\int_{m_\pi}^{\infty}d\omega_p^\pi\cdot\frac{p_\pi^3}{m_\pi^2}v_p^{\pi 2}\sum_Y\frac{|\tilde{K}_\alpha^Y(\omega_p)|^2}{\omega_p^\pi+\Delta m-z}. \qquad (3\cdot3)$$

The coefficients λ_α and $\tilde{\lambda}_\alpha$ are defined in eqs. (2·4) and (2·11), and the summation with respect to Y is done for Λ and Σ. Δm is the mass difference of hyperon and nucleon. The matrix A is

$$A=\tfrac{1}{2}\begin{pmatrix}-1 & 3\\ 1 & 1\end{pmatrix} \qquad (3\cdot4)$$

which satisfies $\sum_\beta A_{\alpha\beta}A_{\beta\gamma}=\delta_{\alpha\gamma}$ and $\sum_\beta A_{\alpha\beta}=1$. $h_\alpha(z)$ and $\tilde{h}_\alpha(z)$ tend to $h_\alpha(\omega)$ and $\tilde{h}_\alpha(\omega)$ respectively when the complex argument z tends to $\omega+i\varepsilon$. The crossing relations are as follows.

$$\lambda_\alpha=-\sum_\beta A_{\alpha\beta}\tilde{\lambda}_\beta,\quad h_\alpha(z)=\sum_\beta A_{\alpha\beta}\tilde{h}_\beta(-z),$$
$$\tilde{\lambda}_\alpha=-\sum_\beta A_{\alpha\beta}\lambda_\beta,\quad \tilde{h}_\alpha(z)=\sum_\beta A_{\alpha\beta}h_\beta(-z). \qquad (3\cdot5)$$

Following CL, we rewrite eq. (3·2) in the form*; (when $\lambda_\alpha\neq0$)

$$h_\alpha(z)=\frac{\lambda_\alpha}{z+\Delta m}\cdot\left[1-\lambda_\alpha\cdot\frac{z+\Delta m}{\pi}\mathfrak{A}_\alpha(z)\right]^{-1} \qquad (3\cdot6)$$

with

---

* Eq. (2) in I corresponds to eq. (3·6) with $\alpha=1$. $A(\omega)$ in I corresponds to $\mathrm{Re}\,\mathfrak{A}_1(\omega)$ where $\mathfrak{A}_1(\omega)=\lim_{z\to\omega+i\varepsilon}\mathfrak{A}_1(z)$. From eq. (3·8) we have

$$\mathcal{W}_1=\tfrac{1}{2}(|\tilde{h}_0|^2+|\tilde{h}_1|^2)/|\tfrac{1}{2}(\tilde{h}_0+\tilde{h}_1)|^2,$$
$$\mathcal{W}_1^\pi=\tfrac{1}{2}(|\tilde{K}_0^\Sigma|^2+|\tilde{K}_1^\Lambda|^2+|\tilde{K}_1^\Sigma|^2)/|\tfrac{1}{2}(\tilde{h}_0+\tilde{h}_1)|^2.$$

Further, using the following relations,

$$\sigma(K^-+p\to K^-+p)=4\pi\cdot|\tfrac{1}{2}(\tilde{h}_0+\tilde{h}_1)|^2,$$
$$\sigma(K^-+p\to\bar{K}^0+n)=4\pi\cdot|\tfrac{1}{2}(\tilde{h}_0-\tilde{h}_1)|^2,$$
$$\sigma(K^-+p\to\pi+\Lambda\text{ and }\pi+\Sigma)=4\pi\cdot(p_\pi^3/pm_\pi^2)\cdot\tfrac{1}{2}(|\tilde{K}_0^\Sigma|^2+|\tilde{K}_1^\Lambda|^2+|\tilde{K}_1^\Sigma|^2),$$

we have

$$\mathcal{W}_1=\sigma(K^-+p\to K^-+p\text{ and }\bar{K}^0+n)/\sigma(K^-+p\to K^-+p),$$
$$(p_\pi^3/pm_\pi^2)\mathcal{W}_1^\pi=\sigma(K^-+p\to\pi+\Lambda\text{ and }\pi+\Sigma)/\sigma(K^-+p\to K^-+p).$$

$$\mathfrak{N}_\alpha(z)=\int_{m_K}^{\infty}d\omega_p\, pv_p^2\left\{\frac{1}{(\omega_p+\Delta m)^2(\omega_p-z)}+\frac{\mathcal{W}_\alpha(\omega_p)}{(\omega_p-\Delta m)^2(\omega_p+z)}\right\}$$
$$+\int_{m_\pi}^{\infty}d\omega_p^\pi\frac{p_\pi^3}{m_\pi^2}v_p^{\pi 2}\frac{\mathcal{W}_\alpha^\pi(\omega_p^\pi)}{\omega_p^{\pi 2}(\omega_p^\pi+\Delta m+z)} \tag{3·7}$$

and

$$\mathcal{W}_\alpha(\omega)=\sum_\beta A_{\alpha\beta}|\tilde{h}_\beta(\omega)|^2/|\sum_\beta A_{\alpha\beta}\tilde{h}_\beta(\omega)|^2,$$
$$\mathcal{W}_\alpha^\pi(\omega)=\sum_{\beta,Y}A_{\alpha\beta}|\tilde{K}_\beta^Y(\omega^\pi)|^2/|\sum_\beta A_{\alpha\beta}\tilde{h}_\beta(\omega^\pi+\Delta m)|^2. \tag{3·8}$$

In the same manner we have from eq. (3·3) (when $\tilde{\lambda}_\alpha\neq 0$)

$$\tilde{h}_\alpha(z)=\frac{\tilde{\lambda}_\alpha}{z-\Delta m}\cdot\left[1-\tilde{\lambda}_\alpha\cdot\frac{z-\Delta m}{\pi}\tilde{\mathfrak{N}}_\alpha(z)\right]^{-1} \tag{3·9}$$

with

$$\tilde{\mathfrak{N}}_\alpha(z)=\int_{m_K}^{\infty}d\omega_p\cdot pv_p^2\left\{\frac{1}{(\omega_p-\Delta m)^2(\omega_p-z)}+\frac{\tilde{\mathcal{W}}_\alpha(\omega_p)}{(\omega_p+\Delta m)^2(\omega_p+z)}\right\}$$
$$+\int_{m_\pi}^{\infty}d\omega_p^\pi\frac{p_\pi^3}{m_\pi^2}v_p^{\pi 2}\frac{\tilde{\mathcal{W}}_\alpha^\pi(\omega_p^\pi)}{\omega_p^{\pi 2}(\omega_p^\pi+\Delta m-z)}, \tag{3·10}$$

and

$$\tilde{\mathcal{W}}_\alpha(\omega)=\sum_\beta A_{\alpha\beta}|h_\beta(\omega)|^2/|\sum_\beta A_{\alpha\beta}h_\beta(\omega)|^2,$$
$$\tilde{\mathcal{W}}_\alpha^\pi(\omega)=\sum_Y|\tilde{K}_\alpha^Y(\omega^\pi)|^2/|\tilde{h}_\alpha(\omega^\pi+\Delta m)|^2. \tag{3·11}$$

In eq. (3·7) the term containing $\mathcal{W}_\alpha$ represents the effect of $\bar{K}$ on the KN scattering, and the term containing $\mathcal{W}_\alpha^\pi$ the effect of pions. In the calculation in Sec. 2, the terms corresponding to these have been dropped. Previously Stapp[6] applied the Chew-Low approach to the KN scattering, but he did not take into account these terms with $\mathcal{W}_\alpha$ and $\mathcal{W}_\alpha^\pi$.

Now as is seen from the forms of the denominators of the integrands in eq. (3·7), the first term in the curly bracket is dominant. In eq. (3·10) the first term in the curly bracket and the last term are important. In both cases the first term is known exactly, while $\mathcal{W}_\alpha$, $\mathcal{W}_\alpha^\pi$, $\tilde{\mathcal{W}}_\alpha$ and $\tilde{\mathcal{W}}_\alpha^\pi$ are unknown. The most natural *first approximation* to make, concerning $\mathcal{W}_\alpha$, etc., is to substitute the Born terms for them. In case of the πN scattering this first approximation is shown to be a good approximation. In the present case too, this approximation will be effective unless the amplitudes containted in $\mathcal{W}_\alpha$, etc., do not bear some singular behavior, e.g. resonance. In Sec. 5, we estimate the amplitudes by this method. The next step of approximation is to substitute the amplitudes obtained by the above first approximation to eqs. (3·8) and (3·11), which then gives a *second approximation*, and so on. Thus we can examine the convergence of this procedure.

Next, we write down the Low equation for $\tilde{K}_\alpha^Y$, which is directly related with the following amplitudes:

$$\begin{aligned}
&\pi+N\to K+Y: \quad -v_q^\pi v_p\cdot 2\pi(q_\pi/m_\pi)(\omega_q^\pi\omega_p)^{-1/2}K_\alpha^Y(\omega_p),\\
&K+Y\to K+Y': \quad -v_q v_p\cdot 2\pi(\omega_q\omega_p)^{-1/2}h_\alpha^{KYY'}(\omega_p),\\
&\pi+N\to \pi+N: \quad -v_q^\pi v_p^\pi\cdot 2\pi(q_\pi p_\pi/m_\pi^2)(\omega_q^\pi\omega_p^\pi)^{-1/2}h_\alpha^{\pi N}(\omega_p^\pi),\\
&\pi+Y\to \pi+Y': \quad -v_q^\pi v_p^\pi\cdot 2\pi(q_\pi p_\pi/m_\pi^2)(\omega_q^\pi\omega_p^\pi)^{-1/2}h_\alpha^{\pi YY'}(\omega_p^\pi).
\end{aligned}\tag{3·12}$$

The isospin index is $\alpha=0$, 1, 2 for $\pi+Y\to\pi+Y'$, and $\alpha=1/2$, $3/2$ for others. In $\pi+Y\to\pi+Y'$, two angular momentum states with $J=1/2$ and $3/2$ appear, in which only the state with $J=1/2$ directly couples with the kaon reactions.

The Low equation for $\tilde{K}_\alpha^Y$ is

$$\begin{aligned}
\tilde{K}_\alpha^Y(z^\pi)=\frac{\tilde{\mu}_\alpha^Y}{z^\pi}+\frac{1}{\pi}\int_{m_K}^\infty d\omega_p\cdot pv_p^2\left\{\frac{\tilde{K}_\alpha^{TY*}(\omega_p)\tilde{h}_\alpha(\omega_p)}{\omega_p-z^\pi-\Delta m}+\sum_{\beta,Y'}\tilde{B}_{\alpha\beta}^Y\frac{h_\beta^{KYY'*}(\omega_p)K_\beta^{Y'}(\omega_p)}{\omega_p+z^\pi+\Delta m}\right\}\\
+\frac{1}{\pi}\int_{m_\pi}^\infty d\omega_p^\pi\frac{p_\pi^3}{m_\pi^2}v_p^{\pi 2}\left\{\sum_{Y'}\frac{h_\alpha^{\pi YY'*}(\omega_p^\pi)\tilde{K}^{Y'}(\omega_p^\pi)}{\omega_p^\pi-z^\pi}\right.\\
\left.+\sum_\beta\tilde{B}_{\alpha\beta}^Y\frac{K_\beta^{T,Y*}(\omega_p^\pi)h_\beta^{\pi N}(\omega_p^\pi)}{\omega_p^\pi+z^\pi}\right\}.
\end{aligned}\tag{3·13}$$

The amplitude with the upper index T represents that for the reversed process in eq. (3·12); namely

$$\begin{aligned}
&\pi+Y\to\bar{K}+N: \quad -v_q^\pi v_p\cdot 2\pi(q_\pi/m_\pi)(\omega_q^\pi\omega_p)^{-1/2}\tilde{K}_\alpha^{TY}(\omega_p),\\
&K+Y\to\pi+N: \quad -v_q v_p^\pi\cdot 2\pi(p_\pi/m_\pi)(\omega_q\omega_p^\pi)^{-1/2}K_\alpha^{TY}(\omega_p^\pi).
\end{aligned}\tag{3·14}$$

Of course, $T(a\to b)=T(b\to a)$ on the energy shell. $\tilde{K}_\alpha^Y(z^\pi)\to\tilde{K}_\alpha^Y(\omega^\pi)$ when $z^\pi\to\omega^\pi+i\varepsilon$. Eq. (3·13) pairs with the equation for K_α^Y, which we do not write here because it is unnecessary in the following. It is sufficient to know only the Born term of K_α^Y:

$$K_\alpha^Y(\omega)_{\text{Born}}=\mu_\alpha^Y/(\omega+\Delta m).\tag{3·15}$$

Here

$$\begin{aligned}
&\tilde{\mu}_0^\Sigma=\sqrt{6}\,(g_\Lambda f_\Lambda-g_\Sigma f_\Sigma), && \tilde{\mu}_0^\Lambda=0,\\
&\tilde{\mu}_1^\Sigma=-2g_\Sigma(f_\Sigma-f_N), && \tilde{\mu}_1^\Lambda=-\sqrt{2}\,(g_\Sigma f_\Lambda-g_\Lambda f_N);\\
&\mu_{1/2}^\Sigma=-g_\Lambda f_\Lambda-2g_\Sigma f_\Sigma+3g_\Sigma f_N, && \mu_{1/2}^\Lambda=-\sqrt{3}\,(g_\Lambda f_N-g_\Sigma f_\Lambda),\\
&\mu_{3/2}^\Sigma=g_\Lambda f_\Lambda-g_\Sigma f_\Sigma, && \mu_{3/2}^\Lambda=0.
\end{aligned}\tag{3·16}$$

The matrix $\tilde{B}^Y$ and the corresponding matrix B^Y in the Low equation for K_α^Y are

$$\tilde{B}^{\Sigma}=-\sqrt{\tfrac{2}{3}}\begin{pmatrix}-1 & 2\\ \sqrt{\tfrac{2}{3}} & \sqrt{\tfrac{2}{3}}\end{pmatrix},\qquad \tilde{B}^{\Lambda}=\sqrt{\tfrac{2}{3}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$
$$B^{\Sigma}=-\sqrt{\tfrac{3}{2}}\begin{pmatrix}-\tfrac{1}{3} & \sqrt{\tfrac{2}{3}}\\ \tfrac{1}{3} & \dfrac{1}{\sqrt{6}}\end{pmatrix},\qquad B^{\Lambda}=\sqrt{\tfrac{3}{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \tag{3·17}$$

The crossing relations are as follows:

$$\tilde{\mu}_{\alpha}^{Y}=-\sum_{\beta}\tilde{B}_{\alpha\beta}^{Y}\mu_{\beta}^{Y},\qquad \tilde{K}_{\alpha}^{Y}(z^{\pi})=\sum_{\beta}\tilde{B}_{\alpha\beta}^{Y}K_{\beta}^{Y}(-z^{\pi}-\Delta m),$$
$$\mu_{\alpha}^{Y}=-\sum_{\beta}B_{\alpha\beta}^{Y}\tilde{\mu}_{\beta}^{Y},\qquad K_{\alpha}^{Y}(z)=\sum_{\beta}B_{\alpha\beta}^{Y}\tilde{K}_{\beta}^{Y}(-z+\Delta m). \tag{3·18}$$

Like eqs. (3·6) and (3·9), eq. (3·13) is rewritten as follows: (when $\tilde{\mu}_{\alpha}^{Y}\neq 0$)

$$\tilde{K}_{\alpha}^{Y}(z^{\pi})=\frac{\tilde{\mu}_{\alpha}^{Y}}{z^{\pi}}\cdot\left[1-\tilde{\mu}_{\alpha}^{Y}\cdot\frac{z^{\pi}}{\pi}\mathfrak{B}_{\alpha}^{Y}(z^{\pi})\right]^{-1} \tag{3·19}$$

with

$$\mathfrak{B}_{\alpha}^{Y}(z^{\pi})=\int_{m_K}^{\infty}d\omega_p\,pv_p^{2}\left\{\frac{\mathfrak{K}_{\alpha}^{Y}(\omega_p)}{(\omega_p-\Delta m)^{2}(\omega_p-z^{\pi}-\Delta m)}+\frac{\mathfrak{Y}_{\alpha}^{Y}(\omega_p)}{(\omega_p+\Delta m)^{2}(\omega_p+z^{\pi}+\Delta m)}\right\}$$
$$+\int_{m_\pi}^{\infty}d\omega_p^{\pi}\frac{p_{\pi}^{3}}{m_{\pi}^{2}}v_p^{\pi 2}\left\{\frac{\mathfrak{K}_{\alpha}^{Y,\pi}(\omega_p^{\pi})}{\omega_p^{\pi 2}(\omega_p^{\pi}-z^{\pi})}+\frac{\mathfrak{Y}_{\alpha}^{Y,\pi}(\omega_p^{\pi})}{\omega_p^{\pi 2}(\omega_p^{\pi}+z^{\pi})}\right\} \tag{3·20}$$

and

$$\mathfrak{K}_{\alpha}^{Y}(\omega)=\tilde{K}_{\alpha}^{TY*}(\omega)h_{\alpha}(\omega)/|\tilde{K}_{\alpha}(\omega-\Delta m)|^{2},$$
$$\mathfrak{K}_{\alpha}^{Y\pi}(\omega^{\pi})=\sum_{Y'}h_{\alpha}^{\pi YY'*}(\omega^{\pi})\tilde{K}_{\alpha}^{Y'}(\omega^{\pi})/|\tilde{K}_{\alpha}(\omega^{\pi})|^{2},$$
$$\mathfrak{Y}_{\alpha}^{Y}(\omega)=\sum_{\beta,Y'}B_{\alpha\beta}^{Y'}h_{\beta}^{KYY'*}(\omega)K_{\beta}^{Y'}(\omega)/|\sum_{\beta}B_{\alpha\beta}^{Y}K_{\beta}^{Y}(\omega+\Delta m)|^{2},$$
$$\mathfrak{Y}_{\alpha}^{Y,\pi}(\omega^{\pi})=\sum_{\beta}B_{\alpha\beta}^{Y}K_{\beta}^{TY*}(\omega^{\pi})h_{\beta}^{\pi N}(\omega^{\pi})/|\sum_{\beta}B_{\alpha\beta}^{Y}K_{\beta}^{Y}(\omega^{\pi})|^{2}. \tag{3·21}$$

The first approximation for eq. (3·21) is given by replacing the amplitudes by the corresponding Born amplitudes. Here it should be noticed that, unlike eqs. (3·7) and (3·10), all terms in eq. (3·20) are unknown. As is seen from the forms of denominators, the terms including $\mathfrak{K}_{\alpha}^{Y}$ and $\mathfrak{K}_{\alpha}^{Y,\pi}$ are predominant. Physically speaking, these two come respectively from the initial and the final interactions in the process $\bar{K}+N\to\pi+Y$. In particular, the final interaction, which is the effect of multiple scatterings of the produced pion by hyperon, should be deliberately examined. If there occurs a resonance in this scattering state, it would not be guaranteed to substitute the Born term for the amplitude in eq. (3·21). As for the initial interaction, the multiple scattering of $K+Y\to K+Y'$, there occurs no strong resonance such as in the πN scattering, as we shall see, and it seems not very misleading to substitute the Born terms. In the next section we shall study the πY scattering.

§ 4. The pion-hyperon scattering

In the πY scattering, effects of virtual kaons are small and, for the qualitative discussion we shall make in this section, the system may be considered as approximately closed with π and Y.

First, we observe the simplest case of $f_N=f_\Sigma=f_\Lambda$, where the global symmetry holds. In this case, as Gell-Mann[2] showed, the πY scattering amplitudes are all expressed by linear combinations of the πN scattering amplitudes. It is therefore apparent that resonance does not occur in the state with $J=1/2$. Consequently the final interaction in the $\bar{K}N$ capture process, where $J=1/2$, is not extremely strong.

By the way we present here the relations between the πY and the πN scattering amplitudes concerning the isospin components. We denote the T-matrix element for $\pi+Y\to\pi+Y'$ in the state with isospin $I(=0, 1, 2)$ by $T_I^{YY'}$, and that for the πN scattering with $I(=1/2, 3/2)$ by T_I. Then the relations are

$$\begin{aligned} &T_2^{\Sigma\Sigma}=T_{3/2}, \quad T_1^{\Sigma\Sigma}=\tfrac{1}{3}(T_{3/2}+2T_{1/2}), \quad T_0^{\Sigma\Sigma}=T_{1/2}, \\ &T_1^{\Sigma\Lambda}=T_1^{\Lambda\Sigma}=\frac{\sqrt{2}}{3}(-T_{3/2}+T_{1/2}), \\ &T_1^{\Lambda\Lambda}=\tfrac{1}{3}(2T_{3/2}+T_{1/2}). \end{aligned} \tag{4·1}$$

From these relations it is readily seen in what case resonance takes place.

Next, we consider the general case where $f_\Sigma\neq f_\Lambda$, using CL's effective range approach. The complicated notation $h_\alpha^{\pi YY'}$ given in eq. (3·12) are simplified in this section as follows:

$$\begin{aligned} &h_\alpha: \quad \pi+\Sigma\to\pi+\Sigma \\ &h_\alpha^a: \quad \pi+\Sigma\to\pi+\Lambda \qquad h_\alpha^{Ta}\equiv h_\alpha^b. \\ &h_\alpha^c: \quad \pi+\Lambda\to\pi+\Lambda \end{aligned} \tag{4·2}$$

The index π of ω^π, etc., are dropped in this section. The Low equations are

$$h_\alpha(z)=\frac{\lambda_\alpha}{z}+\frac{1}{\pi}\int d\omega_p\left(\frac{p^3}{m_\pi^2}\right)v_p^2\left\{\frac{|h_\alpha|^2+\delta_{I1}|h_\alpha^a|^2}{\omega_p-z}+\sum_\beta C_{\alpha\beta}\frac{|h_\beta|^2+\delta_{I1}|h_\beta^a|^2}{\omega_p+z}\right\}, \tag{4·3}$$

$$h_\alpha^a(z)=\frac{\lambda_\alpha^a}{z}+\frac{1}{\pi}\int d\omega_p\left(\frac{p^3}{m_\pi^2}\right)v_p^2\left\{\frac{h_\alpha^{b*}h_\alpha+h_\alpha^{c*}h_\alpha^a}{\omega_p-z}+\sum_\beta C_{\alpha\beta}'\frac{h_\beta^{b*}h_\beta+h_\beta^{c*}h_\beta^a}{\omega_p+z}\right\}, \tag{4·4}$$

$$h_\alpha^b(z)=\frac{\lambda_\alpha^b}{z}+\frac{1}{\pi}\int d\omega_p\left(\frac{p^3}{m_\pi^2}\right)v_p^2\left\{\frac{h_\alpha^*h_\alpha^b+h_\alpha^{a*}h_\alpha^c}{\omega_p-z}+\sum_\beta C_{\alpha\beta}'\frac{h_\beta^*h_\beta^b+h_\beta^{a*}h_\beta^c}{\omega_p+z}\right\}, \tag{4·5}$$

$$h_\alpha^c(z)=\frac{\lambda_\alpha^c}{z}+\frac{1}{\pi}\int d\omega_p\left(\frac{p^3}{m_\pi^2}\right)v_p^2\left\{\frac{|h_\alpha^c|^2+|h_\alpha^b|^2}{\omega_p-z}+\sum_\beta C_{\alpha\beta}''\frac{|h_\beta^c|^2+|h_\beta^b|^2}{\omega_p+z}\right\}, \tag{4·6}$$

with

$$\lambda_\alpha=\frac{1}{3}\begin{pmatrix} 2f_\Sigma^2 & -10f_\Lambda^2 \\ -4f_\Sigma^2 & +2f_\Lambda^2 \\ -7f_\Sigma^2 & +f_\Lambda^2 \\ 2f_\Sigma^2 & -2f_\Lambda^2 \\ -f_\Sigma^2 & -f_\Lambda^2 \\ 2f_\Sigma^2 & +2f_\Lambda^2 \end{pmatrix} \quad \text{for} \quad (I,J)=\begin{pmatrix} 0,\ 1/2 \\ 0,\ 3/2 \\ 1,\ 1/2 \\ 1,\ 3/2 \\ 2,\ 1/2 \\ 2,\ 3/2 \end{pmatrix}$$

$$\lambda_\alpha^a=-\frac{2\sqrt{2}}{3}f_\Sigma f_\Lambda \quad \text{for } (I,J)=(1,1/2) \text{ and } (1,3/2), \tag{4·7}$$

$$\lambda_\alpha^c=\frac{2}{3}\begin{pmatrix} -2 \\ 1 \end{pmatrix}f_\Lambda^2 \quad \text{for } (I,J)=\begin{cases} 1,\ 1/2 \\ 1,\ 3/2 \end{cases},$$

$$C=\frac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}\times\frac{1}{6}\begin{pmatrix} 2 & -6 & 10 \\ -2 & 3 & 5 \\ 2 & 3 & 1 \end{pmatrix},$$

$$C'=-\frac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}, \quad C''=-C'. \tag{4·8}$$

Using the first approximation explained in Sec. 3, we have

$$h_\alpha(\omega)=\frac{\lambda_\alpha}{\omega}\cdot\left[1-\frac{\omega}{\pi}\left(\lambda_\alpha+\delta_{\alpha 1}\frac{(\lambda_\alpha^a)^2}{\lambda_\alpha}\right)\cdot\int d\omega_p\frac{(p^3/m_\pi^2)v_p^2}{\omega_p^2(\omega_p-\omega-i\epsilon)}\right.$$
$$\left.-\frac{\omega}{\pi}\sum_\beta C_{\alpha\beta}\frac{\lambda_\beta^2+\delta_{\beta 1}(\lambda_\beta^a)^2}{\lambda_\alpha}\cdot\int d\omega_p\frac{(p^3/m_\pi^2)v_p^2}{\omega_p^2(\omega_p+\omega)}\right]^{-1}. \tag{4·9}$$

The effective range approximation is to ignore the ω-dependence of the integral in eq. (4·9); we put ω in the integrand to be zero and $\int d\omega_p(p^3/m_\pi^2)v_p^2/\omega_p^2\approx(\omega_{\max}/m_\pi)-1$. Then we have the following result,

$$h_\alpha(\omega)=\frac{\lambda_\alpha}{\omega}\cdot\left[1-\frac{\omega}{m_\pi}\left\{\frac{\omega_{\max}}{m_\pi}-1\right\}f_\Lambda^2\mathcal{H}_\alpha\right]^{-1}, \tag{4·10}$$

where $\mathcal{H}_\alpha$ is a complex constant, of which the real part is given by

$$\mathrm{Re}\,\mathcal{H}_\alpha=\begin{pmatrix} 1.50\rho^2+6.00+35\ /(\rho^2-5) \\ 0.32\rho^2+0.58-0.90\ /(2\rho^2-1) \\ -0.57\rho^2-0.18-1.98\ /(7\rho^2-1) \\ 1.05\rho^2+1.69+0.84\ /(\rho^2-1) \\ -0.32\rho^2+0.11+0.42\ /(\rho^2+1) \\ 1.28\rho^2-1.50+2.98\ /(\rho^2+1) \end{pmatrix} \quad \text{for } (I,J)=\begin{pmatrix} 0,\ 1/2 \\ 0,\ 3/2 \\ 1,\ 1/2 \\ 1,\ 3/2 \\ 2,\ 1/2 \\ 2,\ 3/2 \end{pmatrix} \tag{4·11}$$

with $\rho\equiv f_\Sigma/f_\Lambda$.

Similarly we have

$$h_\alpha^a(\omega)=h_\alpha^b(\omega)=\frac{\lambda_\alpha^a}{\omega}\cdot\left[1-\frac{\omega}{m_\pi}\left\{\frac{\omega_{\max}}{m_\pi}-1\right\}\cdot f_\Lambda^2\mathcal{H}_\alpha^a\right]^{-1}, \tag{4·12}$$

$$\mathrm{Re}\,\mathcal{H}_\alpha^a=\begin{bmatrix}-1.05\rho^2-0.42\\-0.35\rho^2+0.21\end{bmatrix}\quad\text{for }(I,J)=\begin{cases}1,1/2\\1,3/2\end{cases}, \tag{4·13}$$

$$h_\alpha^c(\omega)=\frac{\lambda_\alpha^c}{\omega}\cdot\left[1-\frac{\omega}{m_\pi}\left(\frac{\omega_{\max}}{m_\pi}-1\right)f_\Lambda^2\mathcal{H}_\alpha^c\right]^{-1}, \tag{4·14}$$

$$\mathrm{Re}\,\mathcal{H}_\alpha^c=\begin{bmatrix}0.29\rho^4+0.05\rho^2-0.56\\1.80\rho^4-0.43\rho^2+0.96\end{bmatrix}\quad\text{for }(I,J)=\begin{cases}1,1/2\\1,3/2\end{cases}. \tag{4·15}$$

When ρ^2 is $1/2$ or 1, the Born term coefficient λ_α is zero for $\alpha=(0,3/2)$ and $(1,3/2)$, then this procedure cannot be applied. In the case of $\rho^2=1\cdots$ global symmetry$\cdots$, as was shown previously, the amplitudes of πY scatterings are connected with those of πN scatterings. In the case of $\rho^2=1/2$ there are no such connections as in the case of $\rho^2=1$. Then we examined, by the usual iteration method, the amplitudes for $\alpha=(0,3/2)$ and $(1,3/2)$ which were found to be small. The Born term coefficients λ_α, λ_α^a and λ_α^c and the real parts of $\mathcal{H}_\alpha$, $\mathcal{H}_\alpha^a$ and $\mathcal{H}_\alpha^c$ are plotted in Figs. 2(a), 2(b), 3(a) and 3(b) for $\rho^2=1/2\sim2$.

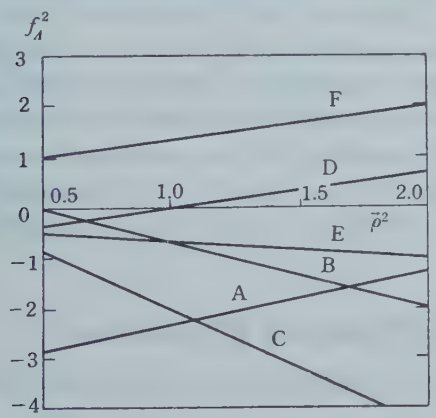

Fig. 2(a). The Born term coefficient λ_α.
A: (0, 1/2), B: (0, 3/2), C: (1, 1/2),
D: (1, 3/2), E: (2, 1/2), F: (2, 3/2).

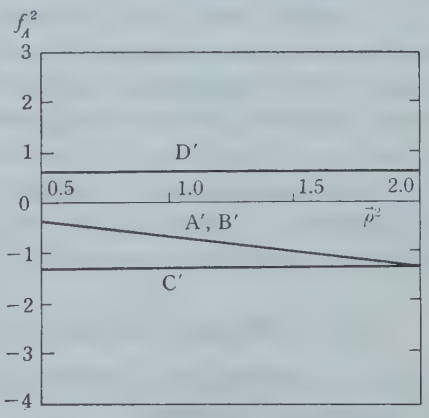

Fig. 2(b). The Born term coefficients λ_α^a and λ_α^c.
A′: (1, 1/2), B′: (1, 3/2) for $\pi+\Sigma\to\pi+\Lambda$.
C′: (1, 1/2), D′: (1, 3/2) for $\pi+\Lambda\to\pi+\Lambda$.

From Figs. 2(a) and 2(b), it is found that the scattering potentials in the states with $(I,J)=(1,3/2)$ and $(2,3/2)$ are attractive and others are all repulsive. These results concerning the signs of the scattering potentials do not depend on the sign of ρ, and are same as in the case of global symmetry.

From Figs. 3(a) and 3(b), we can see what state bears the possibility of occurrence of resonance. If the real part of $\mathcal{H}_\alpha$ is positive, the real part of $1/h_\alpha$ may become to be zero, hence resonance occurs. For example, as to $\pi+\Sigma\to\pi+\Sigma$,

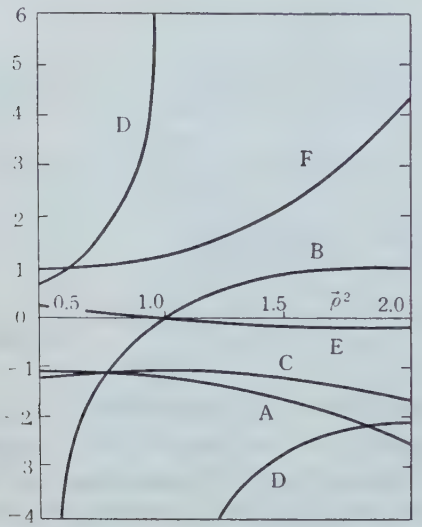

Fig. 3(a). The real part of $\mathcal{H}_\alpha$. The meaning of A, B, C, ··· is same as in Fig. 2(a).

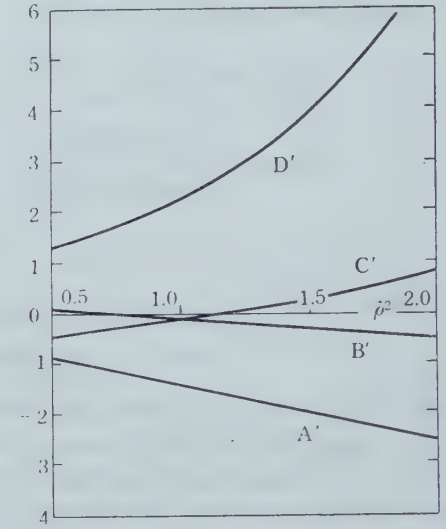

Fig. 3(b). The real partsof $\mathcal{H}^a_\alpha$ and $\mathcal{H}^c_\alpha$. The meaning of A′, B′, C′, ··· is same as in Fig. 2(b).

only two amplitudes with $(I, J)=(0, 3/2)$ and $(1, 3/2)$ have this possibility.

As a result, there is no possibility of resonances in the states with $J=1/2$, while resonances may occur only in the states with $J=3/2$. As far as we take the scalar (NKY)-interaction, the total angular momentum in the KN and the $\bar{K}N$ reactions is always $1/2$ in the static approximation, so that the possibility of resonance does not exist. Hence the KN and the $\bar{K}N$ reactions will be appropriately treated by the approximation method explained in Sec. 3. In the case of the pseudoscalar (NKY)-interaction, since the total angular momentum may take $3/2$, we should be aware of the possibility of resonance.

§ 5. The cross sections for the KN and the $\bar{K}N$ reactions

Prior to entering into detailed analysis we summarize the experimental data for convenience of reference.

(1) The cross section for the K^+p scattering[9)10)] is almost independent of energy or slightly increasing with energy and is about 15 mb, up to 200 Mev of the kaon kinetic energy in the laboratory system. The angular distribution is nearly isotropic in the center-of-mass system; the S-wave is dominant.

(2) The data on the K^+-nucleus scattering show that the cross sections for the processes $K^+ + n \to K^+ + n$ and $K^+ + n \to K^0 + p$ are both about $1/5$, or smaller, of that for K^+p scattering.[12)] This suggests that the scattering amplitude with $I=1$ is dominant.

(3) The K^+p scattering potential is likely to be repulsive.[10)]

(4) The cross section for the process $K^- + p \to K^- + p$[13)] does not appreciablly

depend on energy and is of the order of 50 mb for the kaon kinetic energy 30~100 Mev in the laboratory system. The cross section for the charge exchange process $K^-+p \to K^0+n$[13] is smaller than that for the pure elastic K^-p scattering. The cross section for the capture process $K^-+p \to \pi+Y$[13] is also not sensitive to the variation of energy, and is about 15 mb.

(5) The production ratio of the hyperons in the K^-p capture process[14] is $\Sigma^-:\Sigma^0:\Sigma^+:\Lambda=8:4:4:1$.

Among the data surveyed above, the sign of the potential is apparently incompatible with the scalar interaction model treated in this paper. The sign of the potential is same with that of the coefficient λ_α; the K^+p potential in the present model is attractive. If the experimental data become definite, only this will be sufficient to rule out the scalar interaction. However, we pass over this point and proceed with the analysis.

At first we should determine the coupling constants. Previously we have estimated the kaon coupling constants in II, observing the K^+p scattering, and have found $g_\Lambda^2+g_\Sigma^2 \lesssim 0.7$. The next question is on the ratio of g_Λ and g_Σ. Then we observe the experimental data (2), namely $T_1 \gg T_0$. As is seen from eqs. (2·4), (3·6) and (3·7), this can be explained by taking a larger g_Λ^2 than g_Σ^2. For simplicity we take $g_\Lambda^2=3g_\Sigma^2$; the Born term $\lambda_0=0$, and T_0 becomes very small compared with T_1. There still remains ambiguity on the sign $g_\Lambda=\pm\sqrt{3}\,g_\Sigma$. However,

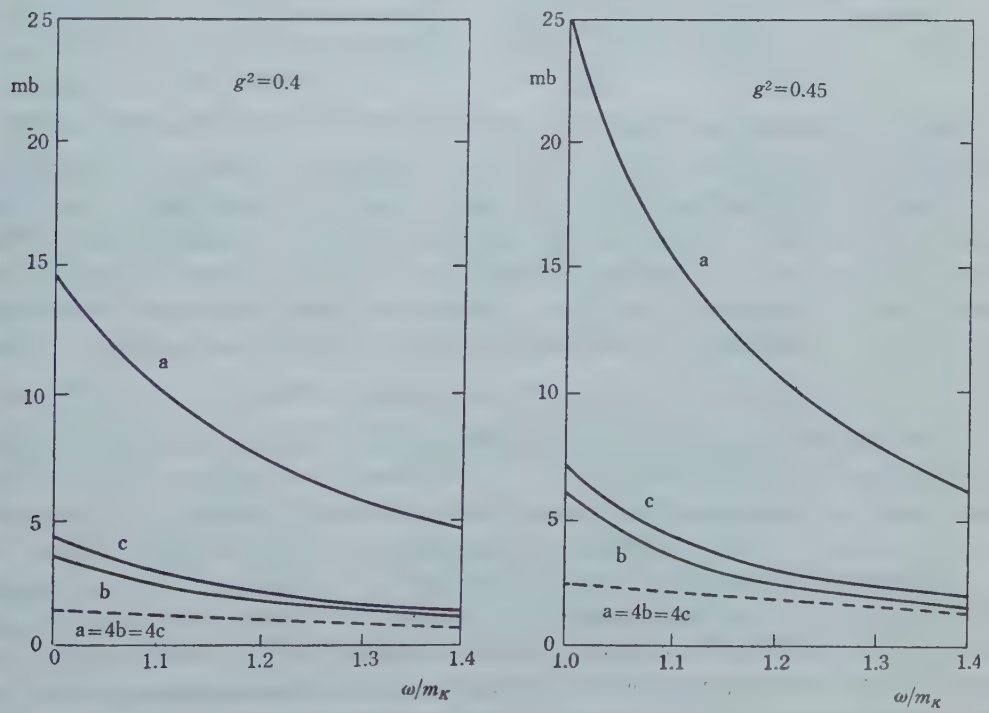

Fig. 4. The cross sections for (a) $K^++p \to K^++p$, (b) $K^++n \to K^++n$, (c) $K^++n \to K^0+p$.

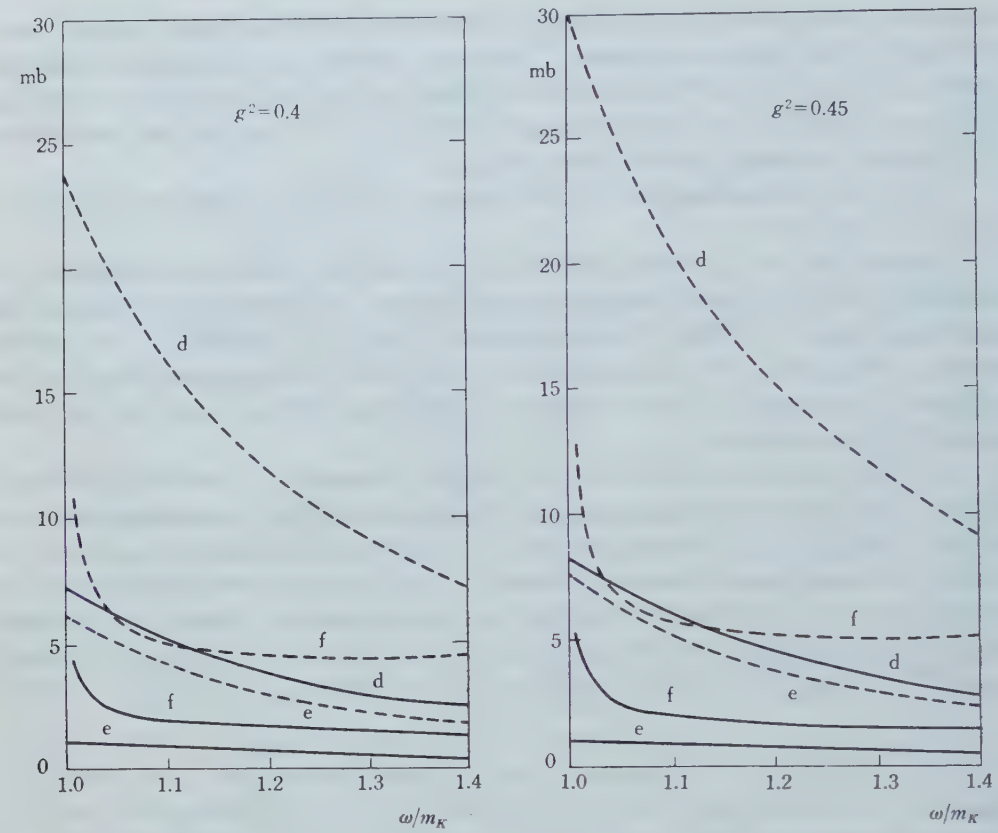

Fig. 5. The cross sections for (d) $K^-+p\to K^-+p$, (e) $K^-+p\to K^0+n$, (f) $K^-+p\to\pi+Y$.

numerical estimate shows that $g_\Lambda=+\sqrt{3}\,g_\Sigma$ is favorable concerning the data (5).

Next the pion coupling constants f_Λ and f_Σ are to be determined. Clues will be found in the data (4) and (5), but those are far more complicated than in the case of g_Λ and g_Σ. A tempting choice is, e.g., $f_\Lambda=\sqrt{3}\,f_N$, then $\tilde{\mu}_0^\Sigma$ and $\tilde{\mu}_1^\Sigma$ are larger than $\tilde{\mu}_1^\Lambda$ provided $g_\Lambda\approx\sqrt{3}\,g_\Sigma$. This choice would be favorable concerning the data (5). However, we take here $f_\Lambda=f_\Sigma=f_N=\sqrt{0.08}$, for simplicity. Thus

$$g\equiv g_\Lambda=\sqrt{3}\,g_\Sigma\,,\quad f\equiv f_\Sigma=f_\Lambda=f_N=\sqrt{0.08}\,. \tag{5·1}$$

The explicit expressions of the amplitudes with this choice of the coupling constants are shown in Appendix B. In this case the coefficients λ_0 and $\tilde{\mu}_1^\Sigma$ are zero, hence the approximation explained in Sec. 3 cannot be applied for the procesess with λ_0 and $\tilde{\mu}_1^\Sigma$. These amplitudes are obtained by the usual iteration method.

Numerical estimates were made for the values of the kaon coupling constants $g^2=0.4$ and 0.45, the cutoff energy $\omega_{\max}=2.5m_K$ ($\approx$hyperon mass) and $\omega_{\max}^\pi=6m_\pi$. In the calculations we dropped the term including the KY scattering amplitude in eq. (3·20), as its contribution is small and, moreover, it bears ambiguity concerning the interaction with Ξ. The cross sections are plotted in Figs. 4 and 5 by solid

lines. The results of the Born approximation are shown in the same Figs. by broken lines, for convenience of comparison. The branching ratio of the hyperon production in the $\bar{K}N$ capture is shown in Table 1.

Table 1 The branching ratio $\sigma_\Sigma/\sigma_\Lambda$ of the processes $K^-+p\to\pi+Y$. In the Born approximation $\sigma_\Sigma/\sigma_\Lambda=3$, independently of g^2 and energy ω.

ω \ g^2	0.4	0.45
1.0	1.73	1.69
1.1	1.71	1.67
1.2	1.71	1.69
1.3	1.76	1-76

From these Figs. and Table it is seen that the higher order effects are fairly large. For the K^+p scattering where the potential is attractive the higher order effects enhance the cross section by about five times. In other processes the higher order effects diminish the cross section by two to three times. Although the pion coupling constant f^2 $(=0.08)$ is smaller than the kaon coupling constant g^2 $(0.4\sim0.45)$, the effects of pions are as large as those of kaons. This is because the pion Compton wave length is larger and that the pion interaction contains derivative.

The results given above are those of the first approximation in the sense mentioned in Sec. 3. Now we examine the reliability of the approximation. The K^+p scattering amplitude is the most sensitive one to the higher order effects, because the potential for this is attractive. Hence we examine this by proceeding further to the second approximation.

In eq. (3·7), the terms including unknown $\mathcal{W}_1$ and $\mathcal{W}_1^\pi$ are rewritten as

$$\int_{m_K}^{\infty}d\omega_p\,\frac{p\mathcal{W}_1+(p_\pi^3/m_\pi^2)\mathcal{W}_1^\pi}{(\omega_p-\Delta m)^2(\omega_p+\omega)}+\int_{m_\pi+\Delta m}^{m_K}d\omega_p\,\frac{(p_\pi^3/m_\pi^2)\mathcal{W}_1^\pi}{(\omega_p-\Delta m)^2(\omega_p+\omega)}\,. \tag{5·2}$$

Here the integration variable has been changed as $\omega_p^\pi\to\omega_p-\Delta m$ in the last term of eq. (3·7). The form factors are all dropped for simplicity. The contribution of the second term of eq. (5·2) is negligibly small as is expected from the fact that the integrand is not singular and the interval of the integration is small. The unknown factor in the integrand of the first term is written as*

$$\mathcal{W}_1+(p_\pi^3/pm_\pi^2)\mathcal{W}_1^\pi=\frac{(K^-p\text{ total cross section})}{(K^-p\text{ pure elastic cross section})}\,. \tag{5·3}$$

The values of eq. (5·3) obtained by the first and the second approximations are plotted in Fig. 6, which shows that the result of the first approximation does not much differ from that of the second one. Then, considering that the main term in eq. (3·7) is obtained exactly, we may say that the first approximation for the K^+p scattering is reliable.

As for other amplitudes the higher order effects appear as damping factors; the amplitudes are not so sensitive to the higher order effects as the K^+p scattering. The first approximation for these is sufficient for the present semiquantitative discussion.

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* See the footnote on p. 157.

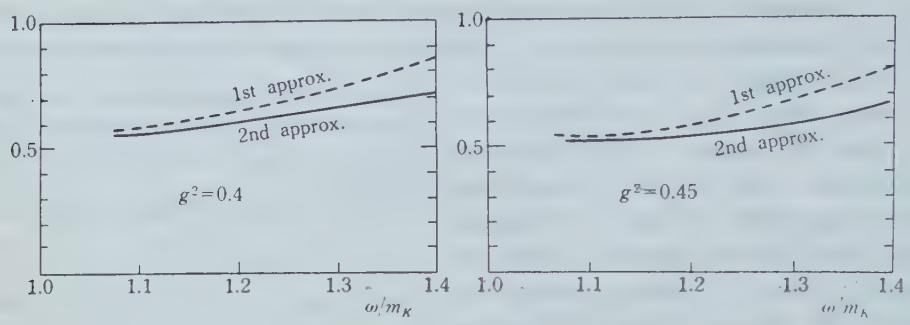

Fig. 6. The ratio: $\dfrac{\sigma(K^-p\to\bar{K}^0n)+\sigma(K^-p\to\pi Y)}{\sigma(K^-p\to K^-p)}$

Now we compare the above results with experiment. First, the coupling constants have been adjusted so as to fit the cross section for the KN scattering. The results shown in Fig. 4 roughly fit the experiment, as is expected. Next, as to the $\bar{K}N$ reactions the cross section for the capture is of an appropriate order of magnitude while the one for the scattering is much smaller than the experiment. This tendency comes from that the higher order effects diminish the amplitude for the $\bar{K}N$ scattering. If we take larger values of the kaon coupling constants, the cross section for the $\bar{K}N$ scattering becomes larger and may be able to fit the experiment. However, the cross section for the K^+p scattering then becomes much larger than the experiment, as we explained previously in II. The discrepancy is, we would like to emphasize, inherent to the scalar interaction model and does not depend on the choice of coupling constants.

As for the branching ratio of the capture processes, the result given in Table 1 is imcompatible with the experiments. Though this would be remedied by an appropriate choice of the pion coupling constants, we do not follow up this further, because we have seen that the present model is not promising as a realistic one.

§ 6. Summary and discussions

We summarize here the results of the preceding sections that are most significant.

(a) The recoil effects are not very appreciable at the center-of-mass energies of kaon below about $1.5m_K$, and the static approximation is effective at these energies.

(b) In the pion-hyperon scattering resonances occur in some states with $J=3/2$ but not in the states with $J=1/2$, provided that the ratio $(f_\Sigma/f_\Lambda)^2$ is $1/2\sim2$.

(c) The experimental cross section for the KN scattering is well reproduced with the coupling constants $g_\Lambda^2=3g_\Sigma^2=0.4\sim0.45$ and $f_\Lambda^2\approx f_\Sigma^2\approx f_N{}^2=0.08$. Then the amplitude with $I=1$ is predominant. There is little arbitrariness on the choice of the kaon coupling constants as far as they are adjusted to fit the experimental cross section for the KN scattering.

(d) The higher order effects of kaons and pions are very important and bear the following qualitative tendency: The amplitudes of the KN scattering are enhanced whereas those of the $\bar{K}N$ scattering and capture are diminished. In other words the potentials for the KN scattering are attractive whereas those for the $\bar{K}N$ reactions are repulsive.

(e) From the factors just mentioned in (d), it necessarily follows that, if we adjust the coupling constants so as to fit the experimental data on the KN scattering, the calculation gives much smaller cross section for the $\bar{K}N$ scattering than the experimental ones. This pessimistic tendency is not sensitively dependent on the choice of the pion coupling constants. Therefore we may conclude that the scalar interaction model treated in this paper cannot reproduce the experimental data, especially those of the $\bar{K}N$ scattering. Thus we have, in addition to the argument on the sign of the potential, another argument *against* the scalar interaction.

The question naturally arises then as to what the type of the kaon interaction is. In view of the sign of the K^+p scattering potential, the case of $\Gamma=i\gamma_5$ is favorable. But for other points nothing definite can be inferred in this case because of difficulties of the pair processes. Recently Barshay[4] discussed on a hypothesis that the interaction of the form $K^+K\pi^2$ is dominant at low energies. We feel, however, this is not the case because it immediately follows from this hypothesis that the KN scattering amplitude with $I=0$ and the one with $I=1$ are nearly equal, and moreover the cross section for the KN scattering and the one for the $\bar{K}N$ scattering are also nearly equal in contradiction with the experiment. The relative parity between Σ and Λ, which we have assumed to be even, might be odd. If experimental information about the reaction $\gamma+p\to K+\Sigma$ will be available, this will, combined with the one about the reaction $\gamma+p\to K+\Lambda$, provide clues to this question.

Acknowledgements

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Appendix A

In this Appendix we obtain the coefficients λ_α, etc., of the Born terms. For this purpose it is convenient to write the interaction Hamiltonian densities in the following form.

$$H_{NN\pi}=i(4\pi)^{1/2}f_N^{(0)}\cdot\sqrt{3}\,(\tfrac{1}{2}1ab|\tfrac{1}{2}c)\,\bar{N}_c\gamma_5 N_a\pi_b$$

$$H_{\Sigma\Sigma\pi}=i(4\pi)^{1/2}f_\Sigma^{(0)}\cdot\sqrt{2}\,(11ab|1c)\,\bar{\Sigma}_c\gamma_5\Sigma_a\pi_b$$

$$H_{\Sigma\Lambda\pi}=i(4\pi)^{1/2}f_{\Lambda}^{(0)}\{(010a|1b)\bar{\Sigma}_b\gamma_5\Lambda\pi_a$$
$$-\sqrt{3}\,(11ab|00)\bar{\Lambda}\gamma_5\Sigma_a\pi_b\} \qquad (A\cdot1)$$
$$H_{N\Sigma K}=(4\pi)^{1/2}g_{\Sigma}^{(0)}\{-\sqrt{3}\,(1\tfrac{1}{2}ab|\tfrac{1}{2}c)\bar{N}_a\Sigma_cK_b\}$$
$$+\sqrt{2}\,(\tfrac{1}{2}\,\tfrac{1}{2}ab\,|\,1c)\bar{\Sigma}_cN_a\bar{K}_b\}$$
$$H_{N\Lambda K}=(4\pi)^{1/2}g_{\Lambda}^{(0)}\{(0\tfrac{1}{2}0a|\tfrac{1}{2}b)\bar{N}_b\Lambda K_a$$
$$+\sqrt{2}\,(\tfrac{1}{2}\,\tfrac{1}{2}ab|00)\bar{\Lambda}N_a\bar{K}_b\}.$$

Here $(\frac{1}{2}1ab|\frac{1}{2}c)$, etc., are the Clebsh-Gordan coefficients. The letters a, b and c indicate the third components of isospins.

For simplicity we ignore the dependence of matrix elements on angular momenta. Then, contributions from various vertex parts of diagrams, except the factors $i(4\pi)^{1/2}f_N^{(0)}\gamma_5$, etc., are given in Fig. A·1. These factors are referred to as vertex factors.

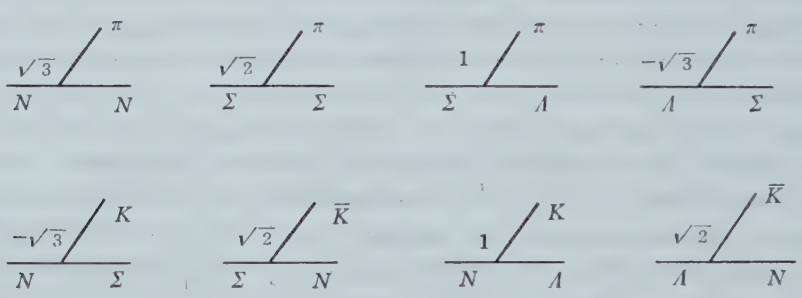


Fig. A.1

A Born term consists, in general, of the two types of diagrams, uncrossed and crossed. Let the two vertex factors of each diagram be denoted by α and β. (Fig. A·2) Then

$$\lambda_{\text{uncrossed}}=-\alpha\beta\cdot(\text{renormalized coupling constants}). \qquad (A\cdot2)$$

Fig. A.2

Of course, $\lambda=0$ if the total isospin is not equal to that of the intermediate particle. The negative sign in eq. (A·1) comes from that in eq. (3·1), etc.

In the uncrossed diagram, we assign isospins a, b, c,$\cdots$ (not the third components) to the lines as Fig. A·2. Then for the process with the total isospin I,

$$\lambda_{\text{crossed}}=\alpha\beta\cdot\sqrt{(2a+1)(2b+1)}\,W(adcb\,;eI)$$
$$\times(\text{renomalized coupling constants}). \qquad (A\cdot 3)$$

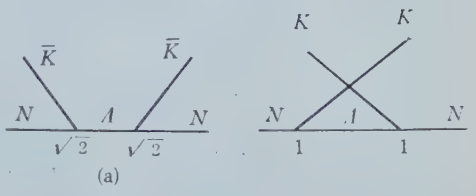
where $W(adcb\,;eI)$ is the Racah coefficient.

For illustration we give some examples.

Ex. 1 $\bar{K}+N\to\bar{K}+N$.

In this case there is only the uncrossed type of diagrams. The vertex factors are all $\sqrt{2}$. Then from the diagram of Fig. A·3(a), $\tilde{\lambda}_0^a=-2g_\Lambda^2$, $\tilde{\lambda}_1^a=0$, and from that of (b), $\tilde{\lambda}_0^b=0$, $\lambda_1^b=-2g_\Sigma^2$. Therefore

$$\tilde{\lambda}_0=-2g_\Lambda^2 \quad \text{and} \quad \tilde{\lambda}_1=-2g_\Sigma^2.$$



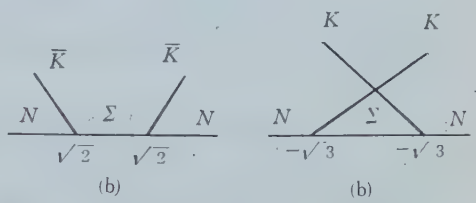


Fig. A.3 Fig. A.4

Ex. 2 $K+N\to K+N$.

In this case there is only the crossed type of diagrams. In the diagram of Fig. A·4(a), $\alpha=\beta=1$, $a=b=c=d=1/2$, $e=0$. Then eq. (A·2) gives

$$\lambda_I^a=1\cdot 2W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,;0I)=\begin{cases}-g_\Lambda^2 & \text{for } I=0\\ g_\Lambda^2 & \text{for } I=1.\end{cases}$$

In the diagram of (b), $\alpha=\beta=-\sqrt{3}$, $a=b=c=d=1/2$, $e=1$. Then

$$\lambda_I^b=3\cdot 2W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,;1I)=\begin{cases}3g_\Sigma^2 & \text{for } I=0\\ g_\Sigma^2 & \text{for } I=1.\end{cases}$$

Therefore

$$\lambda_0=-g_\Lambda^2+3g_\Sigma^3 \quad \text{and} \quad \lambda_I=g_\Lambda^2+g_\Sigma^2.$$

Appendix B

In this Appendix, we present the result of the first approximation for the amplitudes, with the coupling constants

$$g\equiv g_\Lambda=\sqrt{3}\,g_\Sigma, \qquad f\equiv f_\Lambda=f_\Sigma=f_N=\sqrt{0.08}\,.$$

$K+N\to K+N$:

$$h_0(\omega)=\frac{4}{3\pi}g^4F_{-+}(\omega), \qquad h_1(\omega)=\frac{4g^2/3}{\omega+\Delta m}\cdot\left\{R_1(\omega)-i\frac{4g^2p/3}{\omega+\Delta m}\right\}^{-1},$$

$$R_1(\omega)=1-\frac{\omega+\Delta m}{\pi}\{\tfrac{4}{3}g^2F_{+-}(\omega)+\tfrac{5}{3}g^2F_{-+}(\omega)+(4-2\sqrt{3})f^2G_{++}(\omega)\}.$$

$\bar{K}+N\to\bar{K}+N$:

$$\tilde{h}_0(\omega)=\frac{-2g^2}{\omega-\Delta m}\cdot\left\{\tilde{R}_0(\omega)+i\frac{2g^2p+(4-2\sqrt{3})f^2p_\pi^3/m_\pi^2}{\omega-\Delta m}\right\}^{-1},$$

$$\tilde{R}_0(\omega)=1+\frac{\omega-\Delta m}{\pi}\left\{2g^2F_{--}(\omega)+\tfrac{4}{3}g^2F_{+-}(\omega)+(4-2\sqrt{3})f^2G_{--}(\omega)\right\},$$

$$\tilde{h}_1(\omega)=\frac{-2g^2/3}{\omega-\Delta m}\cdot\left\{\tilde{R}_1(\omega)+i\frac{2g^2p/3+(4-2\sqrt{3})f^2p_\pi^3/m_\pi^2}{\omega-\Delta m}\right\}^{-1},$$

$$\tilde{R}_1(\omega)=1+\frac{\omega-\Delta m}{\pi}\left\{\tfrac{2}{3}g^2F_{--}(\omega)+\tfrac{4}{3}g^2F_{+-}(\omega)+(4-2\sqrt{3})f^2G_{--}(\omega)\right\},$$

$$\tilde{K}_0^{\Sigma}(\omega)=\frac{\sqrt{2}(\sqrt{3}-1)gf}{\omega-\Delta m}\cdot\left\{R_0^{\Sigma}(\omega)+i\frac{2g^2p+8f^2p_\pi^3/3m_\pi^2}{\omega-\Delta m}\right\}^{-1},$$

$$R_0^{\Sigma}(\omega)=1+\frac{\omega-\Delta m}{\pi}\left\{2g^2F_{--}(\omega)+\tfrac{8}{3}f^2G_{+-}(\omega)-\tfrac{4}{3}f^2G_{-+}(\omega)\right\},$$

$$\tilde{K}_1^{\Sigma}(\omega)=\frac{-4}{3\pi}\left(1-\frac{1}{\sqrt{3}}\right)gf^3\left\{F_{+-}(\omega)+G_{-+}(\omega)+i\pi\frac{p_\pi^3/m_\pi^2}{(\omega+\Delta m)^2}\right\},$$

$$\tilde{K}_1^{\Lambda}(\omega)=\frac{\sqrt{2}(\sqrt{3}-1)gf/\sqrt{3}}{\omega-\Delta m}\cdot\left\{R_1^{\Lambda}(\omega)+i\frac{2g^2p/3+4f^2p_\pi^3/3m_\pi^2}{\omega-\Delta m}\right\}^{-1},$$

$$R_1^{\Lambda}(\omega)=1+\frac{\omega-\Delta m}{\pi}\left\{\tfrac{2}{3}g^2F_{--}(\omega)+\tfrac{4}{3}f^2G_{+-}(\omega)-\tfrac{8}{3}f^2G_{-+}(\omega)\right\},$$

with

$$F_{\pm\pm}(\omega)\equiv\mathrm{P}\int_{m_K}d\omega_p\frac{v_p^2p}{(\omega_p\pm\Delta m)^2(\omega_p\pm\omega)},$$

and

$$G_{\pm\pm}(\omega)\equiv\mathrm{P}\int_{m_\pi}d\omega_p^\pi\frac{v_p^{\pi2}p_\pi^3/m_\pi^2}{\omega_p^{\pi2}(\omega_p^\pi\pm\Delta m\pm\omega)}.$$

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Collective Motion and Particle Excitation in Many Fermion System

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A relation between collective motion and particle excitation in many fermion system is investigated. A method to construct collective coordinates is presented. Preliminary application to nuclear structure is done. Relations to other theory of collective motion are briefly discussed.

§ 1. Introduction

The nuclear shell model has been proved to be successful in explaining the structure of nuclei, and it seems to be a good approximation to describe the behaviour of the particle in the nucleus as moving freely under an average potential. Accumulated knowledges on the nuclear level scheme, electromagnetic moments of nuclei and so on, however, show that in some nuclei the degree of freedom called collective motion plays an important role.

Bohr[1] derived the collective motion of nuclei through the quantization of the hydrodynamical equation of the nuclear matter treating the latter as a continuous medium. Afterwards many authors have investigated the method of derivation of the collective coordinates in various ways[2]. Some of them quantize the hydrodynamical equation without giving any justification to this procedure and others attempt to justify this procedure and to derive the laws of collective motions in a deductive way. Most of these, however, seem to us to have been performed without any connection with other models for nuclear structure.*)

It is a difficult problem to clarify the origin of the collective motion. Though this can be argued qualitatively, it does not seem to have been solved satisfactorily. In this paper we do not have such an ambitious intention as to give an answer to this difficult question, but here we shall formulate a theory of collective type motion by constructing the latter from the particle motion. We hope it could give some clue to make clear the situation of the above problem.

The nature of the collective motion is an interesting topic in the dynamics of many particle system. We have already known some collective coordinates:

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* Recently, Elliott[7] and Moszkowski[8] have attempted to show the connection between shell structure and the collective model.

Tomonaga[3] generalized Bloch's theory of sound wave which propagates in an assembly of Fermi particles in the case of one-dimensional motion. Brueckner, Gell-Mann and Sawada[4] derived the mode of plasma oscillation of many electron system in an instructive way. The method proposed here is a simple reformulation of their theories to the case of rotationary symmetric system composed of mutually interacting Fermi particles.

Consider an assembly of Fermi particles of finite number. These particles are assumed to be influenced by a common average potential and interact with each other through two-body interactions. If the mutual interaction is absent, each particle occupies each individual level to the top of the Fermi surface. Hereafter we shall tentatively call this state "vacuum". The excited states are described as those in which particles jump from the Fermi sea to states above the top of it. This corresponds to the pure configuration in the shell model. Due to the mutual interaction, even at the ground state of the system, these excited states are mixed into the vacuum. In an excited state there appears a hole of a state α in the Fermi sea and a particle α' above the Fermi sea. The pair state $(\alpha\alpha')$ has a definite frequency but it can be transmitted to other pair state due to the inter-particle interaction. Thus the situation looks like that of the coupled oscillators. The assembly of the coupled oscillators has normal modes of oscillation, and it may happen that the lowest frequency of the normal modes is far smaller than those of pair states. We may call this mode to be "collective" since in this modes oscillators couple with each other in phase. The state in which this mode is excited corresponds to the mixture of the configurations of various single particle excitations with appropriate amplitudes.

If the total system is invariant under the space rotation, we can attribute the definite angular momentum to each individual state α. The pair state $(\alpha\alpha')$ can have various values of angular momentum resulting from the vector addition of two angular momenta associated with the states α and α' respectively. We can, however, separate the pair state into different angular momentum states. The normal modes derived here, therefore, have definite angular momenta.

The inter-particle interaction gives rise to the interaction terms between pair and particles, those between pair and holes and those between particles, and so forth, besides those between two pairs. These terms broaden the levels of the above mentioned modes and give the higher order correction to them. In other words, these normal modes are not independent of the particle motion in a strict sense. But if we do not consider the high excitation, the amplitudes of the states in which more than two pairs are simultaneously excited would be small and these interaction terms may be safely neglected. In this approximation we can solve the problem fairly accurately. Our approximation is considered to correspond to the case of weak coupling in Bohr's theory. The case in which the interaction between pair excitation and particle motion is strong probably corresponds to Bohr's strong coupling case.

§ 2. Derivation of collective variables

For the treatment of the system composed of mutually interacting Fermi particles it is convenient to borrow the technique of quantized amplitudes because the exclusion principle is taken into account automatically. We can write the Hamiltonian in the following form:

$$\begin{aligned} H&=H_0+H_1, \\ H_0&=\sum E_\alpha\, a_\alpha^*\, a_\alpha, \\ H_1&=\tfrac{1}{2}\sum V(\alpha\alpha',\beta\beta')\, a_{\alpha'}^*\, a_{\beta'}^*\, a_\beta\, a_\alpha. \end{aligned} \tag{2·1}$$

a_α^* and a_α are respectively the operators which create and annihilate a fermion into the state α. They satisfy the commutation relations of the Fermi particles:

$$a_\alpha\, a_\beta^* + a_\beta^*\, a_\alpha = \delta_{\alpha\beta}. \tag{2·2}$$

Since the total system is assumed to be rotationally invariant, the state α can be characterized by the set of three quantum numbers $(n_\alpha\ l_\alpha\ m_\alpha)$ and the eigenfunction belonging to the state α can be written as

$$\varphi_\alpha=\frac{f_\alpha(r)}{r}\, Y_{l_\alpha}^{m_\alpha}(\Omega).$$

Here, for the sake of simplicity, we do not consider the spin variable. Its inclusion, however, causes no essential trouble. $f_\alpha(r)$ is the radial wave function and does not depend on azimuthal quantum number m_α. The interaction term $V(\alpha\alpha',\beta\beta')$ in (2.1) is

$$V(\alpha\alpha',\beta\beta')=\int\varphi_{\alpha'}^*(\boldsymbol{x}')\,\varphi_{\beta'}^*(\boldsymbol{x})\,V(\boldsymbol{x},\boldsymbol{x}')\,\varphi_\beta(\boldsymbol{x})\,\varphi_\alpha(\boldsymbol{x}')\,d\boldsymbol{x}d\boldsymbol{x}'$$

where $V(\boldsymbol{x},\boldsymbol{x}')$ is the inter-particle interaction minus the average potential.

Now we define the operator b by the relation $a_\alpha \equiv b_\alpha^*$ when the state α is the occupied state in vacuum. When the state α is an unoccupied one we use the same notation $a_\alpha \equiv a_\alpha$. b_α and b_α^* are, therefore, the annihilation and the creation operator of a hole into state α. Then H_0 can be written in terms of a's and b's apart from the constant term as

$$H_0=\sum{}^{u} E_\alpha\, a_\alpha^*\, a_\alpha-\sum{}^{o} E_\alpha\, b_\alpha^*\, b_\alpha \tag{2·3}$$

where $\sum^u$ and $\sum^o$ denote the sum over the unoccupied states and the occupied states respectively.

The pair formation operator is given by $a_{\alpha'}^*\, b_\alpha^*$ by which a particle is excited from the state α under the Fermi surface into the state α' above it. To separate a definite angular momentum state from this pair state we define the operators

$$\begin{aligned} A_{LM}^*(\alpha\alpha')&=\sum_{m_\alpha}(-1)^{m_\alpha}(l_\alpha\, l_{\alpha'}-m_\alpha\, m_{\alpha'}|LM)\, a_{\alpha'}^*\, b_\alpha^*, \\ A_{LM}(\alpha\alpha')&=\sum_{m_\alpha}(-1)^{m_\alpha}(l_\alpha\, l_{\alpha'}-m_\alpha\, m_{\alpha'}|LM)\, b_\alpha\, a_{\alpha'} \end{aligned} \tag{2·4}$$

where $(l_\alpha\ l_{\alpha'} - m_\alpha\ m_{\alpha'}|LM)$ are the Clebsch-Gordan coefficients. Because of the summation over m_α the operators $A_{LM}(\alpha\alpha')$ and $A^*_{LM}(\alpha\alpha')$ do not depend on m_α and $m_{\alpha'}$ explicitly. The indices α and α' of the operator A and A^* denote the set $(n_\alpha\ l_\alpha)$ and $(n_{\alpha'}\ l_{\alpha'})$ respectively. Hereafter we use α in this meaning.

The commutation relations of these operators are

$$\begin{aligned}[A_{LM}(\alpha\alpha'),\ & A_{L'M'}{}^*(\beta\beta')] \\ &= \delta_{LL'}\ \delta_{MM'}\ \delta_{\alpha\beta}\ \delta_{\alpha'\beta'} \\ &\quad - \sum (l_\alpha\ l_{\alpha'} - m_\alpha\ m_{\alpha'}|LM)\,(l_\alpha\ l_{\beta'} - m_\alpha\ m_{\beta'}|L'M')\,\delta_{\alpha\beta} a_{\beta'}{}^* a_{\alpha'} \\ &\quad - \sum (-1)^{m_\alpha + m_\beta} (l_\alpha\ l_{\alpha'} - m_\alpha\ m_{\alpha'}|LM)\,(l_\beta\ l_{\alpha'} - m_\beta\ m_{\alpha'}|L'M')\,\delta_{\alpha'\beta'} b_\beta{}^* b_\alpha .\end{aligned} \tag{2·5}$$

The last two terms in this expression vanish when operated on vacuum. In the low excitation energy region the contribution from these terms would be small. This is just the case we are now considering. We may drop these terms in good approximation. A and A^* satisfy, therefore, the commutation relations for Bose particles. If we call these Bose particles "excitons", A and A^* are the annihilation and the creation operators of excitons. As can be easily verified, the exciton characterized by (LM) is in the definite angular momentum state L the z-component of which is M.

We can define, in general, the operator $A^*_{LM}(\alpha\alpha')$ as $\sum(-1)^{m_\alpha}(l_\alpha\ l_{\alpha'} - m_\alpha\ m_{\alpha'}|LM)\,a_{\alpha'}{}^*\ a_\alpha$ and $A_{LM}(\alpha\alpha')$ as the conjugate one of $A^*_{LM}(\alpha\alpha')$ where the operators a_α and $a_\alpha{}^*$ are the annihilation and the creation operator of the Fermi particle of α state irrespective of whether it is under or above the Fermi sea. The commutation relation of these A's is

$$\begin{aligned}[A_{LM}(\alpha\alpha'),\ & A_{LM}{}^*(\beta\beta')] \\ &= \sum (-1)^{m_\alpha + m_\beta} (l_\alpha\ l_{\alpha'} - m_\alpha\ m_{\alpha'}|LM)\,(l_\beta\ l_{\beta'} - m_\beta\ m_{\beta'}|L'M') \\ &\qquad \times (\delta_{\alpha'\beta'}\ a_\alpha{}^*\ a_\beta - \delta_{\alpha\beta}\ a_{\beta'}{}^*\ a_{\alpha'}) .\end{aligned} \tag{2·5′}$$

The expectation value of this expression vanishes unless the states α and β are occupied or the states α' and β' are occupied, and when the states α, β, α' and β' are all occupied the expectation value of (2.5′) vanishes. In our approximation of low energy excitation only the combination (2.4) may have a physical meaning. The case in which the level α or α' is partly filled will be considered later.

Now we shall investigate the motion of the excitons. The time variation of A and A^* by free Hamiltonian H_0 are

$$\begin{aligned} \dot{A}_{LM}(\alpha\alpha') &= -i(E_{\alpha'} - E_\alpha)\, A_{LM}(\alpha\alpha') \\ \dot{A}^*_{LM}(\alpha\alpha') &= i(E_{\alpha'} - E_\alpha)\, A^*_{LM}(\alpha\alpha') . \end{aligned} \tag{2·6}*$$

Note that this equation of motion holds exactly in the sense that this does not depend on the approximation made in the derivation of boson-like commutation rules of A's. We introduce the canonical variables q and p in place of A and A^*

$$q_{LM}(\alpha,\alpha')=\sqrt{\frac{1}{2(E_{\alpha'}-E_{\alpha})}}[A_{LM}(\alpha\alpha')+(-1)^{M}A^{*}_{L-M}(\alpha\alpha')],\quad q^{*}_{LM}=(-1)^{M}q_{L-M},$$
$$p_{LM}(\alpha,\alpha')=i\sqrt{\frac{E_{\alpha'}-E_{\alpha}}{2}}[A^{*}_{LM}(\alpha\alpha')-(-1)^{M}A_{L-M}(\alpha\alpha')]. \tag{2·7}$$

q and p satisfy the canonical commutation rules in the same approximation as that for A and A^*

$$[q_{LM}(\alpha\alpha'),\ p_{L'M'}(\beta\beta')]=i\delta_{\alpha\beta}\,\delta_{\alpha'\beta'}\,\delta_{LL'}\,\delta_{MM'}. \tag{2·8}$$

The equations of motion for q and p can be easily derived. We can construct a Hamiltonian $\mathcal{H}_0$ in terms of q and p which determines the motion of excitons:

$$\begin{aligned}&\mathcal{H}_0=\sum\mathcal{H}_0(LM)\\&\mathcal{H}_0(LM)=\tfrac{1}{2}\sum\{|p_{LM}(\alpha\alpha')|^2+(E_{\alpha'}-E_{\alpha})^2|q_{LM}(\alpha\alpha')|^2\}.\end{aligned} \tag{2·9}$$

$\mathcal{H}_0$ represents the assembly of oscillators. Since $\mathcal{H}_0$ gives the same equations of motion of oscillators as those which H_0 gives, $H-\mathcal{H}_0$ contains q and p no longer.

Next we consider the terms in H_1. We pick up those terms which contain two a's and two b's. These terms represent the creation and the annihilation of two pairs and transmutation of pairs:

$$\begin{aligned}H_1=\tfrac{1}{2}[&\sum\{V(\alpha\alpha',\beta\beta')a_{\alpha'}{}^{*}\,b_{\alpha}{}^{*}\,a_{\beta'}{}^{*}\,b_{\beta}{}^{*}+V(\alpha\alpha',\beta'\beta)a_{\alpha'}{}^{*}\,b_{\alpha}{}^{*}\,b_{\beta}\,a_{\beta'}\\&+V(\alpha'\alpha,\beta\beta')b_{\alpha}\,a_{\alpha'}\,a_{\beta'}{}^{*}\,b_{\beta}{}^{*}+V(\alpha'\alpha,\beta'\beta)b_{\alpha}\,a_{\alpha'}\,b_{\beta}\,a_{\beta'}\}\\&-\sum V(\alpha\alpha',\beta\beta')(\delta_{\alpha\beta'}\,a_{\alpha'}{}^{*}\,a_{\beta}+\delta_{\alpha'\beta}\,a_{\beta'}{}^{*}\,a_{\alpha})\\&+\cdots\cdots\qquad\qquad\qquad

$$\times(-1)^{m_\alpha+m_\beta+M}(l_\alpha\, l_{\alpha'}-m_\alpha\, m_{\alpha'}|LM)\,(l_\beta\, l_{\beta'}-m_\beta\, m_{\beta'}|L-M)\,v_L(\alpha\alpha',\,\beta\beta')$$

where

$$v_L(\alpha\alpha',\,\beta\beta')=\int f_{\alpha'}{}^*(r')f_{\beta'}{}^*(r)\,V_L(r,\,r')f_\beta(r)f_\alpha(r')\,dr\,dr'.$$

Due to the factor $(l_\alpha\, l_{\alpha'},\, 00|L0)$ the exciton with even angular momentum is constructed only by the pairs with $l_\alpha+l_{\alpha'}=$even.

As we can choose the radial wave function $f_\alpha(r)$ real, $v_L(\alpha\alpha',\,\beta\beta')=v_L(\alpha\alpha',\,\beta'\beta)=v_L(\alpha'\alpha,\,\beta'\beta)$. Moreover, if $V_L(rr')$ has some symmetric properties with respect to r and r', $v_L(\alpha\alpha',\,\beta\beta')$ has additional properties such as $v_L(\alpha\alpha',\,\beta\beta')=v_L(\beta\beta',\,\alpha\alpha')$

Thus we obtain the final expression for H_c as

$$H_c=\sum u_L(\alpha\alpha',\,\beta\beta')\,q_{LM}(\alpha\alpha')\,q_{LM}{}^*(\beta\beta') \tag{2·12}$$

where

$$u_L(\alpha\alpha',\,\beta\beta')=d_L(\alpha\alpha')\,d_L(\beta\beta')\sqrt{\frac{4\pi}{2L+1}}\sqrt{(E_{\alpha'}-E_\alpha)}\;\sqrt{(E_{\beta'}-E_\beta)}\;v_L(\alpha\alpha',\,\beta\beta'),$$

$$d_L(\alpha\alpha')=\sqrt{\frac{(2l_\alpha+1)\,(2l_{\alpha'}+1)}{4\pi(2L+1)}}(l_\alpha\, l_{\alpha'}\, 00|L0).$$

Putting

$$\begin{aligned}\mathcal{H}_c&\equiv\mathcal{H}_0+H_c=\sum\mathcal{H}_c(LM)\\ \mathcal{H}_c(LM)&=\tfrac{1}{2}\sum\{|p_{LM}(\alpha\alpha')|^2+(E_{\alpha'}-E_\alpha)^2|q_{LM}(\alpha\alpha')|^2\}\\ &\quad+\sum u_L(\alpha\alpha',\,\beta\beta')\,q_{LM}(\alpha\alpha')\,q_{LM}^*(\beta\beta'),\end{aligned} \tag{2·13}$$

we have a familiar form of coupled oscillators.

We can diagonalize $\mathcal{H}_c$ according to the usual procedure. Namely, the normal modes of these coupled oscillators can be found by taking the suitable linear combinations of q. Thus we put

$$q_{LM}(\alpha\alpha')=\sum c_L^n(\alpha\alpha')\,q_{LM}^n,\quad p_{LM}^n=\sum c_L^n(\alpha\alpha')\,p_{LM}(\alpha\alpha'),$$

where $\{c_L^n(\alpha\alpha')\}$ is an orthogonal matrix: $\sum_n c_L^n(\alpha\alpha')\,c_L^{n*}(\beta\beta')=\delta_{\alpha\beta}\,\delta_{\alpha'\beta'}$, $\sum_{(\alpha\alpha')}c_L^n(\alpha\alpha')\,c_L^{m*}(\alpha\alpha')=\delta_{nm}$. Then (2.13) can be written as

$$\mathcal{H}_c=\tfrac{1}{2}\sum_n\{|p_{LM}^n|^2+E_{nL}{}^2|q_{LM}^n|^2\}$$

provided $c^n(\alpha\alpha')$'s satisfy the following equations,

$$(E_{\alpha'}-E_\alpha)^2c_L^n(\alpha\alpha')+\sum(u_L(\alpha\alpha',\,\beta\beta')+u_L(\beta\beta',\,\alpha\alpha'))\,c_L^n(\beta\beta')=E_{nL}{}^2c_L^n(\alpha\alpha').$$

and $E^2{}_{nL}$'s are the roots of the secular equation accompanying the above equations. As stated in § 1, these normal modes may be called collective coordinates.

§ 3. Applications to nuclear matter

The following description is only qualitative since we do not take into account the spin and isospin variables. The detailed investigation of actual nuclei will be done in other places. We shall see, however, that our collective coordinates would play an important role in the nuclear structure.

First we shall treat the nuclei of closed shells. Our vacuum corresponds to the state that the uppermost shell is filled. Since we shall consider only one kind of nucleons, doubly closed shell nuclei will be most suitable. If the lowest excitation energy of our collective mode is far smaller than single particle excitation energy, our mode would have a physical meaning.

We could find the energy of collective mode solving the secular equation. Here we shall, however, follow the procedure similar to that used by Sawada.[4] Suppose that the coupling terms $v_L(\alpha\alpha', \beta\beta')$ are separable: $v_L(\alpha\alpha', \beta\beta') = v_L(\alpha\alpha')\, v_L(\beta\beta')$. Though this assumption looks very strange at first sight, it may not be so unreasonable to assume that for each L value the inter-particle potential could be expressed effectively as proportional to $h_L(r) h_L(r') Y_L^0(\theta)$, where $h_L(r)$ is an appropriate function. Let an excited state of $\mathcal{H}_c$ be Ψ with the eigenvalue E. The ground state is denoted by Ψ_0, its energy being E_0:

$$\mathcal{H}_c \Psi = E\Psi, \quad \mathcal{H}_c \Psi_0 = E_0 \Psi_0 .$$

Since Ψ is an excited state it can be expressed by the linear combination of $A^*\Psi_0$, $A^*A^*\Psi_0$ and so on. The amplitude of $A^*\Psi_0$ in Ψ can be found by computing $\langle \Psi_0 p \Psi \rangle$ and $\langle \Psi_0 q \Psi \rangle$. Now

$$\begin{aligned}
(E-E_0)\langle \Psi_0 p_{LM}(\alpha\alpha')\Psi\rangle &= \langle \Psi_0 [p_{LM}(\alpha\alpha'), \mathcal{H}_c]\Psi\rangle \\
&= -i\Big(\sum (u_L(\alpha\alpha', \beta\beta') + u_L(\beta\beta', \alpha\alpha'))\langle \Psi_0 q^*_{LM}(\beta\beta')\Psi\rangle \\
&\qquad + (E_{\alpha'} - E_\alpha)^2 \langle \Psi_0 q^*_{LM}(\alpha\alpha')\Psi\rangle\Big), \\
(E-E_0)\langle \Psi_0 q^*_{LM}(\alpha\alpha')\Psi\rangle &= i\langle \Psi_0 p_{LM}(\alpha\alpha')\Psi\rangle .
\end{aligned}$$

Thus we have the following equation,

$$\begin{aligned}
&\{(\Delta E)^2 - (E_{\alpha'} - E_\alpha)^2\}\langle \Psi_0 q^*_{LM}(\alpha\alpha')\Psi\rangle \\
&\qquad = \sum (u_L, (\alpha\alpha', \beta\beta) + (u_L(\beta\beta', \alpha\alpha'))\langle \Psi_0 q^*_{LM}(\beta\beta')\Psi\rangle
\end{aligned}$$

where $\Delta E = E - E_0$. Since $u_L(\alpha\alpha', \beta\beta')$ can be written as

$$u_L(\alpha\alpha', \beta\beta') = u_L(\alpha\alpha') u_L(\beta\beta'), \quad u_L(\alpha\alpha') = d_L(\alpha\alpha') \sqrt{E_{\alpha'} - E_\alpha}\left(\frac{4\pi}{2L+1}\right)^{1/4} v_L(\alpha\alpha'),$$

we obtain a dispersion type equation for ΔE:

$$1 = \sum \frac{2u_L^2(\alpha\alpha')}{(\Delta E)^2 - (E_{\alpha'} - E_\alpha)^2} = \sum \sqrt{\frac{4\pi}{2L+1}}\, \frac{2(E_{\alpha'} - E_\alpha) d_L^2(\alpha\alpha') v_L(\alpha\alpha', \alpha\alpha')}{(\Delta E)^2 - (E_{\alpha'} - E_\alpha)^2} . \tag{3·1}$$

If $v_L<0$ this equation has a root ΔE which is positive and smaller than the smallest $E_{\alpha'}-E_\alpha$, provided that $-\sqrt{\frac{4\pi}{2L+1}}\sum\frac{2d_L^2\,v_L}{(E_{\alpha'}-E_\alpha)}<1$. When the latter condition is not satisfied, the lowest root ΔE is larger than the smallest single particle excitation energy.*

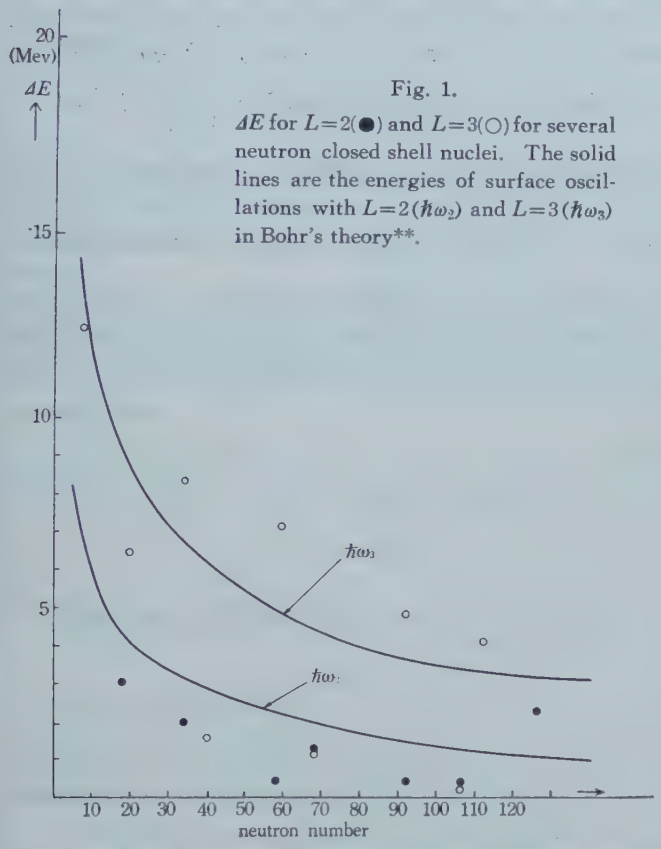

Fig. 1.

ΔE for $L=2$(●) and $L=3$(○) for several neutron closed shell nuclei. The solid lines are the energies of surface oscillations with $L=2(\hbar\omega_2)$ and $L=3(\hbar\omega_3)$ in Bohr's theory**.

As the two-body interaction we take the delta type attractive force $-4\pi g\,\delta(\boldsymbol{x},\boldsymbol{x}')$ and the oscillator type wave functions are chosen for the nuclear wave function for the sake of simplicity. The numerical constants g in the potential and ν in the wave function are taken as the same as those used by Zeldes[5] for the estimation of nuclear energy levels. As the energy levels E_α we take the values used by Marumori, Suekane and Yamamoto[6] in the estimation of surface rigidity.

ΔE with $L=2$ and 3 thus obtained are plotted for several closed shell nuclei in Fig. 1. For comparison the surface oscillation energies $\hbar\omega_2$, $\hbar\omega_3$ in Bohr's hydrodynamical model are also plotted.

The result for $L=2$ shows that if the levels which lie above the closed levels are of the same parity and are close to the closed shells, our collective mode seems to be easy to occur, as it should be. Thus the nuclei in which the $1d_{5/2}$ shell is filled and the $2s_{1/2}$ shell and the higher ones are not filled in the sense of the shell model are the favourable ones for the collective motion. For the nuclei in which

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* In this case, the dispersion relation has an imaginary root which means that our collective mode is a dissipative one. However, in this case our approximation in constructing the collective mode is not good and the terms in H_1 other than H_c are not negligible. Since the coupling term $d_L^2\,v_L$ is not small compared with $E_{\alpha'}-E_\alpha$, we should take other approximation method, e.g., strong coupling approximation.

** His curves are drawn for mass number A. We have reinterpreted them for neutron number N making a correspondence between N and A by the mass formula.

$2d$ shell is just closed or those in which $1h$ shell is just closed, our collective motion may easily occur, too. This seems to be in accordance with the experimental information.

Next we shall consider the case in which the uppermost shell is partly filled. We denote this shell as α_0. Since the energy value does not depned on magnetic quantum number, we take the vacuum state as such that the m_α states are equally occupied. Then the expectation value of commutator $[A_{LM}(\alpha\alpha'),\ A^*_{L'M'}(\alpha\alpha')]$ may be put to be $\delta_{LL'}\ \delta_{MM'}\ \dfrac{N_{\alpha_0}}{2l_{\alpha_0}+1}$ when $\alpha=\alpha_0$ and $\delta_{LL'}\ \delta_{MM'}\left(1-\dfrac{N_{\alpha_0}}{2l_{\alpha_0}+1}\right)$ when $\alpha'=\alpha_0$ where N_{α_0} is the number of nucleons in α_0 level. Thus we substitute $\sqrt{\dfrac{N_{\alpha_0}}{2l_{\alpha_0}+1}}\,q_{LM}(\alpha\alpha')$ and $\sqrt{1-\dfrac{N_{\alpha_0}}{2l_{\alpha_0}+1}}\,q_{LM}(\alpha\alpha')$ in place of $q_{LM}(\alpha\alpha')$ in the coupling term of (2.11) when $\alpha=\alpha_0$ and $\alpha'=\alpha_0$ respectively. Among the terms in H_1 other than H_c we shall only take the diagonal term with respect to the particle state of α_0 level from the interaction terms between exciton and particles in α_0 level. Then our Hamiltonian can be written as

$$\begin{aligned}&\mathcal{H}=\mathcal{H}_c+H'\\&H'=\sum U_{LM}(\alpha\alpha')\,q_{LM}(\alpha\alpha')\end{aligned}\tag{3·2}$$

where

$$U_{LM}(\alpha\alpha')=\frac{N_{\alpha_0}}{2L+1}\sqrt{\frac{(2l_\alpha+1)\,(2l_{\alpha'}+1)}{4\pi(2L+1)}}\,(l_{\alpha_0}\,l_{\alpha_0}\,0\,0|L\,0)\,(l_\alpha\,l_{\alpha'},\,0\,0|L0)$$
$$\times\sum_{m_\alpha}(-1)^{m_\alpha}(l_{\alpha_0}\,l_{\alpha_0}-m_{\alpha_0}-m_{\alpha_0}'|LM)\sqrt{\frac{E_{\alpha'}-E_\alpha}{2}}\,v_L(\alpha_0\,\alpha_0,\,\alpha\alpha').\tag{3·3}$$

To diagonalize this Hamiltonian we perform the following canonical transformation:

$$\begin{aligned}&\Psi=S\Phi,\qquad q'=S^{-1}qS,\qquad p'=S^{-1}pS\\&S=\exp i\{\sum(p_{LM}(\alpha\alpha')\,c_{LM}(\alpha\alpha')+p^*_{LM}(\alpha\alpha')\,c^*_{LM}(\alpha\alpha'))\}.\end{aligned}\tag{3·4}$$

The coefficients $c_{LM}(\alpha\alpha')$ are chosen so as to satisfy the following equations:

$$\begin{aligned}(E_{\alpha'}-E_\alpha)^2(c^*_{LM}(\alpha\alpha')+(-1)^M c_{L-M}(\alpha\alpha'))+\sum u_L(\alpha\alpha',\,\beta\beta')\,(c^*_{LM}(\beta\beta')\\+(-1)^M c_{L-M}(\beta\beta'))-U_{LM}(\alpha\alpha')=0.\end{aligned}\tag{3·5}$$

Then

$$\mathcal{H}=\mathcal{H}_c(q,\ p)+\sum W_{LM}\tag{3·6}$$

where

$$\begin{aligned}W_{LM}=\tfrac{1}{2}\{\sum(E_{\alpha'}-E_\alpha)^2|c_{LM}(\alpha\alpha')+(-1)^M c^*_{L-M}(\alpha\alpha')|^2\\+\sum u_L(\alpha\alpha',\,\beta\beta')\,(c_{LM}(\alpha\alpha')+(-1)^M c^*_{L-M}(\alpha\alpha'))\,(c^*_{LM}(\beta\beta')+(-1)^M c_{L-M}(\beta\beta')\end{aligned}$$

$$-\sum U_{LM}(\alpha\alpha')\,(c_{LM}(\alpha\alpha')+(-1)^M c^*_{L-M}(\alpha\alpha'))$$
$$=-\tfrac{1}{2}\sum U_{LM}(\alpha\alpha')\,(c_{LM}(\alpha\alpha')+(-1)^M c^*_{L-M}(\alpha\alpha')). \tag{3·7}$$

If we approximate c_{LM} as

$$c_{LM}(\alpha\alpha')+(-1)^M c^*_{L-M}(\alpha\alpha')=U_{LM}(\alpha\alpha')/((E_{\alpha'}-E_\alpha)^2+u_L(\alpha\alpha',\ \alpha\alpha'))$$

we have

$$W_{LM}=-\tfrac{1}{2}\sum\frac{U^2{}_{LM}(\alpha\alpha')}{(E_{\alpha'}-E_\alpha)^2+u_L(\alpha\alpha',\ \alpha\alpha')}. \tag{3·8}$$

The diagonalization of $\mathcal{H}_c(q,\ p)$ can be performed as was done in § 2. But, here the meaning of $q_{LM}(\alpha\alpha_0)$ or $q_{LM}(\alpha_0\alpha')$ should be understood as mentioned above. Therefore the energies of normal modes would be different from those obtained before.

Due to the effect of external particles outside the closed shell the nucleus deforms into a stable shape according to this canonical transformation. Thus the intrinsic shape of the nucleus would not be spherically symmetric even in the ground state.

Here we discuss briefly the relation of our arguments to the theory of collective motion of Bohr. When we construct a linear combination of $q_{LM}(\alpha\alpha')$

$$Q_{LM}=\frac{1}{N}\sum a_L(\alpha\alpha')\ q_{LM}(\alpha\alpha')$$

where

$$a_L(\alpha\alpha')=\int f_{\alpha'}{}^*(r)f_\alpha(r)\,r^L\,dr/R_0{}^L$$

we get the effective part of the deformation coordinates α_{LM} of Bohr's theory in our present approximation. Here N is the number of particles and R_0 is a parameter characterizing the radius of the system.

Whether these deformation coordinates α_{LM} are the good collective variables or not, or in other words, whether α_{LM} are the normal coordinates of $\mathcal{H}_c$ or not, depend on the value of the potential parameter $v_L(\alpha\alpha',\ \beta\beta')$. Generally speaking, however, since $v_L(\alpha\alpha',\ \beta\beta')$ would be small when energetically the states α and β lie far apart from α' and β' respectively, the coupling between these far lying states would be small. In the simple case where only the particles in the top level of the Fermi sea are excited to the nearest possible excited state α' Bohr's collective coordinates coincide with ours. When the deformation is small, therefore, Bohr's collective coordinates behaves very similar to ours.

For $L=2$, a formal translation into Bohr's β_μ coordinates can be done in the same way as in his theory. Namely we transform the normal coordinate q_{2M} to $\theta_i(i=1,\ 2,\ 3)$, β and γ according the following operator equations

$$\beta\cos\gamma=\sum_M q_{2M}\,D_{M0}(\theta_i),$$

$$\beta \sin \gamma = \sqrt{2} \sum q_{2M} (D_{M2}(\theta_i) + D_{M-2}(\theta_i)) ,$$

$$0 = \sum q_{2M} D_{M1}(\theta_i) .$$

$D_{MM'}(\theta_i)$ $(M, M' = -2, -1, \cdots, 2)$ are the transformation functions for the spherical harmonics of order 2. Then we have just the same Hamiltonian of rotation and vibration in Bohr's theory. Lastly we shall briefly discuss the approximation made. As can be seen from the formulation we have taken into account the coupling between pairs and particles or holes only partly in determining the collective coordinates. Moreover, our coordinates do not exactly satisfy the commutation relation of Bose particles. This fact as well as the omitted interaction should be taken into account when we want to calculate the higher order terms with respect to collective excitation. Thus our approximation may be valid when the states in which more than two pairs are excited are not mixed into low lying levels with appreciable amplitudes. In the case when the nearest possible α' level lies far higher than the other unoccupied levels, excitation of two pairs into these levels would be another possible mode of excitation which would give lower level with given L than that of our collective modes. For some nuclei near magic one this situation would occur.

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Properties of Nuclear Forces Indicated by $d(\gamma, p)n$

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Continued from the previous works some properties of nuclear forces are illustrated in the analysis of d (γ,p) n process. Especially a rather repulsive property of ^1E,* the strong tensor forces of ^3O and ^3E* are pointed out, and the detailed forms of the potentials are discussed. Comparison with nucleon-nucleon scattering is made. Also the applicability of pion theoretical potential is investigated, and it is shown that its applicability is rather good.

§ 1. Introduction

Up to today there have been done many works on the photodisintegration of the deuteron, and recently we have had many advanced studies both in experiments[1] and theoretical treatments.[2] However the experimental accuracy is not yet very good, and its theoretical explanation is not yet complete.

The investigation into the photodisintegration of deuteron is very important, because it is the simplest nuclear reaction, and also because we may get many knowledges about the nuclear forces from it[3]. In the $d(\gamma, p)n$ process there exist some effects worthy of notice, and we can account for the experiments, when we treat these effects correctly. Therefore we may reveal many properties of nuclear forces by investigating how to explain the experimental results without inconsistency. (It is to be remarked that in nucleon-nucleon scattering there exist so many ambiguities that it is not easy to reveal the properties of nuclear forces from nucleon-nucleon scattering.)

In the previous papers[3]** it has been shown that, to account for the isotropic part in the angular distribution at $E_\gamma \simeq 11$ Mev, the potential of ^1E would be rather repulsive in comparison with what has been estimated and the tensor force of ^3O would be strong at such energy.[4] As a continuation of the previous papers, here we try to investigate the properties of ^1E, ^3O and ^3E indicated by $d(\gamma, p)n$ at various energies. Also it is discussed what effects would appear at what energies. Especially the behavior of the incident γ ray is elucidated.[5] Under such investigation we also intend to research the applicability of the pion potential to $d(\gamma, p)$

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* ^1E, ^3O and ^3E represent the singlet even, the triplet odd, and the triplet even states respectively.

** Symbols used below § 2 are same with those used in the previous works[3].

n. However, this report will not include the contribution from virtual pions.[6]*

Applying the method same as the previous ones[3], we show that, the properties of ${}^1E$ and ${}^3O$ indicated previously are also effective to enhance the isotropic part at higher energies, smaller phase shift of ${}^1D_2$ is favourable at lower energies, the wave function of the deuteron influences on both of *m.d.*† and *e.d.*, and others. Also it is indicated that the applicability of the pion potential[7),8)] is rather good.

In § 2 we shall outline the general relations between $d(\gamma, p)$ and the properties of nuclear forces at various energies. In § 3 we illustrate the properties of ${}^1E$ indicated by m.d., and in § 4 about the properties of ${}^3O$ indicated by e.d. In § 5 we describe the properties of ${}^3E$ known from m.d. and e.d. Then in § 6 some discussions are made concerning the previous calculations. Finally in § 7 the conclusions are summerized.

§ 2. General situation

Here we outline the relations between $d(\gamma, p)n$ and the properties of nuclear forces without proof. The detailed elucideations are made below § 3.

At first we explain the experimental results briefly.

The angular distribution. —The most significant feature in $d(\gamma, p)n$ is that the isotropic part is large,[1] (which is near 5×10^{-29} cm^2 in 20 Mev$\lesssim E_\gamma \lesssim$300 Mev,) and this is the point which have perplexed many researchers.[9] In this paper we pay most attention to this, because we can deduce many properties of nuclear forces from it. Also in the angular distribution there exists fore-back-asymmetry, which increases slowly with the increase of energy.[1] At energies not so high it seems that there is no difficulty to account for it.[3)4)]

The total cross section (σ_T)—There exists a maximum at $E_\gamma - \epsilon^\# \simeq 2$ Mev, and it decreases with the increase of energy, and near $E_\gamma \simeq 250$ Mev there is a resonance. The accuracy of measurement of σ_T is worse than that of the angular distribution, because more detections and knowledges are required in the former.[10] The previous theories have given too small σ_T at $E_\gamma > 50$ Mev.[11] The reason will be explained afterward.

In $d(\gamma, p)n$ the transitions to be considered are *m. d.*, *e. d.*, *m. q.*, and *e. q.* However *m. q.* and *e. q.* are much smaller than the experimental σ_T (See reference 11 and Appendix.), and they become considerable in the angular distribution only at high energies. (Although the influence of *e. q.* appear in the angular distribution at lower energies, we can account for it easily.) Therefore the important

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* Such effect vanishes in the approximation that nucleon recoil is negligible. Therefore such effect is not important at lower energies. The previous calculations predict that its contribution is very small at $E_\gamma < 50$ Mev[6]. Furthermore, note that such effect cannot yet be well treated by present pure meson theory.

† *m. d.* indicates magnetic dipole and so forth.

\# ϵ is the binding energy of the deuteron.

ones are *m.d.* and *e.d.*, and we investigate these two in the following.

The transition matrices of *m. d.* and *e. d.*, are[12]

$$\langle \gamma + d | M^n + M_\pi | {}^1E \rangle \quad \text{for} \quad m.\ d.,^{\#}$$

and

$$\langle \gamma + d | M^n | {}^3O \rangle \quad \text{for} \quad e.\ d.$$

Here d means the ground state of the deuteron, and M^n, M^π represent the matrices due to nucleons and virtual pions respectively. Now we see at once that the elements which are concerned with the $d(\gamma, p)n$ process are the states, ${}^1E$, ${}^3O$, ${}^3E$, the matrices M^n, M^π, and also the incident γ ray. The reason why many researchers have failed to explain this process consistently is owing to that more than one of the above elements are incorrectly treated in their theories.

Now let us see what properties of these elements influence at what energies. Of M^n we know well. For the γ ray also we know in principle that as the energy increases, the electromagnetic wave enters in, and makes the computation very complicated. Therefore we should pay attention to it. Hence in the following we state how the properties of ${}^1E$, ${}^3O$, ${}^3E$ and M^π are reflected, and how the electromagnetic wave behaves at various energies in $d(\gamma, p)n$.

1. $E_\gamma <$ Several Mev.

Near the threshold σ_{md} is the main one, and it decreases as the energy increases. Except at the energies near the threshold σ_{ed} contributes mainly, and there exists a maximum at $E_\gamma - \epsilon \simeq 2$ Mev.

For *m.d.* whose final state is ${}^1E$, the phase shift of ${}^1S_0 (\delta_0{}^0)$* at low energies is well known, and once $\delta_0{}^0$ is given, σ_{md} can be computed easily, because the shape of potential and the radius of hard core influences little at such low energies (only if it satisfies the effective range theory). The calculated σ_{md} can well account for the isotropic part at low energies. In *e. d.* whose final state is ${}^3O$, the phase shifts of ${}^3P$ are yet small, and the cross sections can be well accounted for by adopting plane P wave.

Therefore we may say that at low energies only the property of ${}^1S_0$ is reflected, and there is no difficulty in explaining the data.

2. Several Mev $< E_\gamma \lesssim 20$ Mev.

Because the energy is yet low, the effects worthy of notice are a few. Perhaps we only need to take care of the properties of ${}^1S_0$, ${}^3P$, and f_d(the wave function of the deuteron).

The values of $\delta_0{}^0$ are not yet well confirmed at these energies. Generally speaking, the smaller $\delta_0{}^0$ is, the larger is $\sigma_{md}{}^{ip}$(the isotropic cross section given by *m. d.*). At these energies $\sigma_{md}{}^{ip}$ alone is insufficient to account for σ_{ip}(the isotropic cross section) which is given by the experiments. This fact indicates that $\sigma_{ed}{}^{ip}$ would not be small. To get larger $\sigma_{ed}{}^{ip}$, the three phase shifts of ${}^3P$ should be well separated,

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\# In *m. d.* there exists a transition whose final state is ${}^3E$, but its contribution is negligible.

* Phase shift is denoted by δ_{I^J}.

as we may understand from the formulas of σ_{ed}^{ip}, which means that the tensor force of ${}^3O$ is strong. (See § 3 and § 4.) Also the influence of f_d becomes important as the energy rises, but it is not so decisive as the properties of ${}^1E$ and ${}^3O$.

Therefore we may say that at these energies mainly the properties of ${}^1S_0$ and ${}^3P$ are reflected. Also the influence of f_d is not negligible.

3. 20 Mev $< E_\gamma <$ 50 Mev.

In treating the photodisintegration of the deuteron, it is convenient to regard these energies as one region. In this region, besides the properties of ${}^1S_0$ and ${}^3P$ described above, the influence of ${}^1D_2$ wave, ${}^3F$ wave, and f_d become important, and the effect of M^π may not be negligible. As it will be shown in § 3, the smaller δ_2^2(the phase shift of ${}^1D_2$ wave) is, the larger σ_{md}^{ip} is. As for f_d, it seems that stronger tensor force of ${}^3E$ is favourable, and further the shape of ${}^3E$ potential (in another words the form of f_d) influences. (See § 5.)

4. 50 Mev $< E_\gamma <$ 140 Mev.

When E_γ surpasses 50 Mev, the situation becomes very complicated. *M. d.* and *e. d.* are owing to $j_1(k_\gamma r)$ part of the incident ray. At $E_\gamma < 50$ Mev only the first term of $j_1(k_\gamma r)$ is effective, and we may neglect its higher order terms. However, at $E_\gamma > 50$ Mev it is no longer permissible to neglect them. Owing to the neglect of these terms, the previous authors have given too small cross sections at $E_\gamma > 50$ Mev.[11] (See Fig. 1.) Of course, in the analysis of the properties of nuclear forces from $d(\gamma, p)n$, we have to take them into account. Now at these energies, in addition to many properties of nuclear forces written in 4, the higher order terms of $j_1(k_\gamma r)$ are concerned in the process, hence the situation becomes quite complicated.

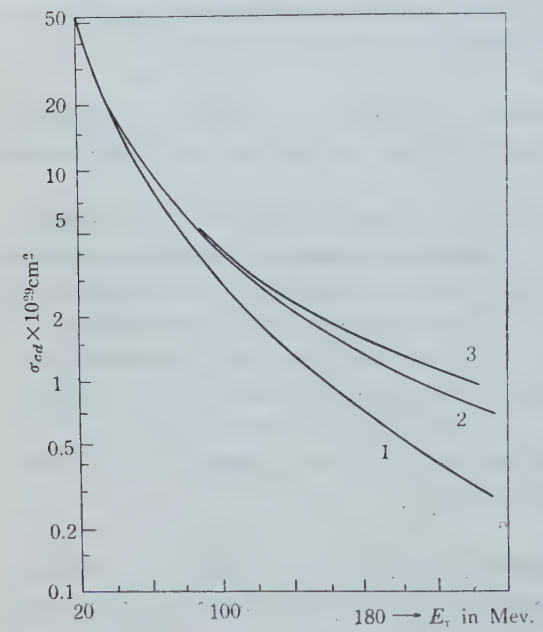

Fig. 1. σ_{ed} as the function of the energy is plotted to indicate the contribution of the higher order terms of $j_1(k_\gamma r)$. 1, 2, and 3 denote that, the first one, two, and three terms of $j_1(k_\gamma r)$ are included in the computation. Here the wave function of the deuteron without including the *D*-state is used, and the final *P* wave is taken as plane wave for simplicity.

5. $E_\gamma > 140$ Mev.

Thers is no doubt that the contribution from the mesons is the most important one at such energies. However, it is yet unknown how much contributions come from the nucleon parts. Notice that the contribution from the nucleon parts is still not negligible even at high energies. (See Fig. 1 and § 3.)

Table 1. The relations between d (γ, p) n and the properties of the nuclear forces are summerized. 0 denotes that considerable effect appears at the corresponding region. The last column indicates the effect of the incident γ ray.

Energy		several Mev	near 20 Mev		50Mev	Main Properties
^1E	$^1S_0$	0	0	0	0	Smaller δ_0^0 is better.
	$^1D_2$		)	0	0	Smaller δ_2^2 is better.
^3O	^3P		0	0	0	V_t is strong.
	^3F		)	0	0	
^3E			)	0	0	V_t is strong,.....
M^π				?	?	yet not well known
The higher order terms of $j_1(k_\gamma r)$.					0	They become important at $E_\gamma > 50$ Mev.

Recently there have been done several attempts to interpret the data from pion theory.[13] Although such attempts are necessary, there may be a doubt whether the present pion theory is powerful to be applied to such phenomenon.

We may make in a table what we have stated above. In Table 1 the relations between $d(\gamma, p)n$ and nuclear forces are summarized, from which we may see the situation at various energies.

§ 3. Properties of ^1E indicated by *M. D.*

The magnetic dipole transitions are represented by $^3S_1 \to {}^1S_0$, and $^3D_1 \to {}^1D_2$ (if we neglect the small contribution whose final state is ^3E)[12]. Therefore we may expect that the properties of $^3S_1 + {}^3D_1$, $^1S_0$ and $^1D_2$ states would be reflected through them. In § 3 we state the properties of $^1S_0$ and $^1D_2$ indicated by *m. d.* without including the contribution from virtual mesons. The contribution of virtual mesons is calculated phenomenologically by Austern et al., and theoretically by Villars et al., and it is found that its effect is small under $E_\gamma = 50$ Mev.[6] Therefore the following statement is decidedly important at lower energies, and at higher energies it is important in the sense that it explains some important effects.

Without including M^π, the formulas of *m. d.* are[12]

$$\sigma_{md} = \frac{\pi}{3}(\mu_n - \mu_p)^2 \frac{e^2}{\hbar c}\left(\frac{\hbar}{Mc}\right)^2 \frac{ME_\gamma}{\hbar^2} k(I_{00}^2 + I_{02}^2) \tag{1}$$

$$\sigma_{md}^{ip} = \frac{\pi}{3}(\mu_n - \mu_p)^2 \frac{e^2}{\hbar c}\left(\frac{\hbar}{Mc}\right)^2 \frac{ME_\gamma}{\hbar^2} k(I_{00}^2 + \sqrt{2}\, I_{00}\, I_{02} \cos(\delta_0^0 - \delta_2^2) + I_{02}^2/2,) \tag{2}$$

where $$I_{00} = \int u_0\, u_d\, dr \tag{3}$$ $$I_{02} = \int u_2\, w_d\, dr, \tag{4}$$

in the approximation neglecting the contribution from the higher order terms of $j_1(k_\tau r)$. Here the meaning of the symbols are as follows.

U_d: Radial wave function for $^3S_1$ state of the deuteron.
w_d: Radial wave function for $^3D_1$ state of the deuteron.
U_0: Radial wave function for $^1S_0$ state.
U_2: Radial wave function for $^1D_2$ state.
M: Mean mass of neutron and proton.
k: Wave number of nucleon. $(k^2=M(E_\tau-\epsilon)/h^2)$,

as shown in the previous paper.[3]

As seen from (2), to get large σ_{md}^{ip}, I_{00} must be large and I_{02} must be small. (Because $I_{00}\gg I_{02}$, I_{02}^2 is not important.) In the following we shall show that if δ_0^0 is smaller I_{00} is larger, and if δ_2^2 is smaller I_{02} is also smaller, and they are favourable to increase σ_{md}^{ip}. We discuss these in the following.

I. When the energy is lower

1. *The property of $^1S_0$ wave.* As we can see from (3), I_{00} represents the overlap of $^1S_0$ wave with U_d. In Fig. 2 the overlap of $^1S_0$ wave with U_d at $E_\tau\simeq$ 17 Mev (It corresponds to 30 Mev nucleon-nucleon scattering in energy.) is shown, from which we may see that, if the energy is not high, I_{00} is determined by the first positive part and the first negative part of $^1S_0$ wave, and it is better, if the overlap of the former with U_d is much larger than the overlap of the latter with

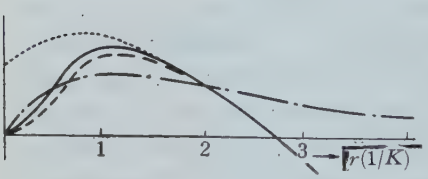

Fig. 2. The overlap of $^1S_0$ wave with $^3S_1$ wave of the deuteron at E_τ =17 Mev is shown. Here the wave function of the deuteron of $F-S$[14] is used.···represents $\sin(kr+\delta_0^0)$, ····· represents $\sin(kr+\delta_0^0)(1-e^{-\zeta r})$, with $\zeta=1.69\ \kappa$, which corresponds to zero energy value of ζ[3]. (It indicates the lower limit.)—would be near to the real $^1S_0$ wave at this energy. shows U_d of $F-S$, and δ_0^0 is put as 50°. (Yukawa type potential of ^1E without hard core gives $\delta_0^0=50°$ at this energy.)

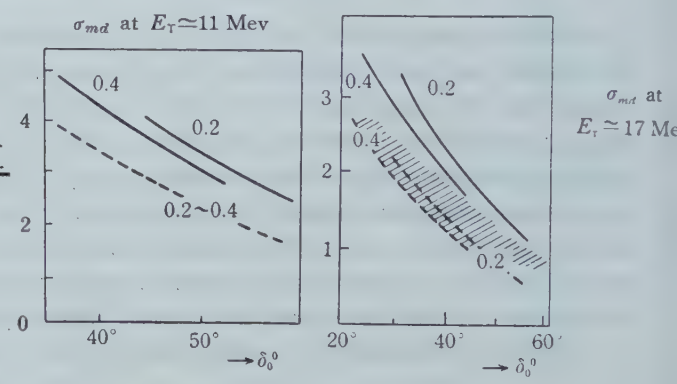

Fig. 3. The relation between δ_0^0 and σ_{md} at $E_\tau\simeq$11 Mev and 17 Mev is shown.—indicates the upper limit, and - - - indicates the lower limit. Here the wave function of the deuteron of $I-O-T-W$[8] is used. The figures, 0.2 and 0.4 denote the corresponding hard core radius of ^1E in unit of $1/K$. At $E_\tau\simeq$11Mev the lower limit for 0.2 is scarcely different from that for 0.4, and σ_{md} is near to the lower limit. At $E_\tau\simeq$ 17 Mev, perhaps, the shadowed part would represent the relation between δ_0^0 and σ_{md}. σ_{md} is in unit of 10^{-29} cm^2

U_d. In the words of nuclear force, it is favourable to give large *m.d.* cross sections, if δ_0^0 is smaller, and therefore $^1S_0$ wave is pushed out.

To make the situation more clearly we have shown σ_{md} as the function of δ_0^0 at $E_\gamma \simeq 11$ Mev and 17 Mev in Fig. 3. (The situation is same also for $\sigma_{md}{}^{ip}$.) Here f_d of $I-O-T-W$[8] is used (Its hard core radius is $0.2/K$.), the D-state is taken in, and the influence of hard core is also shown. In the computation we have taken $^1S_0$ wave in the form,[3]

$$kU_0 = \sin(kr+\delta_0^0)\,(1-e^{-\zeta(r-D_s)}), \qquad (5)$$

and have estimated the upper and the lower limits of σ_{md} by the variation of ζ. Here D_s is the hard core radius of ^1E, and ζ is dependent on D_s. In the upper limit ζ is chosen so that (5) differs from $kU_0=\sin(kr+\delta_0^0)$ by 1% at $r=1/K$. As the lower limit the zero energy value of ζ (satisfying the effective ran range theory) is adopted in (5).[3]

For $E_\gamma \simeq 11$ Mev the lower limit at $D_s=0.2/K$ is scarcely differed from that at $D_s=0.4/K$, and the real value of σ_{md} is near the lower limit, because the energy is low. In Fig. 3 notice that, generally speaking, the smaller δ_0^0 is, the larger the hard core radius is. Perhaps the shadowed part of Fig. 3 would indicate the relation between δ_0^0 and σ_{md} at $E_\gamma \simeq 17$ Mev. (At 30 Mev nucleon-nucleon scattering, $\delta_0^0 \simeq 50°$ for Yukawa type potential of ^1E without hard core, and $\delta_0^0 \simeq 40°$ for the exponential type of ^1E with $D_s \simeq 0.4/K$.)

From Fig. 3 we see that smaller δ_0^0 is favourable to increase σ_{md}, and therefore is favourable for $d(\gamma, p)n$.

2. *The property of $^1D_2$ wave.* Succeedingly we discuss the property of $^1D_2$ wave. As we can see from formulas (2) and (4), if δ_2^2 is larger, I_{02} is also larger. Speaking more in detail, $I_{02}=\int w_d\, U_2\, dr$ is given by the overlap of w_d with U_2, and if δ_2^2 is larger, U_2 wave draws in, and I_{02} is increased. Now because I_{02} interferes with I_{00} to reduce $\sigma_{md}{}^{ip}$, (See (2)) we may say that, if δ_2^2 is larger $\sigma_{md}{}^{ip}$ is smaller. Summarizing our conclusion, δ_2^2 reduces the isotropic part, therefore the smaller δ_2^2 is favourable for $d(\gamma, p)n$.

II. When the energy is higher

1. *The property of $^1S_0$ wave.* Now let us turn to the case in which the energy is higher. We have shown that, if δ_0^0 is smaller, the overlap of the first positive part of U_0 with U_d becomes much larger than the overlap of the first negative part of U_0 with U_d, and therefore *m.d.* cross sections are increased. At high energies $^1S_0$ wave enters in, and the overlap of U_0 with U_d becomes very complicated. Therefore it might be thought that the previous or the similar argument would fail. However, at high energies we can still show that, if δ_0^0 is smaller, *m.d.* cross sections would be increased, about which we shall elucidate in the following.

At high energies the electromagnetic wave draws in, and the second and the succeeding terms of $j_1(k_\tau r)$ cannot be neglected as in (3) and (4). If all terms of $j_1(k_\tau r)$ are taken in, we get

$$I_{00}=\int u_d\, u_0\left\{1-\frac{1}{3!}(k_\tau r)^2+\frac{1}{5!}(k_\tau r)^4-\cdots\right\}dr=\int u_d\, u_0\frac{\sin(k_\tau r)}{k_\tau r}dr \qquad (3)'$$

in stead of (3).

Now because the higher order terms of (3)′ make the contribution from large r unimportant as shown in Fig. 4, we may state as follows, essentially only the first poitive and the first negative parts of U_0 wave are important (especially the first positive part) in I_{00} even at high energies. Therefore we may conjecture that, even at higher energies, if δ_0^0 is smaller, *m.d.* cross sections would be larger. In Fig. 5 we have shown the relation between δ_0^0 and σ_{md} at $E_\tau=100$ Mev using U_d of $I-O-T-W$. From Fig. 5 we see that, the higher order terms of $j_1(k_\tau r)$ enhance σ_{md}, and smaller δ_0^0 is favourable to increase σ_{md}.*

Concerning this, two further arguments are necessary. The contribution of the higher order terms of $j_1(k_\tau r)$ to I_{02} is not given here, and the computation including I_{02} is not done here, because such computation is very complicated. But we might say that the higher order terms of $j_1(k_\tau r)$ would reduce I_{02} to be small, because the higher order terms of $j_1(k_\tau r)$ reduce the contribution of $^1D_2$ wave.

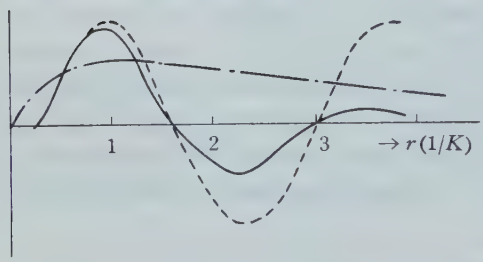

Fig. 4. The contribution of $^1S_0$ wave and $j_1(k_\tau r)$ to *m. d.* at $E_\tau\simeq 110$ Mev is shown. ··· represents U_0 wave taken as $kU_0=\sin(kr+\delta_0^0)(1-e^{-\zeta r})$, with $\zeta=4.6K$[3] and $-\delta_0^0=-30^0$.** represents U_d of $F-S$, and represents $U_0\sin(k_\tau r)/k_\tau r$. It indicates that only the part at shorter distances is effective at high energies.

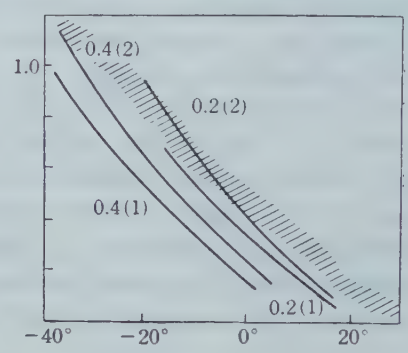

Fig. 5. σ_{md} at $E_\tau=100$ Mev is shown as the function of δ_0^0. Here U_d of $I-O-T-W$ is used, U_0 is taken as $kU_0=\sin(kr+\delta_0^0)(1-e^{-\zeta(r-D_s)})$, with $\zeta=5.8K$[3] for $D_s=0.2/K$, and $\zeta=7.7K$[3] for $D_s=0.4/K$. 0.2 and 0.4 denote the corresponding D_s in unit of $1/K$. (1) denotes the case in which only the first tsrm of $j_1(k_\tau r)$ is taken in. (2) denotes the case in which the first two terms of $j_1(k_\tau r)$ are taken in. Perhaps the shadowed part would represent the relation between δ_0^0 and σ_{md}.** σ_{md} is in unit of 10^{-29} cm^2.

* Notice that at such energies δ_0^0 is much different by the choice of potential.

** $\delta_0^0\simeq -20°$ for exponential type potential of ^1E with $D_s\simeq 0.4/K$, and $\delta_0^0\simeq 30°$ for the Yukawa type potential of ^1E without hard core at the energy corresponding to $E_\tau=100$ Mev.

Next it should be noted that, if separated contributions of M^n and M^π to *m. d.* cross sections which are represented by $(M^n)^2$ and $(M^\pi)^2$ may not be very large, the summed contribution represented by $(M^n+M^\pi)^2$ might be very large. Therefore we should not neglect the contribution of M^n to *m. d.* even at high energies.

2. *The property of* $^1D_2$ *wave.* We have further to discuss the behavior of $^1D_2$ wave and its influence to *m. d.* cross sections at high energies. At first we should notice that at high energies, δ_2^2 would be inevitably large, and I_{02} would not be very small, therefore there is a fear that $\sigma_{m/l}^{ip}$ would be reduced largely. The point is that, if δ_0^0 decreases quickly so that δ_0^0 would be negative at high energies, say above $E_\gamma \simeq 70$ or 80 Mev,[15] then large δ_2^2 is favourable owing to that the interference term $\cos(\delta_0^0-\delta^2_2)I_{00}$ I_{02} becomes small or even negative in this case. (Notice that $\delta_0^0<0$ and $\delta_2^2>0$ in this case.)

III. The ^1E potential

In the following we translate the properties of δ_0^0 and δ_2^2 indicated above into the properties of ^1E potential.

To make δ_0^0 smaller at various energies, especially at high energies, means that we should introduce hard core at short distances. This is in accordance with the requirement from *p-p* scattering which has been indicated by Jastrow.[15] In *p-p* scattering it is shown that, the isotropic part is very large, the cross section decreases rapidly up to about 150 Mev, and it is almost constant in 150 Mev$\sim$400 Mev. To account for it Jastrow has introduced hard core in ^1E and made δ_0^0 decrease rapidly so that $\delta_0^0 \simeq 0°$ at 150 Mev. It is interesting to note that the similar effect is suggested from $d(\gamma, p)n$, but from rather lower energies.

Next let us discuss about δ_2^2. We have seen that at lower energies δ_2^2 should be smaller, because δ_2^2 would reduce the isotropic part of $d(\gamma, p)n$ at least at lower energies. The smallness of δ_2^2 at lower energies means that the outer region of ^1E potential ($r>1/K$) is very shallow. This is also in accordance with the requirement from *p-p* scattering, for δ_2^2 would also reduce the isotropic part of *p-p* scattering.[8)15] It is very interesting to note that δ_2^2 reduce both the isotropic part of *p-p* scattering and $d(\gamma, p)n$.

The above two properties of ^1E potential are mutually supplementary in the sense that, if hard core radius is larger, the outer region of potential would be shallower: this is the natural consequence from the effective range theory. (On the other hand the well depth should be deeper in such case.)[15]

Now we turn to the discussion of ^1E of the pion potential. The pion theoretical potential at $r>1/K$ is almost due to one pion exchange which is represented by[7)8)]

$$V^{(1)} = -\frac{g^2}{4\pi} m_\pi c^2 e^{-Kr}/Kr \qquad (6)$$

If we adopt the Yukawa type potential at all distances, then it is required that $g^2/4\pi \simeq 2.0$[16] On the other hand in the pion theory $g^2/4\pi \simeq 0.08$. Therefore we see that the pion potential of ^1E is very shallow at $r>1/K$ compared with pheno-

menological potentials.

At $r \simeq 1/K$ two pion exchange makes the potential suddenly deep, and at short distances the pion theory indicates the existence of hard core. (But note that the argument on potential is not very effective at short distances.)

Therefore we may state that pion potential of ^1E is favourable to both p-p scattering and $d(\gamma, p)n$ at least in qualitative nature.

§ 4. Properties of ^3O indicated by E. D.

The electric dipole transitions are $^3S_1+{}^3D_2 \to {}^3P_0$, $^3P_1$, $^3P_2+{}^3F_2$, from which we may reveal the properties of the ground state of the deuteron and ^3O. Here we state the properties of ^3O indicated by *e. d.*

The formulas of σ_{ed} and σ_{ed}^{ip} are[12]

$$\sigma_{ed} = \frac{\pi}{3} \cdot \frac{e^2}{\hbar c} \cdot \frac{ME_\gamma}{\hbar^2} k \frac{1}{9} (I_{10}^2 + 3I_{11}^2 + 5I_{12}^2), \tag{7}$$

$$\sigma_{ed}^{ip} = \frac{\pi}{3} \cdot \frac{e^2}{\hbar c} \cdot \frac{ME_\gamma}{\hbar^2} k \left[\frac{1}{9} |e^{i\delta_1^0} I_{10} - e^{i\delta_1^2} I_{12}|^2 + \frac{1}{4} |e^{i\delta_1^1} I_{11} - e^{i\delta_1^2} I_{12}|^2 \right], \tag{8}$$

where

$$I_{10} = \int u_{10} (u_d - \sqrt{2} w_d)\, r dr, \qquad I_{11} = \int u_{11} \left(u_d + \frac{1}{\sqrt{2}} w_d\right) r dr, \tag{9}$$

$$I_{12} = \int u_0 \left(u_d - \frac{1}{5\sqrt{2}} w_d\right) r dr,$$

in the approximation when only the first term of $j_1(k_\gamma r)$ is included. In the following we investigate what properties of ^3O are indicated from $d(\gamma, p)n$, especially what properties of ^3O are required to increase σ_{ed}^{ip}.

I. When the energy is lower

1. *The main property of* ^3O. At first from the above formulas we see that to make σ_{ed}^{ip} larger, the three phase shifts of ^3P, δ_1^0, δ_1^1, and δ_1^2 should not be small, and should be somewhat separated from one another. As an example we have shown the cases at $E_\gamma \simeq 11$ Mev and 17 Mev in Table 2 and Table 3. In these tables some sets of phase shifts which can account for p-p scattering at the corresponding energies are given, and σ_{ed}^{ip} given by these phase shifts are shown. (Here f_d of $I-O-T-W$ is used.) From these tables we can see at once that the larger and separated δ_1^i are favourable to give large σ_{ed}^{ip}. In another words strong tensor force of ^3O enhances σ_{ed}^{ip}, and is favourable for $d(\gamma, p)n$.

2. *On the variation of* ^3O. Succeeding the above discussion, we like to point out that, if the tensor force of ^3O is made somewhat stronger, σ_{ed}^{ip} would not be increased, when the tensor force of ^3O is yet not sufficiently strong. This point may be understood immediately by adopting the set, $\delta_1^0 = \delta_1^1 = \delta_1^2 = 0^0$. The calculated

σ_{ed}^{ip} for the set $\delta_1^{\,i}=0°$ (I'') is also shown in Table 2 and Table 3. We see that there are not much differences between σ_{ed}^{ip} of I' and I'', especially for $E_\gamma \simeq 11$Mev. From these tables we may say as follows, if $\delta_1^{\,0}$, $\delta_1^{\,1}$, and $\delta_1^{\,2}$ are smaller than the dashed sets of our tables, the corresponding σ_{ed}^{ip} are not much different. Further comparing I_1' with I_1'', and I_7' with I_7'', we may say that, at low energies the detailed property of ^3O potential cannot be well known (Compare I_1' with I_1''), whereas at higher energies the influence of potential becomes more sensitive, for example at $E_\gamma \simeq 17$ Mev the sensibility of potential increases by a little (Compare I_7' with I_7'').

3. *The shape of* V_t. We have seen that $d(\gamma, p)n$ requires $|V_t| \gg |V_c|$ for the ^3O state. But the angular distribution of $d(\gamma, p)n$ suggests nothing about the sign of V_t. Nevertheless in the following we shall show that it is not impossible to discern the sign of V_t of ^3O which is physically very important. (Notice that it can give one more test for the applicability of pion potential.)

At first we see that because of $|V_t| \gg |V_c|$ we get

P) $\delta_1^{\,0}>0$, $\delta_1^{\,1}<0$, $\delta_1^{\,2}$ is small, for $V_t>0$, or *Q)* $\delta_1^{\,0}<0$, $\delta_1^{\,1}>0$, $\delta_1^{\,2}$ is small, for

Table 2. Some phase shifts which can fit 18 $p-p$ scattering data given by $B-C-J-V$[17] are shown (in unit of degree), and the corresponding cross sections of $d(\gamma, p)n$ at $E_\gamma \simeq 11$ Mev are indicated. (Here the wave function of $I-O-T-W$ is used.) In σ_{md} the lower limit is used, because σ_{md} is near the lower limit value at this energy. Also in the last column the *e. d.* cross section for the set, $\delta_1^{\,i}=0°$ are shown. σ_{ed} and σ_{md} are in unit of 10^{-29}cm^2.

Set	$\delta_0^{\,0}$	$\delta_1^{\,0}$	$\delta_1^{\,1}$	$\delta_1^{\,2}$	σ_{ed}^{ip}	σ_{ed}	σ_{md}	σ_T
I_1	45.0	23.0	−6.0	1.0	2.45	108.91	2.77	111.68
II_1	〃	−21.0	7.5	1.5	5.01	119.16	2.77	121.93
I_1'	50.5	3.3	−4.3	1.4	1.15	111.10	2.19	113.29
II_1'	〃	−4.8	3.6	1.9	1.83	118.06	2.19	120.25
I_1''		0	0	0	1.25	113.46		

Table 3. Some phase shifts which can fit 30 Mev $p-p$ scattering data given by $C-V-J$[17] are shown (in degree), and the corresponding *e. d.* cross sections of $d(\gamma, p)n$ at $E_\gamma \simeq 17$ Mev are indicated. (The wave function of $I-O-T-W$ is used.) Also in the last column the case for $\delta_1^{\,i}=0°$ is shown. (in unit of 10^{-29} cm^2.)

Set	$\delta_0^{\,0}$	$\delta_1^{\,0}$	$\delta_1^{\,1}$	$\delta_1^{\,2}$	σ_{ed}^{ip}	σ_{ed}
I_7	40	24.5	−4.5	2.0	1.98	65.22
II_7	〃	−20.5	8.5	2.0	2.53	68.96
I_7'	47	14.5	−2.5	2.0	1.33	66.54
II_7'	〃	−10.0	6.0	2.0	1.62	68.82
I_7''		0	0	0	1.03	66.75

$V_t<0$, when the energy is not high. Now from the formulas (9), $I_{1i}=\int r(U_d+x_{1i}\,w_d)U_{1i}\,dr$, and (7), $\sigma_{ed} \propto (I_{10}^{\,2}+3I_{11}^{\,2}+5I_{12}^{\,2})$, we can easily see that, 1) I_{10} gives small contribution to σ_{ed}, because I_{10} is very small (owing to the factor x_{10}), and further its weight to σ_{ed} is also small, 2) the contribution of I_{12} to σ_{ed} is not so different by the different sets of phase shifts, because $\delta_1^{\,2}$ is small, 3) the contribu-

tion of I_{11} is very important, because I_{11} is the largest one, and also it may be much different by the choice of phase shifts. Therefore it suggests that the magnitude of σ_{ed} might be determined by the property of δ_1^1, although not of δ_1^0 and δ_1^2. Next because at lower energies, $\sigma_{ed} \gg \sigma_{md}$, we conjecture that the magnitude of σ_T might be decided by δ_1^1 at lower energies.

However, the above conjecture would be right, only if I_{11} changes much by the variation of δ_1^1. Now at what energies would such requirements be satisfied? We can show that the requirements may be satisfied only at 7 Mev $< E_\gamma <$ 15 Mev, especially at 10 Mev $< E_\gamma <$ 12 Mev. The reasons are as follows. 1) At $E_\gamma <$ 7 Mev, δ_1^i are yet small. 2) At $E_\gamma <$ 15 Mev, the P wave is situated at larger distances (P wave is very small at smaller distances.), and owing to it the displacement of P wave influences I_{1i} very sensitively. (The overlap of U_{1i} with $r(U_d + x_{1i}\, w_d)$ changes much by the shift of P wave.) 3) However, at $E_\gamma >$ 15 Mev, P wave goes in, and then the overlap of P wave with $r(U_d + x_{1i}\, w_d)$ becomes very complicated, and I_{11} is already not sensitive to the variation of δ_1^1. These are the reasons why δ_1^1 is effective to decide the magnitude of σ_T only at 7 Mev $< E_\gamma <$ 15 Mev. The situation may be seen from the actual computation. In Table 2 and Table 3 we see that, at $E_\gamma \simeq 11$ Mev σ_T is somewhat different between the sets P and Q^*, but at $E_\gamma \simeq 17$ Mev σ_T is not so different between the sets P and Q. (Notice that σ_{ip} of the dashed sets are too small, and they cannot account for the angular distribution of $d(\gamma, p)n$.)

In conclusion we may say as follows. There exists some difference between σ_T of the sets P and Q (σ_T of Q is larger than that of P) owing to the different values of δ_1^1, or owing to the sign of V_t at 7 Mev $< E_\gamma <$ 15 Mev. The effect is remarkable especially at 10 Mev $< E_\gamma <$ 12 Mev. (σ_T is differed by about 10% between P and Q^*.) Therefore it is not impossible to know the sign of V_t from the measurement of σ_T of $d(\gamma, p)n$ at such energies. (The error of σ_T must be within about 10% to discern the sign of V_t.)

II. When the energy is higher

1. *On the strength of V_t.* Now we turn to the case in which the energy is high. It is interesting to see what properties of tensor force of ^3O would be indicated at higher energies. Here we have shown the calculation at $E_\gamma \simeq 87$ Mev using the phase shifts of 170 Mev p-p scattering.[17] In Table 4 are written these phase shifts which are got on the assumption that only $^1S_0$ and ^3P waves are effective and the corresponding *e. d.* cross sections are given. In the computation we have used the previous method including only the first term of $j_1(k_\gamma r)$.** Needless to say that there exist many rough points in such calculation.** However,

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* For the analysis at $E_\gamma \simeq 11$ Mev see reference 18. Notice its argument on the D-state probability.

** The discussion of $d(\gamma, p)n$ at such energy is done in § 6. Also see Appendix B.

we only wish to see the order of magnitude and to reveal the main property of ${}^3O$.*

From Table 4 we may easily see that 1) stronger tensor force enhances σ_{ed}^{ip} also at high energies, 2) and further that the potential corresponding to the sets A seems to be more favourable to increase σ_{ed}^{ip}.*

2. *On the shape of V_t.* From Table 4 we also see that, roughly speaking, the potential $V_t > 0$ gives larger σ_{ed}^{ip} and larger σ_{ed} than the potential $V_t < 0$ gives. This is contrary to the situation at lower energies. (See the calculation of I. 3.) Tables 5 and 6 indicate how the contribution of V_t to σ_{ed}^{ip} changes with the energy.

Table 4. Some sets of phase shifts (in degree) fitted to 170 Mev $p-p$ scattering data[17], and the corresponding *e. d.* cross sections at $E_\gamma \simeq 87$ Mev are shown. (Here the wave function of $I-O-T-W$ is used.) Also in the last column the *e. d.* cross sections for the set $\delta_1^1 = 0°$ are shown (in unit of 10^{-29} cm^2).

Set	δ_0^0	δ_1^0	δ_1^1	δ_1^2	σ_{ed}^{ip}	σ_{ed}
A_{45}	45	11	−15	4	0.67	4.68
A_{19}	19	22	−21	10	0.97	4.54
A_0	0	20	−23	14	1.14	4.39
B_{19}	19	−19	24	−7	0.46	3.36
C_{19}	19	51	10	3	0.70	4.56
D_{45}	45	−27	6	0	0.31	3.97
D_{19}	19	−46	15	3	0.33	3.51
E		0	0	0	0.36	4.23

Table 5. Two groups of phase shifts which can account for $p-p$ scattering data[17] are shown (to compute σ_{ed}^{ip}). E denotes proton energy in laboratory system (in degree).

	E	18Mev	30Mev	75Mev	170Mev
α	δ_0^0	52.0	47.0	41.0	19
	δ_1^0	6.5	11.0	19.5	22
	δ_1^1	−3.5	−5.0	−10.5	−21
	δ_1^2	2.0	4.0	6.5	10
β	δ_0^0	52.0	47.0	41.0	19
	δ_1^0	−4.5	−7.5	−15.0	−19
	δ_1^1	5.1	8.5	14.5	24
	δ_1^2	1.0	−1.5	−2.0	−7

Table 6. σ_{ed}^{ip} at corresponding energies of Table 5 are shown. (in unit of 10^{-29} cm^2). In the computation the wave function of $I-O-T-W$ is used, and only the first term of $j_1(k_T r)$ is included.

E_T (Mev)	11.2	17.2	39.7	87.2
α	1.16	1.36	1.43	0.97
β	2.17	2.04	1.06	0.46

* The discussion of $d(\gamma, p)n$ at such energy is done in §6. Also see Appendix B.

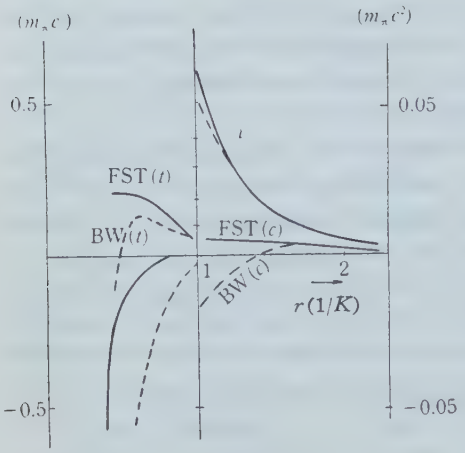

Fig. 6. The static ^3O potentials given by $B-W$[7] and $F-S-T$[7] are shown. In the figure c represents central potential, and t represents tensor potential.

It is important to note that σ_{ed} is differed by about 1/4 between these two cases. (Compare A_{19} and C_{19} with B_{19} and D_{19}.) In spite of that the difference of σ_{ed} by the shape of potential is large at $E_\gamma \simeq 87$ Mev, here we avoid to say further about what properties of V_t would be revealed from the experiments at such energy, because the contribution of M^π at such energy is unknown and our approximation used is very rough.

III. The ^3O potential

Next we turn to the discussion of ^3O potential of pion theory, and its applicability for $d(\gamma, p)n$.

1. *At lower energies.* The one pion exchange potential of ^3O is

$$V^{(1)} = \frac{1}{3} \cdot \frac{g^2}{4\pi} m_\pi c^2 \left[1 + S_{12} \left\{ 1 + \frac{3}{Kr} + \frac{3}{(Kr)^2} \right\} \right] e^{-Kr}/Kr, \qquad (10)$$

from which we can see that the tensor force is much stronger than the central one[7] (See Fig. 6.) Also it is known that the potential in $r > 1/K$ is mainly due to $V^{(1)}$. (See Fig. 6.)

Now we discuss its applicability to $d(\gamma, p)n$. We have shown that the phase shifts of ^3O are already not small, and are well separated one another at $E_\gamma \simeq 11$ Mev and 17 Mev, which indicates that V_t of ^3O is much stronger than V_c at these energies. At these energies the wave number of nucleon (k) are 0.665 K and 0.854 K respectively. From the relation between k and potential we know that the partial wave with the angular momentum l is hardly affected by the potential at $r \lesssim \sqrt{l(l+1)}/K$. Therefore at $E_\gamma \simeq 11$ Mev and 17 Mev the behavior of P wave is mainly affected by the potential at $r \gtrsim 1.2/K$ and $r \gtrsim 0.9/K$ respectively, which is the one pion exchange potential. Thus we may state that the ^3O potential of pion theory is very favourable to $d(\gamma, p)n$, because its large V_t of $V^{(1)}$ would make δ_1^i large, and separated much, which would give large σ_{ed}^{sp}.

For the sign of V_t of ^3O, we have stated that it would not be impossible to discern it by the measurement of σ_T at $E_\gamma \simeq 10$ Mev. However at present it may be yet difficult.

2. *At higher energies.* From the calculation of II we see that stronger tensor force is also favourable for $d(\gamma, p)n$ even at higher energies, and that the potential corresponding to A is more favourable than other three. (Also refer to the discus-

sion of § 6.) From Fig. 6 it seems that such properties of V_t are not inconsistent with the properties of pion potential. However, detailed discussion concerning this point cannot be done here by the following two reasons. 1) In the potential at $r<1/K$ many effects which are not yet well known contribute.[19] 2) There are many rough points in our calculation.

§ 5. Properties of ^3E indicated by *E. D.* and *M. D.*

Continuing the discussion of the properties of ^1E and ^3O, the properties of ^3E indicated by $d(\gamma, p)n$ will be elucidated here.

I. On the *D*-state of the deuteron

1. The *D*-state probability. We can easily see that the larger the *D*-state probability is, the larger σ_{ed}^{ip} is. For σ_{ed}^{ip} is owing to the relative differences of δ_1^2 to δ_1^0 and δ_1^1 on one hand, and to the relative differences of I_{12} to I_{10} and I_{11} on the other hand as may be seen from (8), and the relative differences of I_{12} to I_{10} and I_{11} are increased by the larger *D*-state probability.

To show the situation clear, we have adopted the two groups of phase shifts of Table 5 to the computation of σ_{ed}^{ip}. (They are connected smoothly for the variation of energy, and can account for p-p scattering of 18, 30 and 75 Mev.[17]) In the computation we have used two sorts of wave functions of the deuteron, one is the wave function of $I-O-T-W$ ($f_d(\pi)$) whose *D*-state probability is 6.4%, the other is that of F-S ($f_d(FS)$) whose *D*-state probability is 2.5%.

From Table 7 we see immediately that larger *D*-state probability is favourable to increase σ_{ed}^{ip} as is expected. The important point is that the influence of the *D*-state becomes more evident, as the energy is higher.

2. *The shape of D wave.* We have seen that large *D*-state probability enhances σ_{ed}^{ip}. On one hand we have known from (2) and (4) that to increase σ_{md}^{ip}, $I_{02}=\int U_2 w_d\, dr$ should be smaller, which indicates that large *D*-state probability may be unfavourable to σ_{md}^{ip}. In other words we have the trouble that larger *D*-state probability can increase σ_{ed}^{ip}, but may reduce σ_{md}^{ip}.

However, for the pion potential the situation does not seem to be so. We shall show that W_d of the pion potential (in other words the pion potential of ^3E) can well avoid such difficulty, and is favourable to enhance both σ_{ed}^{ip} and σ_{md}^{ip}. In Fig. 7 it is shown how w_d of F-S ($w_d(FS)$) and w_d of $I-O-T-W$ ($w_d(\pi)$) interact with P wave and D wave at 30 Mev nucleon-nucleon scattering. As easily seen from Fig. 7, $w_d(\pi)$ is much larger than $w_d(FS)$ at $r<2.0/K$. However, at $r>2.5/K$ both are nearly equal, or rather $w_d(\pi)$ becomes a little less than $w_d(FS)$. Because $^1D_2$ wave is an outer quantity we may presume that $I_{02}=\int U_2 w_d\, dr$ is not so different between $w_d(\pi)$ and $w_d(FS)$, although the *D*-state probability of the former (6.4%) is much larger than that of the latter (2.5%). The actual calculation gives $I_{02}=0.1643K^{-3/2}$ for $w_d(\pi)$, and $0.1713\,K^{-3/2}$ for $w_d(FS)$ at $E_T=17.2$ Mev (at the energy indicated in Fig. 7), which indicates that I_{02} by $w_d(\pi)$ is not larger than I_{02} by $w_d(FS)$ at this energy.

On the other hand P wave overlaps with w_d much more (as shown in Fig. 7), from which we may presume that $w_d(\pi)$ would give larger σ_{ed}^{ip} than $w_d(FS)$ gives, which is really verified in 1. We may state that, $w_d(\pi)$ gives larger σ_{ed}^{ip} owing to its larger D-state probability, but it does not reduce σ_{md}^{ip} much, although larger D-state probability seems to be unfavourable to σ_{md}^{ip}.

When the energy becomes high, ${}^1D_2$ wave enters in, and there may be thought $w_d(\pi)$ would make σ_{md}^{ip} reduced largely owing to its large D-state probability. But the situation is not so, for the higher order terms of $j_1(k_\tau r)$ become effective

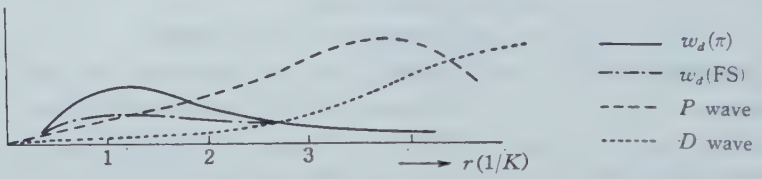

Fig. 7. P wave and D wave at 30 Mev nucleon-nucleon scattering are plotted together with $w_d(\pi)$ and $w_d(FS)$ to indicate their overlaps. The overlap of P wave with w_d gives σ_{ed}^{ip}, and the overlap of D wave with w_d reduces σ_{md}^{ip}.

Table 7. σ_{ed}^{ip} at corresponding energies of Table 5 are shown. σ_{ed}^{ip} (π) denotes σ_{ed}^{ip} given by f_d (π), and σ_{ed}^{ip} (FS) denotes σ_{ed}^{ip} given by f_d (FS). They are in unit of 10^{-29} cm^2.

E_τ(Mev)		11.2	17.2	39.7
α	σ_{ed}^{ip} (π)	1.16	1.36	1.43
	$\sigma_{ed}^{ip}(FS)$	1.02	1.02	0.73
β	σ_{ed}^{ip} (π)	2.17	2.04	1.06
	$\sigma_{ed}^{ip}(FS)$	2.08	1.73	0.62

at high energies, and if ${}^1D_2$ wave went in, its influence would be largely reduced by the higher order terms of $j_1(k_\tau r)$. (Refer to the discussion of § 3.)

II. The ${}^3E$ potential

Now we discuss the properties of ${}^3E$ of pion potential from the view-point more general than what has just now been discussed.

At first, note that the deuteron data is mainly determined by the potential at $r>1/K$.[8] The one pion exchange potential of ${}^3E$ is[7)8)]

$$V^{(1)}=-\frac{g^2}{4\pi}m_\pi c^2\left[1+S_{12}\left(1+\frac{3}{Kr}+\frac{3}{(Kr)^2}\right)\right]e^{-Kr}/Kr \qquad (11)$$

which gives the main contribution at $r>1/K$. The property of this potential is that V_t is much larger than V_c as in ${}^3O$. Further, V_t is very small at $r>2.5/K$, and V_t becomes rapidly deeper at $r\lesssim 2/K$ owing to the factor $1+3/Kr+3/(Kr)^2$ which is absent in the phenomenological potentials. Owing to such characters of $V^{(1)}$, U_d and w_d become much concentrated at $0.5/K\lesssim r\lesssim 2/K$. Such w_d would give larger

D-state probability, and makes σ_{ed}^{ip} larger, on the other hand such w_d can avoid to reduce σ_{md}^{ip} much, although there is a fear that w_d might reduce σ_{md}^{ip} largely, as we have discussed above.

Besides, U_d in this case tend to increase σ_{md}^{ip} at lower energies, for U_d which is much concentrated near $r \simeq 1/K$ makes the overlap of U_d with U_0 larger, and therefore makes σ_{md}^{ip} and σ_{md} larger. (Refer to § 3.) From Table 8 we may see the situation. Namely U_d (π) also tends to enhance $m.\ d.$ cross sections at lower energies, where effects to increase σ_{ip} are only a few. (But such an effect is not so remarkable as the effects of ${}^1S_0$ wave and ${}^3P$ wave.)

Therefore it seems that ${}^3E$ of pion potential is also favourable to $d(\gamma, p)n$.

Table 8. The lower limits (See §3) of σ_{md} at $E_\gamma \simeq 11$ Mev and 17 Mev are shown (in unit of 10^{-29} cm^2). Here the D-state is not included, and D_s is taken as $0.4/K$. (Refer to § 3) We see that U_d (π) enhances σ_{md} somewhat. Notice that σ_{md} is near the lower limit at $E_\gamma \simeq 11$ Mev, but σ_{md} is much larger than the lower limit value at $E_\gamma \simeq$ 17 Mev.

$\delta_0{}^0$	45°	50.5°	54°
$\sigma_{md}(\pi)$	2.70	2.12	1.75
$\sigma_{md}(FS)$	2.26	1.71	1.38

$\delta_0{}^0$	30°	40°	47°
$\sigma_{md}(\pi)$	1.92	1.16	0.71
$\sigma_{md}(FS)$	1.72	0.85	0.60

§ 6. Discussion

I. *$d(\gamma, p)n$ at lower energies.*

From the calculation at $E_\gamma \simeq 11$, 17 and 22 Mev, we may summarize the situation at lower energies as follows.

There is no difficulty in accounting for the total cross section and the $\sin^2\theta \cos\theta$ term in the angular distribution. To account for the isotropic part the following properties of nuclear forces are to be remarked. 1) To increase σ_{md}^{ip}, $\delta_0{}^0$ should be smaller. 2) To enhance σ_{ed}^{ip}, V_t must be strong. 3) When these two effects are added it would not be impossible to explain the isotropic part. Comparing with p-p scattering at lower energies in which only ${}^1S_0$ wave and ${}^3P$ wave are important we may say that, to account for both p-p scattering and $d(\gamma, p)n$ consistently, the above two properties of nulear forces are required. 4) Further it is remarked that the shape of f_d influences above $E_\gamma =$ several Mev. 5) Besides $\delta_2{}^2$ should be very small at lower energies, for $\delta_2{}^2$ reduces σ_{md}^{ip}.

II. *$d(\gamma, p)n$ at higher energies.*

Now let us see the situation at $E_\gamma \simeq 90$ Mev. At such energies σ_{md}^{ip} without including M^π is estimated to be 10^{-29} cm^2 at most, which may be obtained if the hard core radius of ${}^1E$ is taken to be $\simeq 0.4/K$. (See the calculation in § 3. II.)

For σ_{ed}^{ip} the situation is as follows. Neglecting the contribution from F wave and the terms above the second order of $j_1(k_\gamma r)$, we can get $(1 \sim 1.3) \times 10^{-29}$ cm^2

of σ_{ed}^{ip} at the energy in the case when V_t is strong. (See § 4. II.) The second order term of $j_1(k_\tau r)$ makes *e. d.* cross sections increase by about 30%, and the terms higher than the second one influence very little at this energy. (See Fig. 1.) Therefore if we include the higher order terms of $j_1(k_\tau r)$, σ_{ed}^{ip} would be increased to $(1.3\sim1.8)\times10^{-29}\text{cm}^2$ when V_t of ${}^3O$ is strong. Hence we see that, if ${}^1E$ potential has large hard core and V_t of ${}^3O$ is strong, we might be able to get about a half of the isotropic part ($(2\sim2.5)\times10^{-29}\,\text{cm}^2$) at $E_\tau \simeq 90$ Mev in the approximation neglecting the contribution from F wave and M^π.

The contribution of F wave at this energy is very difficult to estimate because the formulas of *e. d.* including F wave is very complicated. (See (2) and (3) of reference 20).) However, the formulas (2) indicates that F wave would act to increase σ_{ed}^{ip}.[20] Therefore we may conclude that without including M^π we would be able to get more than half of the isotropic part at $E_\tau \simeq 90$ Mev in the case when $D_s \simeq 0.4/K$ and V_t of ${}^3O$ is strong.

The important problem left is to see how much part of σ_{ip} can be got without including M^π at the energy which requires very complicated computation.

§ 7. Conclusion

We have elucidated the natures of $d(\gamma, p)n$, and shown many properties of nuclear forces indicated by $d(\gamma, p)n$, and further discussed the applicability of pion potential. In the following we summarize many conclusions in several parts.

I. *On the singlet even state.*

Firstly, the smaller δ_0^0 is required, because the smaller δ_0^0 increase *m. d.* cross sections at all energies. Secondly δ_2^2 should be small at lower energies, for the interference of ${}^1D_2$ wave with ${}^1S_0$ wave acts so as to reduce the isotropic part when the energy is not high.

The above property of δ_0^0 means the existence of hard core at short distances, and the smallness of δ_2^2 at lower energies means that the outer part of ${}^1E$ potential ($r>1/K$) is very shallow.

The above properties of ${}^1E$ are in agreement with the properties required from p-p scattering.

II. *On the triplet odd state.*

The three phase shifts of ${}^3P$ would not be small and would be separated from one another from rather lower energies up to higher energies to enhance σ_{ed}^{ip}, which indicates that the tensor force of ${}^3O$ would be strong in $0.5/K \lesssim r \lesssim 2/K$.

At lower energies *e. d.* cross sections are not very sensitive to the variation of V_t, and at higher energies the dependence of *e. d.* on the shape of ${}^3O$ potential is very sensitive (besides the dependence on the strength of V_t.).

Also it would be possible to discern the sign of V_t, if we can measure the total cross section of $d(\gamma, p)n$ within about 10% error at $E \simeq 10$ Mev.

III. *On the triplet even state.*

At low energies the influence of the wave function of deuteron is little. When the energy rises somewhat, the D-state probability and the shape of the wave function becomes important. The experiments indicate that σ_{et}^{ip} would be large. Generally speaking, large D-state probability is favourable to give large σ_{et}^{ip}. The influence becomes more remarkable as the energy becomes higher.

IV. *On the natures of* $d(\gamma, p)n$.

1) On the Nuclear Potentials. The large isotropic cross section must be accounted for by $\sigma_{md}^{ip}+\sigma_{et}^{ip}$. To get large σ_{mt}^{ip}, δ_0^0 and δ_2^2 should be smaller. To get large σ_{et}^{ip}, firstly the phase shifts of ^3P should be larger and separated from one another, or in other words the tensor force of ^3O must be strong. Besides, large D-state probability of the deuteron is favourable to enhance σ_{et}^{ip}. Furthermore, the detailed form of the wave function of the deuteron, in other words the form of ^3E potential influences.

2) On the Behavior of $j_1(k_\gamma r)$. σ_{md} and σ_{et} are due to the $j_1(k_\gamma r)$ part of the incident γ ray. At $E_\gamma < 50$ Mev only the first term of $j_1(k_\gamma r)$ is effective. But at $E_\gamma > 50$ Mev the contribution of the higher order terms of $j_1(k_\gamma r)$ becomes important as the energy rises. The higher order terms of $j_1(k_\gamma r)$ enhance the magnitude of σ_{md} and σ_{et} at high energies (at $E_\gamma > 50$ Mev).

3) Comparison with Experiment. At lower energies (below $\simeq 20$ Mev) it would not be impossible to account for the isotropic part, and near $E_\gamma \simeq 100$ Mev we would be able to account for more than a half of the isotropic part without including the contribution from virtual pions, in the case in which ^1E potential has large hard core (with $D_s \simeq 0.4/K$) and V_t of ^3O is strong.

V. *The relation with p-p scattering.*

To accout for both p-p scattering and $d(\gamma, p)n$ without inconsistency, δ_0^0 should be smaller, and the phase shifts of ^3P should be larger. Furthermore their indications on the shape of ^1E potential agree.

VI. *The applicability of pion potential.*

1) On ^1E. The pion potential of ^1E is supported in the following two poimts. It gives smaller δ_0^0, which is favourable to account for the large isotropic parts in p-p scattering and $d(\gamma, p)n$. Its outer part $(r > 1/K)$ is very shallow, therefore it gives small δ_2^2 at energies not high. This is also favourable for both p-p scattering and $d(\gamma, p)n$, because δ_2^2 reduce the isotropic parts of both.

2) On ^3O. The strong tensor force of pion potential gives large σ_{et}^{ip}, and is favourable for $d(\gamma, p)n$.

3) On ^3E. The strong tensor force of pion potential gives large D-state probability to the deuteron, which is also favourable for $d(\gamma, p)n$. Its shape of potential is favourable to give large σ_{md}^{ip} and σ_{ed}^{ip}.

Appendix A. On σ_{eq}.

At first we consider the case in which the the tensor forces and the higher order terms of $j_2(k_\tau r)$ are neglected. In such a case,

$$\sigma_{eq} \fallingdotseq \frac{\pi}{240} \cdot \frac{e^2}{\hbar c} \cdot \frac{ME_\tau}{\hbar^2} \left(\frac{E_\tau}{c\hbar} \right)^2 k \, I_2^2$$

where

$$I_2 = \int U_2 r^2 U_{t'} \, dr.$$

Because U_2 is an outer quantity, we may take it as

$$U_2 = r[\cos \delta_2 \, j_2(k \, r) - \sin \delta_2 \, n_2(k \, r)]$$

except at very high energies. Then if we take $U_{t'}$ in the form, $Ae^{-\mu\lambda} - Be^{-\nu\tau}$ which is usually used, we get

$$I_2 = 8Ak^2 \cos \delta_2 \left[\frac{1}{(k^2+\mu^2)^3} - \frac{1}{(k^2+\nu^2)^3} \right]$$
$$+ \frac{B}{k^2} \sin \delta_2 \left[\frac{\mu(3\mu^4+10k^2\mu^2+15k^4)}{(k^2+\mu^2)^3} - \frac{\nu(3\nu^4+10k^2\nu^2+15k^4)}{(k^2+\nu^2)^3} \right]$$

from which we can prove that what value of δ_2 we may take σ_{eq} is always much smaller than the experimental σ_T at all energies. Therefore its contribution to σ_T is negligible.

If we include the D-state of the deuteron, of course the magnitude of σ_{eq} cannot be effectively changed. Because σ_{eq} is always small for any δ_2, it means that the tensor force in the final state does not change σ_{eq} considerably. The contribution of the higher order terms of $j_2(k_\tau r)$ which are not taken in here is not effective to vary the order of σ_{eq}, because such contribution is smaller than the contribution of the higher order terms of $j_1(k_\tau r)$ to σ_{et} which is shown in Fig. 1. Furthermore, if U_2 is much different from the approximate form shown above, the magnitude of σ_{eq} would of course not be considerably changed. Therefore σ_{eq} hardly contributes to σ_T. It contributes to the angular distribution only.

Similarly we can prove that σ_{mq} is much smaller than σ_T at various energies.

Appendix B. On the Approximation of *E. D.* at $E_\tau \simeq 87$ Mev.

We need to examine if our calculation at $E_\tau \simeq 87$ Mev is reliable, because we have used a rough approximation in § 4. In the following we examine the error when U_{1t} is put as

$$r\left[j_1(k\,r)\cos \delta_1^t - n_1(k\,r)\sin \delta_1^t \right] \qquad \text{(B1)}$$

in the integration

$$I_{1i}=\int rU_{1i}(U_d+x_{1i}w_d)\,dr\,. \tag{B2}$$

Notice that the method of proof used in Appendix of reference 3) is not applicable for our case now because P wave enters inward at $E_\gamma\simeq 87$ Mev.

At first, note that although U_{1i} tends to ∞ as r approaches 0, the integrand of (B2) tends to 0 as r approaches D_t, for $U_{1i}\to 1/r$, $U_d\to 0$, and $w_d\to 0$, when $r\to D_t$, where D_t is the hard core radius of $f_d(\pi)$. (In the actual computation we have integrated from $r=D_t=0.2/K$ to $r=\infty$, because the hard core radius of $f_d(\pi)$ is $0.2/K$.)

At $r>2/K$ U_{1i} is nearly equal to (B1), and at $2/K>r>1/K$, U_{1i} is not so deviated from (B1), for at $r=2/K$ $V_t<1$ Mev, and at $r=1/K$ $V_t<10$ Mev in Fig. 6. (The potential at $r<1/K$ is much deeper than that at $r>1/K$.) The important region is $r<1/K$ in where U_{1i} is much different from (B1).

Let us denote the contribution from $r<1/K$ to I_{1i} by I_{1i}'. When $\delta_1^{\,i}=0$ (i. e. when there exists no force) U_{1i} is exactly equal to (B1), i. e. $U_{1i}=r\ j_1(kr)$. We put $I_{1i}'=1$ (in unit of $k^{-5/2}$) for this case. Next we discuss the cases when $\delta_1^{\,i}>0$ and $\delta_1^{\,i}<0$.

1) When $\delta_1^{\,i}>0$.

$$\text{When}\quad \delta_1^{\,i}=20^\circ,\quad \text{we get}\quad I_{1i}'\simeq 1.5.$$

$$\text{When}\quad \delta_1^{\,i}=40^\circ,\quad \text{we get}\quad I_{1i}'\simeq 2.0.$$

From these we see, I_{1i}' is somewhat larger than it would be, but the error would not be very large at least for $\delta_1^{\,i}\lesssim 20^\circ$, for I_{1i}' increases as $\delta_1^{\,i}$ increases. (We may see the situation by plotting the shift of P wave.)

2) When $\delta_1^{\,i}<0$.

$$\text{When}\quad \delta_1^{\,i}=-20^0,\quad \text{we get}\quad I_{1i}'\simeq 0.3.$$

$$\text{When}\quad \delta_1^{\,i}=-40^0,\quad \text{we get}\quad I_{1i}'\simeq -0.4.$$

It is natural that I_{1i}' decreases as δ_{1i}' becomes smaller, but I_{1i}' would be larger than 0. (It may be nearly zero for $\delta_1^{\,i}\simeq -40^0$.) Therefore our I_{1i}' is somewhat smaller than it should be when $\delta_1^{\,i}<0$. But we get $|I_{1i}/I_{1i}'|\simeq 10$, when $-40^0<\delta_1^{\,i}<-20^0$, which indicates that I_{1i}' is very small compared with I_{1i} in these cases.

Therefore we see that our approximation used in § 4, II is not very bad and unreliable. Especially, note that almost $\delta_1^{\,i}$ satisfy $|\delta_1^{\,i}|\lesssim 20^0$ in Table 4.

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Letters to the Editor

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Q-Value Dependence of Decay Probability

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October 5, 1958

We have discovered new strata of laws or structure of matter one after another as the available energy increases. It seems to us to suggest that nature is considered to have a stratified structure in length or energy. For the present, phenomena occurring in a smaller special domain ($<10^{-13}$cm) are beyond the scope of our research. In order to investigate the affairs in this small world, the use of extremely high energy phenomena or weak interactions are probably effective. The validity to use the latter for such a study is not established. But it is expected that the character of the small domain appears through weak interactions in view of its characteristic interaction constant, i.e., $10^{-16}\sim10^{-20}$ cm (in unit of $h=c=1$).[1] Moreover, it seems to us that some novel fact may be hidden in the data in our hands. Here we offer one possibility of existence of a hidden rule by examining all decay processes.

Now, decay processes are restricted by some selection rules partly effective. The empirical law given here has not fragmentary but general applicability, since it is concerned with the mass which is the quantity inherent to all particles.

We begin with some examples: $\pi\to e+\nu$ (which seems to be restrained by derivative coupling) occurs too less compared with $\pi\to\mu+\nu$, and $\Xi\to\Lambda+\pi$ occurs but $\Xi\to\mathcal{N}+\pi$ (which is forbidden by the strangeness rule) does not. These facts allow us to presume that *the decay probability of a particle (into two decay products) is a decreasing function of Q-value.*

We can find that this prophecy is true even in the two body decays of Λ, Σ and K[2], for example, $P(\Lambda\to p+\pi^-+37\ \text{Mev})/P(\Lambda\to n+\pi^0+41\ \text{Mev}\sim 2$ and $P(K^0\to\pi^++\pi^-+214\ \text{Mev})/P(K^0\to 2\pi^0+223\ \text{Mev})\lesssim 2$. Therefore, even a slight difference of Q-value seems to have an effect on the branching ratio of two body decays occurring from a common parent.* (Details are shown in ref. 3).)

Comparing two and three body decay processes, one must be concerned with

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* The apparent disadvantage of K^+ to K^0 ratio ($P(K^+)/P(K^0)\simeq 1/300$) is explained by the duplicity of K^0-lifetime pointed out by Lee, Oehme and Yang[4]. Our prophecy may be applied to the decay processes such as $\pi^0\to 2\gamma$ and $K^0\to 2\gamma$; $\Sigma^0\to\Lambda^0+\gamma$ and $\Sigma^0\to n+\gamma$ but may not explain the ratio of $\Sigma^\pm\to\mathcal{N}+\pi$ and $\Lambda^0\to\mathcal{N}+\pi$.

the phase volume of each channel, as the probability is proportional to the product of absolute square of transition matrix element and final density ($1/\tau \simeq \overline{|M|^2}\cdot\rho_F$). Having many decay branches of two or three decay products, K^+ is especially convenient to exhibit our prediction. The dependency of $\overline{|M|^2}$ on Q-value is approximately expressed as

$$\overline{|M(K^+)|^2}\approx\exp((-Q+b)/a),$$

where $a\approx 36\sim 38$ Mev,* $b\simeq 950\sim 1000$ Mev ($\simeq 2m_K\sim m_{\gamma(}$), and is illustrated

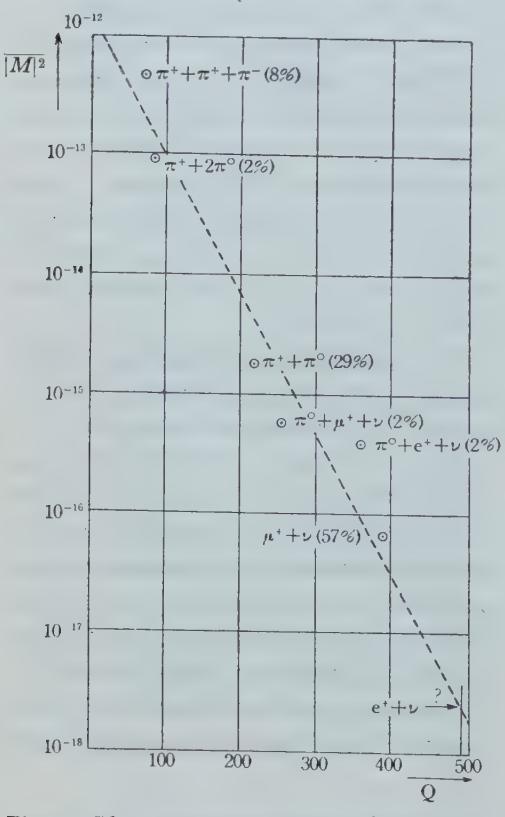

Fig. 1. The experimentally deduced $\overline{|M|^2}$'s are plotted as a function of Q. Masses and lifetimes are quoted from the report of Seventh Rochester Cnoference on High Energy Physics. In the course of calculation, the constant r_0 having the dimension of length is necessary, which is taken to be $r_0=h/m_\pi c=1.4\times 10^{-13}$cm.

in Fig. 1 with experimentally deduced values.[2] From Fig. 1, $K^+\to e^+ +\nu$ is expected to occur with the ratio of $1/100\sim 1/1000$ to $K^+\to\mu^+ +\nu$. $K^+_{\mu 3}$ and K^+_{e3} exceptionally have no sensitive Q-value dependence in spite of the large Q-value difference ($\sim$105 Mev). We have no answer to it.

To calculate the branching ratio of decaying particles, so-called $|\Delta I|=1/2$ *law* has been used but its validity is not yet established.[6] In applying this law, the mass difference between charge multiplets is neglected or treated at most as a small correction. While, in our discussion, this difference must be taken serious. We dare to say that isospin itself is generally not important in decay processes.[7] But our view-point does not necessarily contradict with that of Pais and Treiman.[8]**

Whether or not the character inherent in the small domain ($<10^{-13}$cm) is extracted by our law even partly is a point to be much discussed. For example, it seems to be necessary to estimate the effects of strong interactions participating decay processes.

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* The value of "a" ($\simeq m_\pi/4$) does not seem to be accidental from the point of view of Nambu's law.[5]

** Pais and Treiman pointed out that $|\Delta I|=1/2$ *law* is applicable only for the decay processes with small Q-values and that in future dynamics the transformation property of coupling constant in isospin space must contain energy-momentum-dependent parameters.

their continual interest and stimulating discussions.

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A Possible Symmetry in Sakata's Composite Model

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In Sakata's model[1] of strongly interacting particles, it is assumed that the proton, P, neutron, N, and Λ-particle, Λ, are the basic particles which are able to combine in various manner to compose mesons and other baryons. We want in this note to emphasize a certain equality which exists among these three particles, P, N and Λ, while the usual feeling has been inclined to distinguish the Λ-particle from the other two particles because of Λ's carrying strangeness.

Sakata's theory indicates that the conservation of strangeness quantum number s can be represented by the conservation of the number of Λ, and the conservation of charge q is that of proton's number. In the strong interactions, s and q as well as the baryon's number are all conserved, resulting the conservation of neutron's number. Thus in Sakata's theory the strong interactions is characterized by the following selection rule,

$$\Delta n_\Lambda = \Delta n_P = \Delta n_N = 0. \qquad \text{(A)}$$

According to the expression (A) and from what we know about the nature of these particles, we are tempted to regard each of the three particles as standing on an equal footing. In fact, when the mass difference and the electromagnetic interaction were neglected, we could not expect any difference among three particles, Then we can reasonably expect that a certain symmetry will also be realized in their mutual interactions. This view is maintained by the fact that a kind of symmetry exists in the weak interactions among μ-mesons, electrons and neutrinos in spite of the large splitting of their masses. The symmetry of interaction once noted under the name of *universality*[2] has turned out to be more and more convincing in the magnitude of coupling constant and

in the coupling type.

Now, we have the well-known π-nucleon interaction,

$$\bar{\psi}(1)\tau(1)\psi(1)\phi(1) \ ; \ \psi(1)=\begin{pmatrix}P\\N\end{pmatrix}$$

$$\phi_{(1)}=(\pi^+, \pi^0, \pi^-), \qquad (1)$$

which is derived from the contact (strong Fermi-) interactions between protons and neutrons. According to the above reasoning of the symmetry, we should expect the following interaction

$$\bar{\psi}(2)\tau(2)\psi(2)\phi(2) \ ; \ \psi(2)=\begin{pmatrix}N\\\Lambda\end{pmatrix}$$

$$\phi_{(1)}=(M, M^0, \tilde{M}), \qquad (2)$$

which comes from the same interaction between the neutron and Λ-particle, and similarly we get

$$\bar{\psi}(3)\tau(3)\psi(3)\phi(3) \ ; \ \psi(3)=\begin{pmatrix}\Lambda\\P\end{pmatrix}$$

$$\phi_{(3)}=(\tilde{L}, L^0, L), \qquad (3)$$

obtained from the set of Λ-particle and proton. (Notice that the vector notation of τ and ϕ is only for convenience of description and that the neutral particles π^0, L^0 and M^0 are not necessarily independent of each other.) We should point out here that in the limiting case of equal mass of the particle, any one of these expressions (1), (2) and (3) cannot get precedence over the others, because the particles concerned have a complete equivalence. Of course, we may actually have a deviation from the full symmetry due to the existing mass difference.

In our theory the usual character of charge independence of the interaction is expressed as the invariance of the whole scheme (1), (2) and (3) under the unimodular transformation between protons and neutrons. Our scheme is not always charge independent within the frame work only of the equivalence of Λ, P and N. If necessary, however, we may consistently bring the condition of charge independence into our frame. For instance, the following interaction may serve the purpose,

$$G\sum_{(ijk)}\{2\bar{N}_i N_j \bar{N}_j N_i + 2\bar{N}_j N_i \bar{N}_i N_j - 2\bar{N}_i N_i \bar{N}_j N_j + (\bar{N}_k N_k)^2 + \epsilon[(\bar{N}_k N_k)^2 + 2(\bar{N}_i N_i)(\bar{N}_j N_j)]$$

$$(ijk)=(123) \text{ cyclic}, \qquad (B)$$

where $N_j (j=1, 2, 3)$ mean P, N and Λ, respectively. The Dirac matrix is omitted. ϵ is any real parameter.

Our scheme does not largely deviate from the usual one. Composite particles M and L correspond to the K-meson in the Nakano-Nishijima scheme[3] as

$$(LM)\equiv(K^+K^0),$$

besides which we anticipate the existence of neutral particle states M^0 and L^0 which belong to the strangeness quantum number $s=0$ and when our scheme of the interaction is charge independent and π^0 is taken as a basis of the representation, M^0 and L^0 may not be pure particle but may be combined so as to form the charge singlet state in iso-space. Unfortunately, the more detailed nature of this particle is hardly informed, because of the mass difference between the Λ and nucleon and of the difficulty in solving the composite system practically. The existence of charge singlet π-meson has already been suggested[4] and our proposal is not new. But the way of our reasoning should

be emphasized.

In conclusion we should like to elucidate our view-point a little further. We take the view that there exists a certain origin of the mass of the particle which can never be reduced to the self-action of the usual interactions. The mass difference between the Λ and nucleon should be studied from this side of the problem and the mass splitting in lepton particles will need similar consideration. Another point is concerned with the charge independent character of the interaction. From our standpoint the symmetry (or equality) of the interaction among the three basic particles is of primary importance and charge independence has rather a secondary meaning. When charge independence of strong interaction is finally established, it will be necessary to seek for the reason why such combination of the interaction as (B), for instance, is preferred in nature[5].

The author takes pleasure in expressing his thanks to the collegues of Sakuma laboratory for their helpful discussions.

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Natures of Nuclear Forces Indicated by the Photodisintegration of the Deuteron, IV

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October 11, 1958

It is shown that there appear large differences in the computation of $d(\gamma, p)n$ process at $E_\gamma \simeq 90$ Mev by the different natures of phase shifts which can account for 170 Mev p-p scattering data.[1]

In Table 1 four sets of phase shifts which are fitted to 170 Mev p-p scattering data (without including the polarization data) are shown[1], and the calculated results on the photodisintegration of the deuteron at $E_\gamma = 87$ Mev by these phase shifts are given.[2]

At our energy m.d.* is very small if we assume $\delta_0{}^0 = 19^0$,[3] also the contribution of m.q. and e.q. to σ_{ip}* and σ_T are quite small.[4] Therefore we have only calculated e.d. The method of the computation of e.d. is same with that used in the previous letter.[5]

In Table I we have shown the case in which only the first term of $j_1(k_\gamma r)$ is adopted in the computation. Because the higher order terms of $j_1(k_\gamma r)$ would make σ_{ed} increase by about 20% at our energy,[3] we should multiply 1.2 to σ_{ip} and σ_T of Table 1 in the comparison with experiment.

The experiments give[6]

* Symbols used here are same with those used in the previous papers. (See reference 5.)

$$\sigma_{ip} = (4.5 \sim 5.5) \times 10^{-29} \text{cm}^2,$$

$$\sigma_T = (7 \sim 9) \times 10^{-29} \text{cm}^2$$

at our energy.

We see that *class A can well account for the data, but it is not for class B. Class B gives below one half of* σ_{ip} *of class A.*

Because it seems that the effect of the virtual pions is yet unimportant at our energy,[7] only class A would remain to be favourable. Particularly, the set A is most favourable in the sense that it account for both polarization data and $d(\gamma, p)n$ in the approximation assumed.[1] It is important to note that *the phase shifts of class A is not much different from that to be expected from pion potential.* More detailed analysis and illustration will be published in due time.

Table 1. Phase shifts of 170 Mev p-p scattering (in degree)[1] and the corresponding cross sections (in unit of 10^{-27} cm^2), when we put $j_1(k_T r) \to k_T r/3$, are shown. The higher order terms of $j_1(k_T r)$ increase the cross sections by about 20%.

	$\delta_0{}^0$	$\delta_1{}^0$	$\delta_1{}^1$	$\delta_1{}^2$	σ_T	σ_{ip}
A_1	19	22	−21	10	6.99	4.85
A_2	19	51	−10	3	6.65	4.14
B_1	19	−19	24	−7	4.99	1.75
B_2	19	−46	15	3	4.91	1.82

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A Note on the Theories of Metals and of Pressure Ionization

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October 25, 1958

Consider an assembly of systems at a very high pressure and at a low temperature. At very high pressures, the atoms will come into violent collisions with each other, and as a result, their valency electrons will either switch on to higher quantum orbits, or will be altogether knocked off. Vice-versa, when an ionized atom and an electron come together, the ion may capture the elctron and become a normal or an excited atom. Taking an element M, we may indicate the process of ionization and capture of an electron by the ion by the typical form $\rightleftarrows$ for reversible reaction.

$$M,\ M^{\text{exc}} \rightleftarrows M^+,\ e, \quad \text{(Ionization and Recapture)}$$

where M^+ denotes the ion and e the electron, both considered as gaseous.

The energy relation is given by

$$M \rightleftarrows M^+ + e - \omega, \qquad (1)$$

where ω is called the heat of ionization.

The law of mass action for the assembly of atoms of the element M, ions M^+ and electrons e is given by[1]

$$\alpha_M - \frac{\omega}{kT} = \alpha_{M^+} + \alpha_e, \qquad (2)$$

where $\alpha_i = (\mu_i / kT)$ and μ_i is the chemical potential of the i-th species. Here T is the temperature of the

assembly, k the 'Boltzmann's Constant' and the sub-scripts M, M^+ and e refer to the atom, the ionized atom and the electron respectively.

Since we assume the temperature to be low and the pressure to be very high, the element M and the ions M^+ can be regarded to be weakly degenerate while the electrons e are strongly degenerate. Thus equation (2) yields the following result

$$\ln\left[\frac{n_M^* \cdot h^3}{g_M(2\pi m_M kT)^{3/2}}\right]-\frac{\omega}{kT} = \ln\left[\frac{n_{M^+}^* \cdot h^3}{g_{M^+}(2\pi m_{M^+} kT)^{3/2}}\right] + \frac{\epsilon_0}{kT}\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_0}\right)^2\right], \quad (3)$$

where

$$\epsilon_0 = \frac{h^2}{2m_e}\left(\frac{3n_e^*}{4\pi g_e}\right)^{2/3}.$$

Here g_i, n_i^* and m_i are the weight factor, the equilibrium number per unit volume and mass of the system of i-th species respectively, whereas h is 'Planck's Constant'.

Assuming that $m_{M^+} \approx m_M$ which is very true because $m_e \ll m_M$ and, since the temperature is low we thus have

$$\ln\left(\frac{n_M^*}{n_{M^+}^*}\right)=\frac{\omega}{kT}+\ln\left(\frac{g_M}{g_{M^+}}\right)+\frac{\epsilon_0}{kT}. \quad (4)$$

In the present case the pressure, which effectively depends only on the free-electron concentration in equilibrium (the heavy particles, namely the atoms M and the ions M^+, being non-degenerate make negligible contribution) is given by the usual expression

$$p=\frac{8\pi}{15}\cdot\frac{h^2}{m_e}\left(\frac{3n_e^*}{4\pi g}\right)^{5/3}. \quad (5)$$

Using (5), equation (4) can thus be written as

$$\ln\left(\frac{n_M^*}{n_{M^+}^*}\right)=\frac{\omega}{kT}+\ln\left(\frac{g_M}{g_{M^+}}\right)+\frac{\gamma}{kT}\cdot p^{2/5}, \quad (6)$$

where

$$\gamma=\tfrac{1}{2}\left(\frac{15}{8\pi}\right)^{2/5}\cdot\left(\frac{h^2}{m_e}\right)^{3/5}.$$

If at temperature T and pressure p the element M is r times ionized, we have

$$\ln\left(\frac{1}{r}-1\right)=\frac{\omega}{kT}+\ln\left(\frac{g_M}{g_{M^+}}\right)+\frac{\gamma}{kT}\cdot p^{2/5}. \quad (7)$$

Equation (7) is the expression for the degree of ionization of the metal (or the degenerate matter in the interior of the white dwarf star) in terms of its temperature and pressure.

In conclusion we may point out that one of the major defects of the 'Sommerfeld's Theory of Metals' is overcome with the help of equation (7) because it gives us the number of free electrons per atom in terms of the temperature and the pressure of the metal. We should no longer assume that the free electrons are the valence electrons themselves, unless it is verified to be true by using equation (7). Details of this work will be published subsequently.

Moreover, equation (7) is also true for the degenerate matter in the interior of the white dwarf star. Kothari and Majumdar's[2] theory of pressure ionization, on the basis of which the degree of ionization was found to be independent of temperature, thus seems to be doubtful because the degree of ionization depends not only on pressure but also on the temperature of the degenerate

matter according to eqution (7). Details of this work will also be published subsequently.

My greateful thanks are due to Professor D. S. Kothari for taking interest in the above investigations.

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Drift and Hall Mobilities of Electrons in Nondegenerate Impure *n*-Semiconductors

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Brooks[1] and Herring have obtained the time of relaxation of carriers due to ionized impurity scattering to be given by

$$T=AT^{3/2}\frac{x^3}{f_b(x)} \qquad (1)$$

where

$$A=\frac{2^{1/2}D^2m^{1/2}k^{3/2}}{\pi q^4N_I},$$

$$f_b(x)=\ln(1+bx^2)-\frac{1}{(1+1/bx^2)}$$

$$b=\frac{2\pi Dm(kT)^2}{k^2q^2n}$$

where

D=dielectric constant of the crystal,
m=effective mass of the carriers,
q=the electronic charge,
k=Boltzmann's constant,
$x=(m/2kT)^{1/2}v$,
v=carrier velocity
h=Planck's constant
N_I and n are number of ions and carriers per unit volume

and

T=crystal temperature.

For nondegenerate semiconductors the drift mobility μ is related to time of relaxation T by

$$\mu=(q/m)(Tx^2)/\langle x^2\rangle \qquad (2)$$

where $\langle\ \rangle$ indicates average over the whole distribution given by

$$N(x)\,dx=N_0x^2\exp(-x^2)\,dx. \qquad (3)$$

From equations (1), (2), and (3) we obtain

$$\mu=AT^{3/2}\frac{q}{m}\left[\int_0^\infty\frac{x^7\exp(-x^2)}{f_b(x)}dx\right]\cdot\left[\int_0^\infty x^4\exp(-x^2)\,dx\right]^{-1}. \qquad (4)$$

The usual practice is to replace $f_b(x)$ by a constant $f_b(x_1)$ where $x_1=(7/2)^{1/2}$ makes the rest of integrand viz $x^7\exp\cdot(-x)^2$ maximum.

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Blatt[2] pointed out that this approximation can lead to serious errors and carried out exact numerical integrations.

It has been pointed out by Spitzer and Harm[3] and Sodha and Eastman[4] that electron-electron scattering plays an important part in determining drift and Hall mobilities of n-type semiconductors of high impurity concentrations. In this communication the authors have studied the variation of drift and Hall mobility with temperature taking into account electron-electron scattering and carrying out numerical integrations.

Sodha and Eastman[5] have shown that when $b \gg 1$ ($f_b(x)$ = constant) the effect of electron-electron scattering can be taken into account by substituting $\varphi(x)/x$ for x^3 in equation (1), where $\varphi(x) = ZD(x)/A$ is a function, tabulated by Spitzer and Harm[3]. As a first approximation we can assume that this also holds, when b is not so large as to make $f_b(x)$ = constant. Hence, the time of relaxation for mean ionic charge $Z=1$ is given by

$$T = AT^{3/2} \cdot \varphi(x)/x f_b(x). \qquad (5)$$

In further discussion we shall confine ourselves to the case $Z=1$, which is applicable[4),5] to most semiconductors of interest.

Using equations (2), (3), and (5), we get

$$\mu = (8qAT^{3/2}/3m\pi^{1/2}) F(b) \qquad (6)$$

where

$$F(b) = \int_0^\infty \frac{x^3 \varphi(x) \exp(-x^2)}{f_b(x)} dx.$$

When the number of carriers is small and the temperature is high b has large values; $f_b(x)$ varies slowing with x and, hence, we may replace $f_b(x)$ by $f_b(x_1)$ where $x_1 = 1.61$ makes the rest of integrand viz $x^3\varphi(x)\exp(-x^2)$ maximum. Thus when $b \gg 1$

$$F(b) = 1.746/f_b(1.61) \approx 1.746/\ln b. \qquad (7A)$$

The Hall mobility, μ', is given by

$$\mu' = (q/m) \cdot \langle T^2 x^2 \rangle \langle T x^2 \rangle. \qquad (8)$$

Using equations (3), (5), and (8), we obtain

$$\mu' = \frac{qAT^{3/2}}{m} \Phi(b) \qquad (9)$$

where

$$\phi(b) = \psi(b)/F(b) \qquad (9A)$$

and

$$\psi(b) = \int_0^\infty x^2 \left\{\frac{\varphi(x)}{f_b(x)}\right\}^2 \exp(-x^2)\, dx. \qquad (9B)$$

When $b \gg 1$ it can be shown that

$$\psi(b) = \frac{1}{\{f_b(x=1.85)\}^2} \cdot \int_0^\infty x^2 \varphi^2 \exp(-x^2)\, dx = \frac{5.400}{\{f_b(x=1.85)\}^2}$$

since $x = 1.85$ makes $x^2\varphi^2 e^{-x^2}$ maximum. Hence for $b \gg 1$

$$\Phi(b) \approx 3.093/\ln b. \qquad (9C)$$

Table 1 gives the variation of $F(b)$ and $\phi(b)$ with b obtained by using equations (7) and (9) and performing numerical integrations.

Table 1. Variation of $F(b)$ and $\phi(b)$ with b.

b	$F(b)$	$\phi(b)$
100	0.385	0.727
50	0.456	0.862
10	0.787	1.22
5	1.12	1.71
1	4.14	9.37

It is seen that equations (7A) and (9C) give an error of less than 1% and 8% respectively for $b \geq 100$.

However, Herring[6] has pointed out that the Brooks Herring theory is only a very poor approximation in the case of semiconductors of interest. Hence one should not expect these calculations based on simple band structure to hold very well for commonly studied semiconductors like Germanium.

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A Note on the Physical State of Unstable Particles

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Recently, we have investigated the clothed states of unstable particles.[1),2)] In the second paper,[2)] we have shown that the physical state of the unstable V-particle is given by

$$|V^{\kappa}\rangle \equiv N \cdot F_{\kappa}(H)|V\rangle \qquad (1)$$

with

$$F_{\kappa}(H) \equiv \int_{-\infty}^{0} dT\, \gamma_{\kappa}(T) e^{i(H-E_0)T}, \qquad (2)$$

where the notations are as follows:

$|V\rangle$: bare state,

H : total Hamiltonian,

$E_0 = m_V - i\gamma/2$,

$\gamma_{\kappa}(T)$: production-rate function,[2)]

N : normalization constant.

Though we have so far used mainly the expression of the physical state, it is more convenient to use the expression (1) in the calculation of various quantities. For example, the decay amplitude is simply presented by

$$\langle \boldsymbol{n}^{-}|V^{\kappa}\rangle = N \cdot F_{\kappa}(E_n)\langle \boldsymbol{n}^{-}|V\rangle, \qquad (3)$$

where $|\boldsymbol{n}^{-}\rangle$ is the incoming-wave eigenstate: $H|\boldsymbol{n}^{-}\rangle = E_n|\boldsymbol{n}^{-}\rangle$. $\langle \boldsymbol{n}^{-}|V\rangle$ is just the decay amplitude of the bare V-particle, and the function $F_{\kappa}(E_n)$ is easily calculated (at least numerically) for a given $\gamma_{\kappa}(T)$. The normalization constant N is given by

$$N^{-2} = \langle V|F_{\kappa}^{*}(H)F_{\kappa}(H)|V\rangle = \int dE \rho(E)|F_{\kappa}(E)|^2, \qquad (4)$$

where

$$\rho(E) \equiv \int dn\, \delta(E-E_n)|\langle V|\boldsymbol{n}^{-}\rangle|^2 \qquad (5)$$

is Lehmann's spectral function.[3)] Similarly, the time-development of $|V^{\kappa}\rangle$ is easily calculated, namely

$$\langle \boldsymbol{V}^{\varkappa}|e^{-iHt}|\boldsymbol{V}^{\varkappa}\rangle = N^2\int dE\rho(E)|F_{\varkappa}(E)|^2e^{-iEt} \quad \text{for} \quad t\geqq 0. \tag{6}$$

The main contribution of the integral comes out from the pole $E=E_0$ of $\rho(E)$ in the lower half-plane. Since $F_{\varkappa}(E_0)=\int_{-\infty}^{0}dT\eta_{\varkappa}(T)\equiv 1$, the integral for $E\approx E_0$ is essentially the modified propagator $S_V'(t)$. We therefore obtain[4]

$$\langle \boldsymbol{V}^{\varkappa}|e^{-iHt}|\boldsymbol{V}^{\varkappa}\rangle \approx N^2Z_Se^{-iE_0t}, \tag{7}$$

where

$$Z_S^{-1}=(\partial/\partial E)[S_V'(E)]^{-1}|_{E=E_0}.$$

Finally, we consider the renormalization. The Z-factor is defined by the probability of the bare V-particle, namely

$$Z\equiv|\langle V|\boldsymbol{V}^{\varkappa}\rangle|^2=N^2\left|\int dE\rho(E)F_{\varkappa}(E)\right|^2. \tag{8}$$

We denote the renormalized quantities by affixing the suffix R. Then we have

$$\begin{aligned}\langle \boldsymbol{n}^-|V\rangle_R&=Z^{-1/2}\langle \boldsymbol{n}^-|V\rangle,\\ \rho_R(E)&=Z^{-1}\rho(E),\end{aligned} \tag{9}$$

and (4) and (8) are rewritten as

$$\begin{aligned}(ZN^2)^{-1}&=\int dE\,\rho_R(E)|F_{\varkappa}(E)|^2\\ &=\left|\int dE\,\rho_R(E)F_{\varkappa}(E)\right|^2.\end{aligned} \tag{10}$$

The decay amplitude can, of course, be expressed by the renormalized quantities alone:

$$\langle \boldsymbol{n}^-|\boldsymbol{V}^{\varkappa}\rangle=c\cdot F_{\varkappa}(E_n)\langle \boldsymbol{n}^-|V\rangle_R, \tag{11}$$

where $c^2\equiv ZN^2$ is given by (10). The production-rate function $\eta_{\varkappa}(T)$ can be calculated, at least in principle, by (11) from the theoretical value of $\langle \boldsymbol{n}^-|V\rangle_R$ and the experimental decay spectrum (provided that $\eta_{\varkappa}(T)$ is real).

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Notice to Subscribers

We regret that by an unfortunate error in the process of printing Mr. S. Aono's letter appearing in Vol. 21, No. 1 was repeated in Vol. 21, No. 2.

For the convenience of reference and indexing, it is suggested that in the case of binding, sheet p. **217** be deleted and be replaced by this sheet.

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On a New Approach to Cosmology

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An attempt is made to develop a new theory of cosmology based on a scalar field, in view of several weak points proper to the general relativistic cosmology. In our treatment, a kind of correspondence principle plays the important role, together with the consideration of the scope of applicability of general relativity.

Our cosmological field equation thus proposed not only has an intimate linkage with Einstein's equations, but also satisfies Mach's relation for the universe as a whole in a covariant and intrinsic manner.

§ 1. Introduction

Among modern theories of cosmology, the general relativistic cosmology and the steady-state theory due to Bondi-Gold[1] are most violently opposed to each other. Though we have not yet any decisive evidence for supporting the one superior to the other as for comparison of these theories with the observational data of extra-galaxies, these theories have the following points of difference. As is well-known, the general relativistic cosmology is based on the general theory of relativity which can be regarded in a sense as well established. On the other hand, this theory provides us with such a great diversity of possible model universes that it is almost incompetent to predict any phenomena at all, though it enables us to run after the observational data. In addition, we cannot explain the formation of galaxies within the allowable range of the age of the universe estimated from, say, the evolution of stars, even if we could dissolve the difficulty of age of model universes themselves (cf. § 7).

In contrast with the above situation, the steady-state theory of cosmology is a kind of all-or-nothing-theory in that de Sitter space-time is adopted exclusively and that the theory permits us to predict definitely various relations to compare with observational data of galaxies. But it must be remarked here that this particular space-time is not the one deduced as a solution from any fundamental field equation at all, but the one adopted as a consequence of the fundamental postulate of the theory, i. e. the so-called perfect cosmological principle, together with the concept of a curved space-time.

In these respects, from the standpoint of the general relativistic cosmology, the present authors[2] proposed an idea to overcome various weak points of the theory

mentioned above. On that occasion, the question was aroused concerning the smoothed-out state of cosmic matter and radiation which contributes to the structure of the cosmological background space-time. Because we cannot help analyzing this question in order to restrict a diversity of model universes and to discuss the problem of the formation of galaxies. However, the matter has become serious as soon as we proceeded to smooth out the cosmic matter by means of a thought experiment. In short, we cannot carry out such a thought experiment without assuming, to a certain extent, the smoothed-out state to be attained.

It is however possible to hold the concept of the smoothed-out universe from another view-point, if we take account of a possibility to lay down restrictions on the applicability of general relativity. Such a view-point seems to be more plausible than the one adopted in the previous paper. This compels us, however, to propose a new cosmological field equation to take the place of Einstein's field equation, though the former should have an intimate linkage with the latter. We shall explain details of these points and our new conceptions based on these points in § 2. It will be shown there that our new formalism makes, in some sense, fundamental concepts of the general relativistic cosmology and those of the steady-state cosmology fuse into one. In §§ 3–5 we shall elaborate our new plan. In § 6 we shall investigate the possible model universes, and in § 7 compare them with the recent data of the velocity-distance relation of galaxies.

§ 2. Outline of new idea

In the general theory of relativity the problem of local gravitation such as the solar gravitational field is ordinarily treated with the boundary condition that the space-time is flat at infinity. In the general relativistic cosmology on the other hand, there is required in place of such a boundary condition the so-called cosmological principle which stipulates that the idealized universe, constructed by smoothing out the condensed matter of the real universe, should be spatially homogeneous and isotropic. Then the metric of the cosmological background space-time can be expressed as follows:

$$ds^2=c^2d\tau^2-\Phi^2(\tau)\{dr^2+A^2(r)(d\theta^2+\sin^2\theta d\varphi^2)\}, \qquad (2\cdot1)$$

with

$$A(r)=\begin{cases} r, & \text{for } \textit{euclidean} \text{ 3-space (i)} \\ a\sin(r/a), & \text{for } \textit{elliptic} \text{ 3-space (ii)} \\ a\sinh(r/a), & \text{for } \textit{hyperbolic} \text{ 3-space (iii)}. \end{cases} \qquad (2\cdot2)$$

Here it is to be remarked that the function $A(r)$ cannot be determined by the field equation, but that the alternative forms of $A(r)$ specified by (2·2) are exclusively deduced from the kinematical requirement of the cosmological principle. As a result, $A(r)$ contributes only indirectly, via the field equation, to a scale-factor

$\Phi(\tau)$ (τ=cosmic time) which specifies the cosmic expansion. Thus the essential unknown in (2·1) is the scale-factor $\Phi(\tau)$.

However, the right-hand side of Einstein's field equation, $R_{ij}-\frac{1}{2}g_{ij}R=-\kappa T_{ij}$, contains two unknown quantities, pressure p and density ρ, even if the smoothed-out state of cosmic matter and radiation is considered as a perfect fluid. Therefore we must assume a suitable $p-\rho-$relation. But how to choose this relation contains many complexities.[2] In short, Einstein's equation imposes, on account of its tensorial character of the second rank, too severe restrictions on single unknown $\Phi(\tau)$ reduced from ten g_{ij} by the procedure of smoothing-out. Moreover, the equation is not complete unless we postulate the problematic assumption that the cosmic matter and radiation can be described by the concept of a perfect fluid.

It is also to be remarked that (2·1) with (2·2) is derived only from the kinematical requirement for the universe as a whole and that the material condition is not taken into consideration. We have, however, the following empirical facts: if we estimate the strength of a gravitational field in terms of a dimensionless number $G\rho t^2 \approx GM/c^2L$ (G=Newton's gravitation constant, ρ and t are the density and time, respectively, which characterize the scale of phenomena, and $M\approx\rho L^3$, $L\approx ct$), this quantity amounts to the value for atomic nuclei $\sim 10^{-39}$, for terrestrial, solar and nebular gravitational fields $\sim 10^{-9}$, $\sim 10^{-6}$ and $\sim 10^{-6}$ respectively. On the other hand, for the universe as a whole, it becomes

$$G\rho t^2 \sim 1, \qquad (2\cdot3)$$

where ρ= the mean density of the universe, t= the reciprocal of Hubble's constant h. If we take account of these empirical facts, the dimensionless number $G\rho t^2$ may be regarded as a characteristic quantity specifying the scales of gravitating systems, in a similar manner as the role of Reynolds number in the fluidal motions. Accordingly we shall assume that the empirical relation (2·3) characterizes the universe as a whole in its material aspect.

Among ordinary models of the universe based on the general relativistic cosmology there exists one satisfying (2·3), i.e. Einstein-de Sitter universe, but such a situation is rather accidental. On the other hand, we should like to consider that the standpoint on which the relation (2·3) is first of all taken into the theory is more significant than the previous one of setting up a certain $p-\rho-$relation firstly. And it is interesting that (2·3) satisfies, in some sense, Mach's request that "the inertial field should be determined by the distribution of matter throughout the whole universe". Because the boundary condition specifying the background space-time of a local gravitational field can be interpreted as an idealized representation of the fact that the dimensionless number $G\rho t^2$ is of the order of unity for the universe as a whole in such a way that the relation (2·3) is satisfied even in the limit when $\rho\to 0$ and $t\to\infty$.

From the above consideration it becomes desirable to postulate the formalism which provides only one field equation for $\Phi(\tau)$ and moreover satisfies (2·3)

automatically. This does not, however, mean that general relativity ceases to be applicable to the problem of the local gravitational field specified by the condition $G\rho t^2 \ll 1$. Thus the general relativistic cosmology does not necessarily contradict with our requirement, but it cannot prevent various complexities.[2]

In this respect, we shall propose as our cosmological field equation the following one obtained by contracting Einstein's field equations:

$$R=\kappa T, \tag{2·4}$$

where $\kappa=8\pi G/c^2$. This equation appears to be very plausible, since not only it has by definition an intimate linkage with the field equations in general relativity, but also the right-hand side of (2·4) can be interpreted as: $\kappa T^* \sim G(\rho-3p/c^2) \sim G\rho \sim h^2$ (if we take account of (2·3)).

So far our considerations have been confined to the smoothed-out universe, but it must be required that the formalism of the theory should enable us to deal with the problem of the gravitational condensation which is a starting point for the discussion of the formation of galaxies. Therefore the formalism must allow us at least to derive Newtonian gravitation. For this purpose we have only to adopt the following line-element instead of (2·1):

$$ds^2=\psi^2(t, r, \theta, \varphi)\,(c^2dt^2-dr^2-A^2(r)\,(d\theta^2+\sin^2\theta\,d\varphi^2)). \tag{2·5}$$

Because (2·5) is reduced to (2·1) when ψ depends only on t, i.e. $\psi=\psi_0(t)$, if we use the relation:

$$d\tau=\pm\psi_0(t)\,dt, \quad \{\Psi(\tau)=\psi_0(t(\tau))\}. \tag{2·6}$$

Moreover if we consider the case $\psi=\psi_0(t)+\psi_1(t, r, \theta, \varphi)$, we can make ψ_1/ψ_0 correspond to Newtonian gravitation (in the following we shall designate this as correspondence principle). Generally speaking, ψ-field will determine the combined effects of the cosmic expansion and Newtonian gravitation.

Accordingly we can schematically illustrate our plan as follows:

(*General Theory of Relativity*)		(*Our Theory*)
(*Metric*): $(ds^2=g_{ij}dx^i\,dx^j)$	$\xrightarrow[\text{smoothing-out}]{}$	$(ds^2=\psi^2\{c^2dt^2-dr^2-A^2(r)\,(d\theta^2+\sin^2\theta\,d\varphi^2)\})$
(*Field Equation*): $(R_{ij}-(1/2)g_{ij}R=-\kappa T_{ij})$	$\xrightarrow[\text{contraction}]{}$	$(R=\kappa T)$.

We must not overlook here that the functional form of $A(r)$ is given by the alternative forms of (2·2) due to the cosmological principle. Moreover, the fact that the local gravitation is, in our theory, derived only to the extent of Newtonian

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* The relation $\kappa T \sim G(\rho-3p/c^2)$ was taken from the standpoint of general relativity.

approximation should be regarded as a necessary consequence of the smoothing-out procedure*. In fact, if $A(r)$ of (2·5) is suitably chosen, we can describe the metric of Schwarzschild's space-time in terms of (2·5) with $\psi=\psi(r)$, though the required form of $A(r)$ is inconsistent with the expression (2·2).

Now it is to be remarked that, if we introduce the metric:

$$ds_0^2=c^2dt^2-dr^2-A^2(r)(d\theta^2+\sin^2\theta\, d\varphi^2), \tag{2·7}$$

it is regarded as the metric of an auxiliary space-time on which the invariancy of the field equation is based. Therefore we shall from now on call the space-time specified by (2·7) the "representation space-time". Namely the metric $ds^2=\psi^2 ds_0^2$ of the expanding universe is conformal to this representation space-time. Moreover it is remarkable that our field equation has no such a property satisfied by Einstein's field equation that the conservation of energy-momentum is automatically assured through the identity $(R_j^i-(1/2)g_j^iR)_{;i}\equiv 0$, where the symbol ; denotes covariant derivative. Thus our formalism may have some possibility to allow the conception of continuous creation of matter as envisaged by Bondi-Gold. That is, in our theory there is no doubt that the behavior of the cosmic expansion has an intimate connection with the problem whether matter is created or not. There remains a fundamental problem as to the relation between the creation, if any, mentioned here and the conservation law, the discussion of which will be made on another occasion.

§ 3. Conformal space-time

It is easily seen from the discussion in the last section that the space-time metric (2·5) of our cosmology is conformal to that of the representation space-time. Therefore, using general coordinates (x^i), we can calculate scalar curvature R appeared in (2·5) as follows:

$$R=(6/\psi^3)(\Delta_2\psi+\overset{\circ}{R}\psi/6), \tag{3·1}$$

with

$$\left.\begin{aligned}\Delta_2\psi&\equiv\overset{\circ}{g}{}^{ij}(\psi_{,ij}-\overset{\circ}{\Gamma}{}^a_{ij}\psi_{,a})\\ \Delta_1\psi&\equiv\overset{\circ}{g}{}^{ij}\psi_{,i}\ \ \psi_{,j}*\end{aligned}\right\}, \tag{3·2}$$

where $ds_0^2=\overset{\circ}{g}_{ij}dx^i dx^j$, $\overset{\circ}{\Gamma}{}^a_{ij}$ is Christoffel's symbol for $\overset{\circ}{g}_{ij}$ and $\overset{\circ}{R}$ is scalar curvature of representation space-time. It is not, however, useless to remark that the above two expressions in (3·2) together with ψ itself are all scalars.

Especially if we take (t, r, θ, φ) appeared in (2·7) as (x^i), we can obtain:

$$\overset{\circ}{R}=-(2/A^2)(2AA''+A'^2-1), \tag{3·3}$$

---

* Such a situation is not necessarily proper to our formalism, since the fundamental equation of relativistic cosmology can also be derived formally from the standpoint of Newtonian dynamics.[3]

** This symbol will appear in the next section.

$$\left.\begin{aligned} \Delta_2 &= \partial^2/c^2\partial t^2 - \partial^2/\partial r^2 - 2(A'/A)\partial/\partial r - A^{-2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} \\ \Delta_1\psi &= (\partial\psi/c\partial t)^2 - (\partial\psi/\partial r)^2 - A^{-2}\{(\partial\psi/\partial\theta)^2+(\partial\psi/\sin\theta\,\partial\varphi)^2\} \end{aligned}\right\}, \tag{3·4}$$

where $A'=dA/dr$, etc.

Inserting (3·1) with (3·3) and (2·2) into (2·4), we obtain,

$$(\Delta_2+\varepsilon/a^2)\psi=\sigma\psi^3, \tag{3·5}$$

with

$$\sigma \equiv (1/6)\kappa T=(4\pi/3)\,(GT/c^2), \tag{3·6}$$

$$\varepsilon=\left\{\begin{array}{l} 0 \cdots\cdots\cdots\cdots\cdots\cdots (\mathrm{i}) \\ 1 \cdots\cdots\cdots\cdots\cdots\cdots (\mathrm{ii}) \\ -1 \cdots\cdots\cdots\cdots\cdots\cdots (\mathrm{iii}) \end{array}\right\}, \tag{3·7}$$

according to the different forms of $A(r)$ in (2·2). The equation (3·5) is clearly an invariant equation in the representation space-time.

§ 4. Invariant expression of Mach's relation

Now we must take the condition (2·3)* stated in § 2 into our theory in order to make the field equation (3·5) complete. It is easily seen from the definition of σ, (3·6), and the relation $\kappa T\sim G\rho$ (cf. § 2) that:

$$\sigma\sim(4\pi/3)\,(h/c)^2, \tag{4·1}$$

where h is Hubble's constant.

Next we must express the quantity h on the right-hand side of the above formula in terms of the scalar field ψ. For this purpose we shall at first consider the case in which no gravitational condensation takes place, i. e. $\psi=\psi(t)$. As is well-known, Hubble's constant h is defined by

$$h=(1/\Phi)\,(d\Phi/d\tau) \tag{4·2}$$

in any theory of the expanding universe whose metric is specified by (2·1). Then, if we take account of the relation (2·6), the above expression can be translated as follows:

$$h^2=\left(\frac{1}{\psi^2}\frac{d\psi}{dt}\right)^2. \tag{4·3}$$

This is nothing but the relation between h and ψ in question.

Inserting the above formula in (4·1), we obtain

$$\sigma=\gamma\psi^{-4}(d\psi/cdt)^2, \tag{4·4}$$

---

* We call this "Mach's relation".

where γ is a pure number such that $3\gamma/4\pi\sim 1$. The above formula is derived, however, under a special condition, and further it contradicts with the transformation property of (3·5) as it is, because $(d\psi/cdt)^2$ is not an invariant quantity. In order to remove such a contradiction, we have only to replace it by the invariant differential parameter $\Delta_1\psi$ due to Beltrami (cf. (3·2)). Then, we obtain

$$\sigma=\gamma\psi^{-4}\Delta_1\psi, \tag{4·5}$$

which is nothing but the invariant expression of Mach's relation.

Now it may not be useless to refer to the function of (4·5) in our cosmology. Hönl[4] asserted that "Mach's principle"* cannot be *a constitution principle* for constructing a cosmological theory, but that at most it plays a role only as *a selection principle* for various possible model universes derived from several proposals. In the steady-state theory, on the other hand, "Mach's principle" is practically the constitution principle for its formalism. Because this principle is adopted as a basis for the perfect cosmological principle which is nothing but a kinematical assumption expressing the temporal and spatial homogeneity of the universe. In contrast with both of these extreme opinions, the role of Mach's relation in our theory may be analogous to that of the quantum condition in Bohr's atomic theory.

§ 5. Lagrangian formulation of our cosmological field equation

In this section we shall derive the complete form of our cosmological field equation and then show that Lagrangian formulation is possible.

Inserting (4·5) into (3·5), we obtain

$$(\Delta_2+\varepsilon/a^2)\psi=\gamma\psi^{-1}\Delta_1\psi. \tag{5·1}$$

This is nothing but the field equation in question. From (3·2), however, we can obtain the following identities:

$$\left.\begin{aligned}\Delta_2(\ln\psi)&=\psi^{-1}(\Delta_2\psi-\psi^{-1}\Delta_1\psi) \qquad (\gamma=1),\\ \Delta_2(\psi^{1-\gamma})&=(1-\gamma)\psi^{-\gamma}(\Delta_2\psi-\gamma\psi^{-1}\Delta_1\psi) \quad (\gamma\neq 1).\end{aligned}\right\} \tag{5·2}$$

Taking into account (5·2), we can reformulate (5·1) as follows:

$$\left\{\begin{aligned}&\Delta_2\phi+\varepsilon/a^2=0 && (\phi=\ln\psi,\ \text{for } \gamma=1), && (5·3)\\ &(\Delta_2+(1-\gamma)\varepsilon/a^2)\phi=0 && (\phi=\psi^{1-\gamma},\ \text{for } \gamma\neq 1). && (5·4)\end{aligned}\right.$$

Thus it has been made clear that our non-linear field equation for ψ can be reduced to either of the alternative linear field equations by taking the respective new scalar ϕ corresponding to $\gamma=1$ or $\gamma\neq 1$.

Now, in order to make our theory stand on itself, it is desirable to establish

* We have designated the relation (4·5) as Mach's relation, because there is a diversity of interpretations as for the content of Mach's principle.

the Lagrangian formulation of our field equations (5·3) and (5·4) derived above on the heuristic standpoint. For this purpose we have only to take the following alternative Lagrangian densities:

$$\delta\int \mathcal{L}\, d^4x=0, \tag{5·5}$$

with

$$\begin{aligned}\mathcal{L}&=\{\tfrac{1}{2}\overset{\circ}{g}{}^{ij}\phi_{,i}\ \phi_{,j}\ -(\varepsilon/a^2)\phi\}\,(-\overset{\circ}{g})^{1/2},\quad (\gamma=1),\\ \mathcal{L}&=\{\tfrac{1}{2}\overset{\circ}{g}{}^{ij}\phi_{,i}\ \phi_{,j}\ -\tfrac{1}{2}(1-\gamma)\,(\varepsilon/a^2)\phi^2\}\,(-\overset{\circ}{g})^{1/2},\quad (\gamma\neq 1),\end{aligned} \tag{5·6}$$

where $\overset{\circ}{g}=\det|\overset{\circ}{g}_{ij}|$; variation is to be carried out with respect to ϕ.

Here it is to be remarked that it is a question whether the representation space-time (2·7) itself can be regarded as the background space-time of the universe. If this is the case, it may be unnecessary to regard (2·5) with $\psi=\psi(t)$ as the metric of the smoothed-out universe. On this standpoint, however, we have to interpret the red-shift of the spectral lines of galaxies basing on a new concept other than that of the cosmic expansion. In addition, the correspondence principle of the local gravitational field (which will play an important role when the problem of gravitational condensation is treated) would break down. Accordingly it may be said that we have no positive reason to take such a standpoint.

§ 6. Possible model universes

Now we shall investigate what kind of model universes is allowed in our cosmology. Since a model universe is the one in which condensed matter such as galaxies is completely smoothed out, it is specified by the condition $\psi=\psi(t)$, on account of the correspondence principle in § 2. Therefore we obtain from (5·3) or (5·4) and (4·5)

$$\ddot{\phi}+\varepsilon(c/a)^2=0 \qquad (\phi=\ln\psi;\ \gamma=1), \tag{6·1}$$

$$\ddot{\phi}+(1-\gamma)\varepsilon(c/a)^2\phi=0 \qquad (\phi=\psi^{1-\gamma};\ \gamma\neq 1), \tag{6·2}$$

with

$$\sigma=\gamma\ddot{\psi}^{-4}(\dot{\psi}/c)^2, \tag{6·3}$$

where a dot denotes differentiation with respect to t.

(i) *The case* $\varepsilon=0$. In this case (6·1) and (6·2) are of the same form:

$$\ddot{\phi}=0,$$

so that

$$\phi=\alpha t \quad (\alpha>0)* \tag{6·4}$$

where α is an integration constant and another integration constant is absorbed by t. If we consider the definition of ϕ, (6·4) can be rewritten as

* We can assume this condition without loss of generality.

$$\psi=\begin{cases}\exp(\alpha t) & (\gamma=1),\\ (\alpha t)^{\frac{1}{1-\gamma}} & (\gamma\neq 1).\end{cases} \tag{6·5}$$

Now we are concerned with the cosmic expansion instead of the cosmic contraction, so that it is desirable to choose the sign of (2·6) so as to make $\Phi(\tau)$ of (2·1) an increasing function of the cosmic time $\tau(>0)$ with increasing τ. Then we can obtain from (2·6) with (6·5)

$$\Phi(\tau)=\begin{cases}\alpha\tau, & (\alpha\tau=\exp(\alpha t)), & (\gamma=1) & (6\cdot6)\\ \left\{\left|\dfrac{2-\gamma}{1-\gamma}\right|\alpha\tau\right\}^{\frac{1}{2-\gamma}}, & \left(\alpha\tau=\left|\dfrac{1-\gamma}{2-\gamma}\right|(\alpha t)^{\frac{2-\gamma}{1-\gamma}}\right), & (\gamma\neq 1,\ 2) & (6\cdot7)\\ \exp(\alpha\tau), & (\alpha\tau=-\ln(\alpha t)), & (\gamma=2) & (6\cdot8)\end{cases}$$

in which each integration constant is absorbed by τ without loss of generality. Inserting these quantities into (2·1) and (6·3), we obtain

$$\begin{cases} ds^2=c^2d\tau^2-(\alpha\tau)^2(dr^2+r^2d\Sigma^2)\ ; & \sigma=(c\tau)^{-2}, & (\gamma=1) & (6\cdot9)\\ ds^2=c^2d\tau^2-\left\{\dfrac{2-\gamma}{1-\gamma}\,\alpha\tau\right\}^{\frac{2}{2-\gamma}}(dr^2+r^2d\Sigma^2)\ ; & \sigma=(c\tau)^{-2}\dfrac{\gamma}{(2-\gamma)^2}, & (\gamma\neq 1,\ 2) & (6\cdot10)\\ ds^2=c^2d\tau^2-\exp(2\alpha\tau)\,(dr^2+r^2d\Sigma^2)\ ; & \sigma=2(\alpha/c)^2, & (\gamma=2) & (6\cdot11)\end{cases}$$

where $d\Sigma^2=d\theta^2+\sin^2\theta d\varphi^2$.

As is easily seen, (6·11) is nothing but the metric of de Sitter space-time with positive σ ($\sim$material density). Therefore it becomes clear that the steady-state cosmology is a special case of our theory specified by the condition ($\varepsilon=0$ & $\gamma=2$). On the other hand, (6·9) represents the linear expanding model which is not permissible in the general relativistic cosmology, and (6·10) gives us a family of expanding models similar to (6·9), whose particular case specified by $\gamma=0.5$ corresponds to Einstein-de Sitter model in the general relativistic cosmology.

(ii) *The case* $\varepsilon=1$. In this case, considering the definition of ϕ, we obtain from (6·1) and (6·2)

$$\psi=\begin{cases}\exp\{-\tfrac{1}{2}(ct/a)^2+\alpha t+\beta\}, & (\gamma=1)\\ \{\alpha\exp((ct/a)(\gamma-1)^{1/2})+\beta\exp((-ct/a)(\gamma-1)^{1/2})\}^{\frac{1}{1-\gamma}}, & (\gamma>1)\\ \{\alpha\cos((ct/a)(1-\gamma)^{1/2})+\beta\sin((ct/a)(1-\gamma)^{1/2})\}^{\frac{1}{1-\gamma}}, & (\gamma<1)\end{cases}, \tag{6·12}$$

where α and β are integration constants.

Generally speaking, the integration in (2·6) for finding out $\Phi(\tau)$ which corresponds to the above ψ is not elementary. But in the following cases $\Phi(\tau)$ can be expressed by elementary functions:

$$\begin{cases} (\gamma=2\,;\ \alpha=-\beta=\tfrac{1}{2}) : \Phi(\tau)=\sinh(c\tau/a),\ (c\tau/a=-\ln\tanh(ct/2a)), & (6\cdot13) \\ (\gamma=2\,;\ \alpha=\beta=\tfrac{1}{2}) : \Phi(\tau)=\sin(c\tau/a),\quad (c\tau/a=2\tan^{-1}(e^{ct/a})), & (6\cdot14) \\ (\gamma>1\,;\ \alpha=1,\ \beta=0 \text{ or } \alpha=0,\ \beta=1) : \Phi(\tau)=(c\tau/a)(\gamma-1)^{1/2}, & \\ \qquad [(c\tau/a)(\gamma-1)^{1/2}=\exp\{\mp(ct/a)(\gamma-1)^{1/2}\}], & (6\cdot15) \end{cases}$$

where each integration constant is absorbed by τ. The expressions (2·1) and (6·3) corresponding to the above become as follows:

$$\begin{cases} ds^2=c^2d\tau^2-\sinh^2(c\tau/a)(dr^2+a^2\sin^2(r/a)d\Sigma^{\prime 2})\ ;\ \sigma=(2/a^2)\coth^2(c\tau/a), & (6\cdot16) \\ ds^2=c^2d\tau^2-\sin^2(c\tau/a)(dr^2+a^2\sin^2(r/a)d\Sigma^{\prime 2})\ ;\ \sigma=(2/a^2)\cot^2(c\tau/a), & (6\cdot17) \\ ds^2=c^2d\tau^2-(\gamma-1)(c\tau/a)^2(dr^2+a^2\sin^2(r/a)d\Sigma^{\prime 2})\ ;\ \sigma=\gamma(c\tau)^{-2}. & (6\cdot18) \end{cases}$$

(6·16) is Lanczos' metric and (6·17) represents an oscillating model proper to our theory.

(iii) *The case* $\varepsilon=-1$. In this case we obtain the following results:

$$\psi=\begin{cases} \exp\{(1/2)(ct/a)^2+\alpha t+\beta\}, & (\gamma=1) \\ \{\alpha\cos((ct/a)(\gamma-1)^{1/2})+\beta\sin((ct/a)(\gamma-1)^{1/2})\}^{\frac{1}{1-\gamma}}, & (\gamma>1) \\ \{\alpha\exp((ct/a)(1-\gamma))^{1/2})+\beta\exp((-ct/a)(1-\gamma)^{1/2})\}^{\frac{1}{1-\gamma}}, & (\gamma<1) \end{cases} \quad (6\cdot19)$$

Especially if we seek for the case in which $\Phi(\tau)$ can be represented by an elementary function, we have

$$\left.\begin{aligned} &(\gamma=2) : \Phi(\tau)=(1/\alpha_0)\cosh(\alpha_0 c\tau/a), \\ &c\tau/a=\pm(1/\alpha_0)\ln|\tan((c/2a)(t-t_0))|, \end{aligned}\right\} \quad (6-20)$$

where

$$\alpha=-\alpha_0\sin(ct_0/a),\quad \beta=\alpha_0\cos(ct_0/a).$$

Then, from (2·1) and (6·3), the following metric is obtained:

$$\left.\begin{aligned} &ds^2=c^2d\tau^2-\{(1/\alpha_0)\cosh(\alpha_0 c\tau/a)\}^2(dr^2+a^2\sinh^2(r/a)d\Sigma^{\prime 2}), \\ &\sigma=2(\alpha_0/a)^2\tanh^2(\alpha_0 c\tau/a). \end{aligned}\right\} \quad (6\cdot21)$$

This is nothing but Lanczos' metric of the case $\varepsilon=-1$. But this model must be discarded because σ ($\sim$material density) is an increasing function of τ. And it is to be remarked that this circumstance appears for the first time in the case $\varepsilon=-1$.

§ 7. Discussion

Now, at the present stage of our knowledge, the redshift-magnitude relation of galaxies alone can be relied upon for testing immediately the appropriateness of model universes. In this relation come into question Hubble's constant h (cf. (4·2)) and the deceleration parameter $q\equiv h^{-2}\Phi^{-1}d^2\Phi/d\tau^2$ introduced by Sandage. According to Humason et al.[5] we have the following results:

$$h^{-1}=5.4\times10^{9} \text{ years}, \quad q=-3.0\sim-5.0. \tag{7·1}$$

On the other hand, Baum[6] obtained the result

$$q\approx-0.5\sim-1.5, \tag{7·2}$$

taking advantage of the far more extensive data thanks to a new technique, and pointed out that the q-value of (7·1) contained error. Sandage[7] has recently clarified, however, after a close examination of the cepheids and the brightest star criteria as the distance indicators of the extragalactic space that (1) q-value is more reliable than h-value and that (2) possible range of $1/h$ can be estimated "with very large uncertainty" as

$$h^{-1}\approx(9.9\sim20.0)\times10^{9} \text{ years}. \tag{7·3}$$

So long as such a large permitted limit was suggested for h^{-1}, it is natural to consider that the absolute value of q due to Baum is not necessarily decisive. Therefore we take tentatively the sign of q alone as trustworthy, i.e. we assume that, in place of (7·2),

$$q<0. \tag{7·4}$$

Then how do matters stand in the case of model universes obtained in our theory?

(i) *The case* $\varepsilon=0$. In this case

$$\begin{cases} h=\tau^{-1};\ q=0 & (\gamma=1), \qquad (7\cdot5)\\ h=\tau^{-1}/(2-\gamma);\ q=(\gamma-1) & (\gamma\neq1,\ 2), \qquad (7\cdot6)\\ h=\alpha;\ q=1 & (\gamma=2). \qquad (7\cdot7) \end{cases}$$

The sign of h (at present) should be positive, because h is the parameter specifying the rate of the cosmic expansion. Accordingly it must be at least that

$$0\ll\gamma\leq2, \tag{7·8}$$

because γ is essentially positive (~1).

It is easily seen that (7·5) and (7·7) must be discarded because they are inconsistent with (7·4). On the contrary, it is not the case for (7·6), so long as the following inequality is satisfied:

$$\gamma<1. \tag{7·9}$$

From this and (7·8) we obtain

$$0\ll\gamma<1. \tag{7·10}$$

The case specified by (7·6) corresponds, however, to the exploding model in contrast with the steady-state model. Then the "age of the universe" is the time interval from $\tau=0$ to present value of τ, i.e.

$$\text{"Age of the universe"}=h^{-1}/(2-\gamma), \tag{7·11}$$

where h^{-1} takes its present value. If we take account of (7·10) and (7·11), the following inequality is obtained:

$$h^{-1} > \text{“Age of the universe”} \geqslant \tfrac{1}{2} h^{-1}. \tag{7·12}$$

Accordingly, if we take the minimum and maximum values of (7·3), we have

$$\left\{\begin{matrix} 9.9 \\ \text{or} \\ 20.0 \end{matrix}\right\} > \text{“Age of the universe” in } 10^9 \text{ years units} \geqslant \left\{\begin{matrix} 4.9 \\ \text{or} \\ 10.0 \end{matrix}\right\}. \tag{7·13}$$

It is clear that the above inequality does not contradict with the age of the earth ($\sim 4\times 10^9$ years)[8] and that of heavy elements ($\sim 7\times 10^9$ years).[9]

(ii) *The case* $\varepsilon=1$. In this case if we take the special models specified by (6·13)–(6·15), (6·15) must be discarded because it is similar to (6·6), and for the other two models the following relations are obtained:

$$\begin{cases} h=(c/a)\coth(c\tau/a)\ ;\ q=\tanh^2(c\tau/a), & (\gamma=2\ ;\ \alpha=-\beta=\tfrac{1}{2}), \qquad (7·14) \\ h=(c/a)\cot(c\tau/a)\ ;\ q=-\tan^2(c\tau/a), & (\gamma=2\ ;\ \alpha=\beta=\tfrac{1}{2}). \qquad (7·15) \end{cases}$$

As is easily seen, (7·15) is consistent with (7·4) (& (7·2)), while (7·14) is not. Inserting (7·3) into h of (7·15), we obtain the following relation:

$$\tau=(a/c)\tan^{-1}\{(c/a)\,(9.9\sim 20.0)\times 10^9 \text{ (years)}\}. \tag{7·16}$$

As the symbol a denotes the radius of spatial curvature of the universe, it is at least of the order of 10^9 light-years.* Therefore it is clear that the age of the universe estimated from (7·16) is consistent with that of both stars and heavy elements.

(iii) *The case* $\varepsilon=-1$. In this case h and q cannot be obtained without numerical integration, so that we shall omit this case. But it seems that the contradictory situation appeared in the special case (6·21) suggests the inadequacy of this type.

The above consideration is concerned with the test in terms of the redshifts-magnitude relation for model universes. The next problem is how to treat the gravitational condensation. The reason why this problem is significant in cosmology is that it is of the first importance for the discussion of the formation of galaxies. But it must be remarked that the importance of the problem does not consist solely in this point. Because, for testing an appropriateness of any model universe, it would be useful whether gravitational condensation would take place in harmony with the estimation of the age of stars. This is the case, even when such a situation arises by any chance that the sign of q should be positive in such a way that the steady-state model (say) would not be hopeless.

Considering the above circumstances, we shall, in the next paper, deal with

* The numerical value of a can in principle be deduced from the analysis of nebular counts.

the problem of gravitational condensation in terms of the fundamental equations (5·4) and (4·5). In so doing, the correspondence principle of Newtonian gravitation mentioned previously will play an important role. Such an investigation appears to be useful to estimate not only the reliability of the inequality $q<0$, but also the possible value of γ.

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On The Analogy between Strong Interaction and Electromagnetic Interaction

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According to Sakata's composite hypothesis of heavy particles, we assume that the fundamental particles do not change their kind through strong interactions. A formulation of these interactions is given by requiring the invariance under the gauge transformation of the first kind applied to each baryon field. If we assume that the phase function depends on the space-time coordinate, it leads to the existence of a neutral vector meson analogous to the electromagnetic field the mass of which need not always vanish. Its consequences are examined and found to be quite favourable for the interpretation of the strong interaction. In particular, it leads to the conservation law of parity in strong interactions and the condition for the existence of the composite state.

§ 1. Introduction

According to the composite hypothesis of heavy particles proposed by Sakata[1-4] the following interpretation is possible. That is, the proton, neutron and Λ-particle which are assumed to be the fundamental particles do *not* change their kind through the strong interactions, while they are transformed into each other through the weak interactions. These circumstances are shown schematically in Fig. 1.

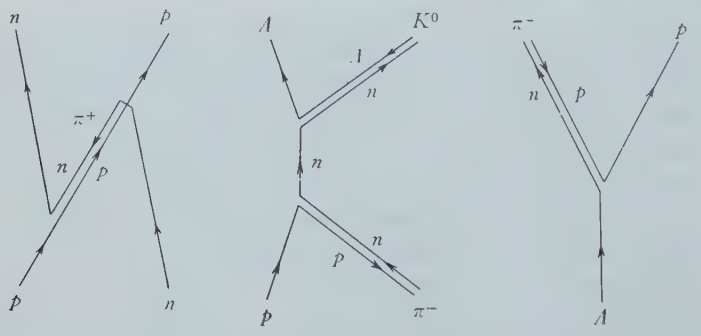

Fig. 1.

The selection rule of this kind is very useful for the lepton processes also. As an example, consider the process

$$K^+ \to e^+ + e^- + \pi^+. \tag{1}$$

Such processes, which contain a pair of lepton and its antiparticle cannot be forbidden by the selection rules which follow from the conservation laws of so-called lepton number and neutrino charge. It has been proposed that the neutrino is the source of weak interactions. We cannot, however, take it as reasonable, because weak processes without neutrinos, such as hyperon and K-meson decays, are really observed. From the fact that the lepton-pair processes like (1) have never been observed at all,* it is necessary to assume a selection rule that a lepton-pair should not couple to any field except for the elctromagnetic field. The above selection rule leads also to the rule that a lepton should always change its kind through the interaction unless some special non-causal interaction is assumed.

On the other hand, we know the electromagnetic intraction as a prototype of interaction through which the kind of particle does not change. As for this interaction there is a remarkable feature that the interaction Lagrangian follows directly from the free Lagrangian by the requirement of invariance under the gauge transformation.

In the following it is attempted to apply this circumstance also to the strong interaction and its consequences are examined.

§ 2. Requirement of gauge invariance and its consequences

Analogous to the case of the electromagnetic interaction, consider the gauge transformation of the first kind which depends on the space-time coordinate

$$\begin{aligned} &\psi_\alpha(x) \to e^{ig_\alpha \Lambda(x)} \psi_\alpha(x), \\ &\bar{\psi}_\alpha(x) \to e^{-ig_\alpha \Lambda(x)} \bar{\psi}_\alpha(x). \\ &(\alpha = p,\ n,\ \Lambda) \end{aligned} \tag{2}$$

If the invariance under these transformations is required, then by the replacement in the free Lagrangian,

$$\partial_\mu \psi_\alpha(x) \to (\partial_\mu - ig_\alpha A_\mu(x))\psi_\alpha(x),$$

the interaction Lagrangian must be introduced in the form

$$j_\mu(x) A_\mu(x), \tag{3}$$

where

$$j_\mu(x) = \sum_\alpha i g_\alpha \bar{\psi}_\alpha(x) \gamma_\mu \psi_\alpha(x). \tag{4}$$

$A_\mu(x)$ is a neutral vector field. Since immediate contradictions to the recent observations will occur as to the vanishing mass, we consider the mass to be non-vani-

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* It is, of course, expected that the electromagnetic correction produces the processes like (1) at the ratio $\alpha^2 \sim 10^{-4}$ to the ordinary weak processes.

shing by the use of the Stückelberg formalism for the neutral vector field.[5]

That is, by introducing a scalar field $B(x)$ in addition to $A_\mu(x)$, the theory is invariant under (2) together with the gauge transformation of the second kind,

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x),$$
$$B(x) \to B(x) - m\Lambda(x), \tag{5}$$

where m denotes the mass of the meson, and $\Lambda(x)$ satisfies the equation

$$(\Box^2 - m^2)\Lambda(x) = 0.$$

This formalism is shown to be equivalent to the ordinary one for the neutral vector field (See Appendix).

We assume that the interaction Lagrangian given by (3) is the only elementary interaction among the strong interactions. This is reminiscent of the electromagnetic interaction and following considerations will be possible.

(i) *Charge independence*

We know that the phenomena of strong interaction is successfully accounted for by the hypothesis of charge independence. Therefore the interaction Lagrangian (3) must be charge independent.

Since the meson here introduced is of single component, the vanishing iso-spin should be assigned. Then, the part of (3) and (4) which contains the proton and neutron, must be the iso-scalar of the form

$$\bar{\psi}_N(x)\gamma_\mu\psi_N(x)A_\mu(x),$$

where $\psi_N(x)$ denotes the usual iso-doublet

$$\psi_N(x) = \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix}.$$

This implies

$$g_p = g_n \tag{6}$$

in (4).

(ii) *Existence of composite particles*

The force resulting from the exchange of mesons here introduced is of opposite sign between baryon and antibaryon as compared with that between baryon and baryon. This is the immediate consequence of the vector coupling. Furthermore, between the particles of the same kind (e.g. p-$\bar{p}$, p-p), the former is attractive and the latter repulsive as long as the adiabatic approximation is valid. This is an exact analogue of the Coulomb interaction (See Appendix, (A. 7)).

As for p-$\bar{n}$, n-$\bar{p}$; p-n, the same is the case owing to (6). This allows the existence of pions as the bound state of nucleon and antinucleon.

Between nucleon and Λ-particle, it is concluded that

$$g_p = g_n \text{ and } g_\Lambda \text{ are of the same sign,} \tag{7}$$

as long as the K-meson exists as the bound state of nucleon and anti-Λ-particle, and no bound state exists between nucleon and Λ-particle.

(iii) *Invariance under space reflection and charge conjugation.*

Since the interaction Lagrangian (3), which is assumed to be the only elementary interaction derived from gauge invariance, does not allow the coexistence of γ_μ and $\gamma_\mu\gamma_5$, it is possible to assign the definite parity to $A_\mu(x)$, irrespective of the relative parity between baryons. Thus, the parity should be conserved in all strong interactions. The same consideration leads also to the invariance under the charge conjugation in strong interactions.

(iv) *Conservation law of baryon number.*

In addition to (6) and (7), we further assume

$$g_p = g_n = g_\Lambda = g, \tag{8}$$

and call it "baryonic charge". This charge has the opposite sign for antibaryon and the conservation law of this charge agrees exactly with the conservation law of so-called baryon number.*

Usually, in order to formulate the conservation law of baryon number in the frame of field theory, it is necessary to require the invariance under the phase transformation which is constant and does not depend on the space-time coordinate. It would, however, be more reasonable for the phase function to depend on the space-time coordinate, from the local character of the field quantities. Although some authors attempted to pursue the analogy between the conservation law of baryon number and that of electric charge and to relate the baryon number to the strength of the strong interaction, they did not always succeed because the associated Bose field was considered to be identified to pion field.[6]

It is remarked that in the above theory the conservation law of strangeness, or of Λ-particle is not the most general consequence of the requirement of gauge invariance. It leads merely to the conservation law of baryon number, or the sum of the number of neutron and Λ-particle. The required conservation law is guaranteed by the more special requirement that the interaction (3) should be the only elementary interaction between fundamental particles. Otherwise, it is necessary to introduce another neutral vector field associated with "strangeness charge", which might cause the mass difference between nucleon and Λ-particle.

These above considerations suggest the possible existence of a close analogy between the electromagnetic interaction and the strong interaction from the point of view of Sakata's hypothesis.

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* For the leptons, $g=0$ should be assigned, the small electromagnetic correction being neglected.

§ 3. Comparison with other model

(i) Tanaka,[2] Maki,[3] and Matsumoto[4] introduced the following forms as the elementary interactions:

$$\sum_{\lambda=1}^{5} f_1^{\lambda}(\bar{N}\Omega^{\lambda}N)(\bar{N}\Omega^{\lambda}N), \quad \sum_{\lambda=1}^{5} f_1'^{\lambda}(\bar{N}\Omega^{\lambda}\tau N)(\bar{N}\Omega^{\lambda}\tau N),$$

$$\sum_{\lambda=1}^{5} f_2^{\lambda}(\bar{N}\Omega^{\lambda}N)(\bar{\Lambda}\Omega^{\lambda}\Lambda),$$

$$\sum_{\lambda=1}^{5} f_3^{\lambda}(\bar{\Lambda}\Omega^{\lambda}\Lambda)(\bar{\Lambda}\Omega^{\lambda}\Lambda),$$

$$N=\begin{pmatrix} p \\ n \end{pmatrix}. \tag{9}$$

Fermi and Yang,[7] and Tanaka[2] showed that the vector coupling, especially the product of iso-scalars, is the most favourable for the existence of the bound state only between baryon and antibaryon. In the present model, if we construct the effective Fermi couplings like (9), the only possibility is evidently the vector coupling. The necessary sign is also supplied (See § 2. (ii)). While in the models proposed by the above authors there is no principle for determining the relation between f_1^{λ}, $f_1'^{\lambda}$, f_2^{λ} and f_3^{λ}, we can determine as follows:

$$f_1^{V}=f_2^{V}=f_3^{V},^{*}$$
$$\text{others}=0, \tag{10}$$

on the basis of its analogy with the electromagnetic interaction.

(ii) Our model is not a theory of Fermion monism which is possible in Sakata's hypothesis. It is noted, however, that the neutral vector fields including the electromagnetic field are the special ones derived from the requirement of gauge invariance, which should not be considered on the same footing as pions and K-mesons.

(iii) According to Fermi and Yang,[7] and Tanaka,[2] who replaced the interaction Hamiltonian (9) by the square-well potential approximately, the potential depth necessary for the existence of pion state amounts to $\sim 26\,M$, where M is the nucleon mass and the nucleon Compton wave length is chosen as the potential range. From this a very crude estimate of the value of g^2 is obtained as

$$g^2/4\pi \sim 26, \tag{11}$$

where the meson mass is assumed to be the order of nucleon mass, though the value may depend significantly on the choice of the range, that is, the mass of the meson. If the relativistic two body problem were solved, the values of g^2 together with the mass could be determined in principle so as to give the correct values of pion mass and the effective pion-baryon coupling constant.

(iv) In the present model, the strong interaction is renormalizable in contrast with

* Mass difference between nucleon and Λ-particle is ignored.

the model starting from (9).

§ 4. Concluding remarks

We have presented a model for the strong interactions, making the best use of the assumption that the fundamental particles do not change their kind through strong interactions. Thus, analogous to the electromagnetic interactions, we have expected that the interaction Lagrangian under which the kind of particles does not change should have a close connection with the free Lagrangian, and have considered that the requirement of gauge invariance is an expression of such a connection. In this way we have been led to the introduction of a neutral vector meson.

The electromagnetic interaction has so far been distinguished from others in that it is invariant under the gauge transformation. As the result of this invariance the form of the interaction Lagrangian is restricted and the electric charge of each particle is equal. According to our formulation, these points are applied also to the strong interactions, at least to the basic ones. In this sense the electromagnetic interaction becomes less special and we could expect to get the unified view-point of the strong interactions including the electromagnetic interaction.

On the other hand, the difference between strong interactions and weak ones becomes more distinctive. Since the kind of particles always changes through the weak interactions as explained in § 1, the interaction Lagrangian for weak interactions can never be obtained from the requirement such as gauge invariance. From our standpoint, therefore, strong and weak interactions must be of quite different origins. It should also be noted that we have found a reason for the parity conservation in strong interactions but no reason in weak ones.

On account of the large coupling constant roughly estimated in (11), the meson here considered will be produced copiously in high energy collision processes and will decay rapidly into other particles (e.g. a number of pions*). In general it is expected to produce some effects both on high energy reactions and virtual processes. In this respect this meson is similar to that introduced phenomenologically by Nambu[8] in order to account for the electromagnetic size of nucleon.

In conclusion, the author wishes to express his sincere thanks to Professor S. Sakata, Professor O. Hara and Dr. T. Goto for their valuable discussions.

Appendix

— On the Stückelberg formalism —

Lagrangian is given by

$$L = L_0 + L_1 + L_F,$$

* In the case of the decay into pions, decays into π^0's only and those into even number of pions are forbidden from the charge conjugation invariance and charge independence.

where

$$L_0 = -(1/4) f_{\mu\nu} f_{\mu\nu} - (1/2) m^2 A_\mu{}^2 - m A_\mu (\partial_\mu B) - (1/2)(\partial_\mu B)^2 - (1/2)\chi^2,$$

$$L_1 = j_\mu A_\mu,$$

$$L_F = -\sum_{\alpha=p,n,\Lambda} \bar{\psi}_\alpha (\gamma_\mu \partial_\mu + M_\alpha)\psi_\alpha, \qquad (A\cdot 1)$$

where

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$j_\mu = \sum_\alpha i g_\alpha \bar{\psi}_\alpha \gamma_\mu \psi_\alpha,$$

$$\chi = \partial_\mu A_\mu + mB, \qquad (\Box^2 - m^2)\chi = 0.$$

In addition to (A·1) the subsidiary condition must be satisfied,

$$\chi \Psi = 0. \qquad (A\cdot 2)$$

The gauge transformation is given by

$$\psi_\alpha(x) \to e^{i g_\alpha \Lambda(x)} \psi_\alpha(x),$$

$$\bar{\psi}_\alpha(x) \to e^{-i g_\alpha \Lambda(x)} \bar{\psi}_\alpha(x),$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x),$$

$$B(x) \to B(x) - m\Lambda(x), \qquad (A\cdot 3)$$

where

$$(\Box^2 - m^2)\Lambda(x) = 0$$

is satisfied.

Lagrangian (A·1) and the subsidiary condition (A·2) is invariant under this transformation. The form for L_0 given above is particularly convenient since the invariance is shown explicitly when put into the form

$$L_0 = -(1/4) F_{\mu\nu} F_{\mu\nu} - (1/2) m^2 U_\mu{}^2 - (1/2)\chi^2,$$

where

$$F_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu,$$

$$U_\mu = A_\mu + (1/m)\partial_\mu B, \qquad (A\cdot 4)$$

which are evidently invariant.

From (A·1) the equations of motion are derived as follows,

$$(\Box^2 - m^2) A_\mu(x) = -j_\mu(x),$$

$$(\Box^2 - m^2) B(x) = 0,$$

$$\gamma_\mu (\partial_\mu - i g_\alpha A_\mu(x))\psi_\alpha(x) + M_\alpha \psi_\alpha(x) = 0.$$

From the invariance of (A·1) under (A·3), the conservation law of current is derived in the well-known procedure, i.e.

$$\begin{aligned}0=\delta L(x)=&[-\partial_\mu j_\mu(x)+m(\Box^2-m^2)B(x)]\delta\Lambda(x)\\&-[(\Box^2-m^2)A_\mu(x)+j_\mu(x)]\partial_\mu\delta\Lambda(x)\\&-f_{\mu\nu}(x)\partial_\mu\partial_\nu\delta\Lambda(x)\\=&-(\partial_\mu j_\mu(x))\delta\Lambda(x),\end{aligned}$$

thus

$$\partial_\mu j_\mu(x)=0. \qquad (A\cdot5)$$

The equations (A·4) represent the relation of this formalism to the conventional one. The interaction Lagrangian is given usually by

$$j_\mu(x)U_\mu(x),$$

which can be put into the form

$$j_\mu(x)A_\mu(x)+(1/m)j_\mu(x)\partial_\mu B(x).$$

The second term is reduced to a 4-divergence owing to (A·5) and has no effect at all, suggesting the equivalence of the two formalisms.

We go further by developing the canonical formalism. Introducing the canonical momenta

$$\begin{aligned}\pi_\mu(x)&=if_{4\mu}(x)+i\delta_{4\mu}\chi(x),\\\pi_B(x)&=\dot{B}(x)+imA_4(x),\\\pi_\alpha(x)&=i\psi_\alpha^*(x),\end{aligned}$$

we can obtain the Hamiltonian,

$$\begin{aligned}H=&(1/2)\pi^2+(1/2)\,(\mathrm{curl}\,\boldsymbol{A})^2+(1/2)m^2\boldsymbol{A}^2+(1/2)\pi_B^2\\&+(1/2)\,(\boldsymbol{\nabla}B)^2+i\boldsymbol{\pi}\cdot(\boldsymbol{\nabla}A_4)-imA_4\pi_B+m\boldsymbol{A}\cdot\boldsymbol{\nabla}B\\&-j_\mu A_\mu+\sum_\alpha\psi_\alpha^*(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+M_\alpha\beta)\psi_\alpha.\end{aligned}$$

We perform the unitary transformation

$$\begin{aligned}&F(x)\to e^{iG}\,F(x)e^{-iG},\\&G=(1/m)\int B(x)[\mathrm{div}\,\boldsymbol{\pi}(x)-ij_4(x)]\,d\boldsymbol{x},\end{aligned}$$

then

$$(1/m)\dot{\chi}(x)\to\pi_B(x).$$

π_B in the new representation is eliminated by using

$$\dot{\chi}\,\Psi=0,$$

then the Hamiltonian reduces to

$$
\begin{aligned}
&H=H_0+H'+H_1+H_F,\\
&H_0=(1/2)\boldsymbol{\pi}^2+(1/2)\,(\mathrm{curl}\ \boldsymbol{A})^2+(1/2)\,m^2\boldsymbol{A}^2+(1/2m^2)\,(\mathrm{div}\ \boldsymbol{\pi})^2,\\
&H'=(1/m^2)\,j_0\ \mathrm{div}\ \boldsymbol{\pi}+(1/2m^2)\,j_0^2,\\
&H_1=-\boldsymbol{j}\cdot\boldsymbol{A},\\
&H_F=\sum_\alpha \psi_\alpha^*(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+M_\alpha\beta)\psi_\alpha,
\end{aligned}
\tag{A·6}
$$

apart from space divergences. This is the same form as the one derived from the conventioanl formalism (for example, in Wentzel's book[9]), proving the equivalence rigorously.

We further perform the unitary transformation,

$$
\begin{aligned}
&F(x)\to e^{iG'}\,F(x)\,e^{-iG'},\\
&G'=\int d\boldsymbol{x}\int d\boldsymbol{x}'\ j_0(x)\,U(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{div}\,\boldsymbol{A}(x'),\\
&U(\boldsymbol{x})=(1/4\pi)\,(e^{-m|\boldsymbol{x}|}/|\boldsymbol{x}|),
\end{aligned}
$$

then

$$
H_0+H'\to H_0+H_C,
$$

where

$$
\int H_C(x)\,d\boldsymbol{x}=(1/2)\int d\boldsymbol{x}\int d\boldsymbol{x}'\ j_0(x)\,U(\boldsymbol{x}-\boldsymbol{x}')\,j_0(\boldsymbol{x}').
\tag{A·7}
$$

which is the analogue of the Coulomb interaction. It is noted that there remains the longitudinal part even at this stage, in contrast with the case of the electromagnetic interaction.

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Quantum Field Theory in terms of Euclidean Parameters

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The quantum field theory is presented in terms of Euclidean parameters. It is shown that this theory is equivalent to the conventional one in terms of the Minkowski parameters, if we confine ourselves to the case of local fields, and it is suggested that the former is more adequate to describe an internal structure of an elementary particle than the latter. There is no necessity that the parameter space of field theory is Minkowski, because it has no direct connection with the macroscopic space. It is enough that expectation values of observable quantities are covariant under Lorentz transformations. The theory shown in this paper satisfies this requirement.

§ 1. Introduction

In the conventional quantum field theory the parameters of the Minkowski space are used to describe behaviors of field variables. This is based on the fact that the macroscopic space we live in is the Minkowski one. But the parameters in field theory are not the quantities that have direct connection with the coordinates of the macroscopic space. The former are connected with the latter only through field variables. Under these circumstances there is no necessity to use the Minkowski parameters to describe microscopic states.

It is enough that the expectation values are covariant with respect to the Lorentz transformations, and it is already shown that the internal structure of an extended particle is described by the Euclidean parameters, but expectation values of observable quantities are covariant with respect to Lorentz transformations.[1)]

In this paper we propose to use the Euclidean parameters to describe microscopic states and show that this theory is equivalent to the conventional one if we confine ourselves to local fields.

When we consider the internal structure of elementary particles, the Minkowski space is very inconvenient, because invariant domains are unbounded. On the contrary, the Euclidean space seems to be adequate to describe the internal structure, as invariant domains are bounded. If the field theory in terms of Euclidean parameters and the theory in terms of Minkowski parameters are equivalent to each other to describe the local fields and the former is more adequate to describe the structure of elementary particles than the latter, we should conclude that field theory should be reconstructed by the use of Euclidean parameters.

Possibilities that we can use the Euclidean parameters instead of the Minkowski

ones have so far appeared in several aspects. On the one hand, tensors and spinors are proper to the Euclidean space, but on the other hand, it is expansors[2] and expinors[3] that are proper to the Minkowski space. From the fact that we know no physical quantity which is represented by an expansor or expinor, it seems enough to use the Euclidean parameters exclusively. Besides, the fact that the Feynman integral[4]

$$\lim_{\varepsilon\to+0}\int d\boldsymbol{k}\int_{-\infty}^{\infty}dk_0(\boldsymbol{k}^2-k_0^2+L-i\varepsilon)^{-n}$$

is equal to the integral with respect to the Euclidean parameters

$$i\int d\boldsymbol{k}\int_{-\infty}^{\infty}dk_4(\boldsymbol{k}^2+k_4^2+L)^{-n}$$

and the fact that the internal motion of two bound particles is described by the Euclidean parameters[5] seem to suggest the possibility of constructing field theory with the Euclidean parameters.

In § 2 we investigate the analytical properties of vacuum expectation values and show that, if we know the vacuum expectation values in terms of Euclidean parameters, we can derive the ones in terms of Minkowski parameters, and vice versa. In § 3 the transformation properties of field variables described by the Euclidean parameters in accordance with the macroscopic Lorentz transformations are discussed. In § 4 the variational principle is displayed and it is shown that the transformation properties of the energy momentum and charge are same as the conventional ones. In § 5 and § 6 we discuss a scalar and a spinor field, respectively, and show that the quantization method and the eigenvalues of the energy, momentum and charge agree with the ones of the customary theory in terms of the Minkowski parameters.

§ 2. Properties of vacuum expectation values of time-ordered field variable products

The properties of vacuum expectation values of field variables were throughly investigated by Wightman.[6] In this section we discuss the analytic properties of vacuum expectation values of time-ordered products of field variables.[7] For simplicity, we treat the case of a neutral scalar field interacting with itself in this section. In succeeding sections we shall treat more general cases.

First, let us consider the product of two variables $\varphi(\boldsymbol{x}_1,\ t_1)$ and $\varphi(\boldsymbol{x}_2,\ t_2)$. The vacuum expectation value is given by[8]

$$\begin{aligned}(\Psi_0,\ \varphi(\boldsymbol{x}_1,\ t_1)\varphi(\boldsymbol{x}_2,\ t_2)\Psi_0) &= i\Delta'^{(+)}(\boldsymbol{x}_1-\boldsymbol{x}_2,\ t_1-t_2)\\ &= -i\Delta'^{(-)}(\boldsymbol{x}_2-\boldsymbol{x}_1,\ t_2-t_1)\end{aligned}\qquad(2\cdot1)$$

where*

$$\Delta'^{(\pm)}(\boldsymbol{x},\ t)=\int_0^{\infty}\Delta^{(\pm)}(\boldsymbol{x},\ t)\rho(m^2)\,d(m^2),$$

$$\Delta^{(\pm)}(\boldsymbol{x},\ t)=\mp\frac{i}{(2\pi)^3}\int d\boldsymbol{k}(2E)^{-1}\exp[i\boldsymbol{k}\boldsymbol{x}\mp iEt], \qquad (2\cdot2)$$

$$E=\sqrt{\boldsymbol{k}^2+m^2}$$

and Ψ_0 is the true vacuum state vector. The relation (2·2) shows that $\Delta^{(+)}(\boldsymbol{x},\ t)$ is analytic when Im $t<0$, and $\Delta^{(-)}$ is analytic when Im $t>0$. Thus, if we introduce the notation

$$z_\theta=\cos\theta\cdot t-i\sin\theta\cdot x_4, \qquad (2\cdot3)$$

$$0<\theta<\pi$$

where t and x_4 are real, $\Delta^{(+)}(\boldsymbol{x},\ z_\theta)$ and $\Delta^{(-)}(\boldsymbol{x},\ z_\theta)$ are analytic when $x_4>0$ and $x_4<0$, respectively. And $\Delta^{(\pm)}(\boldsymbol{x},\ z_\theta)$ is derived from $\Delta^{(\pm)}(\boldsymbol{x},\ t)$ by the relation

$$\Delta^{(\pm)}(\boldsymbol{x},\ z_\theta)=Z(t,\ x_4;\ \theta)\Delta^{(\pm)}(\boldsymbol{x},\ t)$$

$$=\mp\frac{i}{(2\pi)^3}\int d\boldsymbol{k}(2E)^{-1}\exp[i\boldsymbol{k}\boldsymbol{x}\mp iEz_\theta], \qquad (2\cdot4)$$

in which

$$Z(t,\ x_4;\ \theta)=\exp[-i\theta(x_4\partial_t+t\partial_4)] \qquad (2\cdot5)$$

and

$$\partial_t=\partial/\partial t,$$

$$\partial_4=\partial/\partial x_4.$$

Especially when $\theta=\pi/2$, and $x_4>0$ for $\Delta^{(+)}$ and $x_4<0$ for $\Delta^{(-)}$, $\Delta^{(\pm)}(\boldsymbol{x},\ z_\theta)$ results in (see Appendix I).

$$\Delta^{(\pm)}(x_\mu)=\Delta^{(\pm)}(\boldsymbol{x},\ -ix_4)$$

$$=\mp\frac{i}{(2\pi)^3}\int d\boldsymbol{k}(2E)^{-1}\exp[i\boldsymbol{k}\boldsymbol{x}\mp Ex_4]$$

$$=\mp\frac{i}{(2\pi)^4}\int d^4k\frac{\exp[ik_\mu x_\mu]}{k_\mu k_\mu+m^2},\quad x_4\gtrless 0 \qquad (2\cdot6)$$

where

$$\int d^4k=\int_{-\infty}^{\infty}dk_1\int_{-\infty}^{\infty}dk_2\int_{-\infty}^{\infty}dk_3\int_{-\infty}^{\infty}dk_4$$

---

* The units $\hbar=c=1$ are used throughout this paper.

and

$$k_\mu x_\mu = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 .$$

We call the set of real variables (x_1, x_2, x_3, x_4) *Euclidean parameters and the space, whose coordinates are given by* (x_1, x_2, x_3, x_4), *Euclidean parameter space throughout this paper.*

When interactions are absent, $\varphi(\boldsymbol{x}, t)$ is given by

$$\varphi_0(\boldsymbol{x}, t) = V^{-1/2} \sum_k (2E)^{-1/2} \{\varphi_0(\boldsymbol{k}) \exp[i(\boldsymbol{kx} - Et)] + \varphi_0^*(\boldsymbol{k}) \exp[i(-\boldsymbol{kx} + Et)]\}, \tag{2·7}$$

where V is the normalization volume, and $\varphi_0(\boldsymbol{k})$ and $\varphi_0^*(\boldsymbol{k})$ satisfy the following commutation relations.*

$$[\varphi_0(\boldsymbol{k}), \varphi_0^*(\boldsymbol{k}')]_- = \delta_{\boldsymbol{k}\boldsymbol{k}'}$$
$$[\varphi_0(\boldsymbol{k}), \varphi_0(\boldsymbol{k}')]_- = [\varphi_0^*(\boldsymbol{k}), \varphi_0^*(\boldsymbol{k}')]_- = 0, \tag{2·8}$$

in which

$$\delta_{\boldsymbol{k}\boldsymbol{k}'} = \begin{cases} 1, & \text{when } \boldsymbol{k} = \boldsymbol{k}' \\ 0, & \text{when } \boldsymbol{k} \neq \boldsymbol{k}'. \end{cases}$$

When one operates $\varphi_0(\boldsymbol{k})$ on the vacuum state, it vanishes:

$$\varphi_0(\boldsymbol{k}) \Psi_0 = 0. \tag{2·9}$$

The summation $\sum_k$ extends to unlimitedly large values of $\boldsymbol{k}$, so that $\varphi(\boldsymbol{x}, z_\theta)$ has no analytic form in general. However, $\varphi(\boldsymbol{x}, z_\theta)$ is an operator and it is sufficient for our practical calculations that matrix elements of $\varphi(\boldsymbol{x}, z_\theta)$ or products of $\varphi(\boldsymbol{x}, z_\theta)$'s are analytic in some regions of z_θ. For example, one body amplitude

$$(\Phi_0, \varphi(\boldsymbol{x}, z_\theta) \Phi_k) = V^{-1/2} (2E)^{-1/2} \exp[i(\boldsymbol{kx} - Ez_\theta)] \tag{2·10}$$

is analytic everywhere in the z_θ-plane except as $x_4 \to -\infty$. Φ_0 and Φ_k are the free vacuum and one body state vectors, respectively. (See Appendix II). Thus we may write $\varphi_0(\boldsymbol{x}, z_\theta)$ as

$$\varphi_0(\boldsymbol{x}, z_\theta) = Z(t, x_4; \theta) \varphi_0(\boldsymbol{x}, t)$$
$$= V^{-1/2} \lim_{K \to \infty} \sum_k^K (2E)^{-1/2} \{\varphi_0(\boldsymbol{k}) \exp[i(\boldsymbol{kx} - Ez_\theta)] + \varphi_0^*(\boldsymbol{k}) \exp[i(\boldsymbol{kx} + Ez_\theta)]\}, \tag{2·11}$$

here $\sum_k^K$ indicates the summation up to $|\boldsymbol{k}| = K$ and the limit $K \to \infty$ must be taken after completion of calculations. By the use of the relations (2·8) and (2·9) with expression (2·11) we get

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* The bracket symbol is defined by $[A, B]_\pm = AB \pm BA$.

$$(\Phi_0,\ \varphi_0(\boldsymbol{x}_1,\ z_{1\theta})\varphi_0(\boldsymbol{x}_2,\ z_{2\theta})\Phi_0)$$
$$=V^{-1}\lim_{K\to\infty}\sum_{|\boldsymbol{k}|}^{K}(2E)^{-1}\exp[i\boldsymbol{k}(\boldsymbol{x}_1-\boldsymbol{x}_2)-iE(z_{1\theta}-z_{2\theta})]$$
$$=(2\pi)^{-3}\int d\boldsymbol{k}(2E)^{-1}\exp[i\boldsymbol{k}(\boldsymbol{x}_1-\boldsymbol{x}_2)-iE(z_{1\theta}-z_{2\theta})]$$
$$=i\Delta^{(+)}(\boldsymbol{x}_1-\boldsymbol{x}_2,\ z_{1\theta}-z_{2\theta}). \tag{2·12}$$

When $\theta=\pi/2$, (2·11) and (2·12) become

$$\varphi(x_\mu)=\varphi(\boldsymbol{x},\ -ix_4)$$
$$=V^{-1/2}\lim_{K\to\infty}\sum_{|\boldsymbol{k}|}^{K}(2E)^{-1/2}\{\varphi(\boldsymbol{k})\exp[i\boldsymbol{k}\boldsymbol{x}-Ex_4]$$
$$+\varphi^*(\boldsymbol{k})\exp[i\boldsymbol{k}\boldsymbol{x}+Ex_4]\} \tag{2·13}$$

and

$$(\Phi_0,\ \varphi_0(x_{1\mu})\varphi_0(x_{2\mu})\Phi_0)=i\Delta^{(+)}(x_{1\mu}-x_{2\mu}). \tag{2·14}$$

Next we consider the vacuum expectation value of the time-ordered product of two field variables

$$(\Psi_0,\ T(\varphi(\boldsymbol{x}_1,\ t_1),\ \varphi(\boldsymbol{x}_2,\ t_2))\Psi_0)=1/2\,\Delta_F'(\boldsymbol{x}_1-\boldsymbol{x}_2,\ t_1-t_2), \tag{2·15}$$

where

$$\Delta_F'(\boldsymbol{x},\ t)=\begin{cases}2i\Delta'^{(+)}(\boldsymbol{x},\ t), & \text{when } t>0\\ -2i\Delta'^{(-)}(\boldsymbol{x},\ t), & \text{when } t<0.\end{cases} \tag{2·16}$$

Then

$$\Delta_F'(\boldsymbol{x},\ z_\theta)=Z(t,\ x_4;\ \theta)\Delta_F'(\boldsymbol{x},\ t)$$

is analytic in the whole z_θ-plane except the line $x_4=0$ and $\Delta_F'(\boldsymbol{x},\ t)$ is the boundary value of the analytic function $\Delta_F'(\boldsymbol{x},\ z_\theta)$ approaching the line $x_4=0$ in such a way that z_θ gets near to the positive t axis from the half plane where x_4 is positive and to the negative t axis from the one where x_4 is negative (see Fig. 1).

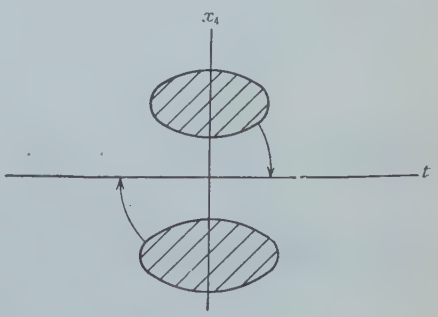

Fig. 1. The directions of continuation to get $\Delta_F'(\boldsymbol{x},\ t)$ from $\Delta_F'(\boldsymbol{x},\ z_\theta)$.

When $\theta=\pi/2$, we get

$$\Delta_F'(x_\mu)=\Delta_F'(\boldsymbol{x},\ -ix_4)$$
$$=\frac{2}{(2\pi)^4}\int d(m^2)\int d^4k\,\frac{\exp[ik_\mu x_\mu]}{k_\mu k_\mu+m^2}\rho(m^2)$$
$$=-\frac{1}{2\pi^2}\int d(m^2)\,\frac{m}{\sqrt{x_\mu x_\mu}}K_1(m\sqrt{x_\mu x_\mu})\rho(m^2). \tag{2·17}$$

The relation (2·17) is also obtained as the vacuum expectation value of field variable product ordered according to the values of x_4's:

$$\begin{aligned}(\Psi_0,\ P(\varphi(x_{1\mu}),\ \varphi(x_{2\mu}))\,\Psi_0)&\\ =\begin{cases}(\Psi_0,\ \varphi(x_{1\mu})\varphi(x_{2\mu})\,\Psi_0), & \text{when } x_{14}>x_{24}\\ (\Psi_0,\ \varphi(x_{2\mu})\varphi(x_{1\mu})\,\Psi_0), & \text{when } x_{24}>x_{14}\end{cases}&\\ =\begin{cases}iA'^{(+)}(x_{1\mu}-x_{2\mu}), & \text{when } x_{14}>x_{24}\\ -iA'^{(-)}(x_{1\mu}-x_{2\mu}), & \text{when } x_{24}>x_{14}\end{cases}&\\ =\tfrac{1}{2}A_F'(x_{1\mu}-x_{2\mu}).&\end{aligned}\tag{2·18}$$

In general we consider the vacuum expectation value of the time-ordered product of n field variables:

$$(\Psi_0,\ T(\varphi(\boldsymbol{x}_1,\ t_1),\ \cdots,\ \varphi(\boldsymbol{x}_n,\ t_n))\,\Psi_0).\tag{2·19}$$

Then, if we fix the order of values of $t_1,\ \cdots,\ t_n$ as, say, $t_1>\cdots>t_n$, (2·19) becomes the Wightman function[6]

$$F^{(n)}(\boldsymbol{x}_1,\ t_1;\cdots;\ \boldsymbol{x}_n,\ t_n)=(\Psi_0,\ \varphi(\boldsymbol{x}_1,\ t_1)\cdots\varphi(\boldsymbol{x}_n,\ t_n)\,\Psi_0).\tag{2·20}$$

From the analysis of Wightman we know that*

$$F^{(n)}(\boldsymbol{x}_1,\ z_1;\ \cdots;\ \boldsymbol{x}_n z_n)=Z(t_1,\ x_{14};\ \theta_1)\cdots Z(t_n,\ x_{n4};\ \theta_n)\,F^{(n)}(\boldsymbol{x}_1,\ t_1;\ \cdots;\ \boldsymbol{x}_n,\ t_n)\tag{2·21}$$

is analytic when

$$x_{14}>x_{24}>\cdots>x_{n4}.$$

And when $\theta_1=\theta_2=\cdots=\theta_n=\pi/2$, (2·21) reduces to

$$F^{(n)}(\boldsymbol{x}_1,\ -ix_{14};\ \cdots;\ \boldsymbol{x}_n,\ -ix_{n4})=(\Psi_0,\ \varphi(x_{1\mu})\cdots\varphi(_{n\mu}x)\,\Psi_0),$$

$$x_{14}>\cdots>x_{n4}.$$

In this way if we know the values of

$$(\Psi_0,\ P(\varphi(x_{1\mu}),\ \cdots,\ \varphi(x_{n\mu}))\,\Psi_0)$$

we can determine the values of (2·17), and vice versa.

§ 3. Lorentz transformations

In this section we discuss transformation properties of field variables with the Euclidean parameters.

We know that the spinor representation of the Lorentz transformation is obtained by the use of the unitary trick** from a unitary representation of rotation

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* Here we omit the suffixes θ_k's of z_k.

** $\psi^\dagger=\psi^*\gamma_4$.

in the Euclidean space. As our parameter space is Euclidean, it is necessary to use the same technique as that we use to obtain the spinor representation.

Consider an infinitesimal inhomogeneous Lorentz transformation in macroscopic Minkowski space:

$$\xi_\mu' = \xi_\mu + \omega_{\mu\nu}\xi_\nu + \varepsilon_\mu, \quad (\mu, \nu = 1, \cdots, 4), \tag{3·1}$$

where $\omega_{\mu\nu}$ and ε_μ are infinitesimal parameters and the $\omega_{\mu\nu}$'s satisfy the relations

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0.$$

While ξ_k, ω_{kl} and ε_k, $(k=1, 2, 3)$ are real, $\xi_4 = i\xi_0$, $\omega_{k4} = i\omega_{k0}$ and $\varepsilon_4 = i\varepsilon_0$ are pure imaginary numbers. As x_μ's are parameters, they should remain unchanged under the transformation (3·1). Field variables, however, are transformed in the following way:

$$q'(x_\mu) = \lambda(\omega, \varepsilon)\Lambda(\omega)q(x_\mu), \tag{3·2}$$

in which $q(x_\mu)$ is a field variable and

$$\lambda(\omega, \varepsilon) = 1 + i/2\omega_{\mu\nu}m_{\mu\nu} - i\varepsilon_\mu p_\mu = 1 + \omega_{\mu\nu}x_\mu\partial_\nu - \varepsilon_\mu\partial_\mu, \tag{3·3}$$

$$\partial_\mu = \partial/\partial x_\mu,$$

$$x_\mu = (x_1, x_2, x_3, x_4) : \text{real}$$

and

$$\Lambda(\omega) = 1 + i/2\omega_{\mu\nu}S_{\mu\nu} \tag{3·4}$$

$S_{\mu\nu}$: spin angular momentum matrix.

On the other hand, the Hermitian conjugate of $q(x)$ is transformed as

$$q'^*(x_\mu) = \lambda^*(\omega, \varepsilon)q^*(x_\mu)\Lambda^*(\omega),$$

where

$$\lambda^*(\omega, \varepsilon) = \lambda(\omega^*, \varepsilon^*) = 1 + \omega_{\mu\nu}^* x_\mu\partial_\nu - \varepsilon_\mu^*\partial_\mu, \tag{3·5}$$

$$\Lambda^*(\omega) = 1 - i/2\omega_{\mu\nu}^* S_{\mu\nu} \tag{3·6}$$

and * means Hermitian conjugation for matrices and complex conjugation for c-numbers. As (3·5) and (3·6) are not inverse operators of (3·3) and (3·4) respectively, we need to introduce an adjoint field of $q(x)$ (see Appendix III)

$$q^\dagger(x_\mu) = q^*(x_\mu^\dagger)D, \tag{3·7}$$

where

$$x_\mu^\dagger = (x_1, x_2, x_3, -x_4)$$

and D satisfies the relations

$$DS_{kl} = S_{kl}D,$$

$$DS_{k4} = -S_{k4}D,$$

$$(k, l=1, 2, 3),$$

$$D^2=1.$$

Then $q^\dagger(x_\mu)$ is transformed as follows:

$$q'^\dagger(x_\mu)=\lambda(\omega, \varepsilon)q^\dagger(x_\mu)\cdot\Lambda^{-1}(\omega), \tag{3·8}$$

in which

$$\Lambda^{-1}(\omega)=1-i/2\,\omega_{\mu\nu}S_{\mu\nu}.$$

By the use of the relations (3·2) and (3·7) the following formulae are obtained:

$$\begin{aligned} I' &= \int d^4x q'^\dagger(x_\mu)q'(x_\mu) \\ &= \int d^4x \lambda(\omega, \varepsilon)q^\dagger(x_\mu)\cdot\lambda(\omega, \varepsilon)q(x_\mu). \\ &= \int d^4x q^\dagger(x_\mu)\lambda^{-1}(\omega, \varepsilon)\lambda(\omega, \varepsilon)q(x_\mu). \quad \text{(by partial integration)} \\ &= \int d^4x q^\dagger(x_\mu)q(x_\mu) \\ &= I, \end{aligned} \tag{3·9}$$

$$\begin{aligned} D_\mu' &= \int d^4x q'^\dagger(x_\mu)\partial_\mu q'(x_\mu) \\ &= \int d^4x q^\dagger(x_\mu)\lambda^{-1}(\omega, \varepsilon)\partial_\mu\lambda(\omega, \varepsilon)q(x_\mu) \\ &= \int d^4x q^\dagger(x_\mu)\partial_\mu q(x_\mu)+\omega_{\mu\nu}\int d^4x q(x_\mu)\partial_\nu q(x_\mu) \\ &= D_\mu+\omega_{\mu\nu}D_\nu, \end{aligned} \tag{3·10}$$

$$\begin{aligned} X_\mu' &= \int d^4x q'^\dagger(x_\mu)x_\mu q'(x_\mu) \\ &= \int d^4x q^\dagger(x_\mu)\lambda^{-1}(\omega, \varepsilon)x_\mu\lambda(\omega, \varepsilon)q(x_\mu) \\ &= \int d^4x q^\dagger(x_\mu)x_\mu q(x_\mu)+\omega_{\mu\nu}\int d^4x q^\dagger(x_\mu)x_\nu q(x_\mu)+\varepsilon_\mu\int d^4x q^\dagger(x_\mu)q(x_\mu) \\ &= X_\mu+\omega_{\mu\nu}X_\nu+\varepsilon_\mu I, \end{aligned} \tag{3·11}$$

where

$$\lambda^{-1}(\omega, \varepsilon)=1-\omega_{\mu\nu}x_\mu\partial_\nu+\varepsilon_\mu\partial_\mu$$

and

$$d^4x=dx_1dx_2dx_3dx_4.$$

It is easily shown that I, D_4 and X_k are Hermitian operators and D_k and X_4 are anti-Hermitian operators: for example,

$$\begin{aligned} I^* &= \{\int d^4x\, q_\dagger(x_\mu)\, q(x_\mu)\}^* \\ &= \int d^4x\, q^*(x_\mu)\, q(x_\mu{}^\dagger) \\ &= \int d^4x\, q^*(x_\mu{}^\dagger)\, q(x_\mu) \\ &= I, \\ X_4{}^* &= \int d^4x\, q^*(x_\mu)\, x_4\, q(x_\mu{}^\dagger) \\ &= \int d^4x\, q^\dagger(x_\mu)\, x_4{}^\dagger\, q(x_\mu) \\ &= -X_4. \end{aligned}$$

For space reflection

$$\xi_k' = -\xi_k, \quad \xi_4' = \xi_4, \tag{3·12}$$

$q(x)$ is transformed as

$$q'(x_\mu) = Pq(-x_\mu{}^\dagger), \tag{3·13}$$

where P is a unitary space reflection matrix for spin variables and commutable with D:

$$PD = DP.$$

Thus

$$\begin{aligned} I' &= \int d^4x\, q^\dagger(-x_\mu{}^\dagger)\, q(-x_\mu{}^\dagger) \\ &= \int d^4x\, q^\dagger(x_\mu)\, q(x_\mu) \\ &= I, \\ D_k' &= \int d^4x\, q^\dagger(-x_\mu{}^\dagger)\, \partial_k\, q(-x_\mu{}^\dagger) \\ &= \int d^4x\, q^\dagger(x_\mu)\, (-\partial_k)\, q(x_\mu) \\ &= -D_k, \end{aligned}$$

etc.

For time inversion (mathematical)

$$\xi_k' = \xi_k, \quad \xi_4' = -\xi_4 \tag{3·14}$$

$q(x)$ is transformed as

$$q'(x_\mu)=Tq(x_\mu^\dagger), \tag{3·15}$$

where T is a unitary time inversion matrix for spin variables and commutable with D for integral spin fields and anticommutable with D for half odd integral spin fields. And for transformation (3·14) the order of field variables in products is assumed to be reversed. Thus, for example,

$$\begin{aligned} I' &= \pm \int d^4x\, q(x_\mu^\dagger) q^\dagger(x_\mu^\dagger) \\ &= \pm \int d^4x\, q(x_\mu) q^\dagger(x_\mu) \\ &= I \end{aligned} \tag{3·16}$$

where $+$ is taken for integral spin fields and $-$ for half odd integral spin fields. Relation (3·16) implies that integral spin fields should be quantized according to Bose-Einstein statistics and half odd integral spin fields should be quantized according to Fermi-Dirac statistics as they are in the ordinary Minkowski variable theory.

The charge conjugate field of $q(x)$ is given by

$$q^C(x_\mu)=Cq^\dagger(x_\mu) \tag{3·17}$$

where C is a charge conjugation matrix for spin variables. Thus, a neutral field variable is not a real function but a self-adjoint function; for example, a neutral scalar field $\varphi(x)$ satisfies the following relation:

$$\varphi^\dagger(x)=\varphi^*(x^\dagger)=\varphi(x). \tag{3·18}$$

§ 4. The variational principle

We consider a Lagrange function L which depends on any c-number functions $q(x)$ of the Euclidean parameters x_μ and their first derivatives

$$q_\mu(x)=\partial_\mu q(x) \tag{4·1}$$

as well as their adjoint functions $q^\dagger(x)$ and $q_\mu^\dagger(x)=\partial_\mu q^\dagger(x)$ but which does not depend explicitly on x_μ. The variation principle

$$\delta \int L(q, q_\mu; q^\dagger, q_\mu^\dagger)\, d^4x=0 \tag{4·2}$$

in which the variation has been assumed to vanish on the boundary of the domain of integration determines the field equations

$$\partial_\mu[\partial L/\partial q_\mu]-\partial L/\partial q=0 \tag{4·3}$$

and

$$\partial_\mu[\partial L/\partial q_\mu^\dagger]-\partial L/\partial q^\dagger=0. \tag{4·4}$$

An energy momentum tensor can be formed from the Lagrange function as

$$T_{\mu\nu}=\partial L/\partial q_\nu\cdot q_\mu+q_\mu^\dagger\partial L/\partial q_\nu^\dagger-\delta_{\mu\nu}L, \tag{4·5}$$

which, because of (4·3) and (4·4) satisfies the continuity equation

$$\partial_\nu T_{\mu\nu}=0. \tag{4·6}$$

If we assume that the Lagrange function L is invariant with respect to a phase transformation

$$q'=e^{i\alpha}q$$

and

$$q'^\dagger=e^{-i\alpha}q^\dagger,$$

where α is a constant number, differentiation of L with respect to the phase α gives the relation

$$\partial L/\partial q\cdot q+\partial L/\partial q_\mu\cdot q_\mu=q^\dagger\partial L/\partial q^\dagger+q_\mu^\dagger\partial L/\partial q_\mu^\dagger. \tag{4·7}$$

Then a current vector defined by

$$j_\mu=i[\partial L/\partial q_\mu\cdot q-q^\dagger\partial L/\partial q_\mu^\dagger] \tag{4·8}$$

satisfies, on account of (4·3), (4·4) and (4·7), the continuity equation

$$\partial_\mu j_\mu=0. \tag{4·9}$$

The three-dimensional volume integrals

$$P_\mu=-i\int T_{\mu 4}\,dv \tag{4·10}$$

and

$$Q=-ie\int j_4\,dv, \tag{4·11}$$

$$dv=dx_1\,dx_2\,dx_3$$

give the energy momentum four vector and charge respectively and they are independent of x_4 owing to (4·6) and (4·9) respectively:

$$\partial_4 P_\mu=\partial_4 Q=0.$$

As the scalar and vector properties of (3·9), (3·10) and (3·11) with respect to the transformation (3·1) are derived with use of partial integrations of the four-dimensional integral, the vector and scalar properties of P_μ and Q which are given by three-dimensional integral cannot be shown in the same way as before. However, owing to the continuity equations, we can show their transformation properties as follows.

With respect to the transformation (3·1) q_μ and $q_\mu^\dagger$ are transformed as

$$q_\mu' = \partial_\mu \lambda(\omega, \varepsilon) \Lambda(\omega) q$$
$$= \lambda(\omega, \varepsilon) \lambda^{-1}(\omega, \varepsilon) \partial_\mu \lambda(\omega, \varepsilon) \Lambda(\omega) q$$
$$= \lambda(\omega, \varepsilon) (\partial_\mu + \omega_{\mu\nu}\partial_\nu) \Lambda(\omega) q, \tag{4·12}$$
$$q_\mu'^\dagger = \partial_\mu \lambda(\omega, \varepsilon) q^\dagger \cdot \Lambda^{-1}(\omega)$$
$$= \lambda(\omega, \varepsilon) (\partial_\mu + \omega_{\mu\nu}\partial_\nu) q^\dagger \cdot \Lambda^{-1}(\omega). \tag{4·13}$$

Then $\partial L/\partial q_\mu$ and $\partial L/\partial q_\mu^\dagger$ are transformed as

$$\partial L'/\partial q_\mu' = \lambda(\omega, \varepsilon)[\partial L/\partial q_\mu + \omega_{\mu\nu}\partial L/\partial q_\nu] \cdot \Lambda^{-1}(\omega), \tag{4·14}$$
$$\partial L'/\partial q_\mu'^\dagger = \lambda(\omega, \varepsilon) \Lambda(\omega)[\partial L/\partial q_\mu^\dagger + \omega_{\mu\nu}\partial L/\partial q_\nu^\dagger]. \tag{4·15}$$

Thanks to (3·2), (3·8), (4·14) and (4·15), we get

$$j_4' = i[\partial L'/\partial q_4' q' - q'^\dagger \partial L'/\partial q_4'^\dagger]$$
$$= \lambda(\omega, \varepsilon) \{j_4 + \omega_{4k} j_k\}.$$
$$= j_4 + \omega_{4k} j_k + (\omega_{\mu\nu} x_\mu \partial_\nu - \varepsilon_\mu \partial_\mu) j_4. \tag{4·16}$$

Putting (4·16) into (4·11), we have

$$Q' = -ie \int j_4' dv$$
$$= -ie \int dv [j_4 + \omega_{4k} j_k + (\omega_{k4} x_k \partial_4 - \varepsilon_4 \partial_4) j_4]$$
$$= -ie \int dv [j_4 + \omega_{4k} j_k - (\omega_{k4} x_k - \varepsilon_4) \partial_l j_l] \quad \text{(owing to (4·9))}$$
$$= -ie \int dv [j_4 + \omega_{4k} j_k + \omega_{k4} j_k] \quad \text{(by partial integration)}$$
$$= -ie \int dv\, j_4$$
$$= Q.$$

Thus we know that Q is a scalar quantity. Similarly, we can prove that P_μ is a four vector in the Minkowski space.

§ 5. A scalar field

In the ordinary Minkowski parameter theory $\varphi^*(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t)$, for example, is a positive definite quantity, but in our Euclidean parameter theory the quantity corresponding to it is given by $\varphi^+(x_\mu)\varphi(x_\mu)$ which has in general no definite sign. In these circumstances it is interesting to investigate the signs of the energy momentum four vector and charge.

The simplest example is the scalar equation without any interaction

$$\Box\varphi - m^2\varphi = 0, \tag{5·1}$$

in which $\Box$ is the operator

$$\Box = \partial_\mu\partial_\mu = \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2. \tag{5·2}$$

The wave equation (5·1) can be derived by the variational principle (4·2) if we use the Lagrange function

$$L = \partial_\mu\varphi^\dagger \cdot \partial_\mu\varphi + m^2\varphi^\dagger\varphi, \tag{5·3}$$

where $\varphi^\dagger(x) = \varphi^*(x^\dagger)$.

From this we get for the energy momentum tensor by eq. (4·5)

$$T_{\mu\nu} = \partial_\mu\varphi^\dagger \cdot \partial_\nu\varphi + \partial_\nu\varphi^\dagger \cdot \partial_\mu\varphi - \delta_{\mu\nu}L \tag{5·4}$$

and for the current vector by eq. (4·8)

$$j_\mu = i[\partial_\mu\varphi^\dagger \cdot \varphi - \varphi^\dagger\partial_\mu\varphi]. \tag{5·5}$$

The solutions of (5·1) and its adjoint equation are written as follows:

$$\varphi(x_\mu) = V^{-1/2} \lim_{K\to\infty} \sum_k^K (2E)^{-1/2} \{\varphi_+(\boldsymbol{k}) \exp[i\boldsymbol{kx} - Ex_4] + \varphi_-^*(\boldsymbol{k}) \exp[-i\boldsymbol{kx} + Ex_4]\}, \tag{5·6}$$

$$\varphi^\dagger(x_\mu) = V^{-1/2} \lim_{K\to\infty} \sum_k^K (2E)^{-1/2} \{\varphi_+^*(\boldsymbol{k}) \exp[-i\boldsymbol{kx} + Ex_4] + \varphi_-(\boldsymbol{k}) \exp[i\boldsymbol{kx} - Ex_4]\}, \tag{5·7}$$

where

$$E = \sqrt{\boldsymbol{k}^2 + m^2}.$$

Putting (5·6) and (5·7) into (5·4) and (5·5) and using the definitions (4·10) and (4·11), we have for the total energy, total momentum and total charge

$$P_0 = -iP_4 = -\int T_{44}\, dv = \sum_k E[\varphi_+^*(\boldsymbol{k})\varphi_+(\boldsymbol{k}) + \varphi_-(\boldsymbol{k})\varphi_-^*(\boldsymbol{k})], \tag{5·8}$$

$$\boldsymbol{P} = \sum_k \boldsymbol{k}[\varphi_+^*(\boldsymbol{k})\varphi_+(\boldsymbol{k}) + \varphi_-(\boldsymbol{k})\varphi_-^*(\boldsymbol{k})], \tag{5·9}$$

$$Q = e\sum_k [\varphi_+^*(\boldsymbol{k})\varphi_+(\boldsymbol{k}) - \varphi_-(\boldsymbol{k})\varphi_-^*(\boldsymbol{k})]. \tag{5·10}$$

These expressions coincide with those of the ordinary theory described in terms of the Minkowski parameters. Then we can quantize the scalar field according to Bose-Einstein's statistics, that is, we may put the commutation relations

$$[\varphi_+(\boldsymbol{k}),\ \varphi_+^*(\boldsymbol{k}')]_- = [\varphi_-(\boldsymbol{k}),\ \varphi_-^*(\boldsymbol{k}')]_- = \delta_{\boldsymbol{k}\boldsymbol{k}'}, \tag{5·11}$$

and the other commutators vanish.

From the relations (5·11) we find that the eigenvalues of

$$\begin{aligned} N_+(\boldsymbol{k}) &= \varphi_+^*(\boldsymbol{k})\varphi_+(\boldsymbol{k}), \\ N_-(\boldsymbol{k}) &= \varphi_-^*(\boldsymbol{k})\varphi_-(\boldsymbol{k}) \end{aligned} \tag{5·12}$$

are the positive integers or zero. When we replace the c-number relation

$$\varphi^+ F\varphi$$

by the q-number relation

$$1/2(\varphi^+ F\varphi+\varphi F\varphi^+),$$

(5·8) to (5·10) can be rewritten as

$$P_0=\sum_k E[N_+(k)+N_-(k)+1], \tag{5·13}$$

$$\boldsymbol{P}=\sum_k \boldsymbol{k}[N_+(k)+N_-(k)+1], \tag{5·14}$$

$$Q=e\sum_k[N_+(k)-N_-(k)]. \tag{5·15}$$

With these definitions, (5·11) and (5·13) to (5·15), we have the relations

$$i[\varphi(x_\mu),\ P_\mu]=\partial_\mu\varphi(x_\mu), \tag{5·16}$$

$$[\varphi(x_\mu),\ Q]=e\varphi(x_\mu), \tag{5·17}$$

etc.

§ 6. A spinor field

The Dirac wave equation in terms of the Euclidean parameters is written as

$$\gamma_\mu\partial_\mu\psi(x)+m\psi(x)=0, \tag{6·1}$$

by using the ordinary Dirac matrices, the Hermitian conjugate equation becomes as

$$\partial_\mu\psi^*(x)\gamma_\mu+m\psi^*(x)=0$$

and the adjoint equation is given by

$$\partial_\mu\psi^\dagger(x)\gamma_\mu-m\psi^\dagger(x)=0, \tag{6·2}$$

where

$$\psi^\dagger(x)=\psi^*(x^\dagger)\gamma_4. \tag{6·3}$$

The wave equations (6·1) and (6·2) can be derived from a Lagrange function

$$L=\tfrac{1}{2}[\psi^\dagger\gamma_\mu\partial_\mu\psi-\partial_\mu\psi^\dagger\cdot\gamma_\mu\psi]+m\psi^\dagger\psi, \tag{6·4}$$

which vanishes if the wave equations are satisfied. Using eqs. (4·5) and (4·8) we get for the energy momentum tensor and the current vector

$$T_{\mu\nu}=\tfrac{1}{2}[\psi^\dagger\gamma_\nu\partial_\mu\psi-\partial_\mu\psi^\dagger\gamma_\nu\psi] \tag{6·5}$$

and

$$j_\mu=i\psi^+\gamma_\mu\psi \tag{6·6}$$

respectively.

The solutions of eqs. (6·1) and (6·2) can be written as

$$\psi(x)=V^{-1/2}\lim_{K\to\infty}\sum_k^K\sum_{r=1,2}\{\psi_+^r(k)a_+^r(k)\exp[i\boldsymbol{kx}-Ex_4]$$

$$+\psi_{-}^{*r}(\boldsymbol{k})a_{-}^{r}(\boldsymbol{k})\exp[-i\boldsymbol{kx}+Ex_4]\}, \tag{6·7}$$

and

$$\psi^{+}(x)=V^{-1/2}\lim_{K\to\infty}\sum_{\boldsymbol{k}}^{K}\sum_{r=1,2}\{\psi_{+}^{*r}(\boldsymbol{k})a_{+}^{r}(\boldsymbol{k})\exp[-i\boldsymbol{kx}+Ex_4]$$
$$+\psi_{-}^{r}(\boldsymbol{k})a_{-}^{r}(\boldsymbol{k})\exp[i\boldsymbol{kx}-Ex_4]\}, \tag{6·8}$$

in which

$$\begin{cases} a_{+}^{r}(\boldsymbol{k})=a_{-}^{*r}(\boldsymbol{k})\gamma_4, \\ a_{-}^{r}(\boldsymbol{k})=a_{+}^{*r}(\boldsymbol{k})\gamma_4 \end{cases} \tag{6·9}$$

and they satisfy the following relations:

$$\begin{cases} a_{+}^{*r}(\boldsymbol{k})a_{-}^{s}(\boldsymbol{k})=\delta_{rs} \\ a_{-}^{*r}(\boldsymbol{k})a_{-}^{s}(\boldsymbol{k})=\delta_{rs} \end{cases}, \quad \begin{cases} a_{+}^{r}(\boldsymbol{k})a_{+}^{s}(\boldsymbol{k})=\delta_{rs} \\ a_{-}^{r}(\boldsymbol{k})a_{-}^{s}(\boldsymbol{k})=-\delta_{rs} \end{cases}, \tag{6·10}$$

and

$$\begin{cases} \sum_{r=1,2}a_{+}^{r}(\boldsymbol{k})a_{+}^{r}(\boldsymbol{k})=(2m)^{-1}(-i\boldsymbol{k\gamma}+E\gamma_4+m), \\ \sum_{r=1,2}a_{-}^{r}(\boldsymbol{k})a_{-}^{r}(\boldsymbol{k})=(2m)^{-1}(i\boldsymbol{k\gamma}-E\gamma_4+m). \end{cases} \tag{6·11}$$

From eqs. (4·10), (4·11), (6·7), (6·8) and (6·10) we get for the energy momentum four vector and for the charge

$$P_0=\sum_{\boldsymbol{k}}\sum_{r=1,2}E[\psi_{+}^{*r}(\boldsymbol{k})\psi_{+}^{r}(\boldsymbol{k})-\psi_{-}^{r}(\boldsymbol{k})\psi_{-}^{*r}(\boldsymbol{k})], \tag{6·12}$$

$$\boldsymbol{P}=\sum_{\boldsymbol{k}}\sum_{r=1,2}\boldsymbol{k}[\psi_{+}^{*r}(\boldsymbol{k})\psi_{+}^{r}(\boldsymbol{k})-\psi_{-}^{r}(\boldsymbol{k})\psi_{-}^{*r}(\boldsymbol{k})], \tag{6·13}$$

$$Q=e\sum_{\boldsymbol{k}}\sum_{r=1,2}[\psi_{+}^{*r}(\boldsymbol{k})\psi_{+}^{r}(\boldsymbol{k})+\psi_{-}^{r}(\boldsymbol{k})\psi_{-}^{*r}(\boldsymbol{k})]. \tag{6·14}$$

These relations suggest that, as in the Minkowski parameter theory, we should quantize the spinor field according to the Pauli exclusion-principle. Thus we put the commutation relations

$$[\psi_{+}^{r}(\boldsymbol{k}),\ \psi_{+}^{*s}(\boldsymbol{k}')]_{+}=[\psi_{-}^{r}(\boldsymbol{k}),\ \psi_{-}^{*s}(\boldsymbol{k}')]_{+}=\delta_{rs}\delta_{\boldsymbol{kk}'} \tag{6·15}$$

and all the remaining brackets with the plus sign of these quantities vanish. Owing to the commutation relations the eigenvalues of

$$N_{+}^{r}(\boldsymbol{k})=\psi_{+}^{*r}(\boldsymbol{k})\psi_{+}^{r}(\boldsymbol{k}), \tag{6·16}$$
$$N_{-}^{r}(\boldsymbol{k})=\psi_{-}^{*r}(\boldsymbol{k})\psi_{-}^{r}(\boldsymbol{k})$$

are 0 and 1.

Antisymmetrizing with respect to ψ and ψ^*, the energy, momentum and charge lead to

$$P_0=\sum_{\boldsymbol{k}}\sum_{r=1,2}E[N_{+}^{r}(\boldsymbol{k})+N_{-}^{r}(\boldsymbol{k})-1], \tag{6·17}$$

$$\boldsymbol{P}=\sum_{\boldsymbol{k}}\sum_{r=1,2}\boldsymbol{k}[N_{+}^{r}(\boldsymbol{k})+N_{-}^{r}(\boldsymbol{k})-1], \tag{6·18}$$

$$Q=e\sum_{\boldsymbol{k}}\sum_{r=1,2}[N_{+}^{r}(\boldsymbol{k})-N_{-}^{r}(\boldsymbol{k})]. \tag{6·19}$$

§ 7. Concluding remarks

We have seen in the preceding sections that the quantum field theory in terms of the Euclidean parameters gives 1) the vacuum expectation values of x_4-ordered field variable products (P-products) which can be analytically continued to the ones of time-ordered products with the Minkowski parameters, and 2) the same eigen values of the energy momentum and charge as those of the conventional theory. In these circumstances we may conclude that the field theory in terms of the Euclidean parameters is equivalent to the customary one and gives the same values to the observable quantities as those of the latter, if we confine ourselves to the local theory.

When we extend our theory to include nonlocal fields or interactions, we meet with various difficulties arising from the incompact property of the Minkowski space. But in our Euclidean theory such difficulties do not seem to occur. For example, $\exp\Box$ is a converging factor in the Euclidean theory but not in the Minkowski one. Thus, if nature requires the nonlocal theory, we may say that the Euclidean theory is more adequate to describe nature than the Minkowski one. In this case the two theories cannot be continued analytically to each other owing to the nonlocal interaction.

In this paper we have been concerned mainly with the case where the parameters are Euclidean. Certainly it is most convenient to describe the internal structure of an elementary particle. If we confine ourselves to the local theory, however, it is possible to construct the equivalent theory in terms of the more general parameter $z_\theta = \cos\theta \cdot t - i \sin\theta \cdot x_4$, where $0 < \theta < \pi$, instead of t as one can see from the relations, e. g., (2·21), (A·4), (A·5), etc.

A method for practical calculations, the connection between our theory and the classical equation of motion of a particle, and so on, will be discussed in the succeeding papers.

In conclusion the author would like to express his sincere thanks to Prof. Y. Yamaguchi for his encouragement and to Dr. T. Murota for valuable discussions and reading of the manuscript.

Appendix I

Evaluation of a definite integral

$$\int_{-\infty}^{\infty} f(k_4) \exp[ik_4 x_4](\boldsymbol{k}^2 + k_4^2 + m^2)^{-1} dk_4$$

When $f(k_4)$ has no pole in the upper half plane of k_4, the only pole of the integrand is

$$k_4 = i\sqrt{\boldsymbol{k}^2 + m^2} = iE.$$

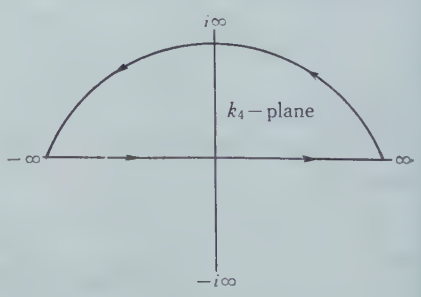

Fig. 2. The path of the integration.

Thus the path of the integration is replaced by a closed loop as shown in Fig. 2 and we obtain the formula, when $x_4>0$,

$$\begin{aligned}&\int_{-\infty}^{\infty} f(k_4)\exp[ik_4x_4](E^2+k_4^2)^{-1}\,dk_4\\&=\int f(k_4)\exp[ik_4x_4](E^2+k_4^2)^{-1}\,dk_4\\&=\pi/Ef(iE)\exp[-Ex_4].\end{aligned}\tag{A·1}$$

Appendix II

A note for integration with respect to x_4

As an example of integrations we often meet in practical calculations, let us consider the Salpeter-Bethe equation[9] for a bound system

$$\psi(1,2)=-i\iiiint d^4x_3\,d^4x_4\,d^4x_5\,d^4x_6\cdot S_F(1,3)S_F(2,4)G(3,4;5,6)\psi(5,6),\tag{A·2}$$

where x_μ's are Euclidean parameters and $G(3,4;5,6)$ is a vacuum expectation value of x_4-ordered field variable products. As seen in § 2, the vacuum expectation values have its Fourier transform (see, e. g., (2·17) and (2·18)). As to $\psi(1,2)=(\Psi_0, P(\psi(1),\psi(2))\Psi_{2\text{-body}})$, with respect to the relative coordinate $x_\mu=x_{1\mu}-x_{2\mu}$ it may be analytic but with respect to the absolute coordinate $x_\mu=ax_{1\mu}+(1-a)x_{2\mu}$ the situation is same as (2·10) :

$$\psi(1,2)=\int d^4k\psi(k)\exp[ik_\mu x_\mu]\exp[\boldsymbol{KX}-EX_4].\tag{A·3}$$

Then the integral of the right-hand side of (A·2) gives rise to the integral of the form

$$I=\int_{-\infty}^{\infty}dX_4'\int_{-\infty}^{\infty}dK_4f(K_4)\exp[iK_4(X_4-X_4')-EX_4'],\tag{A·4}$$

in which E is the total energy. The integration with respect to X_4', if we integrate formally, gives rise to divergence. In this case one should postpone taking the lower limit of X_4' for the integral :

$$\begin{aligned}I&=\lim_{A\to\infty}\int_{-A}^{\infty}dX_4'\int_{-\infty}^{\infty}dK_4f(K_4)\exp[iK_4(X_4-X_4')-EX_4']\\&=-\lim_{A\to\infty}\int_{-\infty}^{\infty}dK_4f(K_4)(E+iK_4)^{-1}\exp[iK_4(X_4+A)+EA].\end{aligned}$$

When $f(K_4)$ has no displaced pole in the upper half plane of K_4, we have

$$I=-2\pi i \lim_{A\to\infty}\{f(iE)\exp[-E|X_4+A|+EA]+\sum_l g_l(\omega_l)\exp[-\omega_l|X_4+A|+EA]\},$$

where ω_l's are the values of $-iK_4$ at the poles of $f(K_4)$ and satisfy the inequality $\omega_l>E$. $g_l(\omega)$'s are the residues of $f(K_4)(E+iK_4)^{-1}$ at the poles $K_4=i\omega_l$. As A tends to infinity, X_4+A becomes positively definite and we finally get

$$I=-2\pi i f(iE)\exp[-EX_4]. \tag{A·5}$$

Comparing (A·3), we see that we can factorize $\exp[-EX_4]$ in the both sides of (A·2).

Appendix III

Interpretation of the formalism in §3 according to the operator $Z(t, x_4; \theta)$

With respect to the Lorentz transformation (3·1), $q(\boldsymbol{x}, t)$ is transformed into

$$q'(\boldsymbol{x}, t)=\lambda_l(\omega, \varepsilon)\Lambda(\omega)q(\boldsymbol{x}, t), \tag{A·6}$$

where

$$\lambda_l(\omega, \varepsilon)=1+\omega_{kl}x_k\partial_l+\omega_{k0}(x_k\partial_l+t\partial_k)-\varepsilon_k\partial_k-\varepsilon_0\partial_l. \tag{A·7}$$

Then, operating $Z(t, x_4; \theta)$ on the both sides of (A·6), we obtain

$$q'(\boldsymbol{x}, z_\theta)=Z(t, x_4; \theta)\lambda_l(\omega, \varepsilon)Z^{-1}(t, x_4; \theta)\Lambda(\omega)q(\boldsymbol{x}, z_\theta), \tag{A·8}$$

and

$$\begin{aligned}&Z(t, x_4; \theta)\lambda_l(\omega, \varepsilon)Z^{-1}(t, x_4; \theta)\\&\quad=1+\omega_{kl}x_k\partial_l+\omega_{k0}\{x_k(\cos\theta\partial_l+i\sin\theta\partial_4)+z_\theta\partial_k\}\\&\quad-\varepsilon_k\partial_k-\varepsilon_0(\cos\theta\partial_l+i\sin\theta\partial_4).\end{aligned} \tag{A·9}$$

Especially when $\theta=\pi/2$, (A·9) is written as

$$\begin{aligned}&Z(t, x_4; \pi/2)\lambda_l(\omega, \varepsilon)Z^{-1}(t, x_4; \pi/2)\\&\quad=1+\omega_{kl}x_k\partial_l+i\omega_{k0}x_k\partial_4-i\omega_{k0}x_4\partial_k-\varepsilon_k\partial_k-i\varepsilon_0\partial_4\\&\quad=\lambda(\omega, \varepsilon).\end{aligned} \tag{A·10}$$

Thus we get the relation (3·2) :

$$q'(x_\mu)=q'(\boldsymbol{x}, -ix_4)=\lambda(\omega, \varepsilon)\Lambda(\omega)q(x_\mu).$$

In the next place, operating $Z(t, x_4; \pi/2)$ on $q^*(\boldsymbol{x}, t)D$, we have

$$Z(t, x_4; \pi/2)q^*(\boldsymbol{x}, t)D=q^*(\boldsymbol{x}, -ix_4)D=q^\dagger(x_\mu). \tag{A·11}$$

On the other hand,

$$q^*(x_\mu)D=\{q(\boldsymbol{x}, -ix_4)\}^*D=q^*(\boldsymbol{x}, ix_4)D. \tag{A·12}$$

Comparing (A·11) and (A·12), we obtain the relation (3·7) :

$$q^\dagger(x_\mu)=q^*(x_\mu{}^\dagger)D.$$

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Radial Dependence of Imaginary Part of Nuclear Optical Potential

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The imaginary part of tht nuclear optical potential for neutrons is calculated by means of a method of the volume direct processes, on the basis of the statistical independent particle model with a eiffuse nuclear surface. It is shown that the localization of the absorption near the surface, which was found by Bjorklund and Fernbach, can be reproduced using reasonable values for the relevant parameters. The effect of the Pauli exclusion principle, of the energy dependence of the two-body collision cross section, of the mass number dependence of the parameters, and of the correlation between nucleons are also discussed.

§ 1. Introduction

Much information on the phenomenological optical potential parameters has been obtained so far, especially from the detailed analyses of the proton-nuclei scattering cross section.[1] It was found that a fairly good fit to the experimental data could be obtained by using the diffuse-boundary shape for both the real and imaginary part of the nuclear optical potential.

On the other hand the surface absorption model had first been suggested by Feshbach, Porter, and Weisskopf[2] from a view-point that the imaginary part would be much less dependent on mass number if it is concentrated in a surface layer. Amster[3] studied the effects of the various surface absorption potential forms at zero energy and also analysed, in collaboration with Emmerich, σ_t and σ_e of the first few Mev neutrons using several potential forms. However, the determination of the best fit surface potential form and the relevant parameters was not aimed at in their analysis.

Recently, Bjorklund and Fernbach[4] have analysed the scattering of 4.1-, 7-, and 14-Mev neutrons by complex nuclei, using the optical potential with the following form* :

* The notation is slightly changed from that used by Bjorklund and Fernbach.

$$V(r)=V_0p(r)+iWq(r)+U(\hbar/\mu c)^2(1/r)\,(dp(r)/dr)\,(\boldsymbol{\sigma}\cdot\boldsymbol{l}), \qquad (1)$$

where

$$p(r)=\{1+\exp[(r-R_0)/s]\}^{-1}, \qquad (1a)$$

$$q(r)=\exp[-(r-R_0)^2/t^2], \qquad (1b)$$

$$R_0=r_0A^{1/3}. \qquad (1c)$$

The characteristic feature of this potential is the use of a Gaussian form centered at the nuclear edge for the imaginary part ($Wq(r)$). At each energy considered, was found a single set of parameters that give a fairly good agreement with experimental data of the elastic, total, and nonelastic cross section. Further, the calculated nonelastic cross sections show better agreement with experimental data when the surface imaginary potential was used instead of the diffuse-boundary (Fermi type) imaginary potential.

Roughly speaking, two types of methods have been developed in order to derive the phenomenological potential mentioned above from the more fundamental two-body interaction; a) the quantum mechanical one[5] which directly relates the complex potential to the two-body potentials as has recently been elaborately developed by Brueckner and his collaborators, and b) the semi-classical one, first proposed by Goldberger,[6] which relates the imaginary part of the complex potential to the empirical two-body cross sections*. While the method a) is clearly more fundamental than the method b), in the former approach we must deal with all of the complications of the nuclear many-body problem, which in fact is very difficult to do at present[7]. On the other hand, the method b), notwithstanding its simplicity, could successfully give the absolute magnitude of the imaginary potential and its dependence upon the energy of the incident particle[8)9)]. However, since all the imaginary potentials so far derived were based on the Fermi gas having a constant density and a sharp boundary, they were constant over the whole nuclear volume.

In a previous paper[10], along the line of method b), we used the Thomas-Fermi gas model with a diffuse nuclear surface to explain the inelastic scattering of protons at intermediate energies by complex nuclei. The inelastic scattering was explained as the direct processes ocurring at the diffuse rim of the nucleus, and it was indicated in that paper that it would be very interesting to derive the radial dependence of the absorption (imaginary) potential from our theory, in which the Pauli principle was exactly taken into account.

We shall show in this work that by using the Thomas-Fermi gas model as in the previous paper, it is possible to reproduce the phenomenological absorption potential of Bjorklund and Fernbach. In this connection, it is to be noted that the simple Thomas-Fermi gas model as used by us is also applicable with a considerable validity to other surface problem[11], such as the derivation of nuclear surface energy.

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* The real part of the optical potential cannot be derived on the method b).

§ 2. Calculation and results

At first we consider the case in which the incident particle is a neutron. It is assumed that the incident neutron interacts individually with the nucleons in the target with a cross section which differs from that for free nucleons only through the restrictions due to the Pauli principle. As the target nucleons are supposed to be the constituents of the Thomas-Fermi gas with a diffuse boundary, the Fermi momentum $p_F(r)$ is radially dependent and can be evaluated from the nucleon density $\rho_p(r)$ and $\rho_n(r)$:

$$p_F^2(r)=[3\pi^2\hbar^3\rho(r)]^{2/3}. \tag{2}$$

When a Fermi type function is assumed for the actual shape of ρ_p and ρ_n,

$$\rho(r)=a\{1+\exp[(r-c)/b]\}^{-1}, \tag{3}$$

we have

$$p_F^2(r)=[3\pi^2\hbar^3 a]^{2/3}\{1+\exp[(r-c)/b]\}^{-2/3}, \tag{4}$$

where c is the half-density radius, b the diffuseness parameter, and a the normalization constant[10]. Then, the kinetic energy of an incident neutron in the nucleus is given by

$$E_1(r)=p_{Fn}^2(r)/2m+S_n+E, \tag{5}$$

where $p_{Fn}(r)$ is the Fermi momentum of the neutrons bound in the nucleus, S_n the separation energy of the neutron, and E the kinetic energy of the incident neutron outside the nucleus. Contrary to the usual theory, $E_1(r)$ is here directly given by the nucleon density distribution $\rho(r)$, without referring to the real part of the optical potential $V(r)$. The assumption that an incident neutron is absorbed from the incident channel when it collides with a target nucleon leads to an expression for the absorption coefficient $K(r)$ which is here radially dependent :

$$\begin{aligned} K(r)&=K_p(r)+K_n(r)\\ &=\rho_p(r)\bar{\sigma}_{np}(r)+\rho_n(r)\bar{\sigma}_{nn}(r), \end{aligned} \tag{6}$$

where $\bar{\sigma}_{np}(r)$ and $\bar{\sigma}_{nn}(r)$ are certain average neutron-proton and neutron-neutron cross section. The averages in $\bar{\sigma}_{np}(r)$ and $\bar{\sigma}_{nn}(r)$ are taken approximately over all possible relative collision momenta $\boldsymbol{p}(r)$, under the restrictions due to the Pauli exclusion principle.

Let the momenta of incident and target nucleons be $\boldsymbol{p}_1(r)$ and $\boldsymbol{p}_2(r)$ and the relative momenta of the initial and final states be $\boldsymbol{p}(r)$ and $\boldsymbol{p}'(r)$, respectively. Then, $\bar{\sigma}(r)$ may be expressed as

$$\bar{\sigma}(r)=\int d\boldsymbol{p}_2(r)\int d\Omega'\, 2p(r)\sigma(\boldsymbol{p}(r),\boldsymbol{p}'(r))/p_1(r)\int d\boldsymbol{p}_2(r), \tag{7}$$

where $d\Omega'$ is the solid angle element for $\boldsymbol{p}'(r)$. The two-body collision cross section $\sigma(\boldsymbol{p}(r),\boldsymbol{p}'(r))$ in the center of mass system is assumed to be isotropic and inversely

proportional to $p^2(r)$[12]. This assumption is appropriate in the present case. In eq. (7), the integral in the denominator is taken over the range of momenta at the radial coordinate r of the target nucleons (i.e. from 0 to $p_F(r)$), and the integral in the numerator is performed under the restriction of the Pauli exclusion principle at the place r. Assuming $\sigma(\boldsymbol{p}(r),\ \boldsymbol{p}'(r))=\sigma_t(p(r))/4\pi=\sigma_0/4\pi p^2(r)$, we get

$$\bar{\sigma}(r)=6\sigma_0\ p_1(r)\ I(r)/p_F^3(r), \tag{8}$$

where

$$I(r)=\int_a^b \frac{x(1+x^2-2\alpha^2)}{\sqrt{2(1+x^2)}}\tanh^{-1}\frac{2x\sqrt{2(1+x^2)}}{1+3x^2}dx, \tag{8a}$$

$$x=p_2(r)/p_1(r),\ \alpha=p_F(r)/p_1(r), \tag{8b}$$

and the integration limits are

$$\begin{aligned} &a=0,\ b=\alpha \qquad \text{for}\quad E_1(r)\geq 2E_F(r),\\ &a=\sqrt{2\alpha-1},\ b=\alpha \quad \text{for}\quad E_1(r)\leq 2E_F(r), \end{aligned} \tag{8c}$$

respectively.

The actual numerical calculation has been carried out for Sn nucleus, where the empirical value for S_n is 9.7 Mev. The parameters b and c in the expression for $\rho_p(r)$ can be determined fairly well from the analysis of the high energy electron scattering[13]; $b=0.57\times10^{-13}$ cm and $c=1.1\times A^{1/3}\times10^{-13}$ cm. Since there is no definite experimental evidence for the density of neutron $\rho_n(r)$, two sets of values of b and c are assumed with the same type of distribution as $\rho_p(r)$: (A) ρ_n is of the same form as ρ_p, and (B) ρ_n is a little extended beyond ρ_p but has the same surface thickness as ρ_p. The values for b and c, and the corresponding Fermi energies $E_F(0)$ at the center of the nucleus are listed in Table 1.

Table 1. Values of a, b, c and $E_F(0)$ for Sn

	protons	neutrons (A)	neutrons (B)
a	0.67×10^{38}	0.94×10^{38}	0.53×10^{38}
b	0.57×10^{-13}	0.57×10^{-13}	0.57×10^{-13}
c	5.42×10^{-13}	5.42×10^{-13}	6.66×10^{-13}
$E_F(0)$	32.8Mev	41.3Mev	28.3Mev

The calculated absorption coefficients are shown in Figs. 1, 2, and 3 for $E=14$ Mev, 7 Mev, 4 Mev, respectively. The dotted curves are the ones calculated from the optical potential determined from the experiment[4], using the following relation between the absorption coefficient and the optical potential,

$$K^2(r)=(4m/\hbar^2)[-(E+V(r))+\sqrt{(E+V(r))^2+W^2(r)}]. \tag{9}$$

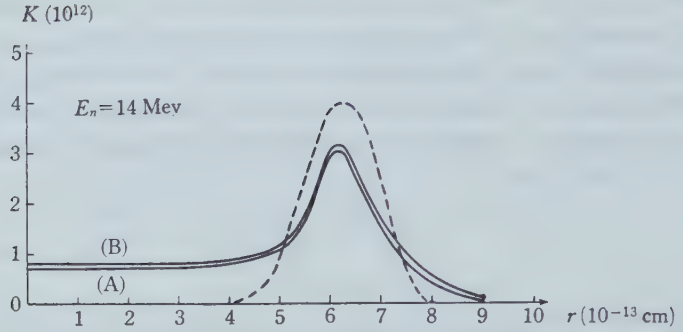

Fig. 1 Full curves are the calculated absorption coefficients for E_n=14 Mev. Dotted curve is the empirical one found by Bjorklund and Fernbach.

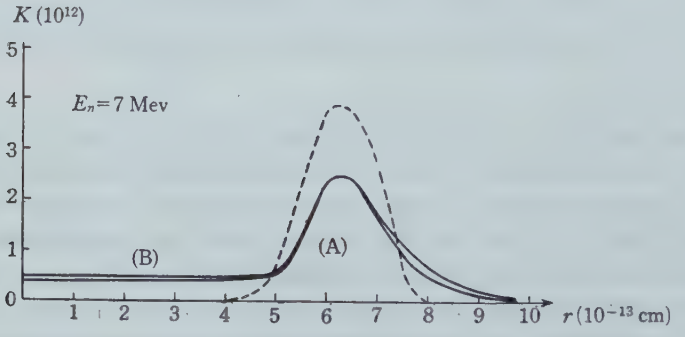

Fig. 2 Full curves are the calculated absorption coefficients for E_n=7 Mev. Dotted curve is the empirical one found by Bjorklund and Fernbach.

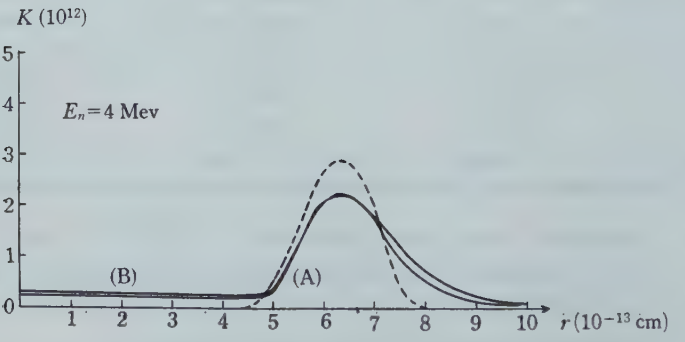

Fig. 3 Full curves are the calculated absorption coefficients for E_n=4 Mev. Dotted curve is the empirical one found by Bjorklund and Fernbach.

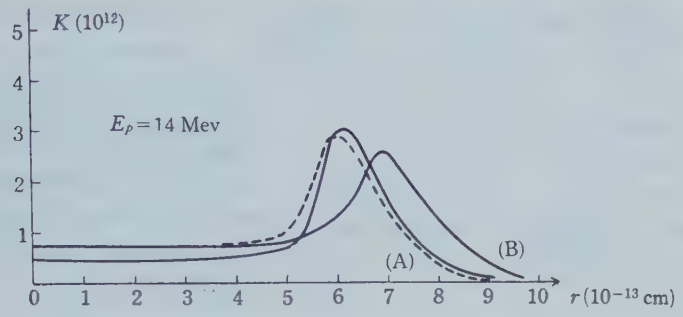

Fig. 4 Full curves are the calculated absorption coefficients for E_p=14 Mev. Dotted curve is the one for E_n=14 Mev.

One sees that the calculated absorption coefficients agree fairly well with the empirical (experimental) results, especially the localization of the absorption at the nuclear surface. It is interesting to note that $K(r)$ is almost indepent of the shape of ρ_n. This is due to the fact that σ_{nn} is much smaller than σ_{np}. In the case of the incident protons it is expected that $K(r)$ will be more sensitive to the shape of ρ_n.

To see this ρ_n dependence more quantitatively, we calculate the absorption coefficient $K(r)$ for the incident proton whose energy is 14 Mev, and show the results in Fig. 4. In this case the empirical value for S_p is 6.2 Mev. One sees that with ρ_p described by a type (A) K's for both the incident neutron and proton have almost the same form, but with ρ_n described by a type (B) the peak of K for the incident proton appreciably shifts towards the outside and its width becomes slightly broader than that of K for the incident neutron.

§ 3. Discussions

i) We shall here examine the problem why the absorptions were localized at the nuclear surface in our calculations on the basis of the statistical independent particle model with a diffuse nuclear surface. As one of the reasons, we may imagine that the relaxation of the Pauli exclusion principle at the surface due to the decrease of $p_F(r)$ plays an important role. To see whether this is actually the case or not, we examined the scattering of 4 Mev neutrons by target protons only. At this energy the Pauli exclusion principle is expected to be most effective. If we assume the two-body cross section to be energy-independent, $\sigma_{np}(\boldsymbol{p}(r), \boldsymbol{p}'(r)) = \sigma_t/4\pi$, we obtain (see Hayakawa et al.,[9] eq. (2.8))

$$\bar{\sigma} = \sigma_t \cdot (E.\ P.), \tag{10}$$

where

$$(E.\ P.) = 1 - \frac{7}{5} \cdot \frac{E_F(r)}{E_1(r)} \qquad \text{for } E_1(r) \geq 2E_F(r),$$

$$=1-\frac{7}{5}\cdot\frac{E_F(r)}{E_1(r)}+\frac{2}{5}\cdot\frac{F_F(r)}{E_1(r)}\left(2-\frac{E_1(r)}{E_F(r)}\right)^{5/2} \quad \text{for } E_1(r)\leq 2E_F(r). \quad (10')$$

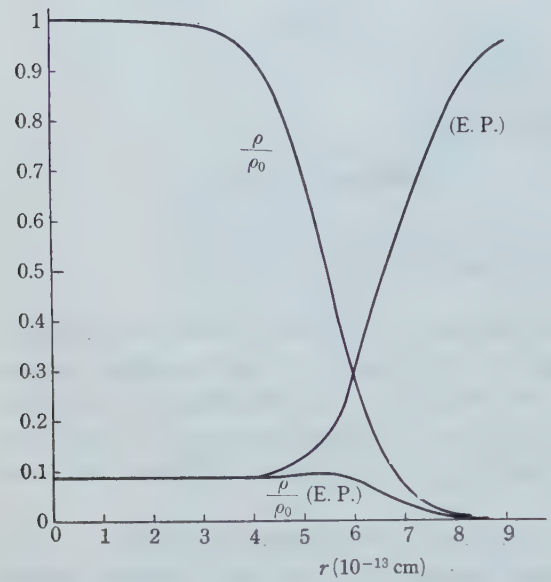

Fig. 5 ρ/ρ_0 is the proton density normalized to unity at the origin. (E. P.) is the factor exprssing the rate of reduction of the free collision cross section σ_{np} due to the exclusion principle for $E_n=4$ Mev.

Here $(E.\ P.)$ is the factor expressing the rate of reduction of the free collision cross section σ_{np} due to the exclusion principle. Then the absorption coefficient $K(r)$ is written as

$$K(r)=\rho(r)\cdot\sigma_t\cdot(E.\ P.). \quad (11)$$

$(E.\ P.)$, ρ/ρ_0 (the proton density normalized to unity at the origin), and $\rho/\rho_0\times(E.\ P.)$ are shown in Fig. 5. From this one sees that the effect of the relaxation of the exclusion principle at the nuclear surface is considerably cancelled out by the decrease of $\rho(r)$. Therefore, using the energy independent cross section, one can not obtain the localization of the absorption at the surface. From the above result it appears that the remarkable localization of the absorption at the nucleon surface in our theory is mainly due to the energy dependence of two-body collision cross section, although the exclusion principle considerably reduce the absorption inside the nuclear surface. The energy dependence of the two-body collision cross sections strikingly affects the results because in our model the momenta of both the target nucleons and an incident nucleon at the nuclear surface are much smaller than inside (see eqs. (2) and (5)).*

ii) On the basis of the Stanford electron scattering experiment[11], the density of protons ρ_p is usually described by a Fermi type function (eq. (3)), having a central region of rather uniform density and a surface region in which the density falls from 90 to 10 per cent of the central value in a distant $D=4.39\,b$, b being independent of the mass number A. Taking account of this fact for $\rho(r)$, we may expect that the calculated absorption coefficients $K(r)$ for nuclei with different A in our theory would have nearly equal magnitude at a fixed incident energy, with the different locations of their maxima corresponding to the different values of radii. This expectation will be supported by the results of Bjorklund and Fernbach that at a fixed incident energy one can get by varying only one parameter, that is the

* The examination of this discussion (i) was done following the suggestion of Dr. Sugie.

radius R_0(eq.(1c)), a fairly good agreement with experimental data on the total and nonelastic cross sections as a function of A (about 8 to 220).

The usual optical potential analysis assumes the same radial form[14] for the real and imaginary potential and the resulting imaginary part shows a remarkable mass number dependence, i.e. one has to decrease the imaginary part about factor two as the mass number increases, in order to obtain a fit to experimental data. The neutron excess in heavier nuclei[15] can produce this tendency, but its effect is too weak to explain the empirical evidence. On the basis of the surface absorption model, this difficulty is not the case.

iii) Since our nuclear model is a statistical independent particle model, the correlations between the nucleons are ignored. If the correlations were taken into account, the higher momentum components of the target nucleons would be increased and the nucleus would be represented by a no-degenerate Thomas-Fermi gas at some temperature. In another place[16]* we gave the detailed calculation of the absorption coefficient with a non-degenerate Fermi gas (having a constant nuclear density and a sharp boundary) in a similar way as in this paper (i.e. as a direct volume process), and the result showed that the non-degeneracy was not so effective in inceasing the absorption coefficient.** It might be, therefore, expected that in the present case with the diffuse surface also the correlation effect will not so much alter the fundamental features of the results. However, the finiteness of temperature is but one result of the correlation effect and a great complexity will be introduced in the calculation through a complete inclusion of the correlation effect, so that we think it a consistent way to work with the classical particle picture together with the Thomas-Fermi gas at zero temperature.

Finally, it may be concluded from the results and disussions given above that even with the present simple semiclassical model the physical reality underlying the nuclear surface problem can be explained successfully to a certain extent.

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* This calculation was done in connection with the sticking probability in the decay process of a compound nulceus (Cohen's paradox)[17], and the residual nucleus was described by a non-degenerate Fermi gas with various excitation energies.

** An analysis of the dependence of the mean free path of nucleon on the nuclear temperature was also done by Kind and Patergnani,[18] from which the similar conclusion may be derived for this case if the nuclear temperature is not so high.

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Nuclear Moments of Inertia

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The problem of nuclear moments of inertia is considered taking into account the effect of the internal motion phenomenologically. The kinetic energy of the internal motion is assumed to take the form of that of a symmetrical top. When the interactions between the irrotational and the internal motions are strong the total system has again a rotational energy spectrum whose moment of inertia is given by the sum of those of two motions. By means of the empirical data for the electric quadrupole excitations their magnitudes are estimated. In consequence the internal motion is shown to be very important. Comparison with other methods is carefully examined, especially with the work by Hayakawa and Marumori.

§ 1. Introduction

It is well known that many nuclei exhibit bands of energy levels with the spectra

$$E_I = h^2 I(I+1)/2\mathcal{J} \qquad (1\cdot1)$$

where I is the spin of a level. Since these spectra resemble those of the symmetrical top, $\mathcal{J}$ is called the nuclear moment of inertia. Present paper offers some theoretical comments on the magnitude of $\mathcal{J}$.

Nowadays these energy levels are believed to be obtained by excitation of certain collective motions of nuclear particles. At first Bohr and Mottelson discussed the collective motion in the 'core' of the nucleus in terms of an irrotational motion of incompressible fluid set in motion by the action of outer particles in the unfilled shell[1]. In the strong coupling limit, the shape of the core is an ellipsoid and the low excitations are the rotational type of (1·1). Then the moment of inertia is given by

$$\mathcal{J}_{\text{ir}} = \mathcal{J}_{\text{rigid}}\,\epsilon^2 \qquad (1\cdot2)$$

where ϵ is the ratio of the difference between major and minor semi-axis of the ellipsoid to the mean radius. The deformation ϵ can be inferred directly from the quadrupole data, and it was found that $\mathcal{J}_{\text{ir}}$ is appreciably smaller than that determined from the level spacing, (1·1). Many authors have made various attempts to elucidate the above situation. Roughly speaking, those attempts may be classified into two groups. One of them has investigated the foundation of Bohr-Mottelson's

phenomenological model developing the theory of quantum-mechanical collective motion of particles and examined the origin of the above-mentioned discrepancy. The others have made a more phenomenological approach. They have remarked the fact that even on the basis of the independent particle model of the nucleus the ground state would generally be deformed by the polarizing action of the outer particles which are in the unfilled shell. Therefore the nucleus could be approximated by the independent particle system in a rotating deformed potential. Thus they have obtained a new formula of the moment of inertia.

In the former method, the collective motions of nuclear particles are assumed to be incompressible since the inter-particle force resists strongly to the compression of the system.[2] Further we assume those to be irrotational, then the displacement potential can be used which gives the particle displacement in those motions. In order to describe those motions quantum-mechanically, we take the displacement potential as the collective coordinate, and by properly defining the momentum canonically conjugate to it, we have to express the hamiltonian of the system in terms of them and the internal variables which are independent of them. When more than two modes of collective motions have to be taken into account as the surface vibration and rotation of the nucleus, canonically conjugate momenta to the displacement potentials do not exist. However, if we employ the approximately conjugate momenta, the hamiltonian can be transformed into the form expressed by the collective and internal variables, which contains the correction terms due to the approximation.[3] In such cases, we extend the degree of freedom of the system introducing new redundant coordinates and momenta canonically conjugate to each other. Instead the state vectors are imposed some subsidiary conditions relating to the new variables. Then we transform the hamiltonian so as to replace the collective variables by the new redundant ones as far as possible. If the collective coordinates and momenta were exactly canonically conjugate to each other, the replacement would be performed completely and the transformed hamiltonian would not involve the collective variables whose role would be played by the new redundant variables ; so we could solve the transformed hamiltonian having nothing to do with the subsidiary conditions.[4] In the actual case the replacement cannot be performed so completely due to the approximate nature of canonical relations, and we must solve the problem taking into account the subsidiary conditions explicitly.

These attempts did not improve the situation ; the resulting moment of inertia was the same as Bohr-Mottelson's phenomenological theory. This means that the assumption of irrotational collective motions is not good enough. Against our expectation they interact strongly with the internal motions, and the collective motions are not stable. Actual nuclei may also execute some other collective motions than the incompressible, irrotational ones—perhaps rotational ones. However, at present, it is not known how to treat the rotational collective motions quantum-mechanically. Thus the program to treat the collective motions quantum-mechanically has encountered with an objection.

On the other hand phenomenological approach was first made by Inglis who introduced the idea of independent particle motion in a rotating deformed potential with the fixed angular velocity Ω[5]. In the reference frame fixed to the potential, the particle system is exerted by a Coriolis force due to the rotation. Treating this force as perturbation, it is seen that the reduction in the energy of the particle system is the same with the increase in the energy of free rotation. Setting this increase as $\mathcal{J}_{cr}\Omega^2/2$, we find the cranking model formula for the moment of inertia,

$$\mathcal{J}_{cr}=2\sum_i|\langle 0|L|i\rangle|^2/(E_i-E_0). \qquad (1\cdot3)$$

The consequences of this formula are discussed by Inglis, Bohr-Mottelson and Moszkowski.[6] Particularly if the potential were assumed to be of a deformed oscillator type, $\mathcal{J}_{cr}$ would give a rigid moment of inertia at the equilibrium configuration. This value could be greatly reduced when the residual interactions between the particles were taken into account.

However, some questions remain also with this model. The first question was pointed out by Lipkin, de-Shalit and Talmi who remarked the fact that the rotation of a nucleus in Inglis' model is externally forced rotation imposed by 'cranking' the external potential, while the collective motion in the actual nucleus is free rotation resulting from the mutual interaction of the nucleons.[7] Moreover, on account of the deformation of the potential the quantum states of Inglis' model are eigenstates of the total angular momentum only in a sense of time average, while actual nuclear states are always the eigenstates of that quantity.

The first problem was discussed by Villars and Hayakawa-Marumori on the basis of quantum-mechanical many body problem.[8] Introducing a reference frame fixed to the particle system Villars expressed the hamiltonian in terms of the coordinates relative to this reference frame and the angles which indicate the orientation of the frame. Thus he obtained a form of the kinetic energy of a many body system, in which the dependence on the total angular momentum is explicitly given. Similar arguments were made by Hayakawa and Marumori who used the angular coordinate of the irrotational collective motion as a collective coordinate and the total angular momentum as the collective momentum and employed the method of canonical transformation. According to their results intrinsic motion has a coupling term with the total angular momentum through L_{in} which is called by them internal angular momentum. This coupling term plays the role of Coriolis force for the intrinsic motion. Hayakawa and Marumori derived the rotational spectra of the energy levels assuming the Coriolis force to be weak enough to be treated by perturbation theory and further assuming the mean square deviation of L_{in} to be negligibly small, the resulting moment of inertia was given by the sum of the hydrodynamical value (1·2) and the term similar to the cranking model formula (1·3). However, some criticism will be given about the consistency of above two assumptions in § 3.

With respect to the second problem about the angular momentum Peierls and Yoccoz made interesting discussions in which they remarked the degeneracy of ground state of independent particle motion in a deformed potential with the orientation of the potential.[9] Forming the linear combinations of these degenerate states properly, they found better trial wave functions which were eigenstates of the total angular momentum. However, we can use more improved trial functions, especially for the excited states, if there are some stable collective motions in the actual nuclei. As an example of the translational motion, the product of the Peierls-Yoccoz trial function for the zero momentum state and the plane wave which expresses the motion of the center-of-mass gives the improved trial function for the translating state which is lower in energy and gives no inconsistency such as the effective mass differs from the total mass. The nuclear collective motions would be under the same circumstances as the case of translational motion.

In this way the approach through the method of quantum-mechanical collective motion and that through the discussion of the phenomenological models have been related to each other and have left some questions respectively. Tomonaga, on the former standpoint, proposed to consider phenomenologically the couplings between the irrotational collective motion and the intrinsic motions.[10] Let us assume that the irrotational surface motion is coupled to one mode of the intrinsic motions. Then the coupling energy between the two motions is given only by the potential energy term and not from the kinetic energy, since the two coordinates which describe the two motions respectively are chosen to be orthogonal to each other. This coupling energy is shown to depend only on the ' relative coordinate ' between the two coordinates and approximated by expanding in its power series in the strong coupling limit. Thus the two modes move together executing relative vibration about an equilibrium point. Consequently, in a rotational motion, the rotating substance in the nucleus increases as compared with the case of surface motion alone, thus making the effective moment of inertia larger than the hydrodynamical value. However, it is not known what intrinsic motion is coupled strongly with the irrotational surface motion in the nuclei. But what motion it may be, its kinetic energy must be given by the square of the angular momentum which generates the intrinsic motion. By these discussions Tomonaga showed that the increase in the moment of inertia was given by a formula similar to that of the cranking model. At first sight this may seem to be the same as Hayakawa and Marumori's work, but they differ not only in appearance but also in substance as is shown in § 3.

The main purpose of this paper is to extend Tomonaga's idea to three dimensional cases and to derive the addition theorem of the moment of inertia in the case of coupled rotational motions. On the basis of these results the experimental data of the energy levels of the nuclei and those of the electric quadrupole excitation yields are analyzed. The results of this analysis indicate importance of the intrinsic motions. In § 2, Tomonaga's idea will be summarized in the two dimen-

sional example. In § 3, the relations with the other methods, especially with that of Hayakawa and Marumori, will be investigated. In § 4, Tomonaga's idea will be applied to separate one mode of intrinsic motions from the others in three dimensional case, and we will solve the problem of surface-internal coupled system in the strong coupling approximation in § 5, deriving the addition theorem of the moment of inertia. In § 6, Coulomb excitation phenomena will be discussed with this example inferring the magnitude of nuclear deformations ϵ and that of the moment of inertia of surface motion by the formula (1·2). Finally in § 7, the magnitude of the moment of inertia of the intrinsic motion will be inferred by means of the results in § 6 and the experimental data of energy levels.

§ 2. Illustration of the method by an example of the two dimensional nucleus

Let us begin with Tomonaga's theory since it is not published in a precise form. As Tomonaga did, we take a simple example of the two-dimensional nucleus. This example is rather simple in many respects, but still contains the very essence of the method and makes it easier to compare the discussions with other two-dimensional ones such as Hayakawa and Marumori's.

Denoting the Cartesian coordinates of the n-th nucleon by x_n, y_n and their conjugate momenta by p_{xn}, p_{yn} respectively, the total hamiltonian of the system is given by

$$H_0=\sum_n (1/2m)\,(p_{xn}^2+p_{yn}^2)+V(x_1, y_1; x_2, y_2; \cdots; x_n, y_n) \qquad (2\cdot1)$$

where m is the nucleon mass and V is the potential depending only on relative coordinates of the particles. First we consider two modes of collective motion corresponding to an irrotational, incompressible flow inside the nucleus which are described by the set of coordinates

$$\xi_1=(AR_0^2)^{-1}\sum_n (x_n^2-y_n^2)$$

$$\xi_2=(AR_0^2)^{-1}\cdot 2\sum_n x_n y_n$$

where A is the total number of the nucleons, and R_0 is the nuclear radius. The separation of the hamiltonian (2·1) into collective and internal parts was performed by Tamura and one of the present authors (T.M.).[3] In order to avoid unnecessary complexity of finding the canonical momenta conjugate to ξ_i, they introduced a set of auxiliary variables α_i and β_i canonically conjugate to each other and the hamiltonian was transformed into the representation in which α_i and β_i play the role of the collective variables. The transformed hamiltonian was given by

$$H_0 \to H=T_c+H' \qquad (2\cdot2a)$$

$$T_c=(1/mAR_0^2)\,\{2(r^2/R_0^2)\,(\beta_1^2+\beta_2^2)+l(R_0^2/r^2)\,(\alpha_1\beta_2-\alpha_2\beta_1)+\cdots\} \qquad (2\cdot2b)$$

$$H'=H_{\text{internal}}+\sum_j H_j\alpha_j+\sum_{j,k} H_{jk}\alpha_j\alpha_k+\cdots \tag{2·2c}$$

where r^2 is defined by $r^2=A^{-1}\sum_n(x_n^2+y_n^2)$ and l by $l=\sum_n(x_np_{yn}-y_np_{xn})$. H_{internal}, H_j and H_{jk} depend only on the internal variables. The first term of (2·2b) is the kinetic energy of the collective motion, while the second term is the coupling term between the internal and collective angular momenta due to the non-commutativity of the momenta taken approximately conjugate to ξ_1 and ξ_2, so this term may be neglected as small compared with the first term. The second term on the right-hand side of (2·2c) gives the coupling between the internal and the collective motions, and the third term gives the effective potential for the surface deformation α_4.

Similarly the total angular momentum was transformed under the above transformation into

$$l\to L=L_c+L_i \tag{2·3a}$$

$$L_c=2(\alpha_1\beta_2-\alpha_2\beta_1) \tag{2·3b}$$

$$L_i=l. \tag{2·3c}$$

L_c is the angular momentum due to the collective motion and L_i is that due to the internal motion.

Instead of characterizing the collective motion by the two coordinates α_1 and α_2, it may also be described by an angular coordinate θ specifying the orientation of the deformed nuclear surface and a parameter determining its shape. They are given by

$$\begin{aligned}\alpha_1&=a\cos 2\theta\\ \alpha_2&=a\sin 2\theta.\end{aligned} \tag{2·4}$$

The kinetic energy of the irrotational collective motion (2·2b) is expressed in terms of these variables and takes the form

$$T_c=T_{\text{rot}}+T_{\text{vib}}. \tag{2·5a}$$

The first term is the rotational kinetic energy which is given by

$$T_{\text{rot}}=L_c^2/2J \tag{2·5b}$$

with the moment of inertia

$$J=2mAR_0^2a^2,$$

while the second term is the vibrational energy independent of θ. We have neglected the second term on the right-hand side of (2·2b) and used the approximation $r^2\simeq R_0^2/2$ which was verified in heavy nuclei.[2)]

With Eq. (2·5a), we can decompose the hamiltonian (2·2a) in the form

$$H=T_{\text{rot}}+H_a \tag{2·6a}$$

$$H_a=H_{\text{internal}}+\sum_j H_j\alpha_j+\sum_{j,k} H_{jk}\alpha_j\alpha_k+\cdots+T_{\text{vib}}. \tag{2·6b}$$

As it was discussed in § 1, we cannot anticipate the collective-internal coupling, the second term on the right-hand side of (2·6b), to be negligibly small; it may be rather strong. The coupled internal motion must be of a rotational nature in order to contribute to the magnitude of the moment of inertia. This internal rotation is supposed to be described by an angular coordinate φ which satisfies the canonical commutation relation with the internal angular momentum as follows

$$[L_{\mathrm{i}}, \varphi]=-i. \tag{2·7}$$

Strictly speaking, it is not a simple matter to see if there exists in the quantum many body system such a quantity as φ that satisfies the commutation relation (2·7). However, if there takes place an internal motion which rotates together with the irrotational collective one, such an angular variable may exist, at least, approximately satisfying the relation (2·7). Moreover, if it is possible to choose φ orthogonal to any other internal coordinates, the kinetic energy of this internal rotation may be given by the square of L_{i}. The moment of inertia of the internal rotation I, being the mass parameter of the kinetic energy, will be defined by means of the following relation

$$[H_{\mathrm{a}}, \varphi]=-(i/I)L_{\mathrm{i}}. \tag{2·8}$$

In order to obtain the explicit formula of I, we make up the representations of the above two relations (2·7) and (2·8) which are diagonal in H_{a} and eliminating the matrix element of φ, I will be given by the following formula,

$$I=2\sum_{n}|(\Psi_0|L_{\mathrm{i}}|\Psi_n)|^2/(E_n-E_0), \tag{2·9}$$

where Ψ_n and E_n are the eigenstates and eigenvalues of H_{a}, respectively. In deriving this formula there seems to be some ambiguity in the choice of the state Ψ_0. However, if the separation of the mode of internal rotation from the other internal motions can be performed fairly well I does not depend on what state is chosen for Ψ_0. We will finally remark the very similarity between I and $\mathcal{J}_{\mathrm{cr}}$ which was given by the cranking model formula (1·3), since Ψ_n is the eigenfunction of the hamiltonian H_{a} which means, being phenomenologically interpreted, to be a wave function in a deformed potential.

By the use of equations (2·7) and (2·8), it can easily be derived that the quantity $H_{\mathrm{a}}-L_{\mathrm{i}}^2/2I$ commutes with the angle φ, that is to say, it does not involve the internal angular momentum L_{i} and, consequently, it is the function of φ only with respect to the internal rotational motion. The φ dependence of H_{a} can be directly inferred from equation (2·6b) in which the second term on the right-hand side, being the coupling term between the internal and the collective motions, must explicitly depend on φ. The remaining terms in (2·6b) may be also φ dependent, but their dependence must be moderate enough to be neglected as compared with that of the second term. After all, H_{a} depends on the collective and the internal rotational motions with the following form,

$$H_a=\alpha_1 H_1+\alpha_2 H_2+L_i^2/2I \qquad (2\cdot10)$$
$$+\text{terms not dependent on the rotation.}$$

Corresponding to (2·4), H_1 and H_2 can be expressed by means of the angle φ as follows:

$$\begin{aligned} H_1&=V\cos 2\varphi \\ H_2&=V\sin 2\varphi, \end{aligned} \qquad (2\cdot11)$$

the origin of the angle φ being properly adjusted. Now we shall show that the quantity V above introduced is rotation independent. As the total angular momentum is a constant of motion, it commutes with the total hamiltonian of the system

$$[H,\ L]=0. \qquad (2\cdot12)$$

According to equations (2·5b) and (2·3b), one obtains the relation

$$[T_{rot},\ L]=[T_{rot},\ L_c]=0. \qquad (2\cdot13)$$

By means of (2·6a), (2·13) and (2·10), we derive from (2·12)

$$[\alpha_1 H_1+\alpha_2 H_2,\ L_c+L_i]=0.$$

If we use the form (2·3b) for L_c, above relation gives the equation

$$\alpha_1[H_1,\ L_i]+\alpha_2[H_2,\ L_i]=2i(\alpha_2 H_1-\alpha_1 H_2).$$

Comparing the factors of α_1 and α_2 on both sides, respectively, we find

$$\begin{aligned} [H_1,\ L_i]&=-2iH_2 \\ [H_2,\ L_i]&=2iH_1\,. \end{aligned} \qquad (2\cdot14)$$

Making use of (2·11), we get finally

$$[V,\ L_i]=0 \qquad (2\cdot15)$$

which means that V is not dependent on the angular coordinate φ. It has already been remarked that V too does not depend on L_i, but we shall prove it again explicitly. For that purpose, we calculate the quantity

$$\begin{aligned} [\varphi,\ [H_a,\ L_i]]&=[\varphi,\ [\alpha_1 H_1+\alpha_2 H_2,\ L_i]] \\ &=[\varphi,\ -2i\alpha_1 H_2+2i\alpha_2 H_1] \\ &=(-2i\alpha_1\sin 2\varphi+2i\alpha_2\cos 2\varphi)[\varphi,\ V] \end{aligned} \qquad (2\cdot16)$$

by means of (2·10), (2·14) and (2·11). Meanwhile we can make the left-hand side of (2·16) vanish using the identity

$$[\varphi,\ [H_a,\ L_i]]=[H_a,\ [\varphi,\ L_i]]+[[\varphi,\ H_a],\ L_i]$$

and Eq. (2·7) and (2·8). Thus one obtains

$$[\varphi,\ V]=0 \qquad (2\cdot17)$$

from (2·16). Consequently we can write down the explicit angular coordinate dependence of the coupling energy between the irrotational collective motion and the internal rotation as follows,

$$\alpha_1 H_1+\alpha_2 H_2=aV\cos 2(\theta-\varphi), \tag{2·18}$$

by means of (2·4). (2·11), (2·15) and (2·17).

Finally the hamiltonian takes the form

$$H=L_c^2/2J+aV\cos 2(\theta-\varphi)+L_i^2/2I \\ +\text{terms not dependent on the rotation.} \tag{2·19}$$

It is convenient to make the further substitution

$$\Theta=(J\theta+I\varphi)/(J+I)$$
$$\Omega=\theta-\varphi,$$

which means respectively the angular center of mass and the relative angle of the two rotators.

Then the total angular momentum is given by

$$L=-i\partial/\partial\Theta$$

by means of (2·3a) and the hamiltonian (2·19) is written in the form

$$H=\frac{1}{2(I+J)}L^2-\frac{I+J}{2IJ}\frac{\partial^2}{\partial\Omega^2}+aV\cos 2\Omega. \tag{2·20}$$

In the case of strong coupling between the irrotational collective and the internal rotations, the absolute value of aV is large compared with the energy of the rotational excitation. In this case the angle Ω will perform small vibration around the equilibrium point where $aV\cos 2\Omega$ takes its minimum value, so we may expand the last term of the right-hand side of (2·20) in the power series in Ω. Taking the series up to the square term in Ω, the hamiltonian (2·20) may be easily diagonalized and give the energy spectrum

$$E(M,n)=aV+M^2/2(I+J)+(n+1/2)\omega_0 \\ \omega_0^2=-4(I+J)aV/IJ \tag{2·21}$$

where M is the magnitude of the total angular momentum and n is zero or the positive integer. Now since the last term of the right-hand side of (2·21) is much larger than the second term, n must take the value zero in the low excitation of the nucleus. Then the excitation is always rotational in which the effective moment of inertia is given by the sum of those of two motions by comparing (2·21) with the expression (1·1). The effective moment of inertia deviates from the hydrodynamical value by the amount (2·9) which becomes greater, as the excited single particle levels in a deformed potential are more closely spaced and the matrix elements of L_i are larger. Therefore, considerable deviation is expected for the

nuclei which lie far from the closed shell and whose deformation are large, as was pointed out by many authors.

The motion of our system may be pictured in the following manner. Suppose that the system is in a configuration corresponding to the lowest value of the coupling energy $aV\cos 2\Omega$, that is a configuration with $\theta=\varphi$ or $\theta=\varphi+\pi/2$ according to the sign of V. Then let us imagine that θ is increased by a small amount δ. This means that our drop of nuclear matter is deformed irrotationally, the change of the drop shape being a tidal rotation of the angle δ. This deformation then causes the increase of energy by the amount $2aV\delta^2$, so that it should be followed by the change of φ of the same angle δ which compensates the increase of energy. Physically this change of φ means the rearrangement of the particles in such a way that the inner stress is eliminated and the lowest value of the coupling energy is restored, the drop shape being kept unchanged. Thus our motion will be, roughly speaking, a succession of deformation and rearrangement, resulting in a rotation of the drop not purely tidal, but, on the other hand, not necessarily rigid because the rearrangement does not necessarily mean the persistence of a fixed relative position of particles in the drop.

§ 3. The connexion with other methods

In order to discuss the connexions between the other methods and that by Tomonaga, we will consider the work by Hayakawa and Marumori[8] which is most suitable for our purpose as an example because their work is very similar to others such as Villars', and is closely related with the idea of cranking model and, in addition, they employed the mathematical methods in common with ours.

They decomposed the hamiltonian of two dimensional particle system in the form:

$$H=H_{\text{rot}}+H_{\text{part}}\,, \tag{3·1a}$$

$$H_{\text{rot}}=1/(2J)\cdot\{(M-\langle L_i\rangle)^2+2(M-\langle L_i\rangle)\langle L_i\rangle+(L_i-\langle L_i\rangle)^2\}, \tag{3·1b}$$

$$H_{\text{part}}=H_a-\omega L_i \tag{3·1c}$$

where $\omega=J^{-1}(M-\langle L_i\rangle)$ and $\langle L_i\rangle$ is the expectation value of L_i by the eigenfunction of H_{part}. This decomposition involves no approximation. In order to solve the Schrödinger equation

$$H_{\text{part}}\psi=\epsilon\psi, \tag{3·2}$$

it was assumed that the last term of the right-hand side of (3·1c) could be treated as a perturbation as in the case of cranking model. As a result of the approximation, one obtains

$$\langle L_i\rangle=MI/(I+J). \tag{3·3}$$

Finally, making the expectation value of H_{rot} in ψ they showed the possibility to derive the rotational spectrum in which the effective moment of inertia was given

by the sum of I and J, if the last term of (3·1b) could be neglected compared with the preceding two terms. Quantitatively the first assumption may be formulated as

$$\sum_{m\neq 0}\left|\frac{\omega}{E_0-E_m}(\Psi_m|L_i|\Psi_0)\right|^2\ll 1, \tag{3·4}$$

where Ψ_m and E_m are the eigenfunction and eigenvalue of H_a respectively, as defined below Eq. (2·9). Meanwhile the second assumption means

$$\begin{aligned}\langle(L_i-\langle L_i\rangle)^2\rangle&=\langle L_i^2\rangle-\langle L_i\rangle^2\ll|(M-\langle L_i\rangle)^2+2(M-\langle L_i\rangle)\langle L_i\rangle|\\&=|M^2-\langle L_i\rangle^2|=M^2-\langle L_i\rangle^2,\end{aligned}$$

by means of (3·3). We may thus write

$$\langle L_i^2\rangle\ll M^2. \tag{3·5}$$

We shall now examine the validity of these two conditions on the basis of the phenomenological model given in § 2. The hamiltonian H_a takes the form

$$\begin{aligned}H_a=&L_i^2/2I+aV-2aV(\varphi-\theta)^2+\cdots\\&+\text{terms not dependent on the rotation}\end{aligned}$$

from (2·10) and (2·18) in the strong coupling approximation. Therefore E_m is given by

$$E_m=aV+(m+1/2)(\alpha^2/I), \tag{3·6}$$

with $\alpha^4=-4aVI$ which is large compared with unity in the strong coupling and m non-negative integer. And Ψ_m now takes the form

$$\begin{aligned}&\Psi_m=N_mH_m[\alpha(\varphi-\theta)]\exp\{-\alpha^2(\varphi-\theta)^2/2\},\\&N_m=\left(\frac{\alpha}{\sqrt{\pi}\,2^m m!}\right)^{1/2}\end{aligned} \tag{3·7}$$

where H_m is the Hermite polynomial of order m.

By means of the definition of ω, (3·3), (3·6) and (3·7), one is able to calculate the left-hand side of (3·4) and the result is given by

$$\sum_{m\neq 0}\left|\frac{\omega}{E_0-E_m}(\Psi_m|L_i|\Psi_0)\right|^2=\left(\frac{IM}{I+J}\right)^2\frac{1}{2\alpha^2},$$

which is small compared with unity in the strong coupling. Thus the condition (3·4) is satisfied by the model in § 2. In the same way, the left-hand side of (3·5) may be shown to be

$$\begin{aligned}\langle L_i^2\rangle&\simeq(\Psi_0|L_i^2|\Psi_0)-2Re\sum_{m\neq 0}\frac{\omega}{E_0-E_m}(\Psi_0|L_i^2|\Psi_m)(\Psi_m|L_i|\Psi_0)\\&=\frac{1}{2}\alpha^2+\frac{3}{4}\left(\frac{IM}{I+J}\right)^2.\end{aligned} \tag{3·8}$$

Since this is large compared with unity, the condition (3·5) cannot be satisfied by the model. This indicates the fundamental difference between the two methods. Moreover, it seems to suggest the inconsistency between the two assumptions (3·4) and (3·5).

By the condition (3·4), it may be expected that the ratio of the two terms on the right-hand side of (3·8) is appreciably small compared with unity which we shall designate by δ. It can be interpreted as an expansion parameter of the perturbation series in the problem of (3·2). Then Eq. (3·8) takes the form

$$\begin{aligned}\langle L_i^2\rangle &\cong (1+\delta)\,(\Psi_0|L_i^2|\Psi_0)\\ &=(1+\delta)\sum_{n\neq 0}|(\Psi_n|L_i|\Psi_0)|^2\\ &=(1+\delta)\,(E_1-E_0)\sum_{n\neq 0}|(\Psi_n|L_i|\Psi_0)|^2/(E_1-E_0)\\ &\geqq(1+\delta)\,(E_1-E_0)\sum_{n\neq 0}|(\Psi_n|L_i|\Psi_0)|^2/(E_n-E_0)\\ &=(1+\delta)\,(E_1-E_0)\,(I/2),\end{aligned}\tag{3·9}$$

by means of (2·9) ; E_1 is the energy of the first excited state of H_a. Meanwhile, the second order perturbation energy of H_{part} is given by $-I\omega^2/2$, therefore we may set

$$\delta^2(E_1-E_0)\cong I\omega^2/2.\tag{3·10}$$

By means of (3·9) and (3·10) one obtains

$$\langle L_i^2\rangle\geqq\frac{1+\delta}{\delta^2}\left(\frac{I\omega}{2}\right)^2=\frac{1+\delta}{4\delta^2}\left(\frac{IM}{I+J}\right)^2.$$

Imposing on $\langle L_i^2\rangle$ the condition (3·5) and assuming the relation $I\cong 4J$ which may be expected from the experimental data, we find the condition for the magnitude of δ

$$25\delta^2-4\delta-4\geqslant 0.$$

Thus one obtains the result

$$\delta\geqslant 0.5$$

which explicitly contradicts our expectation $\delta\ll 1$. Even if δ takes a value between 0.5 and 1, one must give up the application of perturbation theory to solve Eq. (3·2). Then the discussion to derive the rotational spectra may be entirely changed. This seems to mean that it is rather difficult to obtain the rotational spectrum in the case of weak coupling between the irrotational collective and the internal motion.

§ 4. Application to the three dimensional nucleus

In the case of the three dimensional nucleus the total hamiltonian H_0 is given by

$$H_0=(1/2m)\sum_n\sum_\nu p_{\nu n}p^*_{\nu m}+V(\mathbf{r}_1,\ \mathbf{r}_2,\ \cdots,\ \mathbf{r}_A), \qquad (4\cdot1)$$

with

$$p_0=p_z, \qquad p_\pm=\mp\frac{1}{\sqrt{2}}(p_x\pm ip_y).$$

To describe the irrotational collective motion we use the set of coordinates

$$\xi_m=(4\pi/3AR_0^2)\sum_n Y_{2m}(\mathbf{r}_n),$$

where $Y_{2m}(\mathbf{r}_n)$ is the solid harmonics of the second order. The separation of the hamiltonian (4·1) into collective and internal parts gives the result, according to Tamura and one of present authors,[3]

$$H_0\to H=T_c+H', \qquad (4\cdot2a)$$

$$T_c=(4\pi/3mAR_0^4)\sum_{\mu=-2}^{2}\beta^*_\mu R^2\beta_\mu, \qquad (4\cdot2b)$$

$$H'=\sum_{\mu=-2}^{2}\alpha_\mu H_\mu+H_I, \qquad (4\cdot2c)$$

small correction terms in the kinetic energy of the collective motion T_c being neglected. In these expression R is defined by $R^2=(5/3A)\sum_{n=1}^{A}\mathbf{r}_n^2$ and approximately equal to R_0, the nuclear radius. The first term on the right-hand side of (4·2c) gives the coupling between the irrotational collective and the internal motions, the second term contains the energy of internal motion, the potential energy of the collective motion and so on. α_μ are the coordinates of the irrotational collective motion in this transformed representation, and β_μ are the momenta canonically conjugate to α_μ. And H_μ depends only on the internal variables. The total angular momentum takes the form

$$L_m=L^c_m+L^i_m\ ;\ m=-1,\ 0,\ +1, \qquad (4\cdot3)$$

where the angular momentum of irrotational collective motion L^c_m is given by

$$L^c_m=\sqrt{10}\,i\sum_{\lambda,\mu=-2}^{2}(2\lambda2\mu|1m)\alpha_\lambda\beta^*_\mu, \qquad (4\cdot4)$$

and L^i_m is that of the internal rotation.

Instead of characterizing the collective motion by the set of coordinates α_μ, we can use three angular coordinates θ_i specifying the orientation of the deformed nuclear surface and two parameters a_μ determining its shape. The latter coordinates are related to α_μ by the following formula

$$\alpha_\lambda=\sum_{\mu=-2}^{2}D_{\lambda\mu}(\theta_i)\,a_\mu \qquad (4\cdot5)$$

with $a_2=a_{-2}$ and $a_1=a_{-1}=0$, $D_{\lambda\mu}(\theta_i)$ being the transformation functions for the spherical harmonics of order two. Denoting the unit vectors along the each axis

of the frame of reference fixed in space by $(\mathbf{f}_x, \mathbf{f}_y, \mathbf{f}_z)$ and those fixed to the nuclear surface by $(\mathbf{k}_x^c, \mathbf{k}_y^c, \mathbf{k}_z^c)$, we define the transformation coefficients which are the functions of θ_i by

$$K_{uv}(\theta_i) = (\mathbf{f}_u \cdot \mathbf{k}_v^c) \; ; \quad u, \; v = x, \; y, \; z, \tag{4·6}$$

the functional form of $K_{uv}(\theta_i)$ was derived, for example, by Casimir.[11] By means of (4·6) one obtains the component of the collective angular momentum along $\mathbf{k}_u^c$ axis as follows,

$$P_u = \sum_v K_{vu}(\theta_i) L_v^c \; ; \quad u = x, \; y, \; z.$$

Then T_c written in terms of θ_i and a_λ takes the form

$$\begin{aligned} &T_c = T_{rot} + T_{vib} \, , \\ &T_{rot} = (1/2J_x) P_x^2 + (1/2J_y) P_y^2 + (1/2J_z) P_z^2, \end{aligned} \tag{4·7}$$

where T_{vib} is the kinetic energy of surface vibration and J_u are the moments of inertia of surface rotation which are given by

$$\begin{aligned} &J_x = J_y = 9mAR_0^2\beta^2/8\pi = \mathcal{J}_{rigid}\epsilon^2 = \mathcal{J}_{ir} \, , \\ &J_z = 0, \end{aligned} \tag{4·8}$$

where β defined by

$$\beta^2 = a_0^2 + 2a_2^2$$

gives a measure of the degree of deformation of the nuclear surface and is proportional to ϵ. These relations were first derived by Bohr.[1]

With Eq. (4·7), we can decompose the hamiltonian (4·2a) into the form

$$H = T_{rot} + H_a, \tag{4·9a}$$

$$H_u = \sum_{\mu=-2}^{2} \alpha_\mu H_\mu + H_I + T_{vib} \, . \tag{4·9b}$$

Now we shall separate phenomenologically the internal rotation coupled with the collective motion. As in the two dimensional example, it may be described by three eulerian angles, $\varphi_1, \varphi_2, \varphi_3$, which specify the orientation of the principal axes $\mathbf{k}_u^i$ (u=x, y, z) characteristic of the internal rotation. Instead of the angles the transformation coefficients may be used which are defined by

$$K_{uv}(\varphi_i) = (\mathbf{f}_u \cdot \mathbf{k}_v^i) \; ; \quad u, \; v = x, \; y, \; z. \tag{4·10}$$

Since this rotation is generated by the internal angular momentum they must satisfy the set of commutation relations

$$L_x^i K_{yu}(\varphi_i) - K_{yu}(\varphi_i) L_x^i = K_{xu}(\varphi_i) L_y^i - L_y^i K_{xu}(\varphi_i) = iK_{zu}(\varphi_i),$$

.....

u=x, y, z.

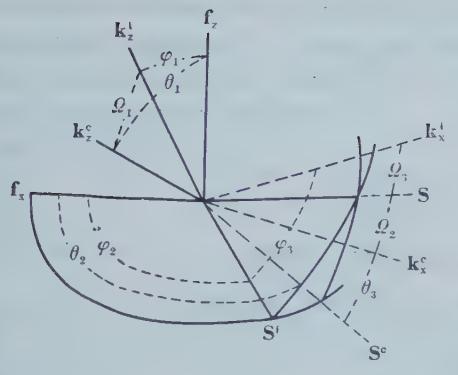

Fig. 1. Eulerian angles of three kinds. θ_1, θ_2, θ_3 denote the orientation of the nuclear surface ($\mathbf{k}_x^c$, $\mathbf{k}_y^c$, $\mathbf{k}_z^c$) relative to the frame of reference ($\mathbf{f}_x$, $\mathbf{f}_y$, $\mathbf{f}_z$) fixed in space. φ_1, φ_2, φ_3 give the directions of the principal axes ($\mathbf{k}_x^i$, $\mathbf{k}_y^i$, $\mathbf{k}_z^i$) characteristic of the internal motion. Finally Ω_1, Ω_2, Ω_3 determine the relative orientation of ($\mathbf{k}_x^c$, $\mathbf{k}_y^c$, $\mathbf{k}_z^c$) and ($\mathbf{k}_x^i$, $\mathbf{k}_y^i$, $\mathbf{k}_z^i$) systems. $\mathbf{S}^c$ is the nodal line, the line of intersection of the $\mathbf{f}_x$, $\mathbf{f}_y$ and the $\mathbf{k}_x^c$, $\mathbf{k}_y^c$ plane. $\mathbf{S}^i$ is that of the $\mathbf{f}$ and $\mathbf{k}^i$ systems and $\mathbf{S}$ is that between $\mathbf{k}^c$ and $\mathbf{k}^i$ systems. The definitions of quantities are the same as those given in the textbook by Casimir[11].

By means of (4·10) the component of internal angular momentum along $\mathbf{k}_u^i$ axis is given by

$$Q_u = \sum_v K_{vu}(\varphi_i) L_v^i .$$

Now, we suppose that the moment of inertia I_u of the internal rotation exists and can be defined by the relations

$$[H_a, K_{ux}(\varphi_i)] = (i/2I_x)\{Q_x K_{uy}(\varphi_i) + K_{uy}(\varphi_i) Q_x\} - (i/2I_y)\{Q_y K_{ux}(\varphi_i) + K_{ux}(\varphi_i) Q_y\}, \qquad (4\cdot11)$$

.....

u=x, y, z,

which express our expectation that the internal kinetic energy takes the form of that of an asymmetrical top whose moments of inertia are I_x, I_y and I_z. In order to gain the explicit form of I_u we multiply the first equation in (4·11) by $K_{uy}(\varphi_i)$ from left and sum them up with respect to the index u. Then by means of the relations

$$\sum_u K_{uv}(\varphi_i) K_{uw}(\varphi_i) = \delta_{vw}$$

one obtains

$$(2i/I_x) Q_x = \sum_n \{K_{uy}(\varphi_i) H_a K_{uz}(\varphi_i) - K_{uz}(\varphi_i) H_a K_{uy}(\varphi_i)\}. \qquad (4\cdot12)$$

Making use of the eigenvalues E_n and eigenfunctions Ψ_n of H_a, I_x may be expressed from (4·12) as

$$I_x = \frac{(\Psi_0 | Q_x | \Psi_0)}{\mathrm{Im} \sum_{nm} E_n (\Psi_0 | K_{uy}(\varphi_i) | \Psi_n)(\Psi_n | K_{uz}(\varphi_i) | \Psi_0)}$$

Similar expressions are valid for I_y and I_z. They are the three dimensional version of the moments of inertia corresponding to (2·9) of the two-dimensional case.

Since $H_a - \sum_u (1/2I_u) Q_u^2$ commutes with $K_{uv}(\varphi_i)$ by means of (4·11), it does

not contain the internal angular momentum, which means that it is a function of angles only with respect to the internal rotational motion. The angular dependence may be inferred from Eq. (4·9b) in which the first term on the right-hand side is most effective as compared with the others. It may therefore be expected that H_a now takes the form

$$H_a=\sum_{\mu}\alpha_{\mu}H_{\mu}+\sum_{u}(1/2I_u)Q_u^2 \\ +\text{terms independent of the rotation.} \qquad (4\cdot13)$$

In order to visualize the mechanism of the motions, we will put

$$H_{\mu}=\sum_{\lambda=-2}^{2}D^*_{\mu\lambda}(\varphi_i)V_{\lambda}\,. \qquad (4\cdot14)$$

Then it is easy to see the rotational invariance of the quantity V_λ above introduced by the discussion similar to that given in § 2. Finally, by means of (4·9a), (4·7), (4·13), (4·5) and (4·14), the hamiltonian takes the form

$$H=\sum_{u}(1/2J_u)P_u^2+\sum_{\lambda\mu\nu}a_{\lambda}V_{\mu}D_{\nu\lambda}(\theta_i)D^*_{\nu\mu}(\varphi_i)+\sum_{u}(1/2I_u)Q_u^2 \\ +\text{terms independent of the rotation.} \qquad (4\cdot15)$$

Now what sort of energy spectrum will this system have in the case of strong coupling? It is too difficult to treat the problem of coupled asymmetrical tops, but fortunately many nuclei seem to have the spheroidal deformation. We shall therefore assume the rotational symmetry of the system from the beginning, that is to say, we put

$$J_x=J_y \quad \text{and} \quad I_x=I_y\,. \qquad (4\cdot16)$$

Although J_z vanishes in this case as shown in (4·8), it will be treated as if it were not zero, since $J_z=0$ is a special case which can be considered more easily.

To solve the problem under the condition of strong coupling, the hamiltonian (4·15) is still inconvenient in two points. One of them is the fact that the interaction term between the two "tops" is too complicated. The other is that, since the nuclear spin does not appear explicitly in the hamiltonian, it is not easy to obtain the rotational spectrum. Thus it is more convenient to use three eulerian angles θ_i specifying the orientation of the system as a whole and three angles Ω_i determining the relative orientation of two tops. It may be natural to define Ω_i as the eulerian angles between $\mathbf{k}_u^c$- and $\mathbf{k}_u^i$ systems; explicitly, they are given by

$$K_{uv}(\Omega_i)=(\mathbf{k}_u^c\cdot\mathbf{k}_v^i)=\sum_{w}K_{wu}(\theta_i)K_{wv}(\varphi_i)\,. \qquad (4\cdot17)$$

Meanwhile it is desirable to define θ_i so as to separate the kinetic energy of the system into two parts as far as possible, one of which represents the motion of the system as a whole and the other represents the relative motion of the two tops If it is possible, θ_i will be chosen as the eulerian angles specifying the orientations of the principal axes of momental ellipsoid of the total system. According to the

above anticipation, θ_i will be defined by the solution of equations

$$\sum_{w}(A_{uw}-A_v\delta_{uw})K_{wv}(\theta_i)=0, \quad u,\ v=x,\ y,\ z, \tag{4·18}$$

where A_{uw} is given by

$$A_{uw}=\sum_{v}(K_{uv}(\theta_i)K_{wv}(\theta_i)J_v+K_{uv}(\varphi_i)K_{wv}(\varphi_i)I_v), \quad u,\ w=x,\ y,\ z,$$

and A_v is determined from Eq. (4·18).

Finding the explicit relations between θ_i, Ω_i and θ_i, φ_i, we can express the nuclear spin (4·3) and the total hamiltonian (4·15) in terms of θ_i and Ω_i. Though the calculation is tedius and lengthy, it can be performed completely without using any approximations.

The components of the spin along the principal axes of whole system defined by

$$R_u=\sum_{v}K_{vu}(\theta_i)L_v$$

take the form

$$\begin{aligned}
R_x&=-i\left[\cos\theta_3\frac{\partial}{\partial\theta_1}+\frac{\sin\theta_3}{\sin\theta_1}\frac{\partial}{\partial\theta_2}-\frac{\cos\theta_1\sin\theta_3}{\sin\theta_1}\frac{\partial}{\partial\theta_3}\right],\\
R_y&=-i\left[-\sin\theta_3\frac{\partial}{\partial\theta_1}+\frac{\cos\theta_3}{\sin\theta_1}\frac{\partial}{\partial\theta_2}-\frac{\cos\theta_1\cos\theta_3}{\sin\theta_1}\frac{\partial}{\partial\theta_3}\right],\\
R_z&=-i\frac{\partial}{\partial\theta_3}.
\end{aligned} \tag{4·19}$$

Thus it becomes apparent that θ_i satisfy the relations between the eulerian angles and the angular momentum.[11] Similarly by means of (4·16) and (4·19) the hamiltonian (4·15) may be written in terms of R_u and Ω_i, whose explicit form will be given in the Appendix.

§ 5. Strong coupling approximation and rotational spectrum

In the case of strong coupling between the two tops, they may execute small relative vibrations about the equilibrium point at which both principal axes coincide. We shall show in this section that the energy spectrum of the coupled system approximately reduces to that of a symmetrical top.

Such vibrations as mentioned above are characterized by

$$\Omega_1\ll 1 \quad \text{and} \quad |\tan\tfrac{1}{2}(\Omega_2+\Omega_3)|\ll 1. \tag{5·1}$$

Therefore it is convenient to make use of three variables

$$\Omega_1, \quad s=\tan\tfrac{1}{2}(\Omega_2+\Omega_3), \quad \vartheta=\Omega_2-\Omega_3, \tag{5·2}$$

instead of eulerian angles Ω_i to describe the relative motions. In terms of these variables the Schrödinger equation is given by

$$H\Psi=E\Psi \tag{5·3}$$

where H is the sum of (A·1) and (A·2) in which variables are transformed by means of (5·2), while the normalization condition for the wave function Ψ now takes the form

$$\int_0^{\pi}\sin\Omega_1 d\Omega_1\int_{-\infty}^{\infty}\frac{ds}{1+s^2}\int_{-2\pi}^{2\pi}d\vartheta\langle\Psi^*\Psi\rangle_{\Theta}=1 \tag{5·4}$$

where $\langle\cdots\rangle_{\Theta}$ is the inner product with respect to θ_i angles. In order to remove the weight function in Eq. (5·4), we make a similarity transformation

$$\Psi\to\Phi=\sqrt{\frac{\sin\Omega_1}{1+s^2}}\Psi,$$

$$H\to\mathfrak{H}=\sqrt{\frac{\sin\Omega_1}{1+s^2}}H\sqrt{\frac{1+s^2}{\sin\Omega_1}},$$

which gives the Schrödinger equation and the normalization condition

$$\mathfrak{H}\Phi=E\Phi,$$

$$\int_0^{\pi}d\Omega_1\int_{-\infty}^{\infty}ds\int_{-2\pi}^{2\pi}d\vartheta\langle\Phi^*\Phi\rangle_{\Theta}=1,$$

from (5·3) and (5·4) respectively. In conformity with the anticipation (5·1), hamiltonian $\mathfrak{H}$ may be expanded in power series in Ω_1 and s. Then it is shown that in the interaction term (A·2) appear linear terms in Ω_1 and s, unless $U_1=U_1'=U_2'=0$. This means the fact that our choice of eulerian angles φ_i was not adequate in general. Therefore we have rather to determine the angles φ_i so that the linear terms of Ω_1 and s vanish in the interaction energy (A·2). The condition is given by

$$U_1=U_1'=U_2'=0,$$

or, by means of (A·3),

$$V_2=V_{-2},\quad V_1=V_{-1}=0.$$

Now the interaction term may be expanded up to the fourth order in Ω_1 and s except those proportional to $\cos\vartheta$ which are taken only up to the second order. These terms must be small enough under the present circumstances because they give the interaction energy variable with the orientation of nodal line $\mathbf{S}$ (Fig. 1), which is physically meaningless. Though Ω_1 is restricted on the range

$$0\leq\Omega_1\leq\pi,$$

it may be approximately extended, in the case of strong coupling, over the whole positive value

$$0\leq\Omega_1<\infty.$$

The expanded form of the hamiltonian now becomes

$$\mathfrak{H}=\mathfrak{H}_0+\mathfrak{H}_1+\mathfrak{H}_2+\mathfrak{H}_3,$$

$$\mathfrak{H}_0=-\frac{1}{2}\Big[\frac{1}{L}\frac{\partial^2}{\partial\Omega_1^2}+\frac{1}{4L\Omega_1^2}+\frac{1}{4D}\frac{\partial^2}{\partial s^2}+\frac{4}{L\Omega_1^2}\frac{\partial^2}{\partial\vartheta^2}+\Big(\frac{1}{B}-\frac{1}{C}\Big)\frac{\partial^2}{\partial s\partial\vartheta}\Big]$$

$$-\frac{i}{2}R_z\Big\{\frac{1}{2B}\frac{\partial}{\partial s}+\frac{4}{L\Omega_1^2}\frac{\partial}{\partial\vartheta}\Big\}+\frac{1}{2L\Omega_1^2}R_z^2$$

$$+a_0V_0+2a_2U_2-\big(\tfrac{3}{2}a_0V_0+a_2U_2\big)\Omega_1^2-16a_2U_2s^2,$$

$$\mathfrak{H}_1=-\frac{1}{2}\Big[\frac{1}{3L}+\frac{\Omega_1^2}{16L}\frac{\partial^2}{\partial s^2}+\frac{1}{D}\Big(s\frac{\partial}{\partial s}+\frac{1}{2}s^2\frac{\partial^2}{\partial s^2}+\frac{1}{4}\Big)+\Big(\frac{1}{D}-\frac{2}{3L}\Big)\frac{\partial^2}{\partial\vartheta^2}\Big]$$

$$-\frac{i}{2}\Big(\frac{1}{3L}-2b\Big)R_z\frac{\partial}{\partial\vartheta}+\frac{b}{2}(R_x^2+R_y^2)+\frac{1}{2}\Big(\frac{1}{3L}-b\Big)R_z^2$$

$$+\Big(\frac{1}{2}a_0V_0+\frac{5}{24}a_2U_2\Big)\Omega_1^4+8a_2U_2s^2\Omega_1^2+32a_2U_2s^4,$$

$$\mathfrak{H}_2=-\frac{ic}{2}\Big(R_x\frac{\partial}{\partial\Omega_1}+\frac{2R_y}{\Omega_1}\frac{\partial}{\partial\vartheta}\Big)+\frac{c}{4\Omega_1}(R_yR_z+R_zR_y)$$

$$-\sqrt{\frac{3}{2}}(a_0U_2+a_2V_0)\Omega_1^2\cos\vartheta,$$

$\mathfrak{H}_3$=the remainders,

where the following notations are employed

$$1/B=1/I_x-1/J_x,\quad 1/C=1/I_z-1/J_z,\quad 1/D=1/I_z+1/J_z,$$

$$b=\frac{1}{(\Delta I+\Delta J)^2}\Big\{\frac{(\Delta I)^2}{I_x}+\frac{(\Delta J)^2}{J_x}\Big\},\qquad c=\frac{2}{\Delta I+\Delta J}\Big(\frac{\Delta I}{I_x}-\frac{\Delta J}{J_x}\Big).$$

$\mathfrak{H}_0$ is the lowest order term in Ω_1 and s, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are the correction terms being of higher order in these variables. $\mathfrak{H}_3$ contains the terms all of which are of higher order than $\mathfrak{H}_1$ and $\mathfrak{H}_2$, and may be seen to be negligible. It is clear that $\mathfrak{H}_0$ and $\mathfrak{H}_1$ commute with the quantities $\partial/\partial\vartheta$, $\sum_u R_u^2$, L_z and R_z which makes it convenient to use the symbols of eigenfunctions defined by

$$\Big(\sum_u R_u^2\Big)|IMK\rangle=I(I+1)|IMK\rangle,$$

$$L_z|IMK\rangle=M|IMK\rangle,$$

$$R_z|IMK\rangle=K|IMK\rangle.$$

Now the equation of the lowest order approximation

$$\mathfrak{H}_0\Phi_0=E_0\Phi_0$$

may be solved by putting

$$\Phi_0=(1/\sqrt{4\pi})f(\Omega_1)g(s)\exp(ik\vartheta/2)|IMK\rangle,$$
$$k=0,\ \pm1,\ \cdots, \tag{5·5}$$

with f and g obeying the equations

$$\left[-\frac{1}{2L}\frac{d^2}{d\Omega_1^2}+\frac{4\rho^2-1}{8L\Omega_1^2}-(\tfrac{3}{2}a_0V_0+a_2U_2)\Omega_1^2-\epsilon_1\right]f(\Omega_1)=0, \tag{5·6}$$

and

$$\left[-\frac{1}{8D}\frac{d^2}{ds^2}-\frac{i}{4}\left(\frac{\rho}{B}-\frac{k}{C}\right)\frac{d}{ds}-16a_2U_2s^2-\epsilon_2\right]g(s)=0, \tag{5·7}$$

respectively, where

$$\rho=K+k,\quad E_0=a_0V_0+2a_2U_2+\epsilon_1+\epsilon_2.$$

(1) Equation (5·6) for $f(\Omega_1)$ is solved easily and gives the result

$$\epsilon_1(mk;\ IMK)=\frac{2\sigma}{L}(2m+1+|\rho|),$$
$$f_m^{\rho}(\Omega_1)=c_m^{\rho}\Omega_1^{(1/2)+|\rho|}\exp(-\sigma\Omega_1^2)\cdot\sum_{n=0}^{m}\binom{m}{n}\frac{(-2\sigma\Omega_1^2)^n}{(n+|\rho|)!},$$
$$(c_m^{\rho})^{-2}=\frac{1}{2(2\sigma)^{|\rho|+1}}\sum_{r,s=0}^{m}\binom{m}{r}\binom{m}{s}\frac{(-1)^{r+s}}{(r+|\rho|)!(s+|\rho|)!}(s+r+|\rho|)!, \tag{5·8}$$
$$\sigma=\left[-\frac{L}{2}(\tfrac{3}{2}a_0V_0+a_2U_2)\right]^{1/2},$$
$$m=0,\ 1,\ 2,\cdots.$$

(2) The solutions for the equation (5·7) are given by

$$\epsilon_2(nk;\ IMK)=(n+\tfrac{1}{2})\frac{\tau}{4D}-\frac{D}{8}\left(\frac{\rho}{B}-\frac{k}{C}\right)^2,$$
$$g_n^{k\rho}(s)=c_nH_n(\sqrt{\tau}\,s)\exp\left\{-iD\left(\frac{\rho}{B}-\frac{k}{C}\right)s-\tfrac{1}{2}\tau s^2\right\},$$
$$c_n=\left(\frac{\sqrt{\tau}}{\sqrt{\pi}\,2^n n!}\right)^{1/2},\quad \tau=8(-2a_2U_2D)^{1/2},$$
$$n=0,\ 1,\ 2,\cdots.$$

In this way we find the eigenvalues of the hamiltonian $\mathfrak{H}_0$ to be

$$E_0(mnk;\ IMK)=a_0V_0+2a_2U_2+(2m+1+|\rho|)\frac{2\sigma}{L}$$
$$+(n+\tfrac{1}{2})\frac{\tau}{4D}-\frac{D}{8}\left(\frac{\rho}{B}-\frac{k}{C}\right)^2, \tag{5·9}$$

and the eigenfunction belonging to the eigenvalue from (5·5)

$$\Phi_0(\Omega_1 s\vartheta:\ mnk;\ IMK)=(1/\sqrt{4\pi})f_m^{\rho}(\Omega_1)g_n^{k\rho}(s)\exp(ik\vartheta/2)|IMK\rangle. \quad (5\cdot10)$$

Since the conditions of strong coupling are quantitatively given by $\sigma\gg1$ and $\tau\gg1$, the relative motions are not excited in the low lying states of actual nuclei. The low lying excited states consist of the rotational states corresponding to $m=n=\rho=0$. In order to find them we consider the effect of the correction terms $\mathfrak{H}_1$ and $\mathfrak{H}_2$ by means of the perturbation theory.

We will use the following simple notation

$$|K\rangle=\Phi_0(\Omega_1 s\vartheta:\ 0\ 0\ -K;\ IMK)$$

to denote a state with $m=n=0$, $k=-K$ (or $\rho=0$). In this notation the parameters to specify the values of I and M are omitted since they are exact constants of motion. Among the states with the definite common values for these quantities, $2I+1$ states

$$|-I\rangle,\quad |-I+1\rangle,\cdots,\ |I\rangle,$$

have almost the same energy values which are given by (5·9) and, further, $2I$ of which are composed of I sets of doubly degenerate states. Therefore, it seems necessary to use the perturbation theory for the degenerate system in order to find the rotational spectrum. But, fortunately, this is not the case for the reason which we shall presently account for.

It is sufficient to determine the energy up to the order $1/L$, the higher orders being no concern of ours, since they give a small correction to the resulting moment of inertia. Then we may neglect the effect of $\mathfrak{H}_3$ and apply the perturbation theory with $\mathfrak{H}_1+\mathfrak{H}_2$ only up to the second order. In these calculations one gets no matrix element of the perturbation energy which connects the different states among $|K\rangle$, and we can use the perturbation formula for the nondegenerate system.

The first order energy is given by

$$E_1(K)\cong\langle K|\mathfrak{H}_1+\mathfrak{H}_2|K\rangle=\langle K|\mathfrak{H}_1|K\rangle$$

$$=-\frac{1}{6L}+\frac{1}{16D}+\frac{\tau}{128L\sigma}+\frac{b}{2}\{I(I+1)-K^2\}$$

$$+\frac{K^2}{8D}+\frac{1}{4\sigma^2}\left(a_0V_0+\frac{5}{12}a_2U_2\right)+\frac{2a_2U_2}{\tau}\left(\frac{1}{\sigma}+\frac{12}{\tau}\right).$$

In the second order calculation $\mathfrak{H}_1$ may be neglected. Further neglecting the terms in the matrix elements of $\mathfrak{H}_2$ which are smaller than $\sqrt{\sigma}/L$ or $\sqrt{\tau}/L$, we obtain

$$\langle\Phi_0(\Omega_1 s\vartheta:\ mn-K;\ IMK+1)|\mathfrak{H}_2|K\rangle=$$

$$\begin{cases}=0\ ; & \text{if}\quad m\geq1,\\ =(ic/16\sigma)c_0{}^0c_0{}^1c_n(\pi/\tau)^{1/4}\sqrt{(I-K)(I+K+1)}\,(iD/\sqrt{\tau}\,B)^n\exp(-D^2/4\tau B^2)\ ; & \text{if}\quad m=0,\end{cases}$$

$$\begin{cases} \cong 0\,; & \text{if} \quad m=0, \quad n\geq 1, \\ \cong ic\sqrt{(I-K)(I+K+1)\sigma/8}\,; & \text{if} \quad m=n=0, \end{cases}$$

$$\langle \Phi_0(\Omega_1 s\vartheta:\ mn-K;\ IMK-1)\,|\mathfrak{H}_2|K\rangle =$$

$$\begin{cases} =0\,; & \text{if} \quad m\geq 1, \\ \cong 0\,; & \text{if} \quad m=0,\ n\geq 1, \\ \cong ic\sqrt{(I+K)(I-K+1)\sigma/8}\,; & \text{if} \quad m=n=0, \end{cases}$$

and

$$\langle \Phi_0(\Omega_1 s\vartheta:\ mn-K\pm 2;\ IMK)\,|\mathfrak{H}_2|K\rangle =$$

$$\begin{cases} =0\,; & \text{if} \quad m\geq 1, \\ \cong 0\,; & \text{if} \quad m=0,\ n\geq 1, \\ \cong -(1/16)\sqrt{3/2}\,(a_0U_2+a_2V_0)\,c_0^{0}c_0^{2}/\sigma^3\,; & \text{if} \quad m=n=0, \end{cases}$$

other matrix elements vanish. Using the energy difference

$$E_0(00\ -K;\ IMK)-E_0(00\ -K;\ IMK\pm 1)\cong -2\sigma/L,$$

$$E_0(00\ -K;\ IMK)-E_0(00\ -K\pm 2;\ IMK)\cong -4\sigma/L,$$

the second order energy may be estimated as

$$E_2(K)\cong -(Lc^2/8)\,\{I(I+1)-K^2\}-(3L/32\sigma^3)\,(a_0U_2+a_2V_0)^2.$$

Finally we will summarize the result. The energy of the system is given up to the order of $1/L$ by

$$\begin{aligned} E(IMK) &= E_0(00\ -K;\ IMK)+E_1(K)+E_2(K) \\ &= \mathfrak{E}_0+\frac{1}{2}\left(b-\frac{Lc^2}{4}\right)\{I(I+1)-K^2\}+\left(\frac{1}{D}-\frac{D}{C^2}\right)\frac{K^2}{8} \qquad (5\cdot 11) \\ &= \mathfrak{E}_0+\frac{1}{2(I_x+J_x)}\{I(I+1)-K^2\}+\frac{1}{2(I_z+J_z)}K^2, \end{aligned}$$

with

$$\mathfrak{E}_0=a_0V_0+2a_2U_2+\left(2\sigma-\frac{1}{6}+\frac{\tau}{128\sigma}\right)\frac{1}{L}+\left(\tau-1-\frac{\tau}{8\sigma}\right)\frac{1}{8D}$$

$$+\frac{1}{4\sigma^2}\left(a_0V_0+\frac{5}{12}a_2U_2\right)-\frac{3L}{32\sigma^3}(a_0U_2+a_2V_0)^2.$$

The wave function takes, up to the order of $1/\sqrt{\sigma}$ or $1/\sqrt{\tau}$, the form

$$\Phi=|K\rangle$$

$$-(iLc/4\sqrt{2\sigma})\,\{\sqrt{(I-K)(I+K+1)}\,\Phi_0(\Omega_1 s\vartheta:\ 00\ -K;\ IMK+1)$$

$$+\sqrt{(I+K)(I-K+1)}\,\Phi_0(\Omega_1 s\vartheta:\ 00\ -K;\ IMK-1)\}$$

$$+(\sqrt{3}L/16\sigma^2)(a_0U_2+a_2V_0)\{\Phi_0(\Omega_1 s\vartheta: 00\ -K+2;\ IMK)$$
$$+\Phi_0(\Omega_1 s\vartheta: 00\ -K-2;\ IMK)\}.$$

It is now known from (5·11) that a pair of the symmetrical tops strongly coupled with each other shows a rotational spectrum as if it were a symmetrical top whose moments of inertia are the sum of those of each ones.

§ 6. Discussion of Coulomb excitation

When a charged nuclear particle impinges upon the nucleus with its energy well below the Coulomb barrier, the nucleus may be excited through the electromagnetic interaction between the particle and the nucleus. Since all excitations identified so far have been found to be of electric quadrupole type, we shall now consider E2 transitions.

In the classical treatment of the impinging particle of the charge Z_1e and velocity v moving along a hyperbolic orbit in the repulsive Coulomb field of the target nucleus with the charge Z_2e, the differential excitation cross section is given by[12]

$$d\sigma_{E2}=(Z_1e/v)^2a^{-2}B(E2)\,df_{E2}(\vartheta,\xi), \tag{6·1}$$

with $a=Z_1Z_2e^2/m_0v^2$, $\xi=Z_1Z_2e^2\Delta E/2vE$, m_0 reduced mass, $E=m_0v^2/2$, ΔE the excitation energy and ϑ being the scattering angle. $f_{E2}(\vartheta,\xi)$ is a definite function of the variables explicitly defined by K. Alder et al.[12] They used the notation $B(E2)$ for the square sum of the matrix elements of electric quadrupole moment between the initial and the final states of the target nucleus

$$B(E2;\ I_i\to I_f)=\sum_{M_f,\mu}|\langle I_iM_i|\mathfrak{M}_0(E2,\mu)|I_fM_f\rangle|^2 \tag{6·2}$$

with

$$\mathfrak{M}_0(E2,\mu)=\sum_k e_kr_k^2Y_{2\mu}(\theta_k,\phi_k). \tag{6·3}$$

By means of (6·1), $B(E2)$ may be estimated from the experimental information for $d\sigma_{E2}$, the results for various nuclei being tabulated in the review work by K. Alder et al.

In order to estimate the right-hand side of (6·2) we now transform (6·3) to the representation in which α_μ and β_μ play the role of the irrotational collective variables, by means of the method given by Tamura and one of present authors.[3] Then the electric quadrupole operator takes the form

$$\mathfrak{M}_0(E2,\mu)\to\mathfrak{M}(E2,\mu)=\mathfrak{M}_0(E2,\mu)+(5R_0^2\alpha_\mu/4\pi R^2)\sum_k e_kr_k^2$$
$$-(dR_0^2/R^2)\sum_{\lambda\nu}(2\lambda 2\nu|2\mu)\mathfrak{M}_0(E2,\lambda)\alpha_\nu, \tag{6·4}$$

with $d=\sqrt{35/8\pi}$, up to the linear order in α_μ. The second term of (6·4) may be interpreted as the irrotational collective part of the electric quadrupole moment and the third term, being due to the approximately conjugate relations between the

collective coordinates and momenta, may be neglected. Assuming that the density of the particle is uniform within the nucleus one may write

$$\mathfrak{M}(E2,\ \mu)=\mathfrak{M}_0(E2,\ \mu)+3Z_2eR_0^2\alpha_\mu/4\pi. \tag{6·5}$$

Since we have not discussed the detailed nature of the internal motion, it is now hard to say something about the electromagnetic excitations of nucleus by means of the internal motion. Therefore, we shall first assume that it has rather smaller effect for the transitions than the irrotational collective motion does. When the mechanism of the internal motion becomes clear, the correction may be easily taken into account.

We obtain from (6·5)

$$\begin{aligned}\langle I_iM_iK_i|\mathfrak{M}(E2,\ \mu)|I_fM_fK_f\rangle&=(3Z_2eR_0^2/4\pi)\langle I_iM_iK_i|\alpha_\mu|I_fM_fK_f\rangle\\&=(3Z_2eR_0^2\beta/4\pi)\langle I_iM_iK_i|D_{\mu0}(\theta_i)|I_fM_fK_f\rangle\end{aligned} \tag{6·6}$$

by means of (4·5) and the condition of rotational symmetry $a_2=0$. Further we may write

$$\langle I_iM_iK_i|D_{\mu0}(\theta_i)|I_fM_fK_f\rangle=(I_iK_i\|D_0(\theta_i)\|I_fK_f)\,(I_fM_f2\mu|I_iM_i)/\sqrt{2I_i+1}\,. \tag{6·7}$$

In order to know the magnitude of $(I_iK_i\|D_0(\theta_i)\|I_fK_f)$, matrix elements of $D_{00}(\theta_i)$ between the states $M_i=0$ and $M_f=0$ are now to be calculated.

$D_{00}(\theta_i)$ is given by

$$\begin{aligned}D_{00}(\theta_i)&=(1/2)\,(3\cos^2\theta_1-1)\\&=\frac{3}{4}\left(\frac{\Delta J+\Delta I\cos 2\Omega_1}{\sqrt{Z(\Omega_1)}}+1\right)\cos^2\theta_1-\frac{3}{4}\left(\frac{\Delta J+\Delta I\cos 2\Omega_1}{\sqrt{Z(\Omega_1)}}-1\right)\sin^2\theta_1\cos^2\theta_3\\&+\frac{3\Delta I}{\sqrt{Z(\Omega_1)}}\sin\Omega_1\cos\Omega_1\sin\theta_1\cos\theta_1\cos\theta_3-\frac{1}{2}\\&\simeq(1/2)\,(3\cos^2\theta_1-1),\end{aligned}$$

up to the lowest order in Ω_1.

Meanwhile the wave functions of the system are slightly changed from (5·10) on account of the symmetry conditions $J_z=0$ and $a_2=0$ and take, in the lowest order approximation, the form

$$\begin{aligned}&\Phi(\Omega_1\Omega_2\Omega_3:\ nm\,;\ IMK)=(1/2\pi)f_n^\rho(\Omega_1)\exp(im\Omega_3)\,|IMK\rangle,\\&n=0,\ 1,\ 2,\cdots;\quad m=0,\ \pm1,\ \pm2,\cdots,\end{aligned}$$

where $f_n^\rho(\Omega_1)$ is defined in (5·8) and $\rho=|K-m|$. Within the energy region of the rotational excitation, n and ρ vanish. The initial state of the target nucleus, being the ground state, is characterized by $I_i=M_i=K_i=0$, the final state by $I_f=2$. Thus we obtain

$$(\Phi(\Omega_i:\ 00\,;000)\,|D_{00}(\theta_i)|\,\Phi(\Omega_i:\ 0K\,;\ 20K))=\delta_{K0}/\sqrt{5}\,,$$

which gives the result, from (6·7)

$$(00\|D_0(\theta_i)\|2K)=\delta_{K0}. \tag{6·8}$$

Finally, by means of (6·2), (6·6), (6·7) and (6·8), we get

$$B(E2\,;\ 0\to 2)=9e^2Z_2^2R_0^4\beta^2/16\pi^2 \tag{6·9}$$

which coincides with the result given phenomenologically by Bohr and Mottelson.[1] Consequently if the internal rotation makes little contribution to the electric quadrupole transition, we can estimate magnitude of J_x by means of (6·9) and (4·8). The results are tabulated in the Table.

§ 7. Discussions

According to our model, the effective moment of inertia $\mathcal{J}$ is given by the sum I_x+J_x. As shown in § 6, J_x is related to the nuclear deformation and experimentally determined by using the Coulomb excitation. On the other hand, $\mathcal{J}=I_x+J_x$ is calculated from the empirical data for the rotational excitation energy. In this way I_x and J_x are determined separately. The results are listed in the Table and drawn in Fig. 2 as a function of the neutron number of the nucleus.

Table. Irrotational and internal moments of inertia J_x and I_x. The energy of the first excited state $3/(I_x+J_x)$ and the reduced transition probability $B(E2)$ give the magnitudes of I_x and J_x according to the formulas (4·8), (5·11) and (6·9). Empirical data are given in the work by K. Alder et al.[12] in which the isotope separation is insufficient for Er and Yb.

Nucleus	$3\hbar^2/(I_x+J_x)$ (kev)	I_x+J_x (10^{-47} gr cm^2)	β	J_x (10^{-47} gr cm^2)	I_x (10^{-47} gr cm^2)	I_x/J_x
Nd^{150}	130	1.60	0.25	0.23	1.37	5.96
Sm^{152}	122	1.71	0.28	0.30	1.42	4.73
Sm^{154}	83	2.50	0.33	0.42	2.08	4.95
Gd^{154}	123	1.69	0.30	0.35	1.34	3.86
Gd^{156}	89	2.34	0.41	0.66	1.68	2.55
Gd^{158}	79	2.64	0.46	0.85	1.79	2.12
Gd^{160}	76	2.73	0.47	0.90	1.83	2.03
Dy^{162}	82	2.54	0.36	0.54	2.00	3.71
Dy^{164}	73	2.84	0.41	0.72	2.12	2.96
Er	80	2.60	0.33	0.48	2.12	4.42
Yb	78	2.66	0.31	0.45	2.21	4.92
Hf^{176}	89	2.34	0.29	0.40	1.94	4.85
Hf^{178}	91	2.28	0.31	0.47	1.81	3.85
Hf^{180}	93	2.23	0.27	0.36	1.87	5.20
W^{182}	100	2.08	0.26	0.34	1.74	5.12
W^{184}	112	1.86	0.24	0.30	1.56	5.22
W^{186}	124	1.67	0.24	0.30	1.37	4.57
Th^{232}	52	4.01	0.25	0.48	3.53	7.36
U^{238}	44	4.75	0.28	0.63	4.12	6.54

In general I_x is four or five times as large as J_x, especially for heavy nuclei with the neutron number larger than 140, the rotational excitations are mainly due to the internal rotation. We thus find that the internal rotation is important. However, there seems to be, at present, little experimental indication which suggests what is the internal motion. We know only that the internal motion may be something of rotational nature but different from the rigid rotation. In our treatment of the internal rotation, we did not assume that the internal hamiltonian has exactly a quadratic term with respect to the internal angular momentum L_i. We only assumed that the internal motion cranked up by the surface rotation may be a rotational one with a definite moment of inertia I when the rotation follows the crank (the surface rotation) with a small phase lag, $\theta-\varphi=\Omega$. The relations (2·7) and (2·8) defining φ are only valid for a range of φ small compared with 2π. Only in this approximation the formula (2·9) is valid too. These relations have no meaning when interpreted too literally, since the quantity I defined by the operator relation (2·8) is not constant strictly. In this connection, it may be worth-while to notice that, when the same procedure as mentioned above is applied to a hamiltonian H which commutes with the total angular momentum L (see eg. Bohr and Mottelson[13]), the formula corresponding to (2·9) has no meaning, because L has no off-diagonal elements. Considering the case of a mechanical top, it is clear that, although such quantities as $\cos\varphi$, $\sin\varphi$, can be defined without any ambiguity, exact definition of the matrix element of the angle φ is impossible. We must be careful in applying such formulas.

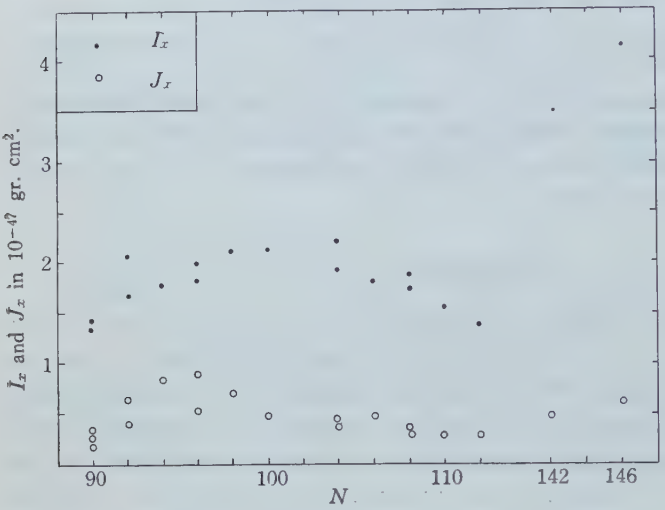

Fig. 2. Irrotational and internal moments of inertia. The figure shows the irrotational moment of inertia J_x (circle) and that of internal one I_x (point) with the neutron number $90 \leqq N \leqq 112$ and $142 \leqq N \leqq 146$. They are determined by means of the quadrupole and the spectral data. The experimental data are taken from the compilation in the review work by K. Alder et al[12].

Consideration about the nature of the internal motion was made by S. Nagata et al. basing upon the analyses of the unsatisfactory points in the usual quantum theory of irrotational collective motions.[14] They showed the necessity of introduction of the rotational collective mode. However, their discussion is a kinematical one having nothing to do with the interaction potential between nucleons. In addition, it contains ambiguity with respect to the nature of the rotational motion.

Thus the discussion seems to be insufficient to clarify the mechanism of the internal motions.

On the other hand Elliott made an interesting study taking into account the correlations of nuclear particles.[15]. He found the level structure of a set of particles occupying levels which are degenerate in harmonic oscillator potential and subject to an internucleon interaction. His interaction takes the form

$$G\sum_{m}\tilde{\xi}^{*}_{m}\tilde{\xi}_{m}$$

in our notations and it is restricted to be effective among the particles in a set of degenerate levels. Since the irrotational collective motion would be generated mainly by the particles in the unfilled shell, the restriction does not seem to be so effective. Elliott's interaction will in the main give couplings among the irrotational motions, which act as the potential energy of the surface vibrations that serves to determine the equilibrium shape of the nuclear surface. Therefore in his model the interactions between the irrotational and internal motions are not so essential. Consequently it may be open to question if it can give the correct order of magnitude for the moment of inertia when we use the value of G determined from the quadrupole data. Hence, it would be necessary to take into account more complicated correlations among the particles in order to explain the deviation of $\mathcal{J}$ from the irrotational value.

There are other arguments[5,9] on the moment of inertia. They are probably more closely connected with the cranking model than with ours, and it may not be easy to say more than that the former corresponds, in a sense, to the introduction of rotating deformed potential (a model wave function) in place of the internal hamiltonian (the internal wave function).

In this paper we discussed only the case of strong coupling between the irrotational and internal rotations. When the coupling is not strong, the spectra will not be rotational in general. It may be of some interest to see if deviation from the rotational spectra can be interpreted in terms of the intermediate coupling.

Acknowledgment

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Appendix

The hamiltonian of the coupled symmetrical tops (4·15) can be written in terms of the nuclear spin $\mathbf{R}$ (4·19) and the relative eulerian angles Ω_i between the two tops. The kinetic energy now takes the form

$$
\begin{aligned}
&(P_x^2+P_y^2)/2J_x+P_z^2/2J+(Q_x^2+Q_y^2)/2I_x+Q_z^2/2I_z= \\
&=-\frac{1}{2}\Big[\frac{1}{J_x}\Big(\frac{\partial^2}{\partial\theta_1^2}+\frac{\cos\theta_1}{\sin\theta_1}\frac{\partial}{\partial\theta_1}+\frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_2^2}-\frac{2\cos\theta_1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_3\partial\theta_2} \\
&\qquad+\frac{\cos^2\theta_1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_3^2}\Big)+\frac{1}{J_z}\frac{\partial^2}{\partial\theta_3^2}\Big] \\
&\quad-\frac{1}{2}\Big[\frac{1}{I_x}\Big(\frac{\partial^2}{\partial\varphi_1^2}+\frac{\cos\varphi_1}{\sin\varphi_1}\frac{\partial}{\partial\varphi_1}+\frac{1}{\sin^2\varphi_1}\frac{\partial^2}{\partial\varphi_2^2}-\frac{2\cos\varphi_1}{\sin^2\varphi_1}\frac{\partial^2}{\partial\varphi_3\partial\varphi_2} \\
&\qquad+\frac{\cos^2\varphi_1}{\sin^2\varphi_1}\frac{\partial^2}{\partial\varphi_3^2}\Big)+\frac{1}{I_z}\frac{\partial^2}{\partial\varphi_3^2}\Big] \\
&=-\frac{1}{2}\Big[\frac{1}{L}\frac{\partial^2}{\partial\Omega_1^2}+\frac{1}{L}\frac{\cos\Omega_1}{\sin\Omega_1}\frac{\partial}{\partial\Omega_1}+\Big(\frac{1}{L\sin^2\Omega_1}-\frac{1}{J_x}+\frac{1}{J_z}\Big)\frac{\partial^2}{\partial\Omega_2^2} \\
&\qquad-\frac{2\cos\Omega_1}{L\sin^2\Omega_1}\frac{\partial^2}{\partial\Omega_2\partial\Omega_3}+\Big(\frac{1}{L\sin^2\Omega_1}-\frac{1}{I_x}+\frac{1}{I_z}\Big)\frac{\partial^2}{\partial\Omega_3^2}\Big] \\
&\quad-\frac{i}{2}\Big[W_1(\Omega_1)R_x\frac{\partial}{\partial\Omega_1}+S_1(\Omega_1)R_x-R_y\Big\{W_2(\Omega_1)\frac{\partial}{\partial\Omega_2}+W_3(\Omega_1)\frac{\partial}{\partial\Omega_3}\Big\} \qquad (A\cdot1) \\
&\qquad+R_z\Big\{S_2(\Omega_1)\frac{\partial}{\partial\Omega_2}+S_3(\Omega_1)\frac{\partial}{\partial\Omega_3}\Big\}\Big] \\
&\quad+\tfrac{1}{2}[T_1(\Omega_1)R_x^2+T_2(\Omega_1)R_y^2+T_3(\Omega_1)\{R_yR_z+R_zR_y\}+T_4(\Omega_1)R_z^2],
\end{aligned}
$$

where following notations are used:

$$1/L=1/I_x+1/J_x,$$

$$W_1(\Omega_1)=\frac{2}{Z(\Omega_1)}\Big[\frac{\Delta I}{I_x}(\Delta I+\Delta J\cos 2\Omega_1)-\frac{\Delta J}{J_x}(\Delta I\cos 2\Omega_1+\Delta J)\Big],$$

$$W_2(\Omega_1)=2\sqrt{2}\cot\Omega_1\Big[-\frac{\Delta I}{I_x\{Z(\Omega_1)+(\Delta J+\Delta I\cos 2\Omega_1)\sqrt{Z(\Omega_1)}\}^{1/2}}+\frac{\Delta J\cos\Omega_1}{J_x\{Z(\Omega_1)+(\Delta I+\Delta J\cos 2\Omega_1)\sqrt{Z(\Omega_1)}\}^{1/2}}\Big],$$

$W_3(\Omega_1)=W_2(\Omega_1)$ in which I_x and J_x, ΔI and ΔJ are exchanged respectively,

$$S_1(\Omega_1)=\frac{\cot\Omega_1}{Z(\Omega_1)}\Big[\frac{\Delta I}{I_x}(\Delta I+\Delta J\cos 2\Omega_1)-\frac{\Delta J}{J_x}(\Delta J+\Delta I\cos 2\Omega_1)\Big]+2\Delta I\Delta J\frac{(\Delta I)^2-(\Delta J)^2}{LZ(\Omega_1)^2}\sin 2\Omega_1,$$

$$S_2(\Omega_1)=\frac{\sqrt{2}}{Z(\Omega_1)^{1/4}\sin^2\Omega_1}\Big[\frac{1}{I_x}\{\Delta J+\Delta I\cos 2\Omega_1+\sqrt{Z(\Omega_1)}\}^{1/2}+\frac{1}{J_x}\{\Delta I+\Delta J\cos 2\Omega_1+\sqrt{Z(\Omega_1)}\}^{1/2}\cos\Omega_1\Big],$$

$S_3(\Omega_1)=-S_2(\Omega_1)$ in which I_x and J_x, ΔI and ΔJ are exchanged respectively,

$$T_1(\Omega_1)=\frac{1}{Z(\Omega_1)^2}\left[\frac{(\Delta I)^2}{I_x}(\Delta I+\Delta J\cos 2\Omega_1)^2+\frac{(\Delta J)^2}{J_x}(\Delta J+\Delta I\cos 2\Omega_1)^2\right],$$

$$T_2(\Omega_1)=\frac{1}{I_x}\frac{2(\Delta I)^2\cos^2\Omega_1}{Z(\Omega_1)+(\Delta I+\Delta J\cos 2\Omega_1)\sqrt{Z(\Omega_1)}}$$

$$+\frac{1}{J_x}\frac{2(\Delta J)^2\cos^2\Omega_1}{Z(\Omega_1)+(\Delta I+\Delta J\cos 2\Omega_1)\sqrt{Z(\Omega_1)}},$$

$$T_3(\Omega_1)=\left(\frac{\Delta I}{I_x}-\frac{\Delta J}{J_x}\right)\frac{\cot\Omega_1}{\sqrt{Z(\Omega_1)}},$$

$$T_4(\Omega_1)=\frac{(\Delta J+\Delta I\cos 2\Omega_1+\sqrt{Z(\Omega_1)})/I_x+(\Delta I+\Delta J\cos 2\Omega_1+\sqrt{Z(\Omega_1)})/J_x}{2\sin^2\Omega_1\sqrt{Z(\Omega_1)}},$$

$$Z(\Omega_1)=(\Delta I)^2+2\Delta I\Delta J\cos 2\Omega_1+(\Delta J)^2,$$

$$\Delta I=I_x-I_z,\quad \Delta J=J_x-J_z.$$

The first term on the right-hand side of (A·1) is the kinetic energy of the relative motion, the second term gives the coupling between the relative motion and the rotation of the system as a whole, and the last term is the kinetic energy of the rotation.

Meanwhile the interaction energy between two tops now takes the form

$$\begin{aligned}
&\sum_{\lambda\mu\nu} a_\lambda V_\mu D_{\nu\lambda}(\theta_i)D^*_{\nu\mu}(\varphi_i)=\\
&=(a_0V_0/2)(3\cos^2\Omega_1-1)-a_0U_2\sqrt{3/2}\sin^2\Omega_1\cos 2\Omega_3-a_2V_0\sqrt{3/2}\sin^2\Omega_1\cos 2\Omega_2\\
&\quad+a_2U_2\{\cos 2\Omega_2\cos 2\Omega_3(1+\cos^2\Omega_1)-2\cos\Omega_1\sin 2\Omega_2\sin 2\Omega_3\}\\
&\quad+\sqrt{6}\,a_0U_1\sin\Omega_1\cos\Omega_1\cos\Omega_3-\sqrt{6}\,a_0U_1'\sin\Omega_1\cos\Omega_1\sin\Omega_3\\
&\quad-(\sqrt{6}/2)a_0U_2'\sin 2\Omega_3\sin^2\Omega_1 \qquad\qquad (A\cdot 2)\\
&\quad-2a_2U_1\{\sin\Omega_1\sin 2\Omega_2\sin\Omega_3-\sin\Omega_1\cos\Omega_1\cos 2\Omega_2\cos\Omega_3\}\\
&\quad-2a_2U_1'\{\sin\Omega_1\sin 2\Omega_2\cos\Omega_3+\sin\Omega_1\cos\Omega_1\cos 2\Omega_2\sin\Omega_3\}\\
&\quad+a_2U_2'\{(1+\cos^2\Omega_1)\cos 2\Omega_2\sin 2\Omega_3+2\cos\Omega_1\sin 2\Omega_2\cos 2\Omega_3\},
\end{aligned}$$

where the hermitian quantities U_μ and U_μ' are defined by

$$V_2=V_{-2}^*=U_2+iU_2',\quad V_1=-V_{-1}^*=U_1'+iU_1. \qquad (A\cdot 3)$$

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On the Representations of Field Quantities, II

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In this series of papers[1] we study whether the framework of a Hilbert space is adequate to the quantum theory of fields. Especially in the present paper we inquire what are the characteristic features of field theory, and study how these features can be described in this framework. For this purpose we examine the character of *additive field quantities*, which are used in this paper in place of field operators, and the relation between a system extending over the whole space and one restricted within a finite volume. After these considerations, we may conclude that there is a sufficient reason to treat the quantum theory of fields within the framework of a Hilbert space.

§ 1. Introduction

The essential results in the previous paper, I, are as follows. Let the set of field quantities of a given physical system be a *-algebra $\mathfrak{A}$, and the states be linear functionals on $\mathfrak{A}$. Putting a set of assumptions each of which seems to us to be very natural from the physical point of view, we have shown that $\mathfrak{A}$ can be represented by an operator algebra $\mathfrak{A}_{\mathfrak{H}}$ in a separable Hilbert space $\mathfrak{H}$[I, § 4]. On any real commutative subalgebra of $\mathfrak{A}$, namely on any commutative subalgebra whose elements are all self-adjoint, every state f can be represented by an element Ψ_f of $\mathfrak{H}$ and the expectation value is given by

$$f(a) = (\Psi_f,\ a_{\mathfrak{H}}\Psi_f),$$

where $a_{\mathfrak{H}}$ is the operator in $\mathfrak{H}$ which corresponds to $a \in \mathfrak{A}$[I, § 5].

All of these arguments are applicable to any quantum mechanical system, and in I no arguments were made about the characteristic features of field theory. In this paper we shall mainly consider this problem and show how these features of field theories can be described. If we study the problem with direct reference to the results of I, namely basing upon the Assumptions 1∼8 in I, there may occur many immaterial complications, which might turn out to keep us from our main purpose. Moreover, we can develop any other theories which are similar to I[I, § 6], but it is very difficult to choose the most appropriate one for our purpose. Therefore, in reference to the results of I but independently of assumptions in it, we introduce in § 2 a set of assumptions each of which seems to be suitable for the present purpose. We may say that this paper is a continuation of I from the

physical point of view, but not from the mathematical one.

What are the characteristic features of field theories? In the classical theory of fields field quantities $q_i(x, t)$, $i=1, 2, \cdots$ are defined at each point (x, t) in the space-time, and each state of the system is settled completely by a set of values of all field quantities. In the usual quantum theory of fields, these field quantities are considered to be operators in a Hilbert space. As stated in I, however, we cannot accept these assumptions from the mathematical point of view. A passable interpretation to avoid this difficulty is the following.[2] Field operators $q_i(x, t)$, $i=1, 2, \cdots$, have only symbolic meanings, and for an appropriate function $\gamma(x)$ is defined an operator $q_i(\gamma, t)$ by the symbolic equation

$$q_i(\gamma, t) = \int q_i(x, t)\gamma(x)\, d^3x .$$

Mathematically, these $q_i(\gamma, t)$ are the very things that have precise meanings. This interpretation can be justified physically. In fact, the field quantity $q_i(x, t)$ *at one point* (x, t) has no meaning, and only its average in some finite domain v has a precise physical meaning. This average is nothing but $q_i(\gamma, t)$ in which $\gamma(x)$ is the characteristic function of v[†]. In this paper we confine ourselves to a system at an instant t omitting the argument t for brevity's sake.

Though the above interpretation has such a physical reason, we cannot adopt these symbolic operators as fundamental objects in our mathematically rigorous treatment. Therefore we must show how such characteristic features of field theories can be expressed in a mathematically precise manner. We say that $q(v)$ is *an additive field quantity*, if a field quantity $q(v)$ is defined for any domain v in the space, and $q(v)=q(v_1)+q(v_2)$ for any v, v_1, v_2, such that $v=v_1 \cup v_2$ and $v_1 \cap v_2=\phi$, the empty set. If we treat these additive field quantities instead of symbolic operators, we can rigorously argue in the framework of a Hilbert space.

In the usual theory of fields it is easy to show that we can consider a system restricted within a finite domain V as an independent physical system. Such a system will be called *a V-system* hereafter, while a system extending over the whole space *a V_∞-system*. It is obviovs that the existence of such V-systems is in close connection with that of additive field quantities. The Assumption 8 in I was considered so as to express the fact that we can determine each state of the system by an enumerable number of experiments, and on this assumption we have proved the separability of $\mathfrak{H}$. As a system extending over the whole space, namely a V_∞-system, will have essentially an infinite number of degrees of freedom, application of the results of I to such a system may be accompanied with some difficulty. On the contrary, we shall be able to apply these results to a V-system. In fact,

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† Though every measurement requires a finite duration of time, it is well known that in quantum mechanics an operator $q(t)$ at a time t has a precise meaning. In this respect it will be of the same circumstances also in the quantum theory of fields, and so we use $\gamma(x)$ but not $\gamma(x, t)$.

we cannot make a thorough research of a V_∞-system, but we can study any V-system as precisely as we want.

Additive field quantities and *V-systems* will be the most characteristic objects in the field theory, and in the following we shall show how these two objects are treated in the framework of a Hilbert space.

§ 2. Basic assumptions

As we have stated in the preceding section, let us begin this section by introducing basic assumptions which are related closely to the results of I.

Assumption 2·1. *The set of field quantities* $\mathfrak{A}$ *forms a* **-algebra with a unit element e. Every state f can be represented by a linear functional on* $\mathfrak{A}$, *and the expectation value of a is given by* $f(a)$.

We denote the set of states by $\mathfrak{S}$. Of course, each state $f \in \mathfrak{S}$ satisfies the conditions of Assumption 2 in I, but it will not be necessary to repeat them here.

Assumption 2·2. $\mathfrak{A}$ *can be represented by an operator algebra* $\mathfrak{A}_\mathfrak{H}$ *in a separable Hilbert space* $\mathfrak{H}$ *as follows: There is a linear subspace* $\mathfrak{H}_0$ *which is everywhere dense in* $\mathfrak{H}$. *For any element* $a \in \mathfrak{A}$, *an operator* $a_\mathfrak{H}$ *is defined, whose domain is* $\mathfrak{H}_0$ *and whose range is contained therein. The set* $\mathfrak{A}_\mathfrak{H}$ *of these operators* $a_\mathfrak{H}$ *forms a* **-algebra which is isomorphic to* $\mathfrak{A}$. *Furthermore, for any element* $\Phi \in \mathfrak{H}_0$, $\|\Phi\| = 1$, *there is a state* $f \in \mathfrak{S}$ *such that* $f(a) = (\Phi, a_\mathfrak{H}\Phi)$ [*I*, § 4].

Here the isomorphism between $\mathfrak{A}$ and $\mathfrak{A}_\mathfrak{H}$ as *-algebras means that $\mathfrak{A}$ and $\mathfrak{A}_\mathfrak{H}$ are isomorphic algebras and $(a_\mathfrak{H})^* \supseteq (a^*)_\mathfrak{H}$, where $(a_\mathfrak{H})^*$ is the adjoint defined in [3), Chap. II]. Hereafter we shall call such an $\mathfrak{A}_\mathfrak{H}$ simply *a representation* of $\mathfrak{A}$.

If $a = a^*$, a is called self-adjoint, and if there is a number $\alpha > 0$ such that $\alpha > |f(a)|$ for every $f \in \mathfrak{S}$, a is called bounded. It is clear that if a is bounded, the corresponding element $a_\mathfrak{H}$ is a bounded operator in $\mathfrak{H}$. Let $\mathfrak{A}_{sa}$ be the set of self-adjoint elements of $\mathfrak{A}$, and $\mathfrak{B}$ the set of bounded elements. From the physical point of view, $\mathfrak{A}_{sa}$ and $\mathfrak{B}_{sa} = \mathfrak{A}_{sa} \cap \mathfrak{B}$ are especially important. In fact, when we measure any field quantity, the immediate object of our measurement is a bounded field quantity. Let us consider, for example, the measurement of energy H. It is obvious that H is not bounded, namely, for any number $\alpha > 0$, there are states whose energies are greater than α. In order to measure the energy in a state it is necessary to select an appropriate apparatus, and once we have selected it, there is an upper bound of energy which can be measured with this apparatus. Therefore, the object of our measurement with this apparatus is some bounded quantity. This means that we have selected an appropriate bounded quantity H_i and that we measure it in place of H. Namely we assume that energy H can be *approximated* by the set of these H_i, $i = 1, 2, \cdots$. More generally, we may say that, as a rule, quantities of $\mathfrak{A}_{sa}$ can be *approximated* by a set of elements of $\mathfrak{B}_{sa}$ in

some appropriate sense. In view of the arguments in § 3 of I, it will be natural to define this approximation as follows:

If the following conditions are satisfied, we say that $a \in \mathfrak{A}_{sa}$ is *B-approximated* by $\{a_i;\ i=1, 2, \cdots\}$, $a_i \in \mathfrak{B}_{sa}$ and write $a_i \to a$ (B):

(i) any two of a, a_i, $i=1, 2, \cdots$, are commutative,

(ii) $a^2 \geq (a_i)^2$, namely $f(a^2) \geq f((a_i)^2)$ for any $f \in \mathfrak{S}$,

(iii) $f(a_i) \to f(a)$ for any $f \in \mathfrak{S}$.

From this definition, it is obvious that we can replace $a \in \mathfrak{A}_{sa}$ by its B-approximation $\{a_i\}$ so far as the expectation values $f(a)$ are concerned, but, with respect to the other properties of a, we cannot tell whether we can use $\{a_i\}$ instead of a. Let $\mathfrak{m}(a)$ be a complete set of commuting field quantities which contains $a \in \mathfrak{A}_{sa}$ [I, § 3]. If, whenever we replace a by $\{a_i\}$ in any $\mathfrak{m}(a)$, we get again a complete set of commuting field quantities, we may say that $\{a_i\}$ is an approximation of a in a more complete sense than the B-approximation. In general, there may be many such approximations. What conditions are necessary to treat these approximations consistently? Let $\mathfrak{C}(a)$ be the set of all elements of $\mathfrak{B}_{sa}$ each of which commutes with a, and $\mathfrak{C}^0(a)$ the set of all elements of $\mathfrak{B}_{sa}$ each of which commutes with any element of $\mathfrak{C}(a)$. If $\{a_i\}$ is an approximation in the complete sense as above, then every a_i belongs to $\mathfrak{C}^0(a)$, and conversely every element of $\mathfrak{C}^0(a)$ may be considered to be something like an element of these $\{a_i\}$'s. Therefore, in order that a can be approximated uniquely and consistently in the above sense, any elements of $\mathfrak{C}^0(a)$ must be pairwise commutative. If a is a physically important element of $\mathfrak{A}_{sa}$, it will be approximated in such a complete sense, namely $\mathfrak{C}^0(a)$ will be commutative.

The above condition is inconvenient for us to treat, because it depends directly on the structure of $\mathfrak{A}$, and we, therefore, rewrite it in the following with slight modification. If a representation of $\mathfrak{A}$ is physically meaningful, then every bounded linear symmetric operator with domain $\mathfrak{H}_0$ will be considered to be a field quantity in a sense, and furthermore each field quantity will be able to be replaced by some of these operators. These operators can be extended uniquely and we get bounded self-adjoint operators. Let $\mathfrak{C}'(a)$ be the set of all bounded self-adjoint operators each of which commutes with $a_{\mathfrak{H}}$, and $\mathfrak{C}'^0(a)$ the set of all bounded self-adjoint operators each of which commutes with any operator of $\mathfrak{C}'(a)$. Then the above condition that $\mathfrak{C}^0(a)$ is commutative will be replaced by the commutativity of $\mathfrak{C}'^0(a)$. At first sight this condition seems to be very complicated, but it is equivalent to the condition that $a_{\mathfrak{H}}$ is normal [4), IV]. As $a_{\mathfrak{H}}$ is symmetric and normal, it is essentially self-adjoint, and has a unique self-adjoint extension $\bar{a}_{\mathfrak{H}}$. Therefore the above condition on $a_{\mathfrak{H}}$ is a natural extension of the condition that $a_{\mathfrak{H}}$ is self-adjoint.

Accordingly we assume

Assumption 2·3. *There is a subset $\mathfrak{A}_0$ of $\mathfrak{A}$ satisfying the following conditions*:

(i) *$\mathfrak{A}$ is generated by $\mathfrak{A}_0$.*

(ii) $\mathfrak{A}_{sa} \supseteq \mathfrak{A}_0$, *and every element of* $\mathfrak{A}_0$ *can be B-approximated.*

(iii) *There is at least one representation of* $\mathfrak{A}$ *such that every operator* $a_{\mathfrak{H}}$ *which corresponds to* $a \in \mathfrak{A}_0$ *is essentially self-adjoint.*

It is obvious that every element of $\mathfrak{B}_{sa}$ satisfies the conditions (ii) and (iii) in any representation. We call any element of $\mathfrak{A}_0$ *a fundamental field quantity*, and any representation which satisfies the condition (iii) *a physically meaningful representation*. Let $\mathfrak{A}_{\mathfrak{H}}$ be a given representation of $\mathfrak{A}$. If we get a new representation $\mathfrak{A}_{\mathfrak{H}}'$ such that $\mathfrak{H}_0' \supseteq \mathfrak{H}_0$ in the same Hilbert space $\mathfrak{H}$ and that $a_{\mathfrak{H}}' \supseteq a_{\mathfrak{H}}$ for any $a \in \mathfrak{A}$, then it is called the extension of $\mathfrak{A}_{\mathfrak{H}}$, and a representation which has no proper extension is called maximal. It is obvious that an extension of a physically meaningful representation remains physically meaningful, and that each representation has a maximal extension. Therefore we have only to consider the maximal and physically meaningful representations for our purpose.

Lemma 2·1. *Let* $a \in \mathfrak{A}_0$, $a_i \to a$ (B), *and the self-adjoint extensions of* $a_{\mathfrak{H}}$, $(a_i)_{\mathfrak{H}}$, $i=1, 2, \cdots$, *be* $\overline{a_{\mathfrak{H}}}$, $\overline{(a_i)_{\mathfrak{H}}}$, $i=1, 2, \cdots$, *respectively in any physically meaningful representation. In order that an element* $\Psi \in \mathfrak{H}$ *may belong to the domain of* $\overline{a_{\mathfrak{H}}}$, *namely* $\Psi \in \mathfrak{D}(\overline{a_{\mathfrak{H}}})$, *it is necessary and sufficient that there is an element* $\Phi \in \mathfrak{H}$ *such that*

$$\overline{(a_i)_{\mathfrak{H}}}\,\Psi \to \Phi \quad \text{(weak)}.$$

If so, then $\overline{a_{\mathfrak{H}}}\,\Psi = \Phi$.

Here, $\Psi_i \to \Psi$ (weak) means that the sequence $\{\Psi_i\}$ converges weakly to Ψ, namely $(\Phi, \Psi_i) \to (\Phi, \Psi)$ for any $\Phi \in \mathfrak{H}$.

Proof. First we shall show that the condition is necessary. As $(\Psi, (a_{\mathfrak{H}})^2\Psi) \geq (\Psi, ((a_i)_{\mathfrak{H}})^2\Psi)$ and $(\Psi, (a_i)_{\mathfrak{H}}\Psi) \to (\Psi, a_{\mathfrak{H}}\Psi)$ for any $\Psi \in \mathfrak{H}_0$, it is easy to show that $(\Psi_1, (a_i)_{\mathfrak{H}}\Psi_2) \to (\Psi_1, a_{\mathfrak{H}}\Psi_2)$ and $\|a_{\mathfrak{H}}\Psi_2\| \geq \|(a_i)_{\mathfrak{H}}\Psi_2\|$ for any $\Psi_1, \Psi_2 \in \mathfrak{H}_0$. These imply that $(a_i)_{\mathfrak{H}}\Psi \to a_{\mathfrak{H}}\Psi$ (weak) for any $\Psi \in \mathfrak{H}_0$. As $a_{\mathfrak{H}}$ is essentially self-adjoint, we can select a set of elements $\Psi_j \in \mathfrak{H}_0$, $j=1, 2, \cdots$, for any $\Psi \in \mathfrak{D}(\overline{a_{\mathfrak{H}}})$, such that $\Psi_j \to \Psi$ and $a_{\mathfrak{H}}\Psi_j \to \overline{a_{\mathfrak{H}}}\Psi$ [3), Chap. II]. Thus $(a_{\mathfrak{H}}\Psi_j, \overline{a_{\mathfrak{H}}}\Psi_j) \to (\overline{a_{\mathfrak{H}}}\Psi, \overline{a_{\mathfrak{H}}}\Psi)$. $\overline{(a_i)_{\mathfrak{H}}}$ is bounded and therefore continuous, and so $(\overline{(a_i)_{\mathfrak{H}}}\Psi_j, \overline{(a_i)_{\mathfrak{H}}}\Psi_j) \to (\overline{(a_i)_{\mathfrak{H}}}\Psi, \overline{(a_i)_{\mathfrak{H}}}\Psi)$. As $a_i \to a$ (B), we have $(\overline{a_{\mathfrak{H}}}\Psi_j, \overline{a_{\mathfrak{H}}}\Psi_j) \geq (\overline{(a_i)_{\mathfrak{H}}}\Psi_j, \overline{(a_i)_{\mathfrak{H}}}\Psi_j)$, and then $(\overline{a_{\mathfrak{H}}}\Psi, \overline{a_{\mathfrak{H}}}\Psi) \geq (\overline{(a_i)_{\mathfrak{H}}}\Psi, \overline{(a_i)_{\mathfrak{H}}}\Psi)$, namely $\|\overline{a_{\mathfrak{H}}}\Psi\| \geq \|\overline{(a_i)_{\mathfrak{H}}}\Psi\|$. On the other hand, for any element $\Phi \in \mathfrak{H}_0$,

$$\lim_i (\Phi, \overline{(a_i)_{\mathfrak{H}}}\Psi) = \lim_i (\overline{(a_i)_{\mathfrak{H}}}\Phi, \Psi) = (\overline{a_{\mathfrak{H}}}\Phi, \Psi) = (\Phi, \overline{a_{\mathfrak{H}}}\Psi).$$

Thus $\overline{(a_i)_{\mathfrak{H}}}\Psi \to \overline{a_{\mathfrak{H}}}\Psi$ (weak) for any $\Psi \in \mathfrak{D}(a_{\mathfrak{H}})$.

Next, we shall show that the condition is sufficient. Let $\mathfrak{D}$ be the set of all elements Ψ such that $\overline{(a_i)_{\mathfrak{H}}}\Psi \to \Phi$ (weak), $\Phi \in \mathfrak{H}$, and $b_{\mathfrak{H}}$ be an operator whose domain is $\mathfrak{D}$ and $b_{\mathfrak{H}}\Psi = \Phi$. It is obvious that $b_{\mathfrak{H}}$ is a linear symmetric operator and coincide with $\overline{a_{\mathfrak{H}}}$ on $\mathfrak{D}(\overline{a_{\mathfrak{H}}})$. As a self-adjoint operator has no proper symmetric

extension [3), Chap. II], $\mathfrak{D}(b_{\mathfrak{H}})=\mathfrak{D}(\overline{a_{\mathfrak{H}}})$, that is $b_{\mathfrak{H}}=\overline{a_{\mathfrak{H}}}$. Q. E. D.

Lemma 2·2. *Let $\mathfrak{A}_{\mathfrak{H}}$ be a physically meaningful representation. If $(\Psi_1, a_{\mathfrak{H}}\Psi_1)=(\Psi_2, a_{\mathfrak{H}}\Psi_2)$ for all $a\in\mathfrak{A}$ imply $\Psi_1=\alpha\Psi_2$, where $|\alpha|=1$, then the representation $\mathfrak{A}_{\mathfrak{H}}$ is irreducible, namely $(\mathfrak{A}_{\mathfrak{H}}\Psi)^c=\mathfrak{H}$ for any $\Psi\in\mathfrak{H}_0$.*

Proof. This lemma is proved by the following *reductio ad absurdum.* If $\mathfrak{A}_{\mathfrak{H}}$ is reducible, then there is an element $\Psi\in\mathfrak{H}_0$ such that $(\mathfrak{A}_{\mathfrak{H}}\Psi)^c=\mathfrak{H}_1\neq\mathfrak{H}$. Put $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$, $\mathfrak{H}_1\cap\mathfrak{H}_0=\mathfrak{H}_{10}$, and $\mathfrak{H}_2\cap\mathfrak{H}_0=\mathfrak{H}_{20}$. Then $\mathfrak{A}_{\mathfrak{H}}\mathfrak{H}_{10}\subseteq\mathfrak{H}_{10}$. As $((a^*)_{\mathfrak{H}}\Psi_1, \Psi_2)=0$ for any $\Psi_1\in\mathfrak{H}_{10}$, $\Psi_2\in\mathfrak{H}_{20}$, and $a\in\mathfrak{A}$, we have $(\Psi_1, a_{\mathfrak{H}}\Psi_2)=0$. This implies that $\mathfrak{A}_{\mathfrak{H}}\mathfrak{H}_{20}\subseteq\mathfrak{H}_{20}$. Put $\Phi_{\pm}=\Psi_1\pm\Psi_2$ for any $\Psi_1\in\mathfrak{H}_{10}$, $\Psi_2\in\mathfrak{H}_{20}$, $\Psi_1, \Psi_2\neq 0$, then $(\Phi_+, a_{\mathfrak{H}}\Phi_+)=(\Phi_-, a_{\mathfrak{H}}\Phi_-)$ for all $a\in\mathfrak{A}$. This is a contradiction, and so $\mathfrak{A}_{\mathfrak{H}}$ is irreducible. Q. E. D.

If a representation of $\mathfrak{A}$ is reducible, we may regard this representation as containing some superfluous part, and Lemma 2·2 shows that to assume the irreducibility of a representation has a sufficient physical reason.

Assumption 2·4. *There is at least one irreducible physically meaningful representation of $\mathfrak{A}$, and it is sufficient from the physical point of view to treat only these representations.*

Therefore, in the remainder of this paper, we shall treat irreducible, physically meaningful representations only. We have assumed in Assumption 2·2 that each element $\Psi\in\mathfrak{H}_0$, $\|\Psi\|=1$, corresponds to a state, while we have shown in I that every state can be represented by an element of $\mathfrak{H}$ on any real commutative subalgebra of $\mathfrak{A}$. In quantum mechanics, any maximal real commutative subalgebra is nothing but a maximal set of simultaneously observable quantities, and every pure state† related to this set can be determined uniquely except for a phase factor by the expectation values of all quantities of this set. Therefore, it will be natural to assume that every state can be represented by an element of $\mathfrak{H}_0$ as in the usual theory. But, in order that such an assumption may be consistent, it is necessary that all maximal and irreducible physically meaningful representations are equivalent to each other.

Assumption 2·5. *Let $\mathfrak{H}'$ and $\mathfrak{H}''$ be the representation spaces of two maximal and irreducible physically meaningful representations. Then there is at least one pair of elements $\Psi_0'\in\mathfrak{H}_0', \Psi_0''\in\mathfrak{H}_0''$, such that $(\Psi_0', a_{\mathfrak{H}'}\Psi_0')=(\Psi_0'', a_{\mathfrak{H}''}\Psi_0'')$ for all $a\in\mathfrak{A}$.*

Theorem 1. *Any two maximal and irreducible physically meaningful representations are equivalent to each other.*

Proof. Let Ψ_0' and Ψ_0'' be the elements described in Assumptions 2·5. As $(\Psi_0', (a^*a)_{\mathfrak{H}'}\Psi_0')=(\Psi_0'', (a^*a)_{\mathfrak{H}''}\Psi_0'')$, we have $\|a_{\mathfrak{H}'}\Psi_0'\|=\|a_{\mathfrak{H}''}\Psi_0''\|$ for any $a\in\mathfrak{A}$, and so it is easy to show that a one-to-one correspondence can be constructed between $\mathfrak{A}_{\mathfrak{H}'}\Psi_0'$ and $\mathfrak{A}_{\mathfrak{H}''}\Psi_0''$, and that they are isomorphic with respect to $\mathfrak{A}$. As

† *Pure state* means H. Weyl's *reiner Fall,*[5] namely a state which is the characteristic element of every quantity of some complete set of commuting quantities, and therefore it does not necessarily exist in a mathematically precise sense.

$(\mathfrak{A}_{\mathfrak{H}'}\Psi_0')^c=\mathfrak{H}'$ and $(\mathfrak{A}_{\mathfrak{H}''}\Psi_0'')^c=\mathfrak{H}''$, we can extend this correspondence and get an isomorphism between $\mathfrak{H}'$ and $\mathfrak{H}''$ as Hilbert spaces. We denote the correspondence by $\Psi'\sim\Psi''$. If $a\in\mathfrak{B}$, then $a_{\mathfrak{H}'}$ and $a_{\mathfrak{H}''}$ have extensions $\overline{a_{\mathfrak{H}'}}$ and $\overline{a_{\mathfrak{H}''}}$ with domain $\mathfrak{H}$. As $\overline{a_{\mathfrak{H}'}}$ and $\overline{a_{\mathfrak{H}''}}$ are bounded, they are continuous, and so $\Psi'\sim\Psi''$ implies $\overline{a_{\mathfrak{H}'}}\Psi'\sim\overline{a_{\mathfrak{H}''}}\Psi''$. Let $a\in\mathfrak{A}_0$, and $a_i\to a(B)$. As the weak convergence of $\{(a_i)_{\mathfrak{H}'}\Psi'\}$ and $\{(a_i)_{\mathfrak{H}''}\Psi''\}$ are equivalent, we can show from Lemma 2·1 that $\mathfrak{D}(\overline{a_{\mathfrak{H}'}})\sim\mathfrak{D}(a_{\mathfrak{H}''})$ and $\overline{a_{\mathfrak{H}'}}\Psi'\sim a_{\mathfrak{H}''}\Psi''$. The rest of the theorem is obvious. Q. E. D.

Theorem 1 shows that the set of all states which can be represented by some elements of $\mathfrak{H}_0$ is the same in these representations. Now, we can consistently assume

Assumption 2·6. *Every state $f\in\mathfrak{S}$ can be represented by an element $\Psi\in\mathfrak{H}_0$ in any maximal and irreducible physically meaningful representation.*

From this we can easily show that if $a_{\mathfrak{H}}$ is bounded then a is also bounded, and that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. In the following we shall call a state which can be represented by $(\Psi, a_{\mathfrak{H}}\Psi)$, $\Psi\in\mathfrak{H}_0$, simply *a state* Ψ.

Lastly, we have a few words to add about the fullness of a subset of $\mathfrak{A}$. We say that a subset $\mathfrak{A}'$ of $\mathfrak{A}$ is full in this system if $f_1(a)=f_2(a)$ for every $a\in\mathfrak{A}'$ imply $f_1=f_2$ for any f_1, $f_2\in\mathfrak{S}$. According to the above consideration, there will be physical ground sufficient to assume that $\mathfrak{B}_{sa}$ is full. In fact, it is easy to show this if we accept the assumptions in § 3 of I. But it is beyond the scope of the present paper to pursue this problem any farther, so we shall discuss it no more.

§ 3. Characteristic features of the field theory

In § 1, we have enumerated the characteristic features of field theory as follows:

(i) There exist additive field quantities, that is, there are defined field quantities $q_i(v)$, $i=1, 2, \cdots$, for any finite domain v, and $q_i(v)=q_i(v_1)+q_i(v_2)$, $i=1, 2, \cdots$, if $v=v_1\cup v_2$ and $v_1\cap v_2=\phi$.

(ii) We can consider the system in any finite domain V as an independent system, and we call it a V-system. The arguments in § 2 can be applied to any V-system. We shall make a mathematical study of these features in the following.

If v and V are taken arbitrarily, there may occur many physically meaningless complications. It is simple and physically reasonable to assume that v and V are all finite intervals, and for simplicity's sake we assume that they are all intervals half open to the right [6), Chap. III]. Let $\mathfrak{F}$ be the set of these intervals, and we write $v=v_1\oplus v_2$ if $v_1\cap v_2=\phi$, $v_1\cup v_2=v$, and v, v_1, $v_2\in\mathfrak{F}$. If $q(v)$ is an additive field quantity, then $q^*(v)$, $(q+q^*)(v)$, and $i(q-q^*)(v)$ are additive, too, and especially $(q+q^*)(v)$ and $i(q-q^*)(v)$ are self-adjoint. As $q(v)$ can be represented by these two self-adjoint quantities, it is sufficient to consider only self-adjoint additive field quantities in the following.

In the usual theory of fields, there are treated only field operators and field

quantities generated by field operators. It is due to the fact that the behaviours of these field operators are assumed to be completely defined by the given commutation relations. We cannot assume in this paper, however, that it is sufficient to treat additive field quantities. In fact, these are not bounded in general, and so we cannot necessarily get sufficiently many bounded quantities as in § 2. Therefore we assume that, there are given not only additive field quantities, but bounded self-adjoint quantities, and that the set of field quantities are generated by all of them. If we assume that the *-algebra of field quantities is closed in any appropriate topology, then we shall be able to get a more unified treatment. But we do not inquire any farther into this situation in this paper.

Assumption 3·1. *The set* $\mathfrak{A}_\infty$ *of all field quantities of a given physical system is a* *-*algebra with a unit element* e, *and it contains self-adjoint additive field quantities* $q_i(v)$, $i=1, 2, \cdots$.

Let the set of these additive field quantities $q_i(v)$, $i=1, 2, \cdots$, be $\mathfrak{E}_\infty$.

Assumption 3·2. *A* V*-system, namely a system which is restricted within a domain* V, *satisfies Assumptions* 2·1∼2·6 *for any* $V \in \mathfrak{F}$. $\mathfrak{A}_V \subseteq \mathfrak{A}_\infty$ *and, if and only if* $v \subseteq V$, $q_i(v) \in \mathfrak{A}_V$.

In the usual theory, the assumption that $\mathfrak{H}_V$ is separable means that in any state there is only a finite number of real particles in a finite domain V.

Assumption 3·3. *Put* $\mathfrak{E}_V = \mathfrak{E}_\infty \cap \mathfrak{A}_V$. *Then*

$$(\mathfrak{A}_V)_0 = \{(\mathfrak{B}_V)_{sa},\ \mathfrak{E}_V\},$$

where $(\mathfrak{A}_V)_0$ *is the one defined in Assumption* 2·3.

As to the relations between different V-systems, it is obvious that $V \subseteq V'$ implies $\mathfrak{E}_V \subseteq \mathfrak{E}_{V'}$. More generally we assume

Assumption 3·4. $V \subseteq V'$, $V, V' \in \mathfrak{F}$ *implies* $\mathfrak{A}_V \subseteq \mathfrak{A}_{V'}$ *and* $\mathfrak{B}_V \subseteq \mathfrak{B}_{V'}$. *Furthermore, if* $a_i \to a$ (B) *in* $\mathfrak{A}_V$ *and* $V \subseteq V'$, *then* $a_i \to a$ (B) *in* $\mathfrak{A}_{V'}$.

For any given field quantity $a \in \mathfrak{A}_\infty$, we shall be able to define a sufficiently large domain V such that the expectation value of a can be determined by experiments in V, and in such a case it is natural to assume that $a \in \mathfrak{A}_V$.

Assumption 3·5. *For any* $a \in \mathfrak{A}_\infty$, *there is at least one* V *such that* $a \in \mathfrak{A}_V$.

Therefore, we can define the boundedness of $a \in \mathfrak{A}_\infty$. If and only if $a \in \mathfrak{B}_V$ for any $V \in \mathfrak{F}$, we say that a is bounded, and we put $\mathfrak{B}_\infty = \cup_{V \in \mathfrak{F}} \mathfrak{B}_V$ and $(\mathfrak{B}_\infty)_{sa} = \cup_{V \in \mathfrak{F}} (\mathfrak{B}_V)_{sa}$. Then we can easily show

Lemma 3·1. $\mathfrak{A}_\infty$ *can be generated by* $(\mathfrak{B}_\infty)_{sa}$ *and* $\mathfrak{E}_\infty$.

Let $V = V_1 \oplus V_2$, and consider the relations among the three systems: V-, V_1-, and V_2-systems. Denote the objects which are related with V_1- or V_2-system by index 1 or 2 respectively. If, for example, we are dealing with the non-local theory, it may happen that some elements $a_1 \in \mathfrak{A}_1$ and $a_2 \in \mathfrak{A}_2$ are not commutative, but we do not deal with such theories in this paper. On the other hand, some of them are anti-commutative in the case of the Fermion field, but it is no matter in this paper. Thus, only for brevity's sake, we assume that $a_1 \in \mathfrak{A}_1$ and

$a_2 \in \mathfrak{A}_2$ are commutative [2), § 21].

As $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are subalgebras of $\mathfrak{A}_V$, the direct product $\mathfrak{A}_1 \times \mathfrak{A}_2$ of them is a subalgebra of $\mathfrak{A}_V$, too. $\mathfrak{A}_1 \times e_2$ and $e_1 \times \mathfrak{A}_2$ are nothing but $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, and we use these notations in order to emphasize that they are subalgebras of $\mathfrak{A}_V$. The relations among $\mathfrak{E}_V$, $\mathfrak{E}_1$, and $\mathfrak{E}_2$ are obvious. For example, if $v = v_1 \oplus v_2$, $v_1 \subseteq V_1$, $v_2 \subseteq V_2$, then

$$q(v) = q(v_1) \times e_2 + e_1 \times q(v_2).$$

From the physical point of view, the essential character of the V-system will be implied in those of V_1- and V_2-systems. It will be too severe, however, to assume that $\mathfrak{A}_V = \mathfrak{A}_1 \times \mathfrak{A}_2$ or $\mathfrak{B}_V = \mathfrak{B}_1 \times \mathfrak{B}_2$. Therefore, we assume

Assumption 3·6. *Let $V = V_1 \oplus V_2$, then any element $a_1 \in \mathfrak{A}_1 \times e_2$ is commutative with any $a_2 \in e_1 \times \mathfrak{A}_2$, and $\mathfrak{A}_1 \times \mathfrak{A}_2$ is full in the V-system.*

Next, we consider the relations among states. According to the above considerations it will be very natural to assume.

Assumption 3·7. *Let $V = V_1 \oplus V_2$, and $(\mathfrak{A}_V)_{\mathfrak{H}_V}$, $(\mathfrak{A}_1)_{\mathfrak{H}_1}$ and $(\mathfrak{A}_2)_{\mathfrak{H}_2}$ be maximal and irreducible physically meaningful representations. Then, there is at least one set of states $\Psi_0 \in (\mathfrak{H}_V)_0$, $\Psi_1 \in (\mathfrak{H}_1)_0$, and $\Psi_2 \in (\mathfrak{H}_2)_0$ such that*

$$(\Psi_0,\ (a_1 \times a_2)_{\mathfrak{H}_V} \Psi_0) = (\Psi_1,\ (a_1)_{\mathfrak{H}_1} \Psi_1)\,(\Psi_2, (a_2)_{\mathfrak{H}_2} \Psi_2)$$

for any $a_1 \in \mathfrak{A}_1$ and $a_2 \in \mathfrak{A}_2$.

Let $\mathfrak{H}_1 \times \mathfrak{H}_2$ be a direct product of these two *linear spaces*, and $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ be the completion of $\mathfrak{H}_1 \times \mathfrak{H}_2$, namely a direct product of two *Hilbert spaces*. In the following we shall show that $\mathfrak{H}_V$ can be equated to $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. Therefore, we may say in any sense that the character of the V-system is implied in those of V_1- and V_2-systems. In order to prove the above argument, we begin with the following lemma.

Lemma 3·2. *Let $V = V_1 \oplus V_2$, $a_1 \in (\mathfrak{A}_1)_0$, and $a_2 \in (\mathfrak{A}_2)_0$. Put $\boldsymbol{a}_1 = (a_1)_{\mathfrak{H}_1} \times (e_2)_{\mathfrak{H}_2}$ and $\boldsymbol{a}_2 = (e_1)_{\mathfrak{H}_1} \times (a_2)_{\mathfrak{H}_2}$. Then, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_1 + \boldsymbol{a}_2$, and $\boldsymbol{a}_1 \boldsymbol{a}_2 = (a_1)_{\mathfrak{H}_1} \times (a_2)_{\mathfrak{H}_2}$ are all essentially self-adjoint operators in $\mathfrak{H}_{12} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ with their domains $(\mathfrak{H}_1)_0 \times (\mathfrak{H}_2)_0$.*

Proof. It is obvious that these operators are all symmetric. As $(a_1)_{\mathfrak{H}_1}$ and $(a_2)_{\mathfrak{H}_2}$ are essentially self-adjoint, their self-adjoint extensions $\overline{(a_1)_{\mathfrak{H}_1}}$ and $\overline{(a_2)_{\mathfrak{H}_2}}$ can be represented by the corresponding resolutions of the identity $E_1(\lambda)$ and $E_2(\mu)$ as $\overline{(a_1)_{\mathfrak{H}_1}} = \int \lambda\, dE_1(\lambda)$, $\overline{(a_2)_{\mathfrak{H}_2}} = \int \mu\, dE_2(\mu)$. It is easy to show that, extending $E_1(\lambda) \times (e_2)_{\mathfrak{H}_2}$ and $(e_1)_{\mathfrak{H}_1} \times E_2(\mu)$, we can get two resolutions of the identity $\boldsymbol{E}_1(\lambda)$ and $\boldsymbol{E}_2(\mu)$ in $\mathfrak{H}_{12}$, and that they are commutative with each other. Therefore,

$$\boldsymbol{E}(z) = \boldsymbol{E}_1(\lambda) \cdot \boldsymbol{E}_2(\mu),\ z = \lambda + i\mu,$$

is a complex resolution of the identity in $\mathfrak{H}_{12}$ [3), Chap. VIII], then

$$\bar{\boldsymbol{a}} = \int (\lambda + \mu)\, d\boldsymbol{E}(z)$$

is a self-adjoint operator in $\mathfrak{H}_{12}$, and $\overline{\boldsymbol{a}} \supseteq \boldsymbol{a}_1+\boldsymbol{a}_2$. Let $\boldsymbol{a}_1+\boldsymbol{a}_2$ be the minimal closed linear extension of $\boldsymbol{a}_1+\boldsymbol{a}_2$. Then $\boldsymbol{a} \supseteq \boldsymbol{a}_1+\boldsymbol{a}_2$.

As $(a_1)_{\mathfrak{H}_1}$ is essentially self-adjoint, we can select a sequence $\Psi_n^1 \in (\mathfrak{H}_1)_0$, $n=1, 2, \cdots$, such that

$$\Psi_n^1 \to \Psi^1, \quad (a_1)_{\mathfrak{H}_1} \Psi_n^1 \to \overline{(a_1)_{\mathfrak{H}_1}} \Psi^1$$

for any $\Psi^1 \in \mathfrak{D}((a_1)_{\mathfrak{H}_1})$. In the same way, we get $\Psi_n^2 \in (\mathfrak{H}_2)_0$, $n=1, 2, \cdots$, such that

$$\Psi_n^2 \to \Psi^2, \quad (a_2)_{\mathfrak{H}_2} \Psi_n^2 \to \overline{(a_2)_{\mathfrak{H}_2}} \Psi^2$$

for any $\Psi^2 \in \mathfrak{D}(\overline{(a_2)_{\mathfrak{H}_2}})$. Then,

$$\Psi_n^1 \times \Psi_n^2 \to \Psi^1 \times \Psi^2,$$

$$[(a_1)_{\mathfrak{H}_1} \Psi_n^1] \times \Psi_n^2 \to [\overline{(a_1)_{\mathfrak{H}_1}} \Psi^1] \times \Psi^2,$$

and

$$\Psi_n^1 \times [(a_2)_{\mathfrak{H}_2} \Psi_n^2] \to \Psi^1 \times [\overline{(a_2)_{\mathfrak{H}_2}} \Psi^2].$$

Therefore,

$$(\boldsymbol{a}_1+\boldsymbol{a}_2)(\Psi_n^1 \times \Psi_n^2) \to [\overline{(a_1)_{\mathfrak{H}_1}} \Psi^1] \times \Psi^2 + \Psi^1 \times [\overline{(a_2)_{\mathfrak{H}_2}} \Psi^2],$$

and this shows that

$$\overline{\boldsymbol{a}_1+\boldsymbol{a}_2} \supseteq \{\overline{(a_1)_{\mathfrak{H}_1}} \times \overline{(e_2)_{\mathfrak{H}_2}} + \overline{(e_1)_{\mathfrak{H}_1}} \times \overline{(a_2)_{\mathfrak{H}_2}}\},$$

where the domain of the operator in the right-hand side is

$$\mathfrak{D}(\overline{(a_1)_{\mathfrak{H}_1}}) \times \mathfrak{D}(\overline{(a_2)_{\mathfrak{H}_2}}).$$

Put $\quad E_1(\Delta_\alpha)=E_1(+\alpha)-E_1(-\alpha), \quad E_2(\Delta_\alpha)=E_2(+\alpha)-E_2(-\alpha),$

and $$\boldsymbol{E}(\Delta_\alpha)=[\boldsymbol{E}_1(+\alpha)-\boldsymbol{E}_1(-\alpha)][\boldsymbol{E}_2(+\alpha)-\boldsymbol{E}_2(-\alpha)].$$

Then $$\boldsymbol{E}(\Delta_\alpha)\mathfrak{H}_{12}=[E_1(\Delta_\alpha)\mathfrak{H}_1] \otimes [E_2(\Delta_\alpha)\mathfrak{H}_2],$$

and $\mathfrak{D}((a_1)_{\mathfrak{H}_1}) \supseteq E_1(\Delta_\alpha)\mathfrak{H}_1$, $\mathfrak{D}(\overline{(a_2)_{\mathfrak{H}_2}}) \supseteq E_2(\Delta_\alpha)\mathfrak{H}_2$. If we restrict $\overline{(a_1)_{\mathfrak{H}_1}}$ and $\overline{(a_2)_{\mathfrak{H}_2}}$ on $E_1(\Delta_\alpha)\mathfrak{H}_1$ and $E_2(\Delta_\alpha)\mathfrak{H}_2$ respectively, we get bounded operators, and so the domain of the minimal closed linear extension of $\overline{(a_1)_{\mathfrak{H}_1}} \times \overline{(e_2)_{\mathfrak{H}_2}} + \overline{(e_1)_{\mathfrak{H}_1}} \times \overline{(a_2)_{\mathfrak{H}_2}}$ contains $\boldsymbol{E}(\Delta_\alpha)\mathfrak{H}_{12}$. Let α_n, $n=1, 2, \cdots$, be a non-decreasing sequence such that $\alpha_n \to +\infty$. Then, for any $\Psi \in \mathfrak{D}(\bar{\boldsymbol{a}})$, $\boldsymbol{E}(\Delta_{\alpha n})\Psi \to \Psi$ and $\bar{\boldsymbol{a}}\boldsymbol{E}(\Delta_{\alpha n})\Psi \to \bar{\boldsymbol{a}}\Psi$. This implies that the domain of the minimal closed linear extension of

$$\overline{(a_1)_{\mathfrak{H}_1}} \times \overline{(e_2)_{\mathfrak{H}_2}} + \overline{(e_1)_{\mathfrak{H}_1}} \times \overline{(a_2)_{\mathfrak{H}_2}}$$

contains that of $\bar{\boldsymbol{a}}$, namely $\overline{\boldsymbol{a}_1+\boldsymbol{a}_2} \supseteq \bar{\boldsymbol{a}}$.

Thus, $\overline{\boldsymbol{a}_1+\boldsymbol{a}_2}=\bar{\boldsymbol{a}}$, and $\boldsymbol{a}_1+\boldsymbol{a}_2$ is essentially self-adjoint. The other parts of the lemma can be proved in the same manner. Q. E. D.

Theorem 2. *Under the same conditions as in Assumption* 3·7, *a one-to-one correspondence can be constructed between* $\mathfrak{H}_r$ *and* $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ *such that they are*

isomorphic to each other with respect to $\mathfrak{A}_1 \times \mathfrak{A}_2$. *Furthermore,* $(a_1 \times a_2)_{\mathfrak{H}_\Gamma}$ *can be completely determined by* $(a_1)_{\mathfrak{H}_1} \times (a_2)_{\mathfrak{H}_2}$ *for any* $a_1 \in \mathfrak{A}_1$ *and* $a_2 \in \mathfrak{A}_2$.

Proof. Let Ψ_0, Ψ_1, and Ψ_2 be those as defined in Assumption 3·7, and put $\mathfrak{A}_\Gamma' = \mathfrak{A}_1 \times \mathfrak{A}_2$. It is easy to show that the closure of $\mathfrak{A}_\Gamma'(\Psi_1 \times \Psi_2) = \mathfrak{A}_1 \Psi_1 \times \mathfrak{A}_2 \Psi_2$ is $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, and that $\mathfrak{A}_\Gamma'$ can be represented isomorphically by an operator algebra on $\mathfrak{A}_1 \Psi_1 \times \mathfrak{A}_2 \Psi_2$. As $\mathfrak{A}_\Gamma \supseteq \mathfrak{A}_\Gamma'$, $\mathfrak{A}_\Gamma'$ can be represented in $\mathfrak{H}_\Gamma$, and from Assumption 3·6 we can prove that $(\mathfrak{A}_\Gamma' \Psi)^c = \mathfrak{H}_\Gamma$ for any $\Psi \in (\mathfrak{H}_\Gamma)_0$ as in Lemma 2·2. Thus, we can construct an isomorphism between $\mathfrak{A}_\Gamma'(\Psi_1 \times \Psi_2)$ and $\mathfrak{A}_\Gamma' \Psi_0$ with respect to $\mathfrak{A}_\Gamma'$ from Assumption 3·7, and can extend it to an isomorphism between $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathfrak{H}_\Gamma$. If $a_1 \in \mathfrak{E}_1$ and $a_1' \to a_1$ (B), then from Lemma 3·2 $(a_1)_{\mathfrak{H}_1} \times (e_2)_{\mathfrak{H}_2}$ is an essentially self-adjoint operator whose domain is $(\mathfrak{H}_1)_0 \times (\mathfrak{H}_2)_0$. Thus we can show as in Lemma 2·1 that a necessary and sufficient condition for $\Psi \in \mathfrak{D}(\overline{(a_1)_{\mathfrak{H}_1} \times (e_2)_{\mathfrak{H}_2}})$ is that the sequence $(a_1')_{\mathfrak{H}_1} \times (e_2)_{\mathfrak{H}_2} \Psi$ converges weakly to an element of $\mathfrak{H}_\Gamma$. From Assumption 3·4, we have $a_1 \times e_2 \in \mathfrak{E}_\Gamma$ and $a_1' \times e_2 \to a_1 \times e_2$ (B). From these, we can show as in Theorem 1 that $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is isomorphic to $\mathfrak{H}_\Gamma$ with respect to $\mathfrak{A}_1 \times \mathfrak{A}_2$. The rest of the theorem is obvious from Lemma 3·2.

Q. E. D.

It is easy to show that all the above arguments can be applied consistently to such a case that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. In order explicitly to express an intimate connection among $\mathfrak{A}_\Gamma$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\cdots$, $\mathfrak{A}_n$, we write $\mathfrak{A}_\Gamma = \mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \cdots \otimes \mathfrak{A}_n = \otimes_{i=1}^n \mathfrak{A}_i$. This notation means that $\mathfrak{A}_\Gamma$ can be considered to be something like a closure of $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_n = \times_{i=1}^n \mathfrak{A}_i$.

Now we shall proceed to study the relations between the V_∞-system and V-systems, and consider how the V_∞-system can be represented by using these relations. Let $\{V_i\}$ be a division of the whole space V_∞ into non-overlapping intervals V_i, $i=1, 2, \cdots$, namely $V_\infty = \oplus_{i=1}^\infty V_i$. For convenience' sake, we assume that there are upper and lower bounds α_1, $\alpha_2 > 0$ of volume V_i, that is $\alpha_1 \geq m(V_i) \geq \alpha_2$, $i=1, 2, \cdots$. Let the direct product $\times_{i=1}^\infty \mathfrak{A}_i$ be an algebra which is generated by such elements as $\times_{i=1}^\infty a_i$, where $a_i \in \mathfrak{A}_i$ and $a_i = e_i$ except for only a finite number of a_i. From the above arguments, it is obvious that the essential characters of $\mathfrak{A}_\infty$ can be implied in those of $\times_{i=1}^\infty \mathfrak{A}_i$, and in order to express this explicitly we write $\mathfrak{A}_\infty = \otimes_{i=1}^\infty \mathfrak{A}_i$. As any element $a \in \mathfrak{A}_\infty$ is contained in some $\otimes_{j=1}^n \mathfrak{A}_{ij}$, $i_j = i_1, i_2, \cdots, i_n$, the above notation will be suitable.

At first sight, it seems to be very natural to take the infinite direct product $\mathfrak{H}_\infty = \otimes_{i=1}^\infty \mathfrak{H}_i$ as the representation space,[7] but there are many difficulties in such a representation. First, $\mathfrak{H}_\infty$ is not separable, and this representation is not irreducible. This can be easily shown from the fact that $(\mathfrak{A}_\infty \Psi)^c$ is von Neumann's *incomplete direct product* $\mathfrak{H}_\sigma$ for any $\Psi \in \mathfrak{H}_\infty$. Second, what is worse, there is no guarantee for $\mathfrak{H}_\infty$ to be independent of the division $\{V_i\}$.

In order to study this problem more closely, we first consider *a state* in the V_∞-system. We have often stated that we can make only a finite number of experiments, and this means that we can study the system only in a finite domain,

though it may be as large as we want. Therefore, every state f in the V_∞-system will be represented with sufficient accuracy by the direct product $\times_{j=1}^{n} \Psi_{ij}$, $\Psi_{ij} \in (\mathfrak{H}_{ij})$. Thus, in the outside of $\oplus_{j=1}^{n} V_{ij}$, it will be in some standard state approximately, and these standard state will be suitable to be called *the vacuum*. The vacuum will be independent of division $\{V_i\}$, and to secure this we assume

Assumption 3·8. *In any V-system, there is the vacuum* $\Psi_V^0 \in (\mathfrak{H}_V)_0$, *and* $\Psi_V^0 = \Psi_1^0 \times \Psi_2^0$ *if* $V = V_1 \oplus V_2$.

Put $\Psi^0 = \times_{i=1}^{\infty} \Psi_i^0$, then it is easy to show that it does not depend on the division $\{V_i\}$, and so we may call it *the vacuum* in the V_∞-system. $\mathfrak{H}_{vac} = (\mathfrak{A}_\infty \Psi^0)'$ is an incomplete direct product, and each element of $\mathfrak{H}_{vac}$ may be considered as approximately a vacuum in the outside of some domain V. Therefore, it will be appropriate to use this $\mathfrak{H}_{vac}$ as the representation space of $\mathfrak{A}_\infty$. When we take this $\mathfrak{H}_{vac}$ as the representation space, the difficulties referred to above are removed. In fact, $\mathfrak{H}_{vac}$ is separable and does not depend on the division $\{V_i\}$ [7), p. 39], and the representation is irreducible.

Let $\mathfrak{H}_{vac}^0$ be the linear subspace of $\mathfrak{H}_{vac}$ which is generated by those elements each of which can be represented by a direct product of an element of $(\mathfrak{H}_V)_0$ for some $V \in \mathfrak{F}$ and the vacuums in other domains. It is obvious that $\mathfrak{H}_{vac}^0$ is everywhere dense in $\mathfrak{H}_{vac}$, and that $\mathfrak{H}_{vac}^0 \supseteq \mathfrak{A}_\infty \Psi^0$. We can uniquely determine a *-algebra of operators in $\mathfrak{H}_{vac}$ such that it is isomorphic to $\mathfrak{A}_\infty$, and the domains of these operators are $\mathfrak{H}_{vac}^0$ and the ranges are contained in it. Furthermore, we can easily show that every operator corresponding to some element of $(\mathfrak{B}_\infty)_{sa}$ or $\mathfrak{E}_\infty$ is essentially self-adjoint. Therefore, from this representation, we can uniquely define a maximal representation. Put the common domain of the maximal representation $(\mathfrak{H}_{vac})_0$. Then $(\mathfrak{H}_{vac})_0 \supseteq \mathfrak{H}_{vac}^0$, and the representation satisfies the conditions of a maximal and irreducible physically meaningful representation in § 2.

Assumption 3·9. *Every state of the* V_∞*-system can be represented by an element* $\Psi \in (\mathfrak{H}_{vac})_0$, $\|\Psi\| = 1$, *and on the contraray every such element corresponds to a state of this system.*

Theorem 3. $\mathfrak{A}_\infty$ *can be represented isomorphically by an operator algebra on* $(\mathfrak{H}_{vac})_0$, *and this representation is nothing but a maximal and irreducible physically meaningful representation.*

We have shown that $\mathfrak{A}_\infty$ is generated by $\mathfrak{E}_\infty$ and $\mathfrak{B}_\infty$, and that $\mathfrak{B}_\infty$ plays an essential role in studying the properties of $\mathfrak{E}_\infty$. But, in general, there has been made no explicit argument about $\mathfrak{B}_\infty$ in the usual field theory. In this respect, the author's opinion is as follows. In the usual field theory, *the field* is assumed to be an assembly of harmonic oscillators. If it is an assembly of a finite number of such oscillators, the properties of the field can be determined completely by their commutation relations except for a unitary transformation.[8)] However, the field is assumed to contain an infinite number of harmonic oscillators in general, and so the properties of the field cannot be determined completely.[9)] In other words, the field is contained in an infinite direct product of separable Hilbert spaces, but it is

not definitive which of the incomplete direct products the field is. As we have stated repeatedly, we can make only a finite number of experiments, and so the physical properties of the field will be independent of the choice of an incomplete direct product. Thus, we may say that the physical properties of the field are determined by the commutation relations. That is, there are many representations of *the field* which are equivalent to one another not mathematically but physically.† This is the reason why $\mathfrak{B}_\infty$ is not treated explicitly in the usual theory.

The dependence of the system on the time t can be treated as in the usual theory, but it will be needless to repeat it here.

§ 4. Additive field quantities

In this section we shall show that additive field quantities are in intimate connection with the symbolic field operators. Let $q(v)$ be an additive field quantity, and put $Q_{\Phi\Psi}(v)=(\Psi,\ q_{\mathfrak{H}}(v)\Phi)$ for any $\Phi\in(\mathfrak{H}_{vac})_0$, $\Psi\in\mathfrak{H}_{vac}$. Then $Q_{\Phi\Psi}(v)$ is an additive interval function [6), p. 59], and can be continued on any figure v_F, namely on any sum of a finite number of intervals [6), p. 61]. It will be natural to assume that $Q_{\Phi\Psi}(v_F)$ is small for sufficiently small figure v_F.

Assumption 4·1. *Corresponding to an arbitrary number $\varepsilon>0$, there is a number $\delta>0$ such that $|Q_{\Phi\Psi}(v_F)|<\varepsilon$, whenever $m(v_F)<\delta$.*

Such function $Q_{\Phi\Psi}(v_F)$ is called *absolutely continuous* [6), p. 93].

Lemma 4·1. *There is a Lebesgue-integrable function $Q_{\Phi\Psi}(x)$ such that*

$$(\Psi,\ q_{\mathfrak{H}}(v)\Phi)=\int_v Q_{\Phi\Psi}(x)\,d^3x$$

for any $v\in\mathfrak{J}$.

Proof. As $Q_{\Phi\Psi}(v_F)$ is absolutely continuous, it can be extended uniquely to an absolutely continuous set function on any bounded Borel set [6), p. 95]. Therefore, there is a Lebesgue integrable function $Q_{\Phi\Psi}(x)$ such that $Q_{\Phi\Psi}(v)=\int_v Q_{\Phi\Psi}(x)\,d^3x$ for any Borel set v [6), p. 36, the theorem of Radon-Nikodym]. Q. E. D.

Theorem 4. *Let $\mathfrak{M}$ be the set of all essentially bounded measurable functions whose carriers are bounded. Then, corresponding to an additive field quantity $q(v)$, we can define linear operator $q_{\mathfrak{H}}(\gamma)$ with domain $(\mathfrak{H}_{vac})_0$ for any $\gamma(x)\in\mathfrak{M}$, such that*

$$(\Psi,\ q_{\mathfrak{H}}(\gamma)\Phi)=\int\gamma(x)Q_{\Phi\Psi}(x)\,d^3x$$

for any $\Phi\in(\mathfrak{H}_{vac})_0$, $\Psi\in\mathfrak{H}_{vac}$.

Proof. It is easy to show that we can define linear operator $q_{\mathfrak{H}}(v_F)$ for any figure v_F corresponding to the given $q(v)$. Let $\mathfrak{J}_B$ be the set of all bounded Borel sets. Then, we can select a sequence $\{v_F^{\,i}\}$ of figures such that

† Of course, we can set up a physical criterion which is sufficient to distinguish a representation from the others, but it is of only formal nature.

$$m(v_F^i) \to m(v), \quad v \subseteq v_F^i, \quad i=1, 2, \cdots,$$

for any $v \in \mathfrak{F}_B$. Thus [10), p. 110],

$$(\Psi, q_{\mathfrak{H}}(v_F^i)\Phi) \to \int_v Q_{\Phi\Psi}(x)\, d^3x.$$

Since Ψ is an arbitrary element of $\mathfrak{H}$, this implies that $\{q_{\mathfrak{H}}(v_F^i)\Phi\}$ converges weakly, and as $\mathfrak{H}$ is weakly complete [11), p. 24], $\{q_{\mathfrak{H}}(v_F^i)\Phi\}$ converges weakly to some element of $\mathfrak{H}$. We write it $q_{\mathfrak{H}}(v)\Phi$, that is, $q_{\mathfrak{H}}(v_F^i)\Phi \to q_{\mathfrak{H}}(v)\Phi$ (weak), then we can easily show that $q_{\mathfrak{H}}(v)$ is a linear operator whose domain is $(\mathfrak{H}_{vac})_0$, and that it does not depend on the sequence $\{v_F^i\}$.

Let $\gamma(x)$ be a simple function such that

$$\gamma(x) = \sum_{k=1}^{n} \alpha_k \chi_k(x),$$

where χ_k is a characteristic function of $v_k \in \mathfrak{F}_B$, $k=1, 2, \cdots, n$, and $v_k \cap v_j = \phi$ for $k \neq j$ [10), pp. 15, 84]. Then, $\gamma(x)Q_{\Phi\Psi}(x)$ is integrable for any $\Phi \in (\mathfrak{H}_{vac})_0$, $\Psi \in \mathfrak{H}_{vac}$, and when we put $q_{\mathfrak{H}}(\gamma) = \sum_k \alpha_k q_{\mathfrak{H}}(v_k)$, $q_{\mathfrak{H}}(\gamma)$ is a linear operator whose domain is $(\mathfrak{H}_{vac})_0$ and

$$(\Psi, q_{\mathfrak{H}}(\gamma)\Phi) = \int \gamma(x) Q_{\Phi\Psi}(x)\, d^3x.$$

Let $\gamma(x)$ be an essentially bounded and measurable function whose carrier is bounded, then $\gamma(x)Q_{\Phi\Psi}(x)$ is integrable [10), p. 113]. We can select a sequence of simple function $\{\gamma_n(x)\}$ such that $|\gamma(x)| \geq |\gamma_n(x)|$ and $\gamma_n(x) \to \gamma(x)$ [10), p. 85]. Then $\gamma_n(x)Q_{\Phi\Psi}(x) \to \gamma(x)Q_{\Phi\Psi}(x)$, and $|\gamma_n(x)Q_{\Phi\Psi}(x)| \leq |\gamma(x)Q_{\Phi\Psi}(x)|$. Thus [10), pp. 102, 110],

$$\lim_{n\to\infty} \int \gamma_n(x) Q_{\Phi\Psi}(x)\, d^3x = \int \gamma(x) Q_{\Phi\Psi}(x)\, d^3x,$$

namely

$$(\Psi, q_{\mathfrak{H}}(\gamma_n)\Phi) \to \int \gamma(x) Q_{\Phi\Psi}(x)\, d^3x.$$

It is easy to show that we can define uniquely an operator $q_{\mathfrak{H}}(\gamma)$ with domain $(\mathfrak{H}_{vac})_0$ such that

$$q_{\mathfrak{H}}(\gamma_n)\Phi \to q_{\mathfrak{H}}(\gamma)\Phi \quad \text{(weak)}. \qquad \text{Q. E. D.}$$

According to the above considerations, we can treat the system in the whole space-time analogously to the usual theory. For example, if the additive field quantity depend on the time t, there is a function $Q_{\Phi\Psi}(x, t)$ such that

$$(\Psi, q_{\mathfrak{H}}(\gamma, t)\Phi) = \int \gamma(x) Q_{\Phi\Psi}(x, t)\, d^3x.$$

If there is an additive field quantity $q^{(s)}(v, t)$ such that

$$(\Psi,\ q^{(i)}(v,\ t)\Phi)=\int_{V}\partial Q_{\Phi\Psi}(x,\ t)/\partial x_i\, d^3x,$$

then and only then we say that $q^{(i)}(v,\ t)$ is a differentiation of $q(v,\ t)$ with respect to x_i and write $q^{(i)}(v,\ t)=\partial_i q(v,\ t)$. A differentiation with respect to t is defined in the same manner. The relativistic invariance of the theory will be reduced to that of functions $Q_{\Phi\Psi}(x,\ t)$.

At first sight, it seems to be natural to define a field operator $q(x,\ t)$ by $(\Psi,\ q(x,\ t)\Phi)=Q_{\Phi\Psi}(x,\ t)$. But, as stated in § 1, the value of $Q_{\Phi\Psi}(x,\ t)$ at only one point $(x,\ t)$ has no meaning, and the above definition is ambiguous. We shall study this problem more closely in the next paper.

§ 5. Concluding remarks

Now, we may conclude as follows. Though we can study the system experimentally only in a finite domain, the V_∞-system can be treated in the same manner as in I, and so we can say that there are sufficient physical reasons to treat the quantum theory of fields in a framework of a Hilbert space. If any of our assumptions are not fulfilled essentially, there may happen a case in which any other theory than that of a Hilbert space plays a most essential role. But this is very unlikely, and if our assumptions or their slight modifications are fulfilled, then the theory of a Hilbert space will be essential.

We cannot treat directly symbolic field operators $q(x,\ t)$ in the framework of a Hilbert space, but the roles of these field operators are only formal and they can be replaced by additive field quantities. Therefore, even if we have succeeded in a mathematical treatment of the symbolic operators $q(x,\ t)$, no essentially important improvement will be achived in the quantum theory of fields solely because of this success. For example, let us consider a Hamiltonian of a system. The dependence of a system on the time t will be expressed by a one-parameter group $U(t)$ of unitary operators. If $U(t)$ is weakly measurable, namely $(\Psi_1,\ U(t)\Psi_2)$ is a measurable function of t for any elements $\Psi_1,\ \Psi_2\in\mathfrak{H}$, then $U(t)$ can be expressed as $\exp(iHt)$ by an appropriate self-adjoint operator H [11), p. 183, Stone's theorem]. Therefore the Hamiltonian will be a self-adjoint operator in a mathematically precise sense, and no symbolic operator will play an essential part in the Hamiltonian.

Though the roles of field operators are only formal, it will nevertheless be very useful and convenient to treat them directly in a strictly mathematical sense, and we can expect that the success of such a treatment will help us to remove many difficulties in the quantum theory of fields. In the next paper we shall deal with this problem.

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The Evolution of Massive Stars. I

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The evolution of stars with mass 15.6 times the solar mass is considered in their early stage. Our model star consists of three regions; a convective core consuming hydrogen, a radiative envelope, and an intermediate zone of continuously varying composition. Radiation pressure is taken into account throughout the three regions. The hydrogen content in the core ranges from 90 per cent to 6.3 per cent by weight. It is found that the convective core retreats as its hydrogen content decreases, setting up the intermediate zone of continuously varying composition and a thin convective unstable region between the radiative envelope and the intermediate zone. The evolutionary track in the $H-R$ diagram is compared with the stars of the $h+\chi$ Persei clusters.

§ 1. Introduction

It is known that many chemical elements observed in our galaxy have been synthesized by nuclear reactions occurring in the interior of stars in the course of their evolution.[1] The details of the synthesizing processes are intimately connected with the evolutionary changes in the stellar structures. Such evolutionary processes of stars have been studied by many authors applying various stellar models, especially by Hoyle and Schwarzschild[2] on the stars of the globular clusters. However, little has been known about the nuclear processes in massive stars which seem to play an important role in element sythesis on account of their rapid evolution.

It is known that stars with different masses evolve differently. Tayler[3] and Kushwaha[4] studied the evolution of massive stars in their early stages and proposed models consisting of a radiative envelope, an intermediate zone in which the mean molecular weight varies continuously, and a convective core. One of the characteristic features of massive stars is the effectiveness of radiation pressure. Although its effect was shown to be negligible for the mass ($\sim 10\,M_\odot$) chosen by Tayler and Kushwaha, it is expected that the radiation pressure becomes effective, as the mass of stars increases. Its effects appear particularly in the condition of convective stability, that is, they make the stellar configuration more unstable against convection. In this respect, Kushwaha's treatment, which neglects the effect of radiation pressure on convective instability, leaves something unsatisfactory.

The purpose of this paper is to investigate the structure of the more massive stars ($15.6\,M_\odot$) in which the effect of radiation pressure is no longer negligible.

Including the initial homogeneous model, five model sequences are obtained in the early stages of the evolution as the hydrogen content in the core decreases. The evolutionary path is compared with the $H-R$ diagram of the $h+\chi$ Persei stars obtained by Johnson and Hiltner[5] which includes the most massive stars observed. These results will give a basis for the study of the later stages of the evolution, the stages of gravitational contraction and helium reactions.

§ 2. Physical assumptions and basic equations

The model here considered consists of a helium enriched core in convective equilibrium, an envelope of almost pure hydrogen in a radiative equilibrium, and an intermediate zone of variable composition in radiative equilibrium. The following subscripts are used to denote the quantities in each part; e=envelope, i=core, c=center, 1=interface between the radiative envelope and the intermediate zone, and f=interface between the intermediate zone and the convective core.

Simpifying assumptions are made as follows.

1. There is no mixing of matter in the envelope, so that the envelope has a constant composition. In the convective core, the mixing caused by the convective turbulence is fully effective so that the composition is homogeneous.

2. The opacity is due to electron scattering only. Kushwaha's calculation showed that the main opacity is electron scattering for the model of ten solar masses. It seems reasonable, therefore, to assume that in the more massive stars the electron scattering opacity is predominant.

3. Steady ejection of mass from the surface of star is neglected, i. e., the total mass of star remains constant throughout the whole stage of evolution.

4. The initial hydrogen and metal contents have been taken as $X_e=X_i=0.900$, and $Z_e=Z_i=0.020$, which give as the mean molecular weight in the envelope, $\mu_e=4/(5X_e-Z_e+3)=0.535$.

The basic equations of mechanical and thermal equilibrium for the radiative zone are

$$\frac{dM_r}{dr}=4\pi r^2\rho,\quad \frac{dP}{dr}=-\frac{GM_r}{r^2}\rho,\quad \frac{dT}{dr}=-\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L_r}{4\pi r^2}. \tag{1}$$

Owing to the assumption 2, the opacity is taken as

$$\kappa=0.19(1+X). \tag{2}$$

The equation of state is given by

$$P=k\rho T/\mu H+aT^4/3. \tag{3}$$

Composition parameters are defined as

$$l\equiv\frac{\mu}{\mu_e},\quad j\equiv\frac{\mu}{\mu_e}\frac{1+X}{1+X_e}. \tag{4}$$

In the homogeneous envelope, the value of l and j are $l=1$, $j=1$. Composition parameters in the intermediate zone are not constant. Following the method of Härm and Schwarzschild,[6] we divide the intermediate zone into a number of thin shells and express l and j in each shell by the formula

$$l=(q/q_n)^{-\nu_n}l_n, \quad j=l^{0.285}, \tag{5}$$

where the suffix n represents the n-th shell and l_n and q_n are the values at its outer edge.

In terms of nondimensional variables

$$P=p\frac{GM^2}{4\pi R^4}, \quad T=t\frac{\mu_e H}{k}\frac{GM}{R}, \quad M_r=qM, \quad r=xR, \tag{6}$$

and the above composition parameters, the basic equations (1) for the radiative regions, the envelope and the intermediate zone, are expressed as

$$\frac{dp}{dx}=-\beta\frac{pq}{x^2t}l, \quad \frac{dq}{dx}=\beta\frac{x^2p}{t}l, \quad \frac{dt}{dx}=-C\beta\frac{p}{t^4x^2}j, \tag{7}$$

with

$$\beta=1-At^4/p,$$

$$A\equiv\frac{4\pi a}{3}\left(\frac{\mu_e H}{k}\right)^4 G^3M^2, \quad C\equiv\frac{3}{4ac}\frac{0.19(1+X_e)}{(4\pi)^2}\left(\frac{k}{\mu_e HG}\right)^4\frac{L}{M^3}. \tag{8}$$

Homology variables in the envelope are given by

$$U_e=\frac{x^3p}{qt}l\beta, \quad V_e=-\frac{q}{xt}l\beta, \quad (n+1)_e=\frac{q}{C}\frac{t^4}{p}. \tag{9}$$

For the convective core, the solutions including the radiation pressure have previously been obtained by Henrich.[7] In terms of Henrich's variables, the homology invariants in the convective core are expressed as

$$U_i=\frac{-\eta\zeta}{dZ/d\eta}, \quad V_i=-\frac{5}{2}\frac{\eta\zeta(dZ/d\eta)}{\pi}, \quad (n+1)_i=\frac{\theta}{\pi}\frac{d\pi}{d\theta}. \tag{10}$$

§ 3. Construction of models

The equations for the envelope have two parameters A and C. Taking the mass equal to 15.6 $M_\odot$ and the mean molecular weight $\mu_e=0.535$, we determine the value of A from the definition (8) as $A=16.0$. Then, we have a series of solutions which are specified by the value of C.

Solutions near the surface ($q\simeq 1$) are given by

$$p=t^4/4C, \quad t=\beta_0(1/x-1)/4, \quad \beta_0=1-4AC, \tag{11}$$

which are used as the starting values for the numerical integrations.

The homogeneous model, which represents the initial stage of evolution, is constructed as follows. With the above starting values, the integration over the

envelope is carried out inward and is stopped at the point where the solution reaches the convective instability. The condition of the convective instability in the homogeneous region is given by

$$(n+1)_{rad} \leqq (n+1)_{ad}, \tag{12}$$

where

$$(n+1)_{rad}=\frac{GM_r\rho}{\kappa L_r}\frac{16\pi acT^4}{3P}, \quad (n+1)_{ad}=\frac{32-24\beta-3\beta^2}{8-6\beta}. \tag{13}$$

At the point of convective instability, the envelope solution is fitted to the core solution continuously. The continuity of physical quantities at the interface between the envelope and the core leads to the relations

$$\beta_{1e}=\beta_{1i}, \quad U_{1i}/U_{1e}=V_{1i}/V_{1e}=1. \tag{14}$$

Among the families of the solutions for the radiative envelope and the convective core, the solution which satisfies the above fitting conditions is determined uniquely. In this way mathematical properties of the model and the values of q_1, x_1, t_1, β_1 and β_c, are completely determined.

The model sequence following the initial model is obtained as follows. In each step of the parameter C, the integration over the envelope is carried out inward as in the case of the initial model and is stopped at the point where $q=q_1^{(0)}$, $q_1^{(0)}$ being the mass fraction of the convective core of the initial model. From this point, the integration over the intermediate zone of variable composition is carried out inward.

It must be noted here that, if the radiative solution of the envelope reaches the point $q=q_1^{(0)}$ before the convective instability sets in, the envelope is surely in radiative equilibrium. Certainly, the models calculated by Kushwaha and Tayler satisfy these conditions. In our case, however, the effect of radiation pressure is more serious on account of the larger mass, and the envelope solution reaches convective instability before it reaches the point $q=q_1^{(0)}$ as shown in Fig. 1.

The above result seems to suggest a sequence of models such that the mass of the convective core grows up as its hydrogen content decreases, the mean molecular weight being discontinuous at the interface between the core and the radiative envelope. However, it is easily shown that this suggestion is not successful in the case in which the opacity is due to electron scattering. In this case the continuity of the radiation flux at the interface requires

$$(1+X_e)(n+1)_e=(1+X_i)(n+1)_i. \tag{15}$$

Since $X_i<X_e$, the above condition gives $(n+1)_i>(n+1)_e$=critical value of convective instability, which means that the core solution near the interface must be in radiative equilibrium contrary to the assumption.

On the basis of evolutionary considerations about the variation of the hydrogen content, it is possible to construct a consistent model as follows; an intermediate radiative zone with variable composition grows inward from the point $q=q_1^{(0)}$, and

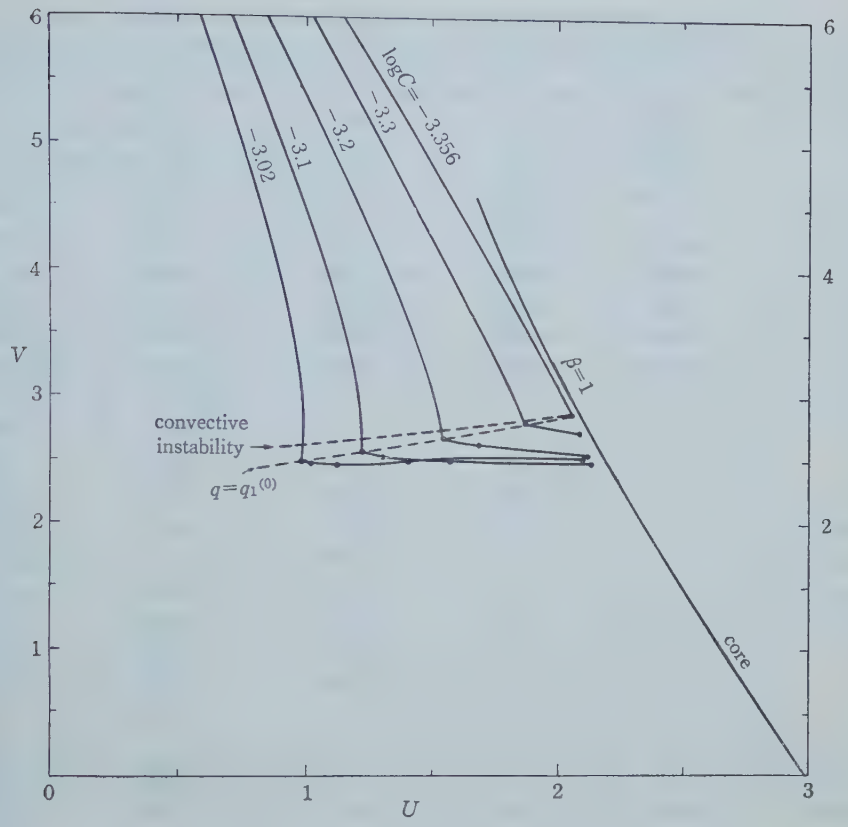

Fig. 1. Representations of model series in terms of the homolgy invariants U and V. The numbers on the envelope solutions denote the values of $\log C$. The fitting points are indicated by dots. The dashed line represents the locus of the points where each envelope solution reaches convective instability.

simultaneously a new convective zone is set up between this intermediate zone and the radiative envelope. In our model, however, the mass fraction of this convective zone is so small that the difference between the convective and radiative equilibrium can be neglected in a good approximation, that is, this region can be treated as an extension of the radiative envelope.

Thus, the radiative envelope is fitted to the intermediate zone at the point where $q=q_1^{(0)}$, and the intermediate zone is fitted to the convective core where it reaches the convective instability. The fitting conditions are the same as in the initial homogeneous case; i. e., the continuity of U, V and β, and in addition the convective instability at the interface are required. Since the mean molecular weight varies continuously in the intermediate zone, the condition of convective instability is slightly altered on account of a gradient of the mean molecular weight. This condition is expressed as

Table 1. Mathematical and physical characteristics of models.

Model	0	1	2	3	4
$\log C$	−3.356	−3.300	−3.200	−3.100	−3.020
U_1	2.06	1.87	1.54	1.22	0.975
V_1	2.87	2.78	2.67	2.57	2.50
$(n+1)_1$	2.80	2.83	2.90	2.98	3.05
$\log x_1$	−0.487	−0.518	−0.575	−0.649	−0.721
$\log p_1$	+1.053	+1.136	+1.295	+1.509	+1.709
$\log t_1$	−0.368	−0.332	−0.265	−0.184	−0.111
$\log q_1$	−0.382	−0.382	−0.382	−0.382	−0.382
β_1	0.952	0.945	0.929	0.909	0.888
U_f		2.08	2.12	2.13	2.09
V_f		2.72	2.55	2.48	2.55
$(n+1)_f$		2.87	2.99	3.14	3.31
$\log x_f$		−0.539	−0.648	−0.803	−1.026
$\log p_f$		+1.193	+1.485	+1.894	+2.470
$\log t_f$		−0.313	−0.201	−0.059	+0.123
$\log q_f$		−0.422	−0.512	−0.627	−0.774
β_f		0.942	0.918	0.881	0.832
ν		0.796	0.830	0.936	0.963
X_i	0.900	0.794	0.575	0.319	0.063
$\log L/L_\odot$	4.203	4.259	4.359	4.459	4.539
$\log R/R_\odot$	0.599	0.642	0.731	0.829	0.937
M_{bol}	−5.89	−6.03	−6.28	−6.53	−6.73
$\log T_e$	4.514	4.506	4.487	4.462	4.429
$\log T_c$	7.499	7.508	7.528	7.555	7.612
B. C.	−3.32	−3.28	−3.18	−3.04	−2.86
$B-V$	−0.42	−0.41	−0.39	−0.37	−0.35
life (year)	0	3.92×10^6	9.79×10^6	1.41×10^7	1.67×10^7

$$\frac{1}{(n+1)_{rad}} \geqq \frac{1}{(n+1)_{ad}} + \frac{\beta}{4-3\beta}\,\frac{d\log\mu}{d\log P}, \qquad (16)$$

where $(n+1)_{rad}$ and $(n+1)_{ad}$ are defined by equations (13). The gradient of mean molecular weight tends to suppress the convection, and consequent-

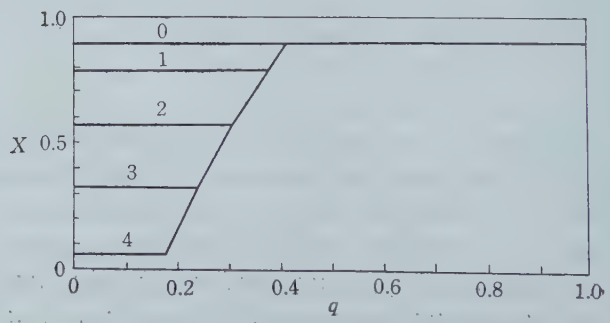

Fig. 2. The variation of the hydrogen content from the center to the surface.

ly guarantees that the intermediate zone is in radiative equilibrium.

The values of the exponent ν_n which appears in equation (5) are determined successively in such a way that the solutions satisfy the above fitting conditions. The mathematical characteristics of these solutions are listed in Table 1, and their $U-V$ curves are shown in Fig. 1. The distribution of the hydrogen content throughout the envelope, the intermediate zone and the core is plotted in Fig. 2.

§ 4. Physical properties and discussions

For each stellar model the physical quantities of interest are obtained from the mathematical ones by equating the rate of the energy production to the stellar luminosity. The energy production in the core results approximately, by replacing the solution near the center by the polytropic one, in the luminosity

$$L \simeq \varepsilon_0 \left\{ \frac{3n_1+3}{2n_1+n_1/n_3} \cdot \frac{k}{2G\mu_i H \beta_c} \right\}^{3/2} \rho_c^{1/2} T_c^{s+3/2}, \tag{17}$$

where n_1 and n_3 are the polytropic indices

$$1+1/n_1 \equiv (d \log P/d \log \rho)_c, \quad 1/n_3 \equiv (d \log T/d \log \rho)_c, \tag{18}$$

and ε_0 and s are constants in the formula of the energy production

$$\varepsilon = \varepsilon_0 \rho T^s. \tag{19}$$

Noting that the main nuclear energy source is due to the $C-N$ cycle, we take as the appropriate values of ε_0 and s

$$\varepsilon_0 = 8 \times 10^{-114} \cdot X X_{CN} \text{ erg cm}^3 \text{g}^{-1} \text{ deg}^{-16}, \quad s=16, \tag{20}$$

with $X_{CN}=(1/3)Z$. The central temperature is determined in such a way that the total energy production obtained from (17) should be equal to the luminosity appeared in the envelope solution. The results of these calculation are shown in Table 1.

We are now in a position to check the validity of the assumptions. The first check is concerned with the stability of the new convective zone which is set up between the radiative envelope and the intermediate zone. If a certain perturbation stirs up hydrogen at the interface between this convective zone and the intermediate zone, mixing may occur throughout both regions, and then the gradient of mean molecular weight may be set up. This gradient suppresses the convection and leads the state to the radiative equilibrium. It is found that the mixing of only a fraction of per cent of hydrogen is enough to lead the state to the radiative one. Thus, the assumption that there is no mixing between the convective zone and the intermediate zone is shown to be valid.

The second check is made on the energy production in the last model. Since the hydrogen content in the last model decreases to six per cent, the assumption that the energy generation is confined to the core may break down. The result

of the integration of energy production throughout the intermediate zone shows that the energy generation outside the core does not exceed five per cent of the core.

The evolutionary path in the $H-R$ diagram is shown in Fig. 3. The star leaving the main sequence evolves rather slowly towards right, i.e., the luminosity increases slightly and the effective temperature decreases. The lifetime of the evolution along the path is longer than that might have been expected. This is because the mass of the convective core relative to the total mass becomes large on account of the effect of radiation pressure. The theoretical evolutionary path and the observed data for the $h+\chi$ Persei stars obtained by Johnson and Hiltner[5] are compared in the $H-R$ diagram.

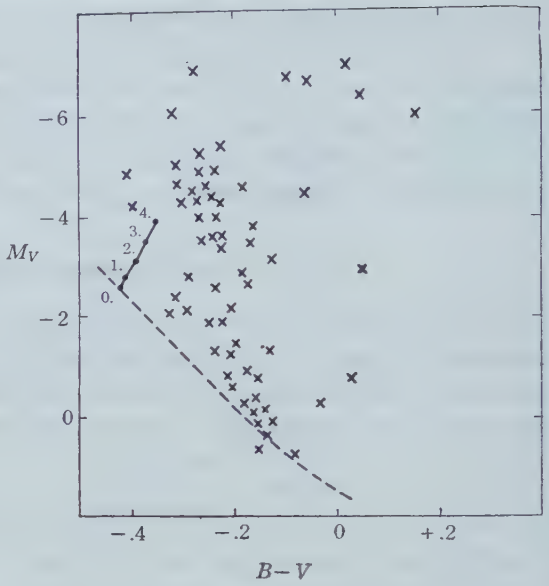

Fig. 3. The evolutionary track in the $H-R$ diagram. Successive evolutionary models are indicated by dots. Crosses represent the observed positions for the $h+\chi$ Persei stars.

It seems that the upper part of the $h+\chi$ Persei is now in a stage of gravitational contraction, having passed through an isothermal core stage. It is inferred that the time spent in the gravitational contraction stage may be rather short for these massive stars on account of the large mass fraction contained in the core. This might explains the gap in the diagram which is found between the early and late type supergiants.

Finally, it is noted that we have assumed no steady ejection of mass from the surface, i. e., the total mass remains constant. Nevertheless, it is found by Slettebak[8] that the line broadening in the supergiant stars of early type is due almost entirely to large scale atmospheric motions. Basing on this evidence, we speculate that the steady ejection from the surface might occur in massive stars. A rough estimate shows that the *loss of about one solar* mass for initial 15.6 $M_\odot$ in last 10^7 years is enough to remove the unstable convection zone in our models.

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Neutrino and Gravity

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Since the neutrino is completely polarized along its travelling direction, a β-decay results in the loss of angular momentum from the universe. As a result of its recoil, the universe begins to rotate, thus violating the mirror symmetry of space. Although the average rotational motion vanishes due to random β-decays, the mean square angular velocity remains finite. This causes the centrifugal acceleration which, according to the equivalence principle, should be equated to the gravitational acceleration due to the loss of neutrino energy. This could take a part in the origin of gravity; the value of the gravitation constant calculated by the above idea is not unreasonable.

The absence of the mirror symmetry in weak interactions must have led many people to infer that the universe itself is asymmetric in some sense, and various speculations must have been born, although few of them have ever been published. None of those ideas may have been convincing enough to be worth publishing. This has been the case with regard to my own idea which seems to be very speculative on one hand and to be so trivial, on the other hand, that somebody else have possibly get the same thing. However, I dare to publish this idea, hoping to stimulate discussions on this problem.

My speculation is guided by the equivalence principle, such that the mirror asymmetry revealed in weak interactions among elementary particles is in some way connected with an asymmetry of the universe in the large and the gravitational force is indistinguishable from the acceleration of a reference frame.

Let us suppose a nucleon to undergo a β-decay, an electron and a neutrino being produced. The neutrino flies away and sets underneath the horizon of the universe,* thereby carrying away angular momentum $\hbar/2$ as well as energy which is of the same order of magnitude as the electron rest energy, mc^2, while the electron is trapped inside the universe.

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* By the universe I mean the space satisfying the cosmological principle and expanding according to Hubble's law. Thus we may regard ourselves as standing at the center of the universe and the horizon as if it were an edge of the universe.

It is quite natural to think that the loss of angular momentum causes an angular momentum change in the system left after the neutrino escape and consequently causes the rotation of the system,† if it were at rest before the β-decay. Such a rotating system is evidently asymmetric with respect to the mirror reflection. In general the universe may be regarded as rotating, but the angular velocity of the rotation fluctuates from time to time because β-decays occur at random in directions so that the time averaged angular velocity vanishes.* However, the mean square angular velocity does not vanish, and one feels centrifugal acceleration. The centrifugal acceleration can be interpreted in the other way around. The energy loss due to the neutrino escape causes the change in the gravitational force that should be felt as acceleration. The equivalence principle demands that these two ways of explanation must give us an identical answer. The above reasoning will be formulated as follows.

The angular velocity caused by the escape of a neutrino is given by

$$\omega=\hbar/2I, \qquad (1)$$

where I is the moment of inertia of the system concerned. At a distance R from the path of the neutrino the centrifugal acceleration is $R\omega^2$. This must be equal to the gravitational acceleration associated with the energy loss mc^2 as

$$R\omega^2=Gm/R^2, \qquad (2)$$

where G is the gravitation constant. If n β-decays occur, the right-hand side of eq. (2) is multiplied by n, while the left-hand side by the sum of squares of the fluctuating angular velocities, the mean square of n which is equal to n. Thus eq. (2) holds as it is.

One may regard that eqs. (1) and (2) account for the origin of gravity. Here the neutrino plays an essential role, as has been thought by many people, but the way played by the neutrino is different from the current one. In our case the structure of the system under consideration has to be specified; the gravitation constant can be calculated, only if the values of I and R are given. I and R are not independent, but are related through

$$I=aMR^2, \qquad (3)$$

where M is the mass** of the system and a a numerical factor depending on the mass distribution. R may be chosen as the gravitational Bohr radius,*** to which the gravitational effect extends,

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† The angular momentum changes due to orbital motions should be nothing to do with the present problem, because of their non-violation of the parity conservation.

* This is consistent with the cosmological principle, according to which only the radial component of velocity exists.

** Since the equivalence principle is always referred to, the inertial and gravitational masses need not be distinguished.

*** The large value of R thus chosen may be consistent with the fact that one can find an inertial system in the universe.

$$R=b\hbar^2/GM^2m, \qquad (4)$$

where b is another numerical factor.

Now the gravitation constant can result from the above arguments, as shown by

$$G=R^3\omega^2/m=R^3\hbar^2/4mI^2=\hbar^2/4a^2mM^2R=G/4a^2b, \qquad (5)$$

provided $4a^2b=1$.

Examining our way of reasoning, one might think that the relation (4) stands on a very shaky basis, although it seems to have some bearing on the inverse square law of gravity. In this case one may replace this by the following argument. The system under consideration is such a simple one that consists of two nucleons separated by the cosmic radius R. If one nucleon undergoes a β-decay, the other receives its gravitational signal, just at the moment when the decay neutrino sets underneath the cosmic horizon. Then the gravitation constant can be evaluated by reference to the fourth expression of eq. (5). Introducing the electron and nucleon masses respectively in m and M and putting $R=6\times10^{27}$ cm, we obtain

$$G=7.4\times10^{-8} \text{ dyne cm}^2 \text{ g}^{-2},$$

which is very close to the observed value, $G=6.7\times10^{-8}$ dyne cm^2 g^{-2}. It may be worth while to remark that the gravitational Bohr radius of this system is practically equal to the cosmic radius, 6.6×10^{27} cm with the observed value of G.*

It is natural to ask how other parity non-conserving processes influence the mirror asymmetry of the universe. I can merely answer this question in such a way that most of the parity non-conserving processes occuring in the universe are nuclear β-decays and these practically govern the structure of the universe somehow like the Mach principle.

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* Since R is chosen as equal to the cosmic radius, the relation (2) may be interpreted in such a way that a detector placed at the edge of the universe feels no acceleration arising from β-decays, because the decrease in the force due to the loss of neutrino is canceled by the force allowing the rotation. This seems to provide with a natural basis to the relation (2), because such a detector looks at the universe as a whole without reference to any other object, thus observing the universe only as non-accelerated. I am much indebted to Professor R. Utiyama who clarified this point.

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On the Method of "Auxiliary Variables"

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The method of "auxiliary variables" (collective coordinates) used by Bogolyubov and Zubarev is formulated in a somewhat different manner. Improvements are made in two points: 1) The Hamiltonian is given in an Hermitian form and 2) the number of the auxiliary variables can be limited, if necessary. We calculate the ground state energy and the excitation energy spectrum of the spinless bosons taking account of phonon-phonon interactions. The expression for the ground state energy is shown to be equivalent to the one derived by Bogolyubov and Zubarev.

§ 1. Introduction

The investigation of the excitation energy of an interacting boson system is closely related to the theory of liquid He^4. Since Landau[1] proposed the so-called "phonon-roton" model for the excitations in liquid He^4, many authors have attempted to derive this model from the first principle. Among them, Bogolyubov and Zubarev[2] have dealt with the boson system using the method of "auxiliary variables", which were thought to be very appropriate for the description of long wave length excitations (phonons).

Several objections may, however, be made to their theory:

1) An infinite number of the auxiliary variables were used in their theory. This may be inconsistent with the fact that the system under consideration has only $3N$ degrees of freedom, N being the particle number of the system.

2) The Hamiltonian derived by them is non-Hermitian. This fact complicates the perturbation calculations.

3) The presence of a strong interaction between two particles may make their theory invalid, because the Fourier expansion of the interaction energy plays an essential rôle in the theory.

4) When the attractive part of the interaction is strong enough compared with the repulsive one, the excitation energy in the zeroth approximation becomes imaginary.

Among the above-mentioned difficulties, the third and the fourth ones will not be discussed in the present note. We shall rather consider a system with the interaction, in connection with which these two objections cannot be raised. The

purpose of the present note is to discuss the first and the second objections. To do this, the method of auxiliary variables will be used in a manner different from that adopted by Bogolyubov and Zubarev.

Abe[3] has already succeeded in deriving the Hermitian Hamiltonian. However, the Hamiltonian used by Abe does not give a reasonable result in the calculation of the energy of the system. As will be shown in later sections, this is due to the fact that he used an infinite number of auxiliary variables from the starting point of the theory.

Takano[4] has introduced the individual coordinates into the theory in addition to the finite number of the auxiliary variables and derived the excitation energy spectrum similar to the phonon-roton model. We will here attempt to construct a theory without use of individual coordinates, since the system considered here can be described in terms of the auxiliary variables only.

We shall consider the system of N spinless identical particles with mass m, which are contained in a cube of edge length L, hence of volume $V=L^3$. The Hamiltonian is assumed to be written as

$$H(\boldsymbol{r})=-\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial \boldsymbol{r}_j^2}+\frac{1}{2}\sum\sum_{i\neq j}\phi(r_{ij})=T(\boldsymbol{r})+V(\boldsymbol{r}), \tag{1·1}$$

where $\phi(r_{ij})$ is the interaction energy between two particles i and j, depending only on the distance r_{ij} between them. And, further, the periodic boundary condition is assumed in the theory.

The auxiliary variables are defined by*

$$\rho_{\boldsymbol{k}}=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j},\quad \boldsymbol{k}\neq 0, \tag{1·2}$$

where the components k_x, k_y and k_z of $\boldsymbol{k}$ are positive or negative integers multiplied by $2\pi/L$. These variables are closely related to the density fluctuation of the system and were used by Bohm and Pines[6] in the case of the electron gas in metals. Real auxiliary variables are defined only for $k_z>0$ as follows:

$$\sigma_{\boldsymbol{k}}=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\cos\,\boldsymbol{k}\cdot\boldsymbol{r}_j \text{ and } \tau_{\boldsymbol{k}}=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\sin\,\boldsymbol{k}\cdot\boldsymbol{r}_j. \tag{1·3}$$

Therefore we have the following relations for $k_z>0$

$$\rho_{\boldsymbol{k}}=\sigma_{\boldsymbol{k}}-i\tau_{\boldsymbol{k}} \quad \text{and} \quad \rho_{-\boldsymbol{k}}=\sigma_{\boldsymbol{k}}+i\tau_{\boldsymbol{k}}. \tag{1·4}$$

We will here make two assumptions:

(I) A wave function to describe the state of the system can be represented as a function of the $\rho_{\boldsymbol{k}}$'s (or $\sigma_{\boldsymbol{k}}$'s and $\tau_{\boldsymbol{k}}$'s) with $0<|\boldsymbol{k}|\leq k_c$ alone. In a formal develop-

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* Ono[5] has pointed out that the variables defined by $\rho_{\boldsymbol{k}}=1/\sqrt{N}\cdot\sum_{j=1}^{N}e^{+i\boldsymbol{k}\boldsymbol{r}_j}$ are more appropriate in connection with the momentum of phonons. We shall here follow the definition of Bogolyubov and Zubarev. If necessary, the signs of all the $\boldsymbol{k}$'s may be changed in the following expressions.

ment of the theory, the value of k_c may be arbitrary.* Such a representation of the wave function will be called hereafter the "ρ_k-representation". It should be noted that the ρ_k-representation is consistent with the periodic boundary condition.

(II) The interaction energy $\phi(r)$ is expanded in a Fourier series

$$\phi(r)=\frac{\nu(0)}{V}+\sum_{k\neq 0}\frac{\nu(k)}{V}e^{ikr},\quad \text{where } \nu(k)=\int\phi(r)e^{-ikr}dr \tag{1·5}$$

and, further, the $\nu(k)$'s for small $|k|$ are positive. The latter part of this assumption is necessary for obtaining real energy values. (cf. eq.(4·6))

Adopting the above two assumptions, we shall derive the Hamiltonian and the total momentum in the ρ_k-representation in § 2. Some functions of ρ_k's appearing in the Hamiltonian will be calculated in § 3 to obtain a concrete form of the Hamiltonian. In § 4, we shall calculate the ground state energy and the excitation energy spectrum to the first approximation. The assumption (I) will be discussed in § 5 in connection with the question on the degrees of freedom of our system.

§ 2. Hamiltonian in the ρ_k-representation

Although Bogolyubov and Zubarev[2] transformed the Schrödinger equation to derive the Hamiltonian in the ρ_k-representation, we will start with the expectation value of the Hamiltonian $H(\boldsymbol{r})$ for a state represented by the wave function $\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)\equiv\Psi(\boldsymbol{r})$, which is written as

$$E=\int\Psi^*(\boldsymbol{r})H(\boldsymbol{r})\Psi(\boldsymbol{r})d\boldsymbol{r}^{(3N)}\Big/\int\Psi^*(\boldsymbol{r})\Psi(\boldsymbol{r})d\boldsymbol{r}^{(3N)}. \tag{2·1}$$

The potential energy part of the Hamiltonian (1·1) is expressed, by the use of Fourier expansion (1·5), as

$$V(\boldsymbol{r})=\frac{1}{2}N(N-1)\frac{\nu(0)}{V}+\frac{1}{2}\sum_{0<|k|\leq k_c}\sum_{i\neq j}^{N}\frac{\nu(k)}{V}e^{ik\cdot(r_i-r_j)}+V'(\boldsymbol{r}), \tag{2·2}$$

where

$$V'(\boldsymbol{r})=\frac{1}{2}\sum_{|k|>k_c}\sum_{i\neq j}^{N}\frac{\nu(k)}{V}e^{ik\cdot(r_i-r_j)}. \tag{2·3}$$

Because of assumption (I) made in the preceding section, we may put

$$\Psi(\boldsymbol{r})=\frac{1}{V^{N/2}}\varphi(\cdots,\rho_k,\cdots)\equiv\frac{1}{V^{N/2}}\varphi(\rho_k). \tag{2·4}$$

Hence we have

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* In fact, any set of k's may be used if it is restricted only by the condition that it does not contain the zero vector but contains the pair k, $-k$ whenever k is a member.

$$T(\boldsymbol{r})\,\Psi(\boldsymbol{r})=\frac{1}{V^{N/2}}\Bigg[\frac{1}{N}\sum_{0<|\boldsymbol{k}|,\,|\boldsymbol{k}'|\leq k_c}\sum\frac{\hbar^2(\boldsymbol{k}\boldsymbol{k}')}{2m}\sum_{j=1}^{N}e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}_j}\frac{\partial^2}{\partial\rho_{\boldsymbol{k}}\partial\rho_{\boldsymbol{k}'}}$$
$$+\sum_{0<|\boldsymbol{k}|\leq k_c}\frac{\hbar^2k^2}{2m}\frac{1}{\sqrt{N}}\sum_{j=1}^{N}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j}\frac{\partial}{\partial\rho_{\boldsymbol{k}}}\Bigg]\varphi(\rho_{\boldsymbol{k}}). \qquad (2\cdot5)$$

The summation and the product over $\boldsymbol{k}$ with $0<|\boldsymbol{k}|\leq k_c$ will be denoted in the present section simply by $\sum_{\boldsymbol{k}}$ and $\prod_{\boldsymbol{k}}$ respectively.

We will define the function of $\rho_{\boldsymbol{k}}$'s (and so of $\sigma_{\boldsymbol{k}}$'s and $\tau_{\boldsymbol{k}}$'s)

$$D(\rho_{\boldsymbol{k}})=\frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}\prod_{\boldsymbol{k}}{}'f_{\boldsymbol{k}}\,g_{\boldsymbol{k}}, \qquad (2\cdot6)$$

where

$$f_{\boldsymbol{k}}=\delta\Big(\sigma_{\boldsymbol{k}}-\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\cos\boldsymbol{k}\cdot\boldsymbol{r}_j\Big)\ \text{and}\ g_{\boldsymbol{k}}=\delta\Big(\tau_{\boldsymbol{k}}-\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\sin\boldsymbol{k}\cdot\boldsymbol{r}_j\Big), \qquad (2\cdot7)$$

and $\prod_{\boldsymbol{k}}{}'$ means the product over $\boldsymbol{k}$ with $k_z>0$ (with $0<|\boldsymbol{k}|\leq k_c$, of course). The following equations are obtained, with the aid of the property of δ-function,

$$\int(d\rho)\,D(\rho_{\boldsymbol{k}})=1 \qquad (2\cdot8)$$

and

$$\int F(\boldsymbol{r})\,d\boldsymbol{r}^{(3N)}=\int(d\rho)\int d\boldsymbol{r}^{(3N)}F(\boldsymbol{r})\prod_{\boldsymbol{k}}{}'f_{\boldsymbol{k}}\,g_{\boldsymbol{k}}\ \text{for an arbitrary}\ F(\boldsymbol{r}), \qquad (2\cdot9)$$

where $\int(d\rho)\cdots$ means $\int_{-\infty}^{+\infty}\cdots\int\prod_{\boldsymbol{k}}{}'(d\sigma_{\boldsymbol{k}}\,d\tau_{\boldsymbol{k}})\cdots$.

By making use of identity (2·9), the expectation values of the kinetic energy and the potential energy are written respectively as (cf. eqs. (2.2) and (2.5))

$$\int\Psi^*(\boldsymbol{r})\,T(\boldsymbol{r})\,\Psi(\boldsymbol{r})\,d\boldsymbol{r}^{(3N)}=\int(d\rho)\,D(\rho_{\boldsymbol{k}})\,\varphi^*(\rho_{\boldsymbol{k}})\Bigg[\frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\frac{\hbar^2(\boldsymbol{k}\boldsymbol{k}')}{2m}A(\boldsymbol{k}+\boldsymbol{k}')\frac{\partial^2}{\partial\rho_{\boldsymbol{k}}\partial\rho_{\boldsymbol{k}'}}$$
$$+\sum_{\boldsymbol{k}}\frac{\hbar^2k^2}{2m}A(\boldsymbol{k})\frac{\partial}{\partial\rho_{\boldsymbol{k}}}\Bigg]\varphi(\rho_{\boldsymbol{k}}) \qquad (2\cdot10)$$

and

$$\int\Psi^*(\boldsymbol{r})\,V(\boldsymbol{r})\,\Psi(\boldsymbol{r})\,d\boldsymbol{r}^{(3N)}=\int(d\rho)\,D(\rho_{\boldsymbol{k}})\,\varphi^*(\rho_{\boldsymbol{k}})\Big[\frac{1}{2}N(N-1)\frac{\nu(0)}{V}-\frac{1}{2}\sum_{\boldsymbol{k}}\frac{N}{V}\nu(\boldsymbol{k})$$
$$+\frac{1}{2}\sum_{\boldsymbol{k}}\frac{N}{V}\nu(\boldsymbol{k})\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}}+V'(\rho_{\boldsymbol{k}})\Big]\varphi(\rho_{\boldsymbol{k}}), \qquad (2\cdot11)$$

where

$$A(\boldsymbol{k})=\frac{1}{D(\rho_{\boldsymbol{k}})}\frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}\frac{1}{\sqrt{N}}\sum_{j=1}^{N}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j}\Big(\prod_{\boldsymbol{k}}{}'f_{\boldsymbol{k}}\,g_{\boldsymbol{k}}\Big) \qquad (2\cdot12)$$

and

$$V'(\rho_k)=\frac{1}{D(\rho_k)}\ \frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}V'(\boldsymbol{r})\,(\prod_{k}{}'\,f_k\,g_k). \tag{2·13}$$

As is seen from eqs. (2·10) and (2·11), the function $D(\rho_k)$ defined by eq. (2·6) may be considered as the Jacobian of the transformation of variables from $\boldsymbol{r}_j$'s to ρ_k's. It may also be considered as the weight function for the inner product in the ρ_k-representation. The operator put in the bracket in eq. (2·10) should be Hermitian, provided that the inner product is defined with the weight function $D(\rho_k)$. It is desirable, however, that the Hermitian nature of an operator is secured in the inner product with no weight function. To carry out this program, we shall introduce the new wave function

$$\Phi(\rho_k)=\sqrt{D(\rho_k)}\,\varphi(\rho_k). \tag{2·14}$$

Then eqs. (2·10) and (2·11) are transformed into the following forms:

$$\int\Psi^*(\boldsymbol{r})\,T(\boldsymbol{r})\,\Psi(\boldsymbol{r})\,d\boldsymbol{r}^{(3N)}=\int(d\rho)\,\Phi^*(\rho_k)\,T(\rho_k)\,\Phi(\rho_k),$$

where

$$\begin{aligned}T(\rho_k)=&\frac{1}{\sqrt{N}}\sum_{k}\sum_{k'}\frac{\hbar^2(\boldsymbol{k}\boldsymbol{k}')}{2m}\Big[\frac{1}{2}\Big\{A(\boldsymbol{k}+\boldsymbol{k}')\cdot\frac{\partial^2}{\partial\rho_k\partial\rho_{k'}}+\frac{\partial^2}{\partial\rho_k\partial\rho_{k'}}\cdot A(\boldsymbol{k}+\boldsymbol{k}')\Big\}\\&-\frac{1}{2}\,\frac{\partial^2A(\boldsymbol{k}+\boldsymbol{k}')}{\partial\rho_k\partial\rho_{k'}}+\frac{3}{4}A(\boldsymbol{k}+\boldsymbol{k}')\frac{1}{D^2}\,\frac{\partial D}{\partial\rho_k}\,\frac{\partial D}{\partial\rho_{k'}}-\frac{1}{2}A(\boldsymbol{k}+\boldsymbol{k}')\,\frac{1}{D}\,\frac{\partial^2D}{\partial\rho_k\partial\rho_{k'}}\Big]\\&-\frac{1}{2}\sum_{k}\frac{\hbar^2k^2}{2m}A(\boldsymbol{k})\frac{1}{D}\,\frac{\partial D}{\partial\rho_k}\\&+\sum_{k}\frac{\hbar^2k^2}{2m}\Big[A(\boldsymbol{k})-\frac{1}{\sqrt{N}}\sum_{k'}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}\Big\{A(\boldsymbol{k}+\boldsymbol{k}')\,\frac{1}{D}\,\frac{\partial D}{\partial\rho_{k'}}+\frac{\partial A(\boldsymbol{k}+\boldsymbol{k}')}{\partial\rho_{k'}}\Big\}\Big]\frac{\partial}{\partial\rho_k},\end{aligned} \tag{2·15}$$

and

$$\int\Psi^*(\boldsymbol{r})\,V(\boldsymbol{r})\,\Psi(\boldsymbol{r})\,d\boldsymbol{r}^{(3N)}=\int(d\rho)\,\Phi^*(\rho_k)\,V(\rho_k)\,\Phi(\rho_k),$$

where

$$V(\rho_k)=\frac{1}{2}N(N-1)\,\frac{\nu(0)}{V}-\frac{1}{2}\sum_{k}\frac{N}{V}\nu(\boldsymbol{k})+\frac{1}{2}\sum_{k}\frac{N}{V}\nu(\boldsymbol{k})\rho_k\rho_{-k}+V'(\rho_k). \tag{2·16}$$

The operators $T(\rho_k)$ and $V(\rho_k)$ are considered as the kinetic energy operator and the potential energy operator in the ρ_k-representation respectively. Because of the presence of the last term in the right-hand side of eq. (2·15), $T(\rho_k)$ is apparently non-Hermitian. However, this term is seen to be identically zero, since the relation

$$A(\boldsymbol{k})=\frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}'}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}\left\{A(\boldsymbol{k}+\boldsymbol{k}')\frac{1}{D}\frac{\partial D}{\partial\rho_{\boldsymbol{k}'}}+\frac{\partial A(\boldsymbol{k}+\boldsymbol{k}')}{\partial\rho_{\boldsymbol{k}'}}\right\} \tag{2·17}$$

does exist, as will be proved in the Appendix. Expression (2·15) is further transformed, by the use of relation (2·17), into

$$\begin{aligned}T(\rho_{\boldsymbol{k}})=&\frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\frac{\hbar^2(\boldsymbol{k}\boldsymbol{k}')}{2m}\frac{1}{2}\left\{A(\boldsymbol{k}+\boldsymbol{k}')\cdot\frac{\partial^2}{\partial\rho_{\boldsymbol{k}}\partial\rho_{\boldsymbol{k}'}}+\frac{\partial^2}{\partial\rho_{\boldsymbol{k}}\partial\rho_{\boldsymbol{k}'}}\cdot A(\boldsymbol{k}+\boldsymbol{k}')\right\}\\&+\sum_{\boldsymbol{k}}\frac{\hbar^2k^2}{2m}\left[-\frac{1}{2}\frac{\partial A(\boldsymbol{k})}{\partial\rho_{\boldsymbol{k}}}-\frac{1}{4}\frac{1}{D}\frac{\partial D}{\partial\rho_{\boldsymbol{k}}}\left\{A(\boldsymbol{k})-\frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}'}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}\frac{\partial A(\boldsymbol{k}+\boldsymbol{k}')}{\partial\rho_{\boldsymbol{k}'}}\right\}\right]\end{aligned} \tag{2·18}$$

Finally, we obtain as the expectation value of the Hamiltonian or the energy

$$E=\int(d\rho)\Phi^*(\rho_{\boldsymbol{k}})H(\rho_{\boldsymbol{k}})\Phi(\rho_{\boldsymbol{k}})\Big/\int(d\rho)\Phi^*(\rho_{\boldsymbol{k}})\Phi(\rho_{\boldsymbol{k}}), \tag{2·19}$$

where

$$\begin{aligned}H(\rho_{\boldsymbol{k}})=&T(\rho_{\boldsymbol{k}})+V(\rho_{\boldsymbol{k}})\\=&\frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\frac{\hbar^2(\boldsymbol{k}\boldsymbol{k}')}{2m}\frac{1}{2}\left\{A(\boldsymbol{k}+\boldsymbol{k}')\cdot\frac{\partial^2}{\partial\rho_{\boldsymbol{k}}\partial\rho_{\boldsymbol{k}'}}+\frac{\partial^2}{\partial\rho_{\boldsymbol{k}}\partial\rho_{\boldsymbol{k}'}}\cdot A(\boldsymbol{k}+\boldsymbol{k}')\right\}\\&+\sum_{\boldsymbol{k}}\frac{\hbar^2k^2}{2m}\left[-\frac{1}{2}\frac{\partial A(\boldsymbol{k})}{\partial\rho_{\boldsymbol{k}}}-\frac{1}{4}\frac{1}{D}\frac{\partial D}{\partial\rho_{\boldsymbol{k}}}\left\{A(\boldsymbol{k})-\frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}'}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}\frac{\partial A(\boldsymbol{k}+\boldsymbol{k}')}{\partial\rho_{\boldsymbol{k}'}}\right\}\right]\\&+\frac{1}{2}N(N-1)\frac{\nu(0)}{V}-\frac{1}{2}\sum_{\boldsymbol{k}}\frac{N}{V}\nu(\boldsymbol{k})+\frac{1}{2}\sum_{\boldsymbol{k}}\frac{N}{V}\nu(\boldsymbol{k})\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}}+V'(\rho_{\boldsymbol{k}}).\end{aligned} \tag{2·20}$$

Expression (2·20) is the Hamiltonian in the $\rho_{\boldsymbol{k}}$-representation, the derivation of which has been the main purpose of the present section. It is clear that this Hamiltonian is Hermitian. It should be noted that the summation $\sum_{\boldsymbol{k}}$ in the present section is to be carried out over $\boldsymbol{k}$ only for $0<|\boldsymbol{k}|\leq k_c$. The Schrödinger equation in the $\rho_{\boldsymbol{k}}$-representation is obtained from eq. (2·19) as

$$H(\rho_{\boldsymbol{k}})\Phi(\rho_{\boldsymbol{k}})=E\Phi(\rho_{\boldsymbol{k}}). \tag{2·21}$$

An explicit dependence of $D(\rho_{\boldsymbol{k}})$, $A(\boldsymbol{k})$ and $V'(\rho_{\boldsymbol{k}})$ upon the $\rho_{\boldsymbol{k}}$'s will be found in the following section. However, the following relations can be derived immediately from definition (2·12) of $A(\boldsymbol{k})$:

$$A(0)=\sqrt{N}\quad\text{and}\quad A(\boldsymbol{k})=\rho_{\boldsymbol{k}}\ \text{ for }\ |\boldsymbol{k}|\leq k_c. \tag{2·22}$$

Therefore, if we take $k_c\to\infty$, relation (2·17) will seem to be reduced to

$$\frac{1}{D}\frac{\partial D}{\partial\rho_{-\boldsymbol{k}}}=-\rho_{\boldsymbol{k}}+\frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}'\neq-\boldsymbol{k}}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}\rho_{\boldsymbol{k}+\boldsymbol{k}'}\frac{1}{D}\frac{\partial D}{\partial\rho_{\boldsymbol{k}'}}. \tag{2·23}$$

This equation was used by Abe[3] in the calculation of $\frac{1}{D}\frac{\partial D}{\partial\rho_{\boldsymbol{k}}}$. He obtained

$$\frac{1}{D}\frac{\partial D}{\partial \rho_{\boldsymbol{k}}}=\sum_{n=0}^{\infty}(-)^{n+1}\frac{1}{n+1}\left(\frac{1}{\sqrt{N}}\right)^{n}\sum_{\boldsymbol{k}_1+\boldsymbol{k}_2+\cdots+\boldsymbol{k}_{n+1}=-\boldsymbol{k}}\rho_{\boldsymbol{k}_1}\rho_{\boldsymbol{k}_2}\cdots\rho_{\boldsymbol{k}_{n+1}}. \tag{2·24}$$

However, the above deduction is not correct. Equation (2·23) cannot be derived from eq. (2·17) even if $k_c\to\infty$, since $\lim_{k_c\to\infty}\frac{\partial A(\boldsymbol{k}+\boldsymbol{k}')}{\partial \rho_{\boldsymbol{k}'}}$ is not equal to $\frac{\partial}{\partial \rho_{\boldsymbol{k}'}}\lim_{k_c\to\infty}A(\boldsymbol{k}+\boldsymbol{k}')=\frac{\partial \rho_{\boldsymbol{k}+\boldsymbol{k}'}}{\partial \rho_{\boldsymbol{k}'}}=0$ as will be seen in the following section.

An expression for the total momentum in the $\rho_{\boldsymbol{k}}$-representation is derived in the same way as the Hamiltonian has been derived. It is

$$\boldsymbol{P}(\rho_{\boldsymbol{k}})=-\sum_{\boldsymbol{k}}\hbar\boldsymbol{k}A(\boldsymbol{k})\left[\frac{\partial}{\partial \rho_{\boldsymbol{k}}}-\frac{1}{2}\frac{1}{D}\frac{\partial D}{\partial \rho_{\boldsymbol{k}}}\right]. \tag{2·25}$$

By the use of eq. (2·22) and the relation

$$\sum_{\boldsymbol{k}}\boldsymbol{k}A(\boldsymbol{k})\frac{\partial D}{\partial \rho_{\boldsymbol{k}}}=0, \tag{2·26}$$

eq. (2·25) is transformed into

$$\boldsymbol{P}(\rho_{\boldsymbol{k}})=-\sum_{\boldsymbol{k}}\hbar\boldsymbol{k}\frac{1}{2}\left[\rho_{\boldsymbol{k}}\cdot\frac{\partial}{\partial \rho_{\boldsymbol{k}}}+\frac{\partial}{\partial \rho_{\boldsymbol{k}}}\cdot\rho_{\boldsymbol{k}}\right], \tag{2·25′}$$

where the fact that $\sum_{\boldsymbol{k}}\boldsymbol{k}=0$ has also been used.

It can be verified that expression (2·25′) commutes with $T(\rho_{\boldsymbol{k}})$ and $V(\rho_{\boldsymbol{k}})$, as it should. Although the proof of this fact is omitted here, the following relation will serve to prove it:

$$\sum_{\boldsymbol{k}}\boldsymbol{k}A(\boldsymbol{k})\frac{\partial A(\boldsymbol{k}')}{\partial \rho_{\boldsymbol{k}}}=\boldsymbol{k}'A(\boldsymbol{k}') \qquad \text{for } 0\leq|\boldsymbol{k}'|<+\infty. \tag{2·27}$$

The proof of relations (2·26) and (2·27) is also omitted here, which is similar to the one of relation (2·17) given in the Appendix.

§ 3. Calculation of $D(\rho_{\boldsymbol{k}})$, $A(\boldsymbol{k})$ and $V'(\rho_{\boldsymbol{k}})$

In order to obtain a concrete form of the Hamiltonian derived in the preceding section, it is necessary to determine the explicit dependence of $D(\rho_{\boldsymbol{k}})$, $A(\boldsymbol{k})$ and $V'(\rho_{\boldsymbol{k}})$ upon the $\rho_{\boldsymbol{k}}$'s. This will be done in the present section. The mathematical technique adopted here is suggested by Zubarev's work[7] of calculating the partition function of a classical system. Similar technique was used for the classical system by Yukhnovsky[8].

We shall begin with the calculation of $D(\rho_{\boldsymbol{k}})$, which has been defined by eq. (2·6). With the aid of the Fourier representation of δ-function

$$\delta(x)=\int_{-\infty}^{\infty}\exp(2\pi i\omega x)\,d\omega,$$

eq. (2·6) is reduced to

$$D(\rho_k)=\int_{-\infty}^{+\infty}\cdots\int(\prod_k{}'d\alpha_k d\beta_k)\frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}\exp\Big[2\pi i\sum_k{}'\Big\{(\sigma_k-\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\cos\boldsymbol{k}\cdot\boldsymbol{r}_j)\alpha_k$$
$$+(\tau_k-\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\sin\boldsymbol{k}\cdot\boldsymbol{r}_j)\beta_k\Big\}\Big].$$

By the use of the variables $\omega_k=\alpha_k+i\beta_k$ and $\omega_{-k}=\alpha_k-i\beta_k$, the above expression is written as

$$D(\rho_k)=\int_{-\infty}^{+\infty}\cdots\int(\prod_k{}'d\alpha_k d\beta_k)[\exp(i\pi\sum_k\rho_k\omega_k)]I(\omega_k),\tag{3·1}$$

where

$$I(\omega_k)=\Big[\frac{1}{V}\int d\boldsymbol{r}\ \exp\Big\{-\frac{i\pi}{\sqrt{N}}\sum_k\omega_k e^{-i\boldsymbol{k}\boldsymbol{r}}\Big\}\Big]^N.\tag{3·2}$$

To calculate the integral on the right-hand side of eq. (3·2), we expand the first exponential function into a power series in its argument and perform the integration term by term. In doing this it is to be noted that the summation $\sum_k$ does not include the term with $\boldsymbol{k}=0$ and is taken over the integers multiplied by $2\pi/L$ (which are not larger than k_c, of course). Then we have

$$\frac{1}{V}\int d\boldsymbol{r}\ \exp\Big\{-\frac{i\pi}{\sqrt{N}}\sum_k\omega_k e^{-i\boldsymbol{k}\boldsymbol{r}}\Big\}=1+\frac{1}{N}\Big\{-\frac{\pi^2}{2}\sum_k\omega_k\omega_{-k}$$
$$-\frac{1}{3!}\frac{(i\pi)^3}{\sqrt{N}}\sum_{k_1+k_2+k_3=0}\omega_{k_1}\omega_{k_2}\omega_{k_3}+\frac{1}{4!}\frac{(i\pi)^4}{N}\sum_{k_1+k_2+k_3+k_4=0}\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4}+o\Big(\frac{1}{N}\Big)\Big\}.$$

Therefore, $I(\omega_k)$ can be written as

$$I(\omega_k)=\exp\Big(-\frac{\pi^2}{2}\sum_k\omega_k\omega_{-k}\Big)\Big[1-\frac{1}{\sqrt{N}}\frac{(i\pi)^3}{6}\sum_{k_1+k_2+k_3=0}\omega_{k_1}\omega_{k_2}\omega_{k_3}$$
$$+\frac{1}{N}\Big\{\frac{(i\pi)^4}{24}\sum_{k_1+k_2+k_3+k_4=0}\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4}-\frac{(i\pi)^4}{8}(\sum_k\omega_k\omega_{-k})^2+\frac{(i\pi)^6}{72}(\sum_{k_1+k_2+k_3=0}\omega_{k_1}\omega_{k_2}\omega_{k_3})^2\Big\}$$
$$+o\Big(\frac{1}{N}\Big)\Big].\tag{3·3}$$

Substitution of eq. (3·3) into eq. (3·1) leads to

$$D(\rho_k)=\Big[1-\frac{1}{\sqrt{N}}\frac{1}{6}\sum_{k_1+k_2+k_3=0}\frac{\partial^3}{\partial\rho_{k_1}\partial\rho_{k_2}\partial\rho_{k_3}}+\frac{1}{N}\Big\{\frac{1}{24}\sum_{k_1+k_2+k_3+k_4=0}\frac{\partial^4}{\partial\rho_{k_1}\partial\rho_{k_2}\partial\rho_{k_3}\partial\rho_{k_4}}$$
$$-\frac{1}{8}\Big(\sum_k\frac{\partial^2}{\partial\rho_k\partial\rho_{-k}}\Big)^2+\frac{1}{72}\Big(\sum_{k_1+k_2+k_3=0}\frac{\partial^3}{\partial\rho_{k_1}\partial\rho_{k_2}\partial\rho_{k_3}}\Big)^2+o\Big(\frac{1}{N}\Big)\Big]D_0(\rho_k),\tag{3·4}$$

where

$$D_0(\rho_k)=\int_{-\infty}^{+\infty}\cdots\int\Big(\prod_k' d\alpha_k d\beta_k\Big)\exp\left\{i\pi\sum_k\rho_k\omega_k-\frac{\pi^2}{2}\sum_k\omega_k\omega_{-k}\right\}. \tag{3·5}$$

The integration on the right-hand side of eq. (3·5) can be readily performed:

$$D_0(\rho_k)=\exp\{-\tfrac{1}{2}\sum_k(\rho_k\rho_{-k}+\log\pi)\}. \tag{3·6}$$

Hereafter the summation $\sum_k$ will be taken over all the integers, including zero, multiplied by $2\pi/L$. As compensation for this we introduce the function ε_k of $\boldsymbol{k}$ with the property

$$\varepsilon_k=\varepsilon_{-k}=\begin{cases}1 & \text{for } 0<|\boldsymbol{k}|\leq k_c\\ 0 & \text{otherwise.}\end{cases} \tag{3·7}$$

With the aid of

$$\partial D_0/\partial\rho_k=-\rho_{-k}D_0,$$

the differentiations on the right-hand side of eq. (3·4) are carried out and the following expression is obtained

$$\begin{aligned}D(\rho_k)=D_0(\rho_k)\Big[1&+\frac{1}{\sqrt{N}}\frac{1}{6}\sum_{k_1+k_2+k_3=0}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\rho_{k_1}\rho_{k_2}\rho_{k_3}\\&+\frac{1}{2}\left\{\frac{1}{\sqrt{N}}\frac{1}{6}\sum_{k_1+k_2+k_3=0}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\rho_{k_1}\rho_{k_2}\rho_{k_3}\right\}^2-\frac{1}{N}\frac{1}{12}\sum_{k_1+k_2+k_3+k_4=0}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\varepsilon_{k_4}\rho_{k_1}\rho_{k_2}\rho_{k_3}\rho_{k_4}\\&+\frac{1}{N}\left\{\frac{1}{8}\sum_{\substack{k_1+k_2+k_3+k_4=0\\k_1+k_2\neq 0}}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\varepsilon_{k_4}(1-\varepsilon_{k_1+k_2})\rho_{k_1}\rho_{k_2}\rho_{k_3}\rho_{k_4}+\frac{1}{2}\sum_k\varepsilon_k\rho_k\rho_{-k}\right.\\&\left.+\frac{1}{4}\sum_{k_1,k_2}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_1+k_2}\rho_{k_1}\rho_{-k_1}-\frac{1}{4}\sum_k\varepsilon_k-\frac{1}{12}\sum_{k_1,k_2}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_1+k_2}\right\}+o\Big(\frac{1}{N}\Big)\Big].\end{aligned} \tag{3·8}$$

Therefore we have

$$\begin{aligned}\frac{1}{D}\frac{\partial D}{\partial\rho_k}=-\rho_{-k}&+\frac{1}{\sqrt{N}}\frac{1}{2}\sum_{k_1+k_2=-k}\varepsilon_{k_1}\varepsilon_{k_2}\rho_{k_1}\rho_{k_2}-\frac{1}{N}\frac{1}{3}\sum_{k_1+k_2+k_3=-k}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\rho_{k_1}\rho_{k_2}\rho_{k_3}\\&+\frac{1}{N}\left\{\frac{1}{2}\sum_{\substack{k_1+k_2+k_3=-k\\k_1+k_2\neq 0}}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}(1-\varepsilon_{k_1+k_2})\rho_{k_1}\rho_{k_2}\rho_{k_3}+\rho_{-k}(1+\tfrac{1}{2}\sum_{k'}\varepsilon_{k'}\varepsilon_{k+k'})\right\}+o\Big(\frac{1}{N}\Big).\end{aligned} \tag{3·9}$$

The calculation of $A(\boldsymbol{k})$ and $V'(\rho_k)$ can be carried out in quite a similar way. We will write only the results (cf. eq. (2·22)):

$$\begin{aligned}A(\boldsymbol{k})&=\sqrt{N}\quad\text{for }\boldsymbol{k}=0\\&=\rho_k\quad\text{for }0<|\boldsymbol{k}|\leq k_c\\&=\frac{1}{\sqrt{N}}\frac{1}{2}\sum_{k_1+k_2=k}\varepsilon_{k_1}\varepsilon_{k_2}\rho_{k_1}\rho_{k_2}+o\Big(\frac{1}{\sqrt{N}}\Big)\quad\text{for }|\boldsymbol{k}|>k_c\end{aligned} \tag{3·10}$$

and

$$V'(\rho_k)=\frac{1}{N}\frac{1}{8}\sum_{\substack{k_1+k_2+k_3+k_4=0\\ k_1+k_2\neq 0}}\frac{N}{V}\nu(\boldsymbol{k}_1+\boldsymbol{k}_2)\,(1-\varepsilon_{k_1+k_2})\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\varepsilon_{k_4}\rho_{k_1}\rho_{k_2}\rho_{k_3}\rho_{k_4}$$

$$-\frac{1}{N}\frac{1}{4}\sum_{k_1+k_2\neq 0}\frac{N}{V}\nu(\boldsymbol{k}_1+\boldsymbol{k}_2)\,(1-\varepsilon_{k_1+k_2})\varepsilon_{k_1}\varepsilon_{k_2}(2\rho_{k_1}\rho_{-k_1}-1)+o\left(\frac{1}{N}\right). \quad (3\cdot 11)$$

It can be readily confirmed that the above-derived $(1/D)$ $(\partial D/\partial\rho_k)$ and $A(\boldsymbol{k})$ satisfy relation $(2\cdot 17)$ and that $(\partial A(\boldsymbol{k}+\boldsymbol{k}')/\partial\rho_{k'})$ does not vanish even if $k_c\to\infty$ (cf. the discussion after eq. $(2\cdot 24)$ in the preceding section). If we take $k_c\to\infty$ in the right-hand side of eq.$(3\cdot 9)$, the second line is reduced to

$$(1/N)\rho_{-k}(1+\tfrac{1}{2}\sum_{k'}\varepsilon_{k'}\varepsilon_{k+k'}).$$

The first line agrees with eq.$(2\cdot 24)$ as far as the terms with a smaller order of magnitude than $1/N$ are neglected.

Substituting eqs. $(3\cdot 9)$, $(3\cdot 10)$ and $(3\cdot 11)$ into eq. $(2\cdot 20)$, we can write down the Hamiltonian in the ρ_k-representation, neglecting the terms with a smaller order of magnitude than $1/N$*, in the following form :

$$H(\rho_k)=H_0+H_1+H_2+o(1/N), \quad (3\cdot 12)$$

where

$$H_0=\sum_k\varepsilon_k\frac{\hbar^2k^2}{2m}\left[-\frac{\partial^2}{\partial\rho_k\partial\rho_{-k}}-\frac{1}{2}+\frac{1}{4}\rho_k\rho_{-k}\right]$$

$$+\frac{1}{2}N(N-1)\frac{\nu(0)}{V}-\frac{1}{2}\sum_k\varepsilon_k\frac{N}{V}\nu(\boldsymbol{k})+\frac{1}{2}\sum_k\varepsilon_k\frac{N}{V}\nu(\boldsymbol{k})\rho_k\rho_{-k}, \quad (3\cdot 13)$$

$$H_1=\frac{1}{\sqrt{N}}\sum_{k_1}\frac{\hbar^2k_1^2}{2m}\sum_{k_2}\varepsilon_{k_2}\varepsilon_{k_1+k_2}\left[\frac{(\boldsymbol{k}_1\boldsymbol{k}_2)}{k_1^2}\rho_{k_1+k_2}\frac{\partial^2}{\partial\rho_{k_1}\partial\rho_{k_2}}-\frac{1}{8}\rho_{k_1}\rho_{k_2}\rho_{-k_1-k_2}\right]$$

$$=\frac{1}{\sqrt{N}}\sum_{k_1,k_2}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_1+k_2}\frac{\hbar^2(\boldsymbol{k}_1\boldsymbol{k}_2)}{2m}\left[\rho_{k_1+k_2}\frac{\partial^2}{\partial\rho_{k_1}\partial\rho_{k_2}}+\frac{1}{4}\rho_{k_1}\rho_{k_2}\rho_{-k_1-k_2}\right] \quad (3\cdot 14)$$

and

$$H_2=\frac{1}{N}\sum_{k_1+k_2+k_3+k_4=0}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\varepsilon_{k_4}\frac{\hbar^2k_1^2}{2m}\frac{1}{12}\rho_{k_1}\rho_{k_2}\rho_{k_3}\rho_{k_4}$$

$$+\frac{1}{N}\sum_{k_1+k_2=k_3+k_4\neq 0}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\varepsilon_{k_4}(1-\varepsilon_{k_1+k_2})\frac{\hbar^2(\boldsymbol{k}_1\boldsymbol{k}_2)}{2m}\frac{1}{4}\left[\frac{\partial^2}{\partial\rho_{k_1}\partial\rho_{k_2}}\cdot\rho_{k_3}\rho_{k_4}+\rho_{k_3}\rho_{k_4}\cdot\frac{\partial^2}{\partial\rho_{k_1}\partial\rho_{k_2}}\right]$$

$$-\frac{1}{N}\sum_{\substack{k_1+k_2+k_3+k_4=0\\ k_1+k_2\neq 0}}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\varepsilon_{k_4}(1-\varepsilon_{k_1+k_2})\frac{\hbar^2k_1^2}{2m}\frac{1}{8}\rho_{k_1}\rho_{k_2}\rho_{k_3}\rho_{k_4}$$

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* Strictly speaking, this expresson may be of little sense. However, the Hamiltonian $(3\cdot 12)$ is sufficient for the calculation of the energy per particle to the order of magnitude of $(V/N)^2$ formally, as will be seen in the following section. (cf. the discussion after eq. $(4\cdot 10)$)

$$-\frac{1}{N}\sum_{k}\varepsilon_k\frac{\hbar^2k^2}{2m}\frac{1}{2}\rho_k\rho_{-k}\Big(1+\frac{1}{2}\sum_{k'}\varepsilon_{k'}\,\varepsilon_{k+k'}\Big)$$

$$+\frac{1}{N}\,\frac{1}{8}\sum_{\substack{k_1+k_2+k_3+k_4=0\\ k_1+k_2\neq 0}}\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_3}\varepsilon_{k_4}(1-\varepsilon_{k_1+k_2})\,\frac{N}{V}\nu(\boldsymbol{k}_1+\boldsymbol{k}_2)\rho_{k_1}\rho_{k_2}\rho_{k_3}\rho_{k_4}$$

$$-\frac{1}{N}\,\frac{1}{4}\sum_{k_1+k_2\neq 0}\varepsilon_{k_1}\varepsilon_{k_2}(1-\varepsilon_{k_1+k_2})\frac{N}{V}\nu(\boldsymbol{k}_1+\boldsymbol{k}_2)\,(2\rho_{k_1}\rho_{-k_1}-1). \tag{3·15}$$

As will be seen in the following section, H_0 can be considered to represent free phonons. Then H_1 and H_2 give the phonon-phonon interactions.

§ 4. Ground state energy and excitation energy spectrum

We will calculate the ground state energy and the excitation energy spectrum for the system under consideration. To diagonalize H_0, we introduce the following operators

$$b_k=\lambda_k\frac{\partial}{\partial\rho_{-k}}+\frac{1}{2\lambda_k}\rho_k \quad\text{and}\quad b_k^+=-\lambda_k\frac{\partial}{\partial\rho_k}+\frac{1}{2\lambda_k}\rho_{-k} \tag{4·1}$$

or, conversely,

$$\rho_k=\lambda_k(b_k+b_{-k}^+) \qquad\text{and}\qquad \frac{\partial}{\partial\rho_k}=\frac{1}{2\lambda_k}(b_{-k}-b_k^+), \tag{4·1′}$$

where $\lambda_k=\lambda_{-k}>0$ is a function of $\boldsymbol{k}$ to be determined. The operators b_k and b_k^+ satisfy the commutation relations

$$[b_k,\ b_{k'}^+]=\delta_{k,k'} \quad\text{and}\quad [b_k,\ b_{k'}]=[b_k^+,\ b_{k'}^+]=0, \tag{4·2}$$

from which it is seen that the eigenvalues of each of $b_k^+b_k$ are positive integers including zero.

We adopt a representation in which all the $b_k^+b_k$ are diagonal. The total momentum is written from eq. (2·25′) in terms of b_k's and b_k^+'s as

$$\boldsymbol{P}(\rho_k)=\sum_k\varepsilon_k\hbar\boldsymbol{k}b_k^+b_k. \tag{4·3}$$

Therefore we can interpret the operators b_k and b_k^+ as the annihilation and creation operator of a phonon with momentum $\hbar\boldsymbol{k}$, respectively.

When we take as

$$\lambda_k{}^4=\frac{\hbar^2k^2}{4m}\Big/\Big(\frac{\hbar^2k^2}{4m}+\frac{N}{V}\nu(\boldsymbol{k})\Big), \tag{4·4}$$

H_0, which is given by eq. (3·13), is written in the diagonal form:

$$H_0=\sum_k\varepsilon_kE(\boldsymbol{k})b_k^+b_k+E_0^{(0)}, \tag{4·5}$$

where

$$E(\boldsymbol{k})=\frac{\hbar^2k^2}{2m}/\lambda_k^2=\sqrt{\frac{N}{V}\nu(\boldsymbol{k})\frac{\hbar^2k^2}{m}+\left(\frac{\hbar^2k^2}{2m}\right)^2} \tag{4·6}$$

and

$$E_0^{(0)}=\frac{1}{2}N(N-1)\frac{\nu(0)}{V}+\frac{1}{2}\sum_k\varepsilon_k\left[E(\boldsymbol{k})-\frac{\hbar^2k^2}{2m}-\frac{N}{V}\nu(\boldsymbol{k})\right]. \tag{4·7}$$

The above-written $E_0^{(0)}$ and $E(\boldsymbol{k})$ are the ground state energy and the excitation energy spectrum in the zeroth approximation, respectively. These expressions were already derived by Bogolyubov and Zubarev[2]. As is seen from eq. (4·5), H_0 can be considered to represent the free phonons.

To obtain the ground state energy and the excitation energy spectrum to the next approximation, the phonon-phonon interactions must be taken into account, which are given by H_1 and H_2. The perturbation calculations to the second order in H_1 and to the first order in H_2 are enough for this purpose. Higher order perturbation calculations are of no use, unless we derive the Hamiltonian with higher accuracy than is given in eq. (3·12).

After a lengthy but straightforward calculation, we obtain, by taking account of phonon-phonon interactions, the ground state energy E_0 and the excitation energy spectrum E_k to the first approximation as follows:

$$E_0=E_0^{(0)}+E_0^{(1)}+N\cdot o(V/N)^2 \quad \text{and} \quad E_k=E(\boldsymbol{k})+E_k^{(1)}+o(V/N)^2, \tag{4·8}$$

where

$$\begin{aligned}
E_0^{(1)}=&-\frac{1}{24}\frac{1}{N}\sum_{\boldsymbol{k}_1}\sum_{\boldsymbol{k}_2}\left(\frac{\hbar^2}{2m}\right)^2\frac{\varepsilon_{k_1}\varepsilon_{k_2}\varepsilon_{k_1+k_2}}{E(\boldsymbol{k}_1)+E(\boldsymbol{k}_2)+E(\boldsymbol{k}_1+\boldsymbol{k}_2)}\times\\
&\times\Big[-(\boldsymbol{k}_1\boldsymbol{k}_2)^2\frac{(1-\lambda^2_{k_1})(1-\lambda^2_{k_2})}{\lambda^2_{k_1}\lambda^2_{k_2}}\{2(1-\lambda^2_{k_1+k_2})(\lambda^2_{k_1}+\lambda^2_{k_2})-\lambda^2_{k_1+k_2}(1-\lambda^2_{k_1})\\
&\qquad\times(1-\lambda^2_{k_2})\}\\
&-(\boldsymbol{k}_1,\boldsymbol{k}_1+\boldsymbol{k}_2)^2\frac{(1-\lambda^2_{k_1})(1-\lambda^2_{k_1+k_2})}{\lambda^2_{k_1}\lambda^2_{k_1+k_2}}\{2(1-\lambda^2_{k_2})(\lambda^2_{k_1}+\lambda^2_{k_1+k_2})-\lambda^2_{k_2}(1-\lambda^2_{k_1})\\
&\qquad\times(1-\lambda^2_{k_1+k_2})\\
&-(\boldsymbol{k}_2,\boldsymbol{k}_1+\boldsymbol{k}_2)^2\frac{(1-\lambda^2_{k_2})(1-\lambda^2_{k_1+k_2})}{\lambda^2_{k_2}\lambda^2_{k_1+k_2}}\{2(1-\lambda^2_{k_1})(\lambda^2_{k_2}+\lambda^2_{k_1+k_2})-\lambda^2_{k_1}(1-\lambda^2_{k_2})\\
&\qquad\times(1-\lambda^2_{k_1+k_2})\}\\
&+2\{(\boldsymbol{k}_1\boldsymbol{k}_2)(\boldsymbol{k}_1,\boldsymbol{k}_1+\boldsymbol{k}_2)+(\boldsymbol{k}_1\boldsymbol{k}_2)(\boldsymbol{k}_2,\boldsymbol{k}_1+\boldsymbol{k}_2)-(\boldsymbol{k}_1,\boldsymbol{k}_1+\boldsymbol{k}_2)(\boldsymbol{k}_2,\boldsymbol{k}_1+\boldsymbol{k}_2)\}\\
&\times\frac{(1-\lambda^2_{k_1})(1-\lambda^2_{k_2})(1-\lambda^2_{k_1+k_2})}{\lambda^2_{k_1}\lambda^2_{k_2}\lambda^2_{k_1+k_2}}(\lambda^2_{k_1}\lambda^2_{k_2}+\lambda^2_{k_1}\lambda^2_{k_1+k_2}+\lambda^2_{k_2}\lambda^2_{k_1+k_2}\\
&\qquad+\lambda^2_{k_1}\lambda^2_{k_2}\lambda^2_{k_1+k_2})\Big]\\
&+\frac{1}{4}\frac{1}{N}\sum_{\boldsymbol{k}_1+\boldsymbol{k}_2\neq0}\sum\varepsilon_{k_1}\varepsilon_{k_2}(1-\varepsilon_{k_1+k_2})\frac{N}{V}\nu(\boldsymbol{k}_1+\boldsymbol{k}_2)(1-\lambda^2_{k_1})(1-\lambda^2_{k_2})\\
&+\frac{1}{4}\frac{1}{N}\sum_k\varepsilon_k\frac{\hbar^2k^2}{2m}(1-\lambda_k^2)^2
\end{aligned} \tag{4·9}$$

and

$$E_k^{(1)}=-\frac{1}{8}\frac{1}{N}\sum_{k'}\left(\frac{\hbar^2}{2m}\right)^2\frac{\varepsilon_{k'}\,\varepsilon_{k+k'}}{E(\boldsymbol{k})+E(\boldsymbol{k}')+E(\boldsymbol{k}+\boldsymbol{k}')}$$

$$\times\Big[-(\boldsymbol{k}\boldsymbol{k}')^2\frac{(1-\lambda_k^2)(1-\lambda_{k'}^2)}{\lambda_k^2\lambda_{k'}^2}\{2(1-\lambda_{k+k'}^2)(\lambda_k{}^2+\lambda_{k'}^2)-\lambda_{k+k'}^2(1-\lambda_k{}^2)\times(1-\lambda_{k'}^2)\}$$

$$-(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{k}')^2\frac{(1-\lambda_k^2)(1-\lambda_{k+k'}^2)}{\lambda_k^2\lambda_{k+k'}^2}\{2(1-\lambda_{k'}^2)(\lambda_k{}^2+\lambda_{k+k'}^2)-\lambda_{k'}^2(1-\lambda_k^2)\times(1-\lambda_{k+k'}^2)\}$$

$$-(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{k}')^2\frac{(1-\lambda_{k'}^2)(1-\lambda_{k+k'}^2)}{\lambda_{k'}^2\lambda_{k+k'}^2}\{2(1-\lambda_k^2)(\lambda_{k'}^2+\lambda_{k+k'}^2)-\lambda_k^2(1-\lambda_{k'}^2)\times(1-\lambda_{k+k'}^2)\}$$

$$+2\{(\boldsymbol{k}\boldsymbol{k}')(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{k}')+(\boldsymbol{k},\boldsymbol{k}')(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{k}')-(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{k}')(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{k}')\}$$

$$\times\frac{(1-\lambda_k^2)(1-\lambda_{k'}^2)(1-\lambda_{k+k'}^2)}{\lambda_k^2\lambda_{k'}^2\lambda_{k+k'}^2}(\lambda_k^2\lambda_{k'}^2+\lambda_k^2\lambda_{k+k'}^2+\lambda_{k'}^2\lambda_{k+k'}^2+\lambda_k^2\lambda_{k'}^2\lambda_{k+k'}^2)\Big]$$

$$-\frac{1}{8}\frac{1}{N}\sum_{k'}\left(\frac{\hbar^2}{2m}\right)^2\frac{\varepsilon_{k'}\,\varepsilon_{k+k'}}{E(\boldsymbol{k}')+E(\boldsymbol{k}+\boldsymbol{k}')-E(\boldsymbol{k})}$$

$$\times\Big[(\boldsymbol{k}\boldsymbol{k}')^2\frac{(1+\lambda_k^2)(1-\lambda_{k'}^2)}{\lambda_k^2\lambda_{k'}^2}\{\lambda_{k+k'}^2(1-\lambda_k^2)(1+\lambda_{k'}^2)+2(\lambda_k^2-\lambda_{k'}^2)\}$$

$$+(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{k}')^2\frac{(1+\lambda_k^2)(1-\lambda_{k+k'}^2)}{\lambda_k{}^2\lambda_{k+k'}^2}\{\lambda_{k'}^2(1-\lambda_k^2)(1+\lambda_{k+k'}^2)+2(\lambda_k^2-\lambda_{k+k'}^2)\}$$

$$+(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{k}')^2\frac{(1-\lambda_{k'}^2)(1-\lambda_{k+k'}^2)}{\lambda_{k'}^2\lambda_{k+k'}^2}\{\lambda_k^2(1+\lambda_{k'}^2)(1+\lambda_{k+k'}^2)+2(\lambda_{k'}^2+\lambda_{k+k'}^2)\}$$

$$-2\{(\boldsymbol{k}\boldsymbol{k}')(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{k}')+(\boldsymbol{k}\boldsymbol{k}')(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{k}')-(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{k}')(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{k}')\}$$

$$\times\frac{(1+\lambda_k^2)(1-\lambda_{k'}^2)(1-\lambda_{k+k'}^2)}{\lambda_k^2\lambda_{k'}^2\lambda_{k+k'}^2}(\lambda_k^2\lambda_{k'}^2+\lambda_k^2\lambda_{k+k'}^2-\lambda_{k'}^2\lambda_{k+k'}^2+\lambda_k^2\lambda_{k'}^2\lambda_{k+k'}^2)\Big]$$

$$-\frac{1}{N}\lambda_k{}^2\sum_{k'\neq-k}\varepsilon_{k'}(1-\varepsilon_{k+k'})\frac{N}{V}\nu(\boldsymbol{k}+\boldsymbol{k}')(1-\lambda_{k'}^2)$$

$$-\frac{1}{N}\lambda_k^2\frac{\hbar^2k^2}{2m}(1-\lambda_k^2). \tag{4·10}$$

The last terms in the right-hand side of eqs. (4·9) and (4·10) may be omitted, since these terms vanish in the limit $N\to\infty$ while keeping the value of N/V fixed. In the same limit, $\sum_k$ can be replaced by $(V/(2\pi)^3)\int d\boldsymbol{k}$. Therefore, $E_0^{(1)}/N$ and $E_k^{(1)}$ may be said to be "formally" proportional to $(V/N)^2$ and V/N respectively, where the word "formally" means that the coefficients of $(V/N)^2$ or V/N themselves depend on V/N in complicated manners. If we take account of the terms which have been neglected in the right-hand side of eqs. (3·9), (3·10) or (3·11), we shall obtain the Hamiltonian with higher accuracy than is given in eq.

(3·12). Then we shall obtain a formal expansion of the ground state energy or the excitation energy spectrum in terms of V/N. It is rather doubtful whether our expressions (4·9) and (4·10) are useful ones in a case of large values of V/N. However, it can be said that our expressions are useful ones in a case where the number of excited phonons is small.

Our expressions (4·9) and (4·10) vanish for the non-interacting system where $\lambda_k=1$, as they should. The Hamiltonian derived by Abe[3] has not led to such a result. This is because of his adoption of eq. (2·24) for $(1/D)\ \partial D/\partial\rho_k$.

Bogolyubov and Zubarev[2] also calculated a correction to the ground state energy. It was found that the correction was of the order $(1-\lambda^2_k)^2$, while our expression (4·9) is of the order $(1-\lambda_k{}^2)^3$. (Note that $\lambda_k=1$ for the non-interacting system.) This apparent discrepancy is owing to the presence of redundant terms in the expression of Bogolyubov and Zubarev, which are to vanish by the summation over $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Subtracting these redundant terms from their expression, we can obtain the identical expression with eq. (4·9) except for the last two terms in the right-hand side, which vanish in the limit $k_c\to\infty$ and in the limit $N\to\infty$, with the value of N/V kept constant, respectively.

§ 5. Discussion

So far we have proceeded with the assumption (I), which was made in § 1 — that is, we have restricted the ρ_k's only for $0<|\boldsymbol{k}|\leq k_c$. However, it is clear that this assumption is not correct, since the ρ_k's with $0<|\boldsymbol{k}|<\infty$ are to be used in expressing the wave function of our system as a function of the auxiliary variables. Conversely, an arbitrary wave function of our system can be written as the function of the ρ_k's with $0<|\boldsymbol{k}|<\infty$, as was proved by Abe[3]. (This may be almost self-evident, because the wave function can always be expanded in terms of permanents consisting of the plane waves.) Therefore, we are led to the conclusion that the limit $k_c\to\infty$ should be taken in the final expressions such as eqs. (4·9) and (4·10).

On the other hand, the above conclusion may be inconsistent with the degrees of freedom of our system as has been mentioned in § 1. Although no definite answer to this question can be given at present, the reasoning adopted by Takano[4] may be applied also to our case. It is as follows:

In the representation used in § 4, the wave functions

$$|\boldsymbol{k}_1,\ \boldsymbol{k}_2,\cdots,\ \boldsymbol{k}_m\rangle=b^+_{\boldsymbol{k}_1}\,b^+_{\boldsymbol{k}_2}\cdots b^+_{\boldsymbol{k}m}|0\rangle,\quad m=0,\ 1,\ 2,\cdots \tag{5·1}$$

form a complete orthogonal set, where $|0\rangle$ is the vacuum state defined by $b_k|0\rangle=0$ for all the $\boldsymbol{k}$'s. The function (5·1) represents a state in which phonons with the wave numbers $\boldsymbol{k}_1$, $\boldsymbol{k}_2,\cdots$, and $\boldsymbol{k}_m$ are excited. On the other hand, the functions

$$\Psi_{\boldsymbol{k}_1,\boldsymbol{k}_2,\cdots,\boldsymbol{k}_N}(\boldsymbol{r})=\sum_P P\exp[i(\boldsymbol{k}_1\boldsymbol{r}_1+\boldsymbol{k}_2\boldsymbol{r}_2+\cdots+\boldsymbol{k}_N\boldsymbol{r}_N)] \tag{5·2}$$

form a complete orthogonal set in the original representation, where $\sum_P$ means the sum over all permutations P of $\boldsymbol{r}_1$, $\boldsymbol{r}_2$,$\cdots$, and $\boldsymbol{r}_N$. In order to make one-to-one correspondence between the sets $|\boldsymbol{k}_1, \boldsymbol{k}_2, \cdots, \boldsymbol{k}_m\rangle$ and $\Psi_{\boldsymbol{k}_1,\boldsymbol{k}_2,\cdots\boldsymbol{k}_N}(\boldsymbol{r})$, the number m of the $\boldsymbol{k}$'s appearing in eq. (5·1), that is, the number of the excited phonons, must not exceed N. It is not necessarily equal to N, since some of the $\boldsymbol{k}$'s may be zero in eq. (5·2) while $\boldsymbol{k}=0$ is excluded in eq. (5·1). Therefore, we obtain

$$\sum_{\boldsymbol{k}\neq 0} b_{\boldsymbol{k}}^{+} b_{\boldsymbol{k}} = \sum_{\boldsymbol{k}\neq 0} n_{\boldsymbol{k}} \leq N. \tag{5·3}$$

In other words, we are making a conjecture that the number of phonons are limited by taking account of the degrees of freedom.

If we admit that eq. (5·3) is reasonable, it is certain that a system of non-interacting particles is correctly treated also in the $\rho_{\boldsymbol{k}}$-representation. And, further, we obtain the second order phase tansition in a system of interacting particles as was shown by Takano[4].

Percus and Yevick[9] have treated a classical system by means of the auxiliary variables $\rho_{\boldsymbol{k}}$'s and carefully investigated a rôle of the set of $\boldsymbol{k}$'s used in the theory. If k_c must be a finite value in our theory, their method can be used also in our case.

The method used in the present note will be able to be generalized to the case where strong interactions are present and also to the case of a fermion system under adequate assumptions. Such generalization will be attempted in forthcoming notes.

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Appendix

The proof of relation (2·17) will be given here. We obtain from eq. (2·7), with the aid of eq. (1·4), the following equations

$$\frac{\partial}{\partial \rho_{\boldsymbol{k}'}}\left(\prod_{\boldsymbol{k}}{}' f_{\boldsymbol{k}} g_{\boldsymbol{k}}\right) = \begin{cases} \frac{1}{2}\left(f_{\boldsymbol{k}'} g_{\boldsymbol{k}'} + i f_{\boldsymbol{k}'} g_{\boldsymbol{k}'}'\right)\left(\prod_{\boldsymbol{k}\neq\boldsymbol{k}'}{}' f_{\boldsymbol{k}} g_{\boldsymbol{k}}\right) & \text{for } k_z' > 0 \\ \frac{1}{2}\left(f'_{-\boldsymbol{k}'} g_{-\boldsymbol{k}'} - i f_{-\boldsymbol{k}'} g'_{-\boldsymbol{k}'}\right)\left(\prod_{\boldsymbol{k}\neq-\boldsymbol{k}'}{}' f_{\boldsymbol{k}} g_{\boldsymbol{k}}\right) & \text{for } k_z' < 0 \end{cases} \tag{A·1}$$

and

$$\frac{\partial}{\partial \boldsymbol{r}_j}\left(\prod_{\boldsymbol{k}}{}' f_{\boldsymbol{k}} g_{\boldsymbol{k}}\right) = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}'}{}' \boldsymbol{k}' \left(f_{\boldsymbol{k}'}' g_{\boldsymbol{k}'} \sin \boldsymbol{k}'\cdot\boldsymbol{r}_j - f_{\boldsymbol{k}'} g_{\boldsymbol{k}'}' \cos \boldsymbol{k}'\cdot\boldsymbol{r}_j\right)\left(\prod_{\boldsymbol{k}\neq\boldsymbol{k}'}{}' f_{\boldsymbol{k}} g_{\boldsymbol{k}}\right), \tag{A·2}$$

where f_k' or g_k' denotes the derivative of f_k or g_k with respect to its argument. By the use of eqs. (A·1) and (2·12), we obtain

$$\frac{1}{\sqrt{N}}\sum_{k'}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}\frac{\partial}{\partial\rho_{k'}}\{A(\boldsymbol{k}+\boldsymbol{k}')D(\rho_k)\}$$

$$=\frac{1}{N}\frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}\sum_{j=1}^{N}\sum_{k'}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}_j}\frac{\partial}{\partial\rho_{k'}}\left(\prod_k{}' f_k g_k\right)$$

$$=\frac{1}{N}\frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}\sum_{j=1}^{N}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j}\sum_{k'}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}\frac{1}{2}e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}_j}$$

$$\times\begin{cases}(f_{k'}'\,g_{k'}+if_{k'}\,g_{k'}')\left(\prod_{k\neq k'}{}' f_k g_k\right) & \text{for } k_z'>0\\ (f_{-k'}'\,g_{-k'}-if_{-k'}\,g'_{-k'})\left(\prod_{k\neq -k'}{}' f_k g_k\right) & \text{for } k_z'<0\end{cases}$$

$$=-i\frac{1}{N}\frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}\sum_{j=1}^{N}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j}\sum_{k'}{}'\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}$$

$$\times(f_{k'}'\,g_{k'}\sin\boldsymbol{k}'\cdot\boldsymbol{r}_j-f_{k'}\,g_{k'}'\cos\boldsymbol{k}'\cdot\boldsymbol{r}_j)\left(\prod_{k\neq k'}{}' f_k g_k\right).$$

Substitution of eq. (A·2) into the last expression above leads to

$$=-i\frac{1}{\sqrt{N}}\frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}\sum_{j=1}^{N}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j}\frac{1}{k^2}\left(\boldsymbol{k}\,\frac{\partial}{\partial\boldsymbol{r}_j}\right)\left(\prod_k{}' f_k g_k\right).$$

The integration by parts gives

$$=\frac{1}{V^N}\int d\boldsymbol{r}^{(3N)}\frac{1}{\sqrt{N}}\sum_{j=1}^{N}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j}\left(\prod_k{}' f_k g_k\right)$$

$$=A(\boldsymbol{k})D(\rho_k).$$

Therefore, we obtain the relation

$$\frac{1}{\sqrt{N}}\sum_{k'}\frac{(\boldsymbol{k}\boldsymbol{k}')}{k^2}\frac{\partial}{\partial\rho_{k'}}\{A(\boldsymbol{k}+\boldsymbol{k}')D(\rho_k)\}=A(\boldsymbol{k})D(\rho_k),\qquad(\text{A}\cdot3)$$

which is equivalent to relation (2·17).

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Gauge Invariance and the Meissner Effect

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A gauge invariant theory of a many electron system in an external magnetic field, interacting through arbitrary velocity dependent two-body forces, is presented. The energy spectrum for a "free" electron is also assumed to be arbitrary to take into account the self-energy due to its interaction with the phonon field. If the two-body forces are neglected, it is shown that no Meissner effect is obtained so long as the energy spectrum is continuous together with its first order derivative. Bardeen's energy gap model is reconsidered from our standpoint and is shown to be drastically changed.

§ 1. Introduction

Questions have been raised[1] whether an energy-gap model of superconductivity, such as that of Bardeen, Cooper and Schrieffer,[2] can explain the Meissner effect. The idealized Hamiltonian which they used is not gauge invariant because of the momentum dependent cut-off on the effective interaction between electrons. As has been argued especially by Schafroth and Buckingham,[1] a proof of gauge invariance lies at the core of the problem of superconductivity, since the violation of this invariance does very easily lead to the Meissner effect.

Anderson has shown by introducing the collective excitations of the system that Bardeen's theory is approximately gauge invariant in the weak coupling case, but the examination of the equation of continuity alone is not sufficient as was stressed by Schafroth.[1] Wentzel[3] and Rickayzen[4] have started with Fröhlich's[5] Hamiltonian, with Coulomb interactions omitted, in order to keep the gauge invariance correctly and have taken the Bogolyubov[6] approach to obtain the ground-state energy and excitation energies of the superconducting state. It should be noted, however, that the gauge invariance is to be maintained at any stage of approximations in order to keep the theory gauge invariant and that the Bogolyubov approximation essentially violates this invariance since his approximate Hamiltonian can create or annihilate two electrons.

The Hamiltonian used by BCS is based on the one derived by Bardeen and

Pines[7] with use of a canonical transformation which replaces the electron-phonon interaction by an effective interaction between electrons. This canonical transformation introduces extra terms in the expression for the magnetic interaction and current density; one term is associated with the self-energy of an electron and the other with velocity dependent potentials. We may perform any further canonical transformations, but we should add the extra terms for the magnetic interaction and current density corresponding to these transformations in order to keep the theory gauge invariant.

The purpose of the present paper is to show how a manifestly gauge invariant theory is obtained once the spectrum for an electron and the effective potentials between two electrons are defined in the absence of a magnetic field. The postulate of gauge invariance almost uniquely determines the magnetic interaction and current even in our general case, if the charge density is defined in the usual way, disregarding the ambiguity of the path in the line integral of vector potential defined below; the Pauli term need not be considered since the original electron-phonon interaction does not depend on the spin of an electron.

In Section 2, a gauge invariant theory of a many electron system in an external magnetic field, interacting through velocity dependent potentials, is formulated. In Section 3, a free electron model with arbitrary spectrum is considered. If the spectrum is continuous with its first order derivative with respect to momentum, no Meissner effect is obtained. A free electron model with the energy gap gives London's diamagnetism in the non-gauge invariant theory as shown by Bardeen,[2] but in our theory this term is exactly cancelled and we have a peculiar singularity for the current. In the Appendix, it is shown that the magnetic interaction and current defined in Section 2 are identical with conventional ones if the spectrum is a quadratic function of momentum. To this end, the path in the line integral of vector potential should be chosen linear.

§ 2. Gauge invariant theory

We consider a system of electrons interacting through an arbitrary velocity dependent potential V. The spectrum of an electron is also assumed to be arbitrary. As has been stated in the Introduction, such a system will follow from the original electron-phonon system with use of a suitable canonical transformation. The field theoretical Hamiltonian for our system is in general written as

$$H=\int d^3x'd^3x f(\mathbf{x}'-\mathbf{x})\psi^*(\mathbf{x}')\psi(\mathbf{x})$$

$$+\tfrac{1}{2}\int d^3xd^3x'd^3yd^3y'V(\mathbf{x},\mathbf{x}';\ \mathbf{y},\mathbf{y}')\psi^*(\mathbf{x})\psi^*(\mathbf{x}')\psi(\mathbf{y})\psi(\mathbf{y}'), \qquad (2\cdot1)$$

where $\psi(\mathbf{x})$ describes the electron wave. The Fourier transform of $f(\mathbf{x})$ gives the energy spectrum of an electron:

$$\int d^3x f(\mathbf{x}) e^{-\frac{i}{\hbar}\mathbf{p}\mathbf{x}} = \varepsilon(\mathbf{p}), \tag{2·2}$$

which, for the sake of simplicity, is assumed to be independent of the direction of the wave vector $\mathbf{p}$. On account of the anti-commutativity of ψ's or ψ^*'s, the potential V is taken to be antisymmetric with respect to $\mathbf{x}$ and $\mathbf{x}'$, or $\mathbf{y}$ and $\mathbf{y}'$ without loss of generality. The real property of the Hamiltonian (2·1) is guaranteed by the conditions

$$f^*(\mathbf{x}) = f(-\mathbf{x}), \quad V^*(\mathbf{x}, \mathbf{x}'; \mathbf{y}, \mathbf{y}') = V(\mathbf{y}', \mathbf{y}; \mathbf{x}', \mathbf{x}) \tag{2·3}$$

where the first condition follows from the reality of $\varepsilon(\mathbf{p})$ which is assumed hereafter. The translational and rotational invariance of the system imposes the further condition on V that it is a function only of the differences of position vectors.

Next, we consider the interaction of our system with an external magnetic field $\mathbf{H}$ derived from a vector potential $\mathbf{A}$ ($\mathbf{H} = \mathrm{rot}\,\mathbf{A}$). The Hamiltonian $H(\mathbf{A})$ in this case is determined from the following conditions. First of all, $H(\mathbf{A})$ does not contain the Pauli term ($\sigma\mathbf{H}$) as has been stated in the Introduction and $H(\mathbf{A}\equiv 0) = H$. Second, there should exist an unitary operator corresponding to a gauge transformation such that

$$UH(\mathbf{A})U^{-1} = H(\mathbf{A} - \nabla\varphi), \tag{2·4}$$

where $\varphi(\boldsymbol{x})$ is the usual gauge function, since if $\Psi_{\mathbf{A}}$ is a Schroedinger wave function satisfying the Schroedinger equation

$$i\hbar \cdot \partial\Psi_{\mathbf{A}}/\partial t = H(\mathbf{A})\Psi_{\mathbf{A}}, \tag{2·5}$$

then $\Psi_{\mathbf{A}}' \equiv U\Psi_{\mathbf{A}}$ satisfies

$$i\hbar \cdot \partial\Psi_{\mathbf{A}}'/\partial t = H(\mathbf{A} - \nabla\varphi)\Psi_{\mathbf{A}}', \tag{2·6}$$

from which we have $\Psi_{\mathbf{A}}' = \Psi_{\mathbf{A}-\nabla\varphi}$ disregarding a phase factor. Third, the charge density $\rho(\mathbf{x})$ is assumed to be defined as usual by*

$$\rho(\mathbf{x}) = -e\psi^*(\mathbf{x})\psi(\mathbf{x}). \tag{2·7}$$

Then the unitary operator U should have the form

$$U = \exp\left[\frac{ie}{\hbar c}\int \psi^*(\mathbf{x})\psi(\mathbf{x})\varphi(\mathbf{x})\, d^3x\right], \quad U\psi(\mathbf{x})U^{-1} = \psi(\mathbf{x})e^{-\frac{ie}{\hbar c}\varphi(\mathbf{x})}. \tag{2·8}$$

From the above conditions we immediately see that

$$H(\mathbf{A}) = \int d^3x' d^3x f(\mathbf{x}' - \mathbf{x})\psi^*(\mathbf{x}')\psi(\mathbf{x})\exp\left[-\frac{ie}{\hbar c}\int_{\mathbf{x}}^{\mathbf{x}'}\mathbf{A}(\mathbf{s})\, d\mathbf{s}\right]$$

$$+\tfrac{1}{2}\int d^3x d^3x' d^3y d^3y' V(\mathbf{x}, \mathbf{x}'; \mathbf{y}, \mathbf{y}')\psi^*(\mathbf{x})\psi^*(\mathbf{x}')\psi(\mathbf{y})\psi(\mathbf{y}')$$

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* This is not always true after a canonical transformation which does not commute with U is performed. In this sense our gauge invariant theory is purely phenomenological.

$$\times\exp\left[-\frac{ie}{\hbar c}\int_{\mathbf{y}}^{\mathbf{x}'}\mathbf{A}(\mathbf{s})\,d\mathbf{s}-\frac{ie}{\hbar c}\int_{\mathbf{y}'}^{\mathbf{x}}\mathbf{A}(\mathbf{s})\,d\mathbf{s}\right], \tag{2·9}$$

where the integral of $\mathbf{A}$ means a line integral.* The real property of $H(\mathbf{A})$ follows from eq. (2·3). It seems that another gauge invariant theory is obtained by interchanging $\mathbf{x}$ and $\mathbf{x}'$ and $\mathbf{y}$ and $\mathbf{y}'$ in the second term of eq. (2·8), but turns out to be identical with eq. (2·8) on account of the antisymmetric property of V. On the other hand, by choosing the path in the line integral arbitrarily an infinite set of gauge invariant theories are obtained. Ambiguities discussed by Schafroth[1] belong to a special case. There is no definite way in our phenomenological approach to determine which path should be taken, but as shown in the Appendix a straight line leads to the ordinary magnetic interaction in the free electron model with $\varepsilon(\mathbf{p})=\hbar^2\mathbf{p}^2/2m$.

The electric current is defined in the usual way by

$$\begin{aligned} j_\mu(\mathbf{x}) &= -c\frac{\partial H(\mathbf{A})}{\partial A_\mu(\mathbf{x})}=\frac{ie}{\hbar}\int f(\mathbf{y}'-\mathbf{y})\psi^*(\mathbf{y}')\psi(\mathbf{y})\exp\left[-\frac{ie}{\hbar c}\int_{\mathbf{y}}^{\mathbf{y}'}\mathbf{A}d\mathbf{s}\right] \\ &\cdot\int_{\mathbf{y}}^{\mathbf{y}'}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu\cdot d^3y d^3y'+\frac{ie}{2\hbar}\int d^3z d^3z' d^3y d^3y' V(\mathbf{z},\mathbf{z}';\mathbf{y},\mathbf{y}') \\ &\cdot\psi^*(\mathbf{z})\psi^*(\mathbf{z}')\psi(\mathbf{y})\psi(\mathbf{y}')\exp\left[-\frac{ie}{\hbar c}\int_{\mathbf{y}}^{\mathbf{z}'}\mathbf{A}d\mathbf{s}-\frac{ie}{\hbar c}\int_{\mathbf{y}'}^{\mathbf{z}}\mathbf{A}d\mathbf{s}\right] \\ &\cdot\left\{\int_{\mathbf{y}}^{\mathbf{z}'}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu+\int_{\mathbf{y}'}^{\mathbf{z}}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu\right\}. \end{aligned} \tag{2·10}$$

The equation of continuity in the Schroedinger representation

$$\partial j_\mu(\mathbf{x})/\partial x_\mu-(i/\hbar)[H(\mathbf{A}),\,\rho(\mathbf{x})]=0, \tag{2·11}$$

follows immediately from the above definition of the electric current. For the actual application, we may expand eq. (2·10) in terms of e or $\mathbf{A}$ and retain terms up to second order. Then we have

$$j_\mu(\mathbf{x})=j_\mu^p(\mathbf{x})+j_\mu^d(\mathbf{x}), \tag{2·12}$$

where

$$\begin{aligned} j_\mu^p(\mathbf{x}) &= +\frac{ie}{\hbar}\int d^3y d^3y' f(\mathbf{y}'-\mathbf{y})\psi^*(\mathbf{y}')\psi(\mathbf{y})\int_{\mathbf{y}}^{\mathbf{y}'}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu \\ &+\frac{ie}{2\hbar}\int d^3z d^3z' d^3y d^3y' V(\mathbf{z},\mathbf{z}';\mathbf{y},\mathbf{y}')\psi^*(\mathbf{z})\psi^*(\mathbf{z}')\psi(\mathbf{y})\psi(\mathbf{y}') \\ &\cdot\left\{\int_{\mathbf{y}}^{\mathbf{z}'}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu+\int_{\mathbf{y}'}^{\mathbf{z}}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu\right\}, \end{aligned} \tag{2·13}$$

* This type of electromagnetic interaction was considered by many authors, e.g. by Heisenberg in positron theory, by Peierls in non-local field theory and by Capps and Holladay in static meson theory. See, for example, Phys. Rev. **99** (1955), 931.

$$j_\mu^q(\mathbf{x}) = \frac{e^2}{\hbar^2 c}\int d^3y d^3y' f(\mathbf{y}'-\mathbf{y})\psi'^*(\mathbf{y}')\psi'(\mathbf{y})\int_{\mathbf{y}}^{\mathbf{y}'}\mathbf{A}ds\cdot\int_{\mathbf{y}}^{\mathbf{y}'}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu$$

$$+\frac{e^2}{2\hbar^2 c}\int d^3z d^3z' d^3y d^3y' V(\mathbf{z},\mathbf{z}';\mathbf{y},\mathbf{y}')\psi^*(\mathbf{z})\psi^*(\mathbf{z}')\psi(\mathbf{y})\psi(\mathbf{y}')$$

$$\cdot\left\{\int_{\mathbf{y}}^{\mathbf{z}'}\mathbf{A}ds+\int_{\mathbf{y}'}^{\mathbf{z}}\mathbf{A}ds\right\}\left\{\int_{\mathbf{y}}^{\mathbf{z}'}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu+\int_{\mathbf{y}'}^{\mathbf{z}}\delta(\mathbf{x}-\mathbf{s})\,ds_\mu\right\}. \tag{2·14}$$

Correspondingly, the Hamiltonian becomes

$$H(\mathbf{A}) = H-\frac{1}{c}\int d^3x(\mathbf{j}^p(\mathbf{x})\mathbf{A}(\mathbf{x})), \tag{2·15}$$

where the terms of higher order than e^2 are neglected.

§ 3. Free electron model with arbitrary energy spectra

We now consider an example of a free electron model with arbitrary spectra. The electric current is given by eqs. (2·12), (2·13) and (2·14) in which the effective potential V is put equal to zero. When the field operators ψ^* and ψ are expressed in terms of creation and annihilation operators $a^*_{\mathbf{p}m}$, $a_{\mathbf{p}m}$ for the electron of momentum $\hbar\mathbf{p}$ and spin direction specified by m, the Fourier transform of the current takes the form

$$j_\mu^p(\mathbf{k}) = \frac{1}{\sqrt{\Omega}}\int e^{-i\mathbf{k}\mathbf{x}} j_\mu^p(\mathbf{x})\,d\mathbf{x}$$

$$= -\frac{e}{\hbar\sqrt{\Omega}}\sum_{\mathbf{p}m} a^*_{\mathbf{p}-\mathbf{k}m}a_{\mathbf{p}m}\int_0^1 dt\frac{p_\mu-k_\mu t}{|\mathbf{p}-\mathbf{k}t|}\varepsilon'(|\mathbf{p}-\mathbf{k}t|), \tag{3·1}$$

$$j_\mu^q(\mathbf{k}) = -\frac{e^2}{\hbar^2 c\Omega}\sum_{\mathbf{p}\mathbf{q}m} a^*_{\mathbf{p}+\mathbf{q}-\mathbf{k}m}a_{\mathbf{p}m}\int_0^1 dtdt'\left\{\frac{\epsilon'(z)}{z}A_\mu(\mathbf{q})+(z\epsilon''(z)-\epsilon'(z))\frac{\mathbf{z}\cdot\mathbf{A}(\mathbf{q})}{z^3}z_\mu\right\}, \tag{3·2}$$

where Ω the normalization volume, ε the energy spectrum of an electron defined in eq. (2·2), $\mathbf{z}$ the vector $\mathbf{p}-\mathbf{k}t+\mathbf{q}t'$ and $\mathbf{A}(\mathbf{q})$ is the Fourier transform of the vector potential. Here we have chosen a straight path for the line integrals in (2·13) and (2·14), since it is simplest and seems to be reasonabe as shown in the Appendix.

In the absence of the magnetic field, the system takes the Fermi distribution with the Fermi momentum $\hbar k_F$. Then by the effect of the magnetic field the electron distribution changes through the interaction in eq. (2·15) and the current flows. The wave function for the system may be written as

$$\Psi(\mathbf{A}) = \Phi_0+\Psi_1+(\text{terms of order } \mathbf{A}^2\cdots), \tag{3·3}$$

where Ψ_1 takes the form

$$\Psi_1 = \sum_{i\neq 0}\frac{(\Phi_i|H_I|\Phi_0)}{W_0-W_i}\Phi_i, \tag{3·4}$$

according to the usual perturbation theory. H_I means the interaction energy (the second term) on the right-hand side of eq. (2·15). To the lowest order in $\mathbf{A}(\mathbf{x})$, the expectation value of the current operator is then given by

$$\langle j_\mu(\mathbf{k})\rangle = \langle j_\mu^p(\mathbf{k})\rangle + \langle j_\mu^d(\mathbf{k})\rangle, \tag{3·5}$$

$$\langle j_\mu^p(\mathbf{k})\rangle = (\Psi_1 j_\mu^p(\mathbf{k})\Phi_0) + (\Phi_0 j_\mu^p(\mathbf{k})\Psi_1), \tag{3·6}$$

$$\langle j_\mu^d(\mathbf{k})\rangle = (\Phi_0 j_\mu^d(\mathbf{k})\Phi_0), \tag{3·7}$$

By means of eqs. (3·1), (3·4) and (3·6), we obtain for the paramagnetic part of the current density

$$\begin{aligned}\langle j_\mu^p(\mathbf{k})\rangle &= -\frac{1}{c}\sum_\lambda \sum_{i\neq 0} \{(\Phi_0|j_\mu^p(\mathbf{k})|\Phi_i)(\Phi_i|j_\lambda^p(-\mathbf{k})|\Phi_0) \\ &\quad + (\Phi_0|j_\lambda^p(-\mathbf{k})|\Phi_i)(\Phi_i|j_\mu^p(\mathbf{k})|\Phi_0)\} A_\lambda(\mathbf{k})/(W_0-W_i) \\ &= \frac{e^2}{2\pi^3\hbar^2 c}\sum_\lambda A_\lambda(\mathbf{k}) \int_{\substack{|\mathbf{p}-\mathbf{k}|>k_F \\ p<k_F}} d\mathbf{p} \int_0^1 dt dt' \frac{(p_\lambda - k_\lambda t')(p_\mu - k_\mu t)}{|\mathbf{p}-\mathbf{k}t'||\mathbf{p}-\mathbf{k}t|} \\ &\quad \cdot \frac{\varepsilon'(|\mathbf{p}-\mathbf{k}t'|)\varepsilon'(|\mathbf{p}-\mathbf{k}t|)}{\varepsilon(\mathbf{p}-\mathbf{k})-\varepsilon(p)}.\end{aligned} \tag{3·8}$$

We can calculate the right-hand side of (3·8) in the limit $k\to 0$, by expanding the integrand in the power series of $\mathbf{k}$. The result is given by

$$\begin{aligned}\langle j_\mu^p(\mathbf{k})\rangle = \frac{e^2}{\pi^2\hbar^2 c}\sum_\lambda \Big[&\frac{k_F^2\varepsilon'(k_F)}{3}\delta_{\lambda\mu} + \frac{k^2}{180}\{k_F^2\varepsilon'''(k_F) - 8k_F\varepsilon''(k_F) - 7\varepsilon'(k_F)\} \\ &\cdot\Big(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2}\Big) + \frac{k^2}{60}\{k_F^2\varepsilon'''(k_F) + 2k_F\varepsilon''(k_F) - 2\varepsilon'(k_F)\}\frac{k_\lambda k_\mu}{k^2}\Big] A_\lambda(\mathbf{k}),\end{aligned} \tag{3·9}$$

up to the second order of $\mathbf{k}$.

On the other hand, we have, for the diamagnetic part,

$$\begin{aligned}\langle j_\mu^d(\mathbf{k})\rangle = -\frac{e^2}{2\pi^3\hbar^2 c}\int_{p<k_F} d\mathbf{p}\int_0^1 dt(1-t)\Big\{&\frac{\epsilon'(z)}{z}A_\mu(\mathbf{k}) \\ &+ (z\epsilon''(z)-\epsilon'(z))\frac{(p_\mu - k_\mu t)}{z^3}\sum_\lambda (p_\lambda - k_\lambda t)A_\lambda(\mathbf{k})\Big\}\end{aligned} \tag{3·10}$$

where z means the quantity $|\mathbf{p}-\mathbf{k}t|$. Following the same method applied to eq. (3·8), the integral can be estimated in the limit $k\to 0$ and gives the result

$$\begin{aligned}\langle j_\mu^d(\mathbf{k})\rangle = -\frac{e^2}{\pi^2\hbar^2 c}\sum_\lambda \Big[&\frac{k_F^2\varepsilon'(k_F)}{3}\delta_{\lambda\mu} + \frac{k^2}{180}\{k_F^2\varepsilon'''(k_F) + 2k_F\varepsilon''(k_F) - 2\varepsilon'(k_F)\} \\ &\cdot\Big(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2}\Big) + \frac{k^2}{60}\{k_F^2\varepsilon'''(k_F) + 2k_F\varepsilon''(k_F) - 2\varepsilon'(k_F)\}\frac{k_\lambda k_\mu}{k^2}\Big] A_\lambda(\mathbf{k}).\end{aligned} \tag{3·11}$$

Thus, we see that the current density vanishes in the limit $k \to 0$, that is, the current does not flow in a free electron system with arbitrary continuous spectra. This is quite reasonable, since the continuous energy spectrum of the system will not secure the stability of the electric current. Our result is a generalization of the well-known fact that the free electron system does not give the Meissner effect.

Now it becomes interesting to inquire if the system with some discontinuous energy spectrum may be able to account for the Meissner effect, guaranteeing the stability of the current. In fact, Bardeen[2] has shown the Meissner effect with his simple energy gap model by means of a non-gauge invariant discussion. We will reconsider this from our gauge invariant standpoint.

The Hamiltonian of the energy gap model may be written as

$$H=\sum_{\substack{k>k_F \\ m}}\left(\frac{k^2}{2m}+\varepsilon\right)a^*_{\mathbf{k}m}a_{\mathbf{k}m}+\sum_{\substack{k\leqq k_F \\ m}}\frac{k^2}{2m}a^*_{\mathbf{k}m}a_{\mathbf{k}m}\,, \tag{3·12}$$

where ε is the finite energy gap necessary for the excitation of an electron from below the Fermi surface. In eq. (3·12), the term proportional to ε may be considered as an interaction term and expressed in the form

$$H'=\varepsilon\int F(\mathbf{x}-\mathbf{y})\psi^*(\mathbf{x})\psi(\mathbf{y})\,d\mathbf{x}d\mathbf{y}, \tag{3·13}$$

with

$$F(\mathbf{x})=\frac{1}{\Omega}\sum_{k>k_F}e^{i\mathbf{k}\mathbf{x}}. \tag{3·14}$$

Then, following the procedure given in Section 2 one obtains the current density operators and the magnetic interaction energy as follows;

$$j^p_\mu(\mathbf{k})=-\frac{e\hbar}{2m\sqrt{\Omega}}\sum_{\mathbf{p}m}\left\{2p_\mu-k_\mu+\frac{2m\varepsilon}{\hbar^2k_F}\int_0^1 dt\,(p_\mu-k_\mu t)\,\delta(|\mathbf{p}-\mathbf{k}t|-k_F)\right\}a^*_{\mathbf{p}-\mathbf{k}m}a_{\mathbf{p}m}, \tag{3·15}$$

$$j^a_\mu(\mathbf{k})=-\frac{e^2}{mc\Omega}\sum_{\mathbf{p}\mathbf{q}m}a^*_{\mathbf{p}+\mathbf{q}-\mathbf{k}m}a_{\mathbf{p}m}\left\{A_\mu(\mathbf{q})+\frac{m\varepsilon}{\hbar^2}\frac{\partial}{\partial p_\mu}\left(\mathbf{A}(\mathbf{q})\frac{\partial}{\partial\mathbf{p}}\right)\right\}\int_0^1 dt\,dt' \cdot\epsilon(|\mathbf{p}-\mathbf{k}t+\mathbf{q}t'|)\Big\}\,, \tag{3·16}$$

$$H_I=-\frac{1}{c}\sum_{\lambda\mathbf{k}}j^p_\lambda(-\mathbf{k})A_\lambda(\mathbf{k}), \tag{3·17}$$

up to first order of $\mathbf{A}(\mathbf{x})$. Here, we have used a straight path for the line integrals in eqs. (2·13) and (2·14), and $\epsilon(p)$ is equal to unity for $p>k_F$ and zero otherwise.

Applying the same method as in the former example, we can obtain the expectation value of the electric current. The paramagnetic part of the current density is given by

$$\langle j_\mu^p(\mathbf{k})\rangle = \frac{e^2}{4\pi^3 mc}\sum_\lambda' A_\lambda(\mathbf{k}) \int_{\substack{|\mathbf{p}+\mathbf{k}|>k_F \\ p<k_F}} d\mathbf{p}\ \frac{1}{2\mathbf{pk}+k^2+(2m\varepsilon/\hbar^2)}$$
$$\cdot\left\{2p_\lambda+k_\lambda+\frac{2m\varepsilon/\hbar^2}{\mathbf{pk}+k^2x}(p_\lambda+k_\lambda x)\right\}\cdot\left\{2p_\mu+k_\mu+\frac{2m\varepsilon/\hbar^2}{\mathbf{pk}+k^2x}(p_\mu+k_\mu x)\right\}, \tag{3·18}$$

where

$$x=\{-\mathbf{pk}+\nu\sqrt{(\mathbf{pk})^2-k^2p^2+k^2k_F^2}\}/k^2. \tag{3·19}$$

If we introduce the simple notations as

$$\alpha=k/2k_F,\quad \beta=m\varepsilon/\hbar^2kk_F, \tag{3·20}$$

the right-hand side of eq. (3·18) gives the result

$$\langle j_\mu^p(\mathbf{k})\rangle = \frac{e^2k_F^3}{16\pi^2 mc}\Big[-2\alpha^3+\frac{10}{3}\alpha+2\alpha\beta\Big(-\frac{19}{3}+\frac{5}{9}\alpha^2\Big)+\frac{2}{3}\alpha\beta^2-\frac{2}{3}\alpha\beta^3$$
$$+8\alpha\beta\Big(1+\frac{1}{3}\alpha^2\Big)\log\frac{1+\beta-\alpha}{\beta}$$
$$+\Big\{1+\frac{16}{3}\beta+4\beta^2+\frac{1}{3}\beta^4+4\alpha\beta+\frac{4}{3}\alpha^3\beta-2\alpha^2+\alpha^4\Big\}\log\frac{1+\beta+\alpha}{1+\beta-\alpha}$$
$$+4\beta^2\int_{-\alpha}^{1-2\alpha}dq\ \frac{\log(q+2\alpha)}{q+\alpha+\beta}-4\beta^2\int_{-\alpha}^{1}dq\ \frac{\log|q|}{q+\alpha+\beta}\Big]\sum_\lambda'\Big(\delta_{\lambda\mu}-\frac{k_\lambda k_\mu}{k^2}\Big)A_\lambda(\mathbf{k})$$
$$+\frac{e^2k_F^3}{m\pi^2c}\Big\{\frac{1}{3}+\beta\Big(\frac{1}{2}-\frac{1}{6}\alpha^2\Big)\Big\}\sum_\lambda'\frac{k_\lambda k_\mu}{k^2}A_\lambda(\mathbf{k}). \tag{3·21}$$

On the other hand, we have for the diamagnetic part

$$\langle j_\mu^d(\mathbf{k})\rangle = -\frac{e^2k_F^3}{3m\pi^2c}A_\mu(\mathbf{k})-\frac{e^2\varepsilon}{2\pi^3\hbar^2c}\int_{\substack{p<k_F \\ |\mathbf{p}+\mathbf{k}|>k_F}} d\mathbf{p}\Big[(1-x)\frac{\delta_{\lambda\mu}}{k_F}+\frac{(p_\lambda+k_\lambda x)(p_\mu+k_\mu x)}{k_F(\mathbf{pk}+k^2x)}$$
$$-\frac{(1-x)}{k_F}\Big\{\frac{1}{\mathbf{pk}+k^2x}\{k_\lambda(p_\mu+k_\mu x)+k_\mu(p_\lambda+k_\lambda x)\}$$
$$-\frac{(p_\lambda+k_\lambda x)(p_\mu+k_\mu x)}{(\mathbf{pk}+k^2x)^2}k^2\Big\}\Big]\frac{k_F}{\mathbf{pk}+k^2x}A_\lambda(\mathbf{k}) \tag{3·22}$$
$$=-\frac{e^2k_F^3}{3m\pi^2c}A_\mu(\mathbf{k})-\frac{e^2k_F^3}{m\pi^2c}\beta\sum_\lambda' A_\lambda(\mathbf{k})\Big[-\frac{1}{2}\Big(\log\alpha+\frac{1}{2}-\frac{\alpha^2}{2}\Big)\Big(\delta_{\lambda\mu}-\frac{k_\lambda k_\mu}{k^2}\Big)$$
$$+\Big(\frac{1}{2}-\frac{1}{6}\alpha^2\Big)\frac{k_\lambda k_\mu}{k^2}\Big]. \tag{3·23}$$

Finally one obtains the total current density by summing eqs. (3·21) and (3·23), non-gauge invariant terms being exactly cancelled. Moreover, the term of London's diamagnetism which played an essential role to account for the electro-

magnetic properties of the superconductors in Bardeen's simple energy gap model is shown to vanish in our gauge invariant treatment. On the other hand, the current has a peculiar singularity in the limit $k \to 0$. We can see this by expanding the expressions of eqs. (3·21) and (3·23) in the power series of α and β^{-1}, that is, when k is smaller than k_F and $m\varepsilon/\hbar^2 k_F$. Then one obtains

$$\langle j_\mu(\mathbf{k})\rangle \simeq \frac{3e^2 n}{8mc}\beta\left[4-\frac{13}{9}\alpha^2\right]\sum_\lambda\left(\delta_{\lambda\mu}-\frac{k_\lambda k_\mu}{k^2}\right)A_\lambda(\mathbf{k}), \tag{3·24}$$

where n is the number of electrons in the unit volume. This means that when the external magnetic field is applied to our system, an infinite electric current flows inside the system. Such an unphysical solution clearly indicates the break down of the simple energy gap model. Thus it seems to be almost conclusive that any one electron model with arbitrary continuous or discontinuous spectra may not account for the Meissner effect. Some other types of correlations between electrons would play an essential role in the theory of superconductivity as was first pointed out by Blatt, Butler and Schafroth.[8]

§ 4. Acknowledgement

We would like to express our thanks to Professor S. Tomonaga and Professor T. Miyazima for their kind interest in this work. We have been benefitted from illuminating discussions with Mr. F. Takano.

Appendix
Gauge invariance in the free electron model

We would like to show that the magnetic interaction introduced in eq. (2·9) is identical with the conventional in the free electron model with $\varepsilon(p)=\hbar^2 p^2/2m$. In this case the Hamiltonian becomes

$$\begin{aligned} H(\mathbf{A}) &= \int d^3x' d^3x f(\mathbf{x}'-\mathbf{x})\psi^*(\mathbf{x}')\psi(\mathbf{x})\exp\left[-\frac{ie}{\hbar c}\int_{\mathbf{x}}^{\mathbf{x}'}\mathbf{A}(\mathbf{s})\,d\mathbf{s}\right] \\ &= \frac{\hbar^2}{2m}\int d^3x' d^3x\left[\nabla_{x'}\nabla_{x'}\delta(\mathbf{x}-\mathbf{x}')\right]\psi^*(\mathbf{x}')\psi(\mathbf{x})\exp\left[-\frac{ie}{\hbar c}\int_{\mathbf{x}}^{\mathbf{x}'}\mathbf{A}(\mathbf{s})\,d\mathbf{s}\right] \\ &= \frac{\hbar^2}{2m}\int d^3x' d^3x\,\delta(\mathbf{x}-\mathbf{x}')\nabla_{x'}\nabla_{x'}\left\{\psi^*(\mathbf{x}')\psi(\mathbf{x})\exp\left[-\frac{ie}{\hbar c}\int_{\mathbf{x}}^{\mathbf{x}'}\mathbf{A}(\mathbf{s})\,d\mathbf{s}\right]\right\}. \end{aligned} \tag{A·1}$$

If we choose a straight line for the path integral, we may write

$$\int_{\mathbf{x}}^{\mathbf{x}'}\mathbf{A}(\mathbf{s})\,d\mathbf{s}=\int_0^1 dt\,(\mathbf{x}'-\mathbf{x})\,\mathbf{A}((\mathbf{x}'-\mathbf{x})t+\mathbf{x})\equiv G(\mathbf{x},\mathbf{x}'), \tag{A·2}$$

and

$$\nabla_x G(\mathbf{x}, \mathbf{x}') = -\int_0^1 dt \mathbf{A}((\mathbf{x}'-\mathbf{x})t+\mathbf{x}) + \int_0^1 dt(1-t)(\mathbf{x}'-\mathbf{x})\cdot\nabla_z \mathbf{A}(\mathbf{z}), \quad (A\cdot 3)$$

$$\nabla_{x'} G(\mathbf{x}, \mathbf{x}') = \int_0^1 dt \mathbf{A}((\mathbf{x}'-\mathbf{x})t+\mathbf{x}) + \int_0^1 dt\, t(\mathbf{x}'-\mathbf{x})\cdot\nabla_z \mathbf{A}(\mathbf{z}), \quad (A\cdot 4)$$

$$\nabla_x \nabla_{x'} G(\mathbf{x}, \mathbf{x}') = \int_0^1 dt(1-2t)\,\mathrm{div}\,\mathbf{A}(\mathbf{z}) + \int_0^1 dt\, t(1-t)(\mathbf{x}'-\mathbf{x})\cdot\nabla_z^2 \mathbf{A}(\mathbf{z}), \quad (A\cdot 5)$$

where $\mathbf{z}=(\mathbf{x}'-\mathbf{x})t+\mathbf{x}$ and $\nabla\mathbf{A}$ means to take gradient of each component of $\mathbf{A}$.

Making use of the above equations, we have

$$\begin{aligned}
H(\mathbf{A}) = \frac{\hbar^2}{2m}\int d^3x' d^3x\, \delta(\mathbf{x}-\mathbf{x}') \Big\{ & \nabla\psi^*(\mathbf{x}')\cdot\nabla\psi(\mathbf{x})\, e^{-\frac{ie}{\hbar c}G} \\
& - \frac{ie}{\hbar c}\psi^*(\mathbf{x}')\nabla\psi(\mathbf{x})(\nabla_{x'}G(\mathbf{x}, \mathbf{x}'))e^{-\frac{ie}{\hbar c}G} - \frac{ie}{\hbar c}\nabla_{x'}\psi^*(\mathbf{x}')\cdot\psi(\mathbf{x})(\nabla_x G(\mathbf{x}, \mathbf{x}'))e^{-\frac{ie}{\hbar c}G} \\
& - \frac{ie}{\hbar c}\psi^*(\mathbf{x}')\psi(\mathbf{x})(\nabla_x\nabla_{x'}G(\mathbf{x}, \mathbf{x}'))e^{-\frac{ie}{\hbar c}G} \\
& - \frac{e^2}{\hbar^2 c^2}\psi^*(\mathbf{x}')\psi(\mathbf{x})(\nabla_x G(\mathbf{x}, \mathbf{x}'))(\nabla_{x'}G(\mathbf{x}, \mathbf{x}'))e^{-\frac{ie}{\hbar c}G}\Big\} \\
= \frac{\hbar^2}{2m}\int d\mathbf{x} \Big\{ & \nabla\psi^*(\mathbf{x})\nabla\psi(\mathbf{x}) - \frac{ie}{\hbar c}\psi^*(\mathbf{x})\nabla\psi(\mathbf{x})\mathbf{A}(\mathbf{x}) + \frac{ie}{\hbar c}\nabla\psi^*(\mathbf{x})\cdot\psi(\mathbf{x})\mathbf{A}(\mathbf{x}) \\
& + \frac{e^2}{\hbar^2 c^2}\mathbf{A}^2(\mathbf{x})\psi^*(\mathbf{x})\psi(\mathbf{x})\Big\} \\
= \frac{\hbar^2}{2m}\int d\mathbf{x} & \left(\nabla - \frac{ie}{\hbar c}\mathbf{A}(\mathbf{x})\right)\psi^*(\mathbf{x})\cdot\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}(\mathbf{x})\right)\psi(\mathbf{x}), \qquad (A\cdot 6)
\end{aligned}$$

where we noticed

$$\lim_{\mathbf{x}\to\mathbf{x}'} \nabla_{x'}\nabla_x G(\mathbf{x}, \mathbf{x}') = 0. \quad (A\cdot 7)$$

Eq. (A·6) is the conventional Hamiltonian for the free electron system in the presence of a magnetic field.

It is difficult to see what will happen when we change the path. In this case $G(\mathbf{x}, \mathbf{x}')$ is changed into

$$G'(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}, \mathbf{x}') + \phi(\mathbf{x}, \mathbf{x}'), \quad (A\cdot 8)$$

where $\phi(\mathbf{x}, \mathbf{x}')$ is a magnetic flux passing through inside a contour made up by a straight line $\overline{\mathbf{x}\,\mathbf{x}'}$ and a new path connecting $\mathbf{x}$ and $\mathbf{x}'$. If the new paths are chosen such that $\phi(\mathbf{x}, \mathbf{x}')$ and its first and second derivatives tend to zero as $\mathbf{x}$ approaches $\mathbf{x}'$, then we will get the same result as eq. (A·6).

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Letters to the Editor

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On the Geometrical Structure of Benzene-Iodine Complex

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October 22, 1958

It is a very interesting problem whether the strutcure of the benzene-iodine complex belongs to the symmetry of C_{2v} or C_6. The recent experiment[1] of the infrared spectra of this complex supports the C_6 symmetry. On the other hand, Mulliken[2] has proposed the C_{2v} symmetry of this complex to be energetically most stable from the charge-transfer mechanism. It is needless to say that the charge-transfer of this complex occurs from benzene to iodine. The stabilization energy of this complex is expressed as eq. (1), by the second order perturbation method,

$$\Delta E = -2\Sigma_i H_{Bi,I}^2/(H_{Bi}-H_I), \quad (1)$$

where H_{Bi} is the orbital energy of i-th occupied orbital of benzene and H_I is that of the vacant orbital of iodine. Mulliken has presented the geometrical structure of this complex as Fig. a, "R" model, considering the symmetrical properties of molecular orbitals of both molecules. In this case the denominator of eq. (1) is equal to the difference between the orbital energy of b_1 orbital of benzene and that of $b_2\sigma_u$ orbital of iodine, and this mechanism of charge-transfer is favourable in the sense that an electron in the highest occupied orbital of benzene transfers to the lowest vacant orbital of iodine. However, the precise estimation of interaction term $H_{Bi,I}$ will deny this model. As expressed in Fig. a, the atomic p orbital constructing the $b_2\sigma_u$ molecular orbital of iodine shows + and − signs on the either side of the line joining the adjacent benzene carbon atoms. Here notice the interatomic distance of iodine molecule nearly equals to the width of the benzene ring. If we assume $b_2\sigma_u$ molecular orbital of

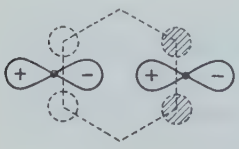

Fig. a

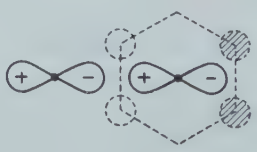

Fig. b

Fig. c

The solid line represents $b_2\sigma_u$ molecular orbital of iodine. The dotted line represents b_1 molecular orbital benzene in Fig. a and Fig. b, and a_1 molecular orbital in Fig. c.

iodine to be constructed by pure p atomic orbitals of iodine ($s-p$ hybridization not considered), $H_{Bi,I}$ is nearly zero, and ΔE will vanish.

Therefore, the author cannot but consider that the translated structure of Fig. b or the perpendicular structure of symmetry of C_6 as Fig. c is the suitable form of the benzene-iodine complex. In the case of Fig. c, the charge-transfer occurs from a_1 molecular orbital of benzene to the $b_2\sigma_u$ molecular orbital of iodine, and the denominator of eq. (1) of this case is larger than that of the case of Fig. b. But the interaction term $H_{Bi,I}$ of Fig. c seems to be so much larger than that of Fig. b, that it excells in the losing of the separation between orbital energies. The interaction term $H_{Bi,I}$ needs to be calculated by separating it into σ- and π-types,[3] according to the orientations of the atomic orbitals of benzene and iodine molecules. The precise calculation may support the C_6 symmetry of this complex. The details will be reported in the near future.

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On the Possibility of Helium Burning in Giant Stars in a Globular Cluster

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The problem of what phenomena occur in the innermost part of stars at the tip of the giant branch of a globular cluster is very important from the point of view of the evolution of stars and of nucleogenesis. Hoyle and Schwarzchild[1] treated the problem for the first time, using their stellar models with partially degenerate isothermal cores, and concluded that the central temperature of the helium core of the star, near the tip of the giant branch, would rise much higher than that of the hydrogen burning shell on account of the released gravitational energy and that the helium core would rapidly become nondegenerate owing to helium burning, newly set up by high temperature of the core. By means of a skillful method, for the star of 1.2 solar masses in a globular cluster such as M3, they estimated the total luminosity to be about 5,000 times the luminosity of the sun and the core mass necessary for the central temperature to rise enough for the onset of helium burning to be about 52 per cent of the total mass. However, it has been found by us that they committed an error in their approximation and so their numerical result must be abandoned. According to our computation, the rise of the central temperature of the core is so little that helium burning is quite impossible even for a star having a larger core mass and more luminous than the above star.

In order to consider the gravitational energy released in each part of the core, an exact equation between energy transport and energy production must be used rather than the approximated one generally used in static stellar models. From the law of energy conservation, the following equation is obtained for the

spherically symmetric structure:

$$\frac{3}{2}\rho^{2/3}\frac{d}{dt}\left(\frac{P}{\rho^{5/3}}\right)=\varepsilon-\frac{dL_r}{dM_r}$$

where t, ρ, P, and ε are respectively time, density, pressure and nuclear energy production per unit mass and unit time; L_r is the energy flux transported through the spherical surface of radius r; M_r is the mass within the sphere of radius r. Recently, Schwarzschild[2] has proposed to construct a more exact model for the gravitationally contracting stage than those ever calculated under a rough approximation, using this equation and the rest of basic equations generally used in stellar models. In our calculation, the left side of the above equation is computed from the sequence of models for the isothermal core obtained by Hoyle and Schwarzschild, by means of simple numerical differenciation. Then it is integrated numerically to give L_r as a function of M_r, putting $\varepsilon=0$. L_r thus obtained is not exact, because the existence of nonvanishing L_r means the existence of temperature gradient, contradicting the isothermal structure of the core. However, this L_r may be expected to give the upper limit of true L_r as will be shown below. Moreover, this approximation exempts us from a tremendous volume of the calculation necessary for the exact estimation of L_r which only an electronic computer can do well. The temperature of the hydrogen burning shell is controlled almost only by the luminosity and little affected by other conditions, so that the growth of temperature gradient in the helium core means the rise of the central temperature. This increase of the temperature and the higher pressure gradient accompanied will prevent the gravitational contraction of the core. This is the reason why L_r obtained from models with the isothermal structure may be thought of as the upper limit of true L_r. From L_r, the temperature in the core is calculated through the following equation,

$$dT/dM_r=\{3/4ac(4\pi)^2\}\kappa L_r/T^3r^4,$$

where opacity κ is given as

$$\kappa=0.19+3.68\times10^{22}\rho/T^{3.5}$$

for the outer layer of the core where electron scattering and free-free transition contribute to opacity, while

$$\kappa=4.26\times10^{-6}(T\ \rho)^{1.8571}\ \rho$$

for the inner part of the core where degenerate electrons contribute to thermal conduction. These opacity formulas are same as what Haselgrove and Hoyle[3] used in their calculation of models of type II stars. The numerical integration of obtaining temperature T is started from the surface of the core, by using r given by the suitable model for the isothermal core by Hoyle and Schwarzschild, and is stopped at some point where thermal conduction sufficiently short circuits the opacity caused by photoelectric processes to vanish temperature gradient.

As a result of the computation, it is found that the central temperature of the core does not exceed 56×10^6 °K for the star of 1.2 solar masses, having a helium core of 0.61 the total mass, the shell temperature of which is 50×10^6 °K, giving the luminosity of 13,500 times that of the sun. It is also known

that, for the star having a helium core smaller than about 0.45 the total mass, the temperature difference between the shell and the center is below 2 per cent of the shell temperature; hence the isothermal structure may be assumed in good approximation. As the temperature necessary for the core becoming non-degenerate by helium burning, is above 110×10^6 °K from our estimation, it is apparent that the above star is far from the possibility of helium burning and still it should go up to the giant branch.

Thus, the problem of how the star evolves hereafter, has been presented before us, and its thorough solution must be waited for until the model of the star with a larger helium core is constructed. The luminosity of 13,500 times that of the sun is about 3 times as large as that of the brightest star in M3, due to the increasing discrepancy, as the core gets larger, between the luminosity of Hoyle-Schwarzschild's model and that observed in M3. Since the central temperature is far below 110×10^6 °K even in such a bright star, it is expected that helium burning will not occur until the star consumes a considerable part of the total hydrogen.[4] The result of further calculations with respect to the model of the star with a larger core will be published in this journal before long.

1) F. Hoyle and M. Schwarzschild, Ap. J. Supple. 2 No. 13 (1955).
2) M. Schwarzschild, *Structure and Evolution of the Stars*, Chap. 3, (1958), Princeton.
3) C. B. Haselgrove and F. Hoyle, M. N. **116** (1956), 515.
4) A. Sandage, Ap. J. **126** (1957), 326.

On the Gauge-Invariant Meissner Effect

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November 26, 1958

As suggested by Bardeen[1] and emphasized by Anderson,[2] it is essential to take account of plasma excitations in order to obtain the gauge-invariant Meissner effect. Indeed, Blatt and Matsubara[3] have recently shown that the gauge-invariant Meissner effect results even from Fröhlich's Hamiltonian if longitudinal collective excitations (phonon like in this case) are included. To discuss the problem properly, however, we should go back to the Bardeen-Pines theory[4] on decoupling of the plasma mode in the electron-ion system. There the Bohm-Pines method has been applied, but Tomonaga's method[5] seems to be more suitable for our purpose.

Tomonaga's method leads not only to the partition of the Hamiltonian, but also to that of the electron momentum density into collective and individual modes. To the first order of collective variables, the momentum density is thus written as

$$\boldsymbol{p}(k)=\boldsymbol{p}_0(k)-[mn]^{1/2}(i\boldsymbol{k}/k)P_{-k} \quad (1)$$

where P_k is the canonical momentum of the plasma mode and

$$\boldsymbol{p}_0(k)=\tfrac{1}{2}\sum_i\{e^{i\boldsymbol{k}\cdot\boldsymbol{x}_i}(\boldsymbol{p}_i-(\boldsymbol{k}/k^2)(\boldsymbol{k}\cdot\boldsymbol{p}_i))+(\boldsymbol{p}_i-(\boldsymbol{k}/k^2)(\boldsymbol{k}\cdot\boldsymbol{p}_i))e^{i\boldsymbol{k}\cdot\boldsymbol{x}_i}\}$$

is the individual part. We thus see that the latter part is coupled only with

the transverse part of the vector potential in the magnetic perturbation $-(e/mc)\boldsymbol{A}(\boldsymbol{k})\cdot\boldsymbol{p}(-\boldsymbol{k})$. Applying B C S theory[6] to the individual mode, the diamagnetic current induced by the magnetic field is given as

$$\sum_{\nu=1}^{3}\{-(ne^2/mc)\delta_{\mu\nu}+(e^2/m^2c)P_{\mu\nu}(\boldsymbol{k})\}A_\nu(\boldsymbol{k}) \qquad (2)$$

with

$$P_{\mu\nu}(\boldsymbol{k})=mn(k_\mu k_\nu/k^2)+2h^2\sum_{\boldsymbol{l}}\{l_\mu l_\nu-(\boldsymbol{k}\cdot\boldsymbol{l})(k_\mu l_\nu/k^2)\}\times L(\varepsilon_{\boldsymbol{l}},\ \varepsilon_{\boldsymbol{l}+\boldsymbol{k}}). \qquad (3)$$

Here the first term is the contribution from the plasma mode, and $L(\varepsilon, \varepsilon')$ is the function defined by B C S. In superconductors we have the finite energy gap for individual excitations and $L(\varepsilon_{\boldsymbol{l}}, \varepsilon_{\boldsymbol{l}+\boldsymbol{k}})\to 0$ as $k\to 0$ at 0°K, so that we obtain the gauge-invariant Meissner effect.

On the other hand, in normal metals, i. e., in the limit of vanishing energy gap, we have

$$L(\varepsilon_{\boldsymbol{l}},\ \varepsilon_{\boldsymbol{l}+\boldsymbol{k}})=-\frac{\partial f(\varepsilon_{\boldsymbol{l}})}{\partial\varepsilon}-\frac{1}{2}\frac{\partial^2 f(\varepsilon_{\boldsymbol{l}})}{\partial\varepsilon^2}(\varepsilon_{\boldsymbol{l}+\boldsymbol{k}}-\varepsilon_{\boldsymbol{l}})-\cdots\cdots,$$

where f is the normal Fermi distribution at 0°K. The first term on the right makes the contribution $mn(\delta_{\mu\nu}-(k_\mu k_\nu/k^2))$ to the individual part of (3). The first term of this contribution cancels the first term of (2), whereas the second term just cancels the contribution from the plasma mode. Hence only the normal diamagnetism remains in this case. The argument is similar at a finite temperature.

We believe that in this way Bardeen's original suggestion is confirmed to be correct and also that the non gauge-invariant term in B C S theory introduces a minor error in the normal part of the diamagnetic current. The detailed disdussion will be made in a forthcomming paper.

I should like to express thanks to Professor Matsubara and Dr. Yosida for stimulating discussions.

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2) P. W. Anderson, Phys. Rev. **110** (1958), 985.
3) J. M. Blatt and T. Matsubara, Prog. Theor. Phys. to be published.
4) J. Bardeen and D. Pines, Phys. Rev. **99** (1955), 1140.
5) S. Tomonaga, Prog. Theor. Phys. **13** (1955), 482.
T. Kasuya, Busseiron Kenkyu, **2-2** (1957), 490.
6) J. Bardeen, L. N. Cooper, J. R. Schrieffer, Phys. Rev. **108** (1957), 1175.

On Critical Magnetic Fields of Superconductors

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December 1, 1958

Bogoljubov[1] has applied Landau's criterion of superfluidity to superconducting metals. Thus the perfect conductivity is a direct consequence of the fact

that electronic excitation quanta have a finite energy gap. The critical velocity is given by

$$v_c = \varepsilon_0 / p_F = (\hbar / \pi m \xi) \qquad (1)$$

where ε_0 is the energy gap, p_F the electron momentum at the Fermi surface, and ξ the Pippard coherence length as given by Bardeen et al.[2]

The present note is a remark on the magnitude of (1). Suppose a wire in which the persistent current is flowing with the critical velocity (1). We assume the Meissner effect so that the current flows within a thin surface layer of width λ. Then the magnetic field produced by the current at the surface of the wire is given as

$$(H_c^2/8\pi) = (\lambda/\lambda_L)\,(n\varepsilon_0^2/4E_F) \qquad (2)$$

where n is the electron density, E_F the Fermi energy, and $\lambda_L = [mc^2/4\pi ne^2]^{1/2}$ the London value of the penetration depth. The critical field obtained in this way is a dynamical one, so to say, and to be distinguished from the usual static value determined by thermodynamics of perfect diamagnetism. The later is given as[2]

$$(H_0^2/8\pi) = (N(0)/2)\,\varepsilon_0^2 \qquad (3)$$

where $N(0)$ is the state density at the Fermi surface. The static H_0 is primarily the critical field at which the Meissner effect is destroyed. From (2) and (3), however, we obtain

$$(H_c/H_0)^2 = (\lambda/\lambda_L)^2 (n/2N(0)E_F) = (3/2)\,(\lambda/\lambda_L)^2. \qquad (4)$$

Since λ/λ_L is somewhat between 1 and 3, the dynamical critical field is a few times as large as the static one. In other words, in agreement with observed facts, the parsistent current is destroyed together with the Meissner effect at H_0, before the critical velocity (1) is reached.

1) N. N. Bogoljubov, JETP **34** (1958), 58.
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Progress of Theoretical Physics, Vol. 21, No. 3, March 1959

Theory of Classical Fluids: Hyper-Netted Chain Approximation. II

——*Formulation for Multi-Component Systems*——

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Formulae of the free energy and the radial distribution function in the hyper-netted chain approximation are obtained for multi-component systems. The expansion formulae by means of the hyper-netted chains are also given for them, and also for the potentials of average force of n particles.

Another expansion scheme is given which is suitable for comparison with the theories of ionic solutions in the past. The ring and watermelon approximations in this scheme are discussed in detail for the Coulombic potential and the Coulombic potential with hard core.

§ 1. Introduction

We have presented in the previous paper[1] the hyper-netted chain approximation for the case of one-component system. There we have made comparison of our results with the results obtained in the past for the ionic solutions. However the ionic solutions are essentially multi-component systems of positive and negative ions even when we may replace the effect of the solvent by dielectric constant. In this paper, we will derive the formulae of the free energy and the radial distribution functions for multi-component systems in the approximation presented in the previous paper, so as to be applicable for the systems of ionic solutions or of mixtures of fluids.

In § 2, the formula of the free energy in our approximation will be derived, with the aid of a functional $L(\alpha, \beta, \cdots, \tau; F_{\lambda\mu}^{(n)}(\boldsymbol{k}))$. The explicit form of this functional will be obtained in § 3, with the aid of a lemma, the proof of which is given in Appendix I. The formula of the radial distribution function in our approximation is given in § 4. The expansion formulae for the free energy and the radial distribution function by means of the "hyper-netted chains", containing the results in § 2 and § 4 as the leading terms, are given in § 5. The corresponding formula for the potential of average force of n particles—the logarithm of the n particle distribution function—is given in Appendix II. In § 6, another expansion scheme is presented which is suited for the purpose of comparison with the theories of ionic solutions in the past.

§ 2. Free energy

We consider a uniform fluid in a volume V and at temperature T, consisting

of N particles in which N_α are of the species α, N_β are of the species β, $\cdots$, and N_τ are of the species τ. We assume that the potential energy of our system is given by the sum of potential energies between a pair of particles:

$$\frac{1}{2}\sum_{\substack{i=1}}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\phi_{\nu_i\nu_j}(r_{ij}) \tag{1}$$

where ν_i denotes the species of the particle at $\boldsymbol{r}_i$.

The virial expansion formulae of the free energy A is given by[2]

$$\frac{A}{VkT}=\sum_{\nu=\alpha}^{\tau}\rho_\nu\ln\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\frac{\rho_\nu}{e}+\frac{A'}{VkT} \tag{2}$$

$$-\frac{A'}{VkT}=\sum_{n_\alpha=0}^{\infty}\cdots\sum_{n_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu^{n_\nu}}{n_\nu!}\cdot\frac{1}{V}\int\int\cdots\int d\boldsymbol{r}_1\cdots d\boldsymbol{r}_{n_\alpha+\cdots+n_\tau}\sum_{n_\alpha+\cdots+n_\tau\geq i>j\geq 1}\prod f_{ij}^{(0)} \tag{2'}$$

Sum over all products which are more than singly connected.

with

$$f_{ij}^{(0)}=f_{\nu_i\nu_j}^{(0)}(r_{ij})=\exp\{-\phi_{\nu_i\nu_j}(r_{ij})/kT\}-1, \tag{3}$$

where $\rho_\nu=N_\nu/V$ $(\nu=\alpha, \beta, \cdots, \tau)$ and $\rho=\sum_{\nu=\alpha}^{\tau}\rho_\nu=N/V$.

As in I, we make a correspondence of the terms in (2′) with graphs, group them by the times of "identification"* needed to reduce them to a line or a ring, and calculate the contribution to (2′) of each group. The graphs which do not reduce to a line or a ring by the identifications will be considered in § 5.

We insert here a comment on our notations. In the following, one of double subscripts attached to a letter will be dropped in suitable occasions if they are the same: e. g. $f_{\nu\nu}^{(n)}(\boldsymbol{r})$ and $f_{\nu}^{(n)}(\boldsymbol{r})$ represent the same thing.

Now, we proceed to the calculation of (2′). We calculate the contribution to (2′) of the graphs which are a line without identification—given by B in Fig. 1. The contribution of a graph with ends of the species ν and ν' is

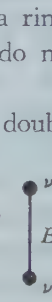

Fig. 1

$$\frac{1}{1+\delta_{\nu,\nu'}}\rho_\nu\rho_{\nu'}\frac{1}{V}\int\int d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,f_{\nu\nu'}^{(0)}(r_{12})=\frac{1}{1+\delta_{\nu,\nu'}}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(0)}(0), \tag{4}$$

where $\delta_{\nu,\nu'}$ is the Kronecker symbol. Then, the contributions to (2′) of all the graphs of the type B are summed up to

$$\left[-\frac{A'}{VkT}\right]_B=\sum_{\nu=\alpha}^{\tau}\frac{1}{2}\rho_\nu^2F_\nu^{(0)}(0)+\sum_{\nu>\nu'}\sum\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(0)}(0)=\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(0)}(0). \tag{5}$$

Here and in the following, $F_{\nu\nu'}^{(n)}(\boldsymbol{k})$ and $H_{\nu\nu'}^{(n)}(\boldsymbol{k})$, $n=0, 1, 2, \cdots$, denote the Fourier transforms of $f_{\nu\nu'}^{(n)}(\boldsymbol{r})$ and $h_{\nu\nu'}^{(n)}(\boldsymbol{r})$, respectively (cf. (24)).

* The terminology in this paper follows that in I.

Next, we proceed to rings. The contribution of the graphs in Fig. 2 is somewhat complex for systems of many species, and so we will postpone the explicit calculation to the following section and only cite the general form of the result:

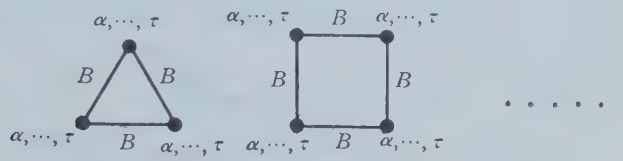

Fig. 2. R_B: Rings of B.

$$\left[-\frac{A'}{VkT}\right]_{R_B}=\frac{1}{2V}\sum_{\boldsymbol{k}}\{-\ln L(\alpha,\beta,\cdots,\tau;\ F^{(0)}_{\nu\nu'}(\boldsymbol{k}))$$

$$-\sum_{\nu=\alpha}^{\tau}\rho_\nu F^{(0)}_{\nu}(\boldsymbol{k})-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}F^{(0)}_{\nu\nu'}(\boldsymbol{k})^2\}. \qquad (6)^*$$

Next, we proceed to watermelons. First, we introduce the functions $H^{(0)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k})^{**}$, representing the propagation by means of the graphs B_S in Fig. 3. Then

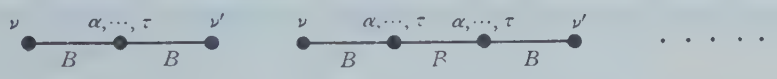

Fig. 3. B_S: sequence of B with ends, $\nu, \nu'=\alpha, \beta, \cdots, \tau$.

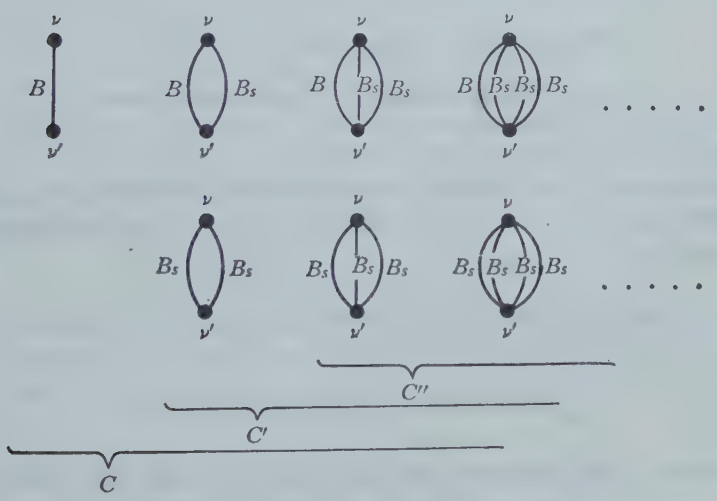

Fig. 4. C, C' and C'': watermelons of B.

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* Here and in the following, $L(\alpha,\beta,\cdots,\tau;\ F^{(n)}_{\nu\nu'}(\boldsymbol{k}))$, $n=0, 1, 2, \cdots$, is to read as the abbreviation of $L(\alpha,\beta,\cdots,\tau;\ F^{(n)}_{\alpha\alpha}(\boldsymbol{k}),\ F^{(n)}_{\alpha\beta}(\boldsymbol{k}),\cdots,F^{(n)}_{\tau\tau}(\boldsymbol{k}))$.

** $(\alpha,\beta,\cdots,\tau)$ in the subscript representing the species in the midway points will be dropped in suitable occasions.

we construct the watermelons of B and B_s as in Fig. 4. The graphs of C'' are those which reduce to a line by one time of identification. Their contribution can be calculated in the same way as in I and we have

$$\left[-\frac{A'}{VkT}\right]_{C''}=\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}\left\{f_{\nu\nu'}^{(0)}(r)\left[e^{h_{\nu\nu'}^{(0)}(\boldsymbol{r})}-1-h_{\nu\nu'}^{(0)}(\boldsymbol{r})\right]\right.$$

$$\left.+e^{h_{\nu\nu'}^{(0)}(\boldsymbol{r})}-1-h_{\nu\nu'}^{(0)}(\boldsymbol{r})-\frac{1}{2!}h_{\nu\nu'}^{(0)}(\boldsymbol{r})^2\right\}$$

$$=\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\left\{F_{\nu\nu'}^{(1)}(0)-F_{\nu\nu'}^{(0)}(0)\right.$$

$$\left.-\int d\boldsymbol{r}\,f_{\nu\nu'}^{(0)}(r)h_{\nu\nu'}^{(0)}(\boldsymbol{r})-\frac{1}{2}\int d\boldsymbol{r}\,h_{\nu\nu'}^{(0)}(\boldsymbol{r})^2\right\}, \tag{7}$$

where we have used $F_{\nu\nu'}^{(1)}(\boldsymbol{k})$ which represents the propagation by means of the graphs C:

$$f_{\nu\nu'}^{(1)}(\boldsymbol{r})=f_{\nu\nu'}^{(0)}(r)\,e^{h_{\nu\nu'}^{(0)}(\boldsymbol{r})}+e^{h_{\nu\nu'}^{(0)}(\boldsymbol{r})}-1-h_{\nu\nu'}^{(0)}(\boldsymbol{r}). \tag{8}$$

Next, we must calculate the contribution of the terms corresponding to the graphs which reduce to a ring by one time of identification; denoted as R_C. R_C

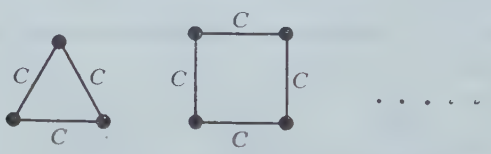

Fig. 5. R_C*, ● occupied by a particle of the species, $\alpha, \beta, \cdots, \tau$.

is the whole rings of C, R_C* in Fig. 5, minus those for which all C are B and those for which one of C is C' and others are B. The contribution of R_C* is obviously given by

$$\left[-\frac{A'}{VkT}\right]_{R_C^*}=\frac{1}{2V}\sum_{\boldsymbol{k}}\left\{-\ln L(\alpha,\beta,\cdots,\tau\,;\,F_{\nu\nu'}^{(1)}(\boldsymbol{k}))\right.$$

$$\left.-\sum_{\nu=\alpha}^{\tau}\rho_\nu F_\nu^{(1)}(\boldsymbol{k})-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(1)}(\boldsymbol{k})^2\right\}. \tag{9}$$

The contributions to be subtracted are calculated as follows: we define $\Delta F_{\nu\nu'}^{(1)}(\boldsymbol{k})$ by

$$F_{\nu\nu'}^{(1)}(\boldsymbol{k})=F_{\nu\nu'}^{(0)}(\boldsymbol{k})+\Delta F_{\nu\nu'}^{(1)}(\boldsymbol{k}), \tag{10}$$

substitute it in the right-hand side of (9) and calculate the terms of zeroth and first order in $\Delta F_{\nu\nu'}^{(1)}(\boldsymbol{k})$, which are the contributions to be subtracted. The term of the zeroth order is obviously equal to $[-A'/VkT]_{R_B}$ and that of the first order is obtained as

$$\frac{1}{2V}\sum_{\boldsymbol{k}}\sum_{\nu\geq\nu'}\Delta F_{\nu\nu'}^{(1)}(\boldsymbol{k})\frac{\delta}{\delta F_{\nu\nu'}^{(0)}(\boldsymbol{k})}\Big\{-\ln L(\alpha,\cdots,\tau;\,F_{\nu\nu'}^{(0)}(\boldsymbol{k}))$$
$$-\sum_{\nu=\alpha}^{\tau}\rho_\nu F_\nu^{(0)}(\boldsymbol{k})-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(0)}(\boldsymbol{k})^2\Big\}$$
$$=\frac{1}{2V}\sum_{\boldsymbol{k}}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}[F_{\nu\nu'}^{(1)}(\boldsymbol{k})-F_{\nu\nu'}^{(0)}(\boldsymbol{k})]H_{\nu\nu'}^{(0)}(\boldsymbol{k})$$
$$=\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}[f_{\nu\nu'}^{(1)}(\boldsymbol{r})-f_{\nu\nu'}^{(0)}(r)]h_{\nu\nu'}^{(0)}(\boldsymbol{r}), \tag{11}$$

where use has been made of the results which will be proved in the next section:

$$\rho_\nu^2 H_{\nu(\alpha,\cdots,\tau)}^{(n)}(\boldsymbol{k})=-\frac{L_\nu(\alpha,\cdots,\tau;\,F_{\nu\nu'}^{(n)}(\boldsymbol{k}))}{L(\alpha,\cdots,\tau;\,F_{\nu\nu'}^{(n)}(\boldsymbol{k}))}-\rho_\nu-\rho_\nu^2F_\nu^{(n)}(\boldsymbol{k}) \tag{12}$$

$$\rho_\nu\rho_{\nu'}H_{\nu\nu'(\alpha,\cdots,\tau)}^{(n)}(\boldsymbol{k})=-\frac{L_{\nu\nu'}(\alpha,\cdots,\tau;\,F_{\nu\nu'}^{(n)}(\boldsymbol{k}))}{2L(\alpha,\cdots,\tau;\,F_{\nu\nu'}^{(n)}(\boldsymbol{k}))}-\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(n)}(\boldsymbol{k}) \tag{13}$$

where

$$L_{\nu\nu'}(\alpha,\cdots,\tau;\,F_{\nu\nu'}^{(n)}(\boldsymbol{k}))\equiv\frac{\delta}{\delta F_{\nu\nu'}^{(n)}(k)}L(\alpha,\cdots,\tau;\,F_{\nu\nu'}(\boldsymbol{k})). \tag{14}$$

As the consequence, we have

$$\left[-\frac{A'}{VkT}\right]_{R_C}=\left[-\frac{A'}{VkT}\right]_{R_C^*}-\left[-\frac{A'}{VkT}\right]_{R_B}$$
$$-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}[f_{\nu\nu'}^{(1)}(\boldsymbol{r})-f_{\nu\nu'}^{(0)}(\boldsymbol{r})]h_{\nu\nu'}^{(0)}(\boldsymbol{r}). \tag{15}$$

The contribution of the graphs which reduce to a line by two times of identification is calculated as follows: We construct the graphs C_S^* which represent the graphs in Fig. 3 if we replace B by C there, define $C_S=C_S^*-B_S$, then construct the propagation of C_S as

$$\Delta H_{\nu\nu'}^{(1)}(\boldsymbol{k})=H_{\nu\nu'}^{(1)}(\boldsymbol{k})-H_{\nu\nu'}^{(0)}(\boldsymbol{k}), \tag{16}$$

from the propagation $H_{\nu\nu'}^{(0)}(\boldsymbol{k})$ by means of B_S and the propagation $H_{\nu\nu'}^{(1)}(\boldsymbol{k})$ by means of C_S^*, then the watermelons of C are represented by Fig. 4 if we replace the letters B and C by C and D respectively. The contribution just being calculated is obtained as:

$$\left[-\frac{A'}{VkT}\right]_{D''}=\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\Big\{F_{\nu\nu'}^{(2)}(0)-F_{\nu\nu'}^{(1)}(0)$$
$$-\int d\boldsymbol{r}\,f_{\nu\nu'}^{(1)}(\boldsymbol{r})\Delta h_{\nu\nu'}^{(1)}(\boldsymbol{r})-\frac{1}{2}\int d\boldsymbol{r}[\Delta h_{\nu\nu'}^{(1)}(\boldsymbol{r})]^2\Big\}. \tag{17}$$

The contribution of rings of D is evaluated in the same way as for the case of R_C, and so forth.

In general, the contribution of the graphs which reduce to a line by n times of identification, denoted as $B^{(n)}-B^{(0)}=B$, $B^{(1)}=C''$, $B^{(2)}=D''$, $\cdots$ ——is obtained to be

$$\left[-\frac{A'}{VkT}\right]_{B^{(n)}}=-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\left\{F_{\nu\nu'}^{(n)}(0)-F_{\nu\nu'}^{(n-1)}(0)\right.$$
$$\left.-\int d\boldsymbol{r}\, f_{\nu\nu'}^{(n-1)}(\boldsymbol{r})\,\Delta h_{\nu\nu'}^{(n-1)}(\boldsymbol{r})-\frac{1}{2}\int d\boldsymbol{r}[\Delta h_{\nu\nu'}^{(n-1)}(\boldsymbol{r})]^2\right\} \tag{18}$$

and the contribution of the graphs which reduces to a ring by n times of identification, denoted as $R^{(n)}-R^{(0)}=R_B$, $R^{(1)}=R_C$, $\cdots$ ——is

$$\left[-\frac{A'}{VkT}\right]_{R^{(n)}}=\left[-\frac{A'}{VkT}\right]_{R^{(n)}*}-\left[-\frac{A'}{VkT}\right]_{R^{(n-1)}*}$$
$$-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}[f_{\nu\nu'}^{(n)}(\boldsymbol{r})-f_{\nu\nu'}^{(n-1)}(\boldsymbol{r})]h_{\nu\nu'}^{(n-1)}(\boldsymbol{r}) \tag{19}$$

with

$$\left[-\frac{A'}{VkT}\right]_{R^{(n)}*}=\frac{1}{2V}\sum_{\boldsymbol{k}}\left\{-\ln L(\alpha,\cdots,\tau;\, F_{\nu\nu'}^{(n)}(\boldsymbol{k}))\right.$$
$$\left.-\sum_{\nu=\alpha}^{\tau}\rho_\nu F_\nu^{(n)}(\boldsymbol{k})-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(n)}(\boldsymbol{k})^2\right\}. \tag{20}$$

Summing these up to $B^{(n)}$ and $R^{(n)}$, we have

$$\left[-\frac{A'}{VkT}\right]_{\sum_{s=0}^{n}[B^{(s)}+R^{(s)}]}=\left[-\frac{A'}{VkT}\right]_{R^{(n)}*}+\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\left\{F_{\nu\nu'}^{(n)}(0)\right.$$
$$\left.-\int d\boldsymbol{r}\, f_{\nu\nu'}^{(n)}(\boldsymbol{r})h_{\nu\nu'}^{(n-1)}(\boldsymbol{r})-\frac{1}{2}\int d\boldsymbol{r}\sum_{s=0}^{n-1}[\Delta h_{\nu\nu'}^{(s)}(\boldsymbol{r})]^2\right\}. \tag{21}$$

Summing these up to $R^{(n)}$ and $B^{(n+1)}$, we have

$$\left[-\frac{A'}{VkT}\right]_{\sum_{s=0}^{n}[B^{(s)}+R^{(s)}]+B^{(n+1)}}=\left[-\frac{A'}{VkT}\right]_{R^{(n)}*}+\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\left\{F_{\nu\nu'}^{(n+1)}(0)\right.$$
$$\left.-\int d\boldsymbol{r}\, f_{\nu\nu'}^{(n)}(\boldsymbol{r})h_{\nu\nu'}^{(n)}(\boldsymbol{r})-\frac{1}{2}\int d\boldsymbol{r}\sum_{s=0}^{n}[\Delta h_{\nu\nu'}^{(s)}(\boldsymbol{r})]^2\right\}. \tag{22}$$

In the limit of $n\to\infty$, these both tend to

$$\left[-\frac{A'}{VkT}\right]_{\mathrm{HNC}}=\frac{1}{2V}\sum_{\boldsymbol{k}}\left\{-\ln L(\alpha,\cdots,\tau;\,\mathcal{J}_{\nu\nu'}(\boldsymbol{k}))\right.$$
$$\left.-\sum_{\nu=\alpha}^{\tau}\rho_\nu\mathcal{J}_\nu(\boldsymbol{k})-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\mathcal{J}_{\nu\nu'}(\boldsymbol{k})^2\right\}$$
$$+\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}\left\{\mathcal{J}_{\nu\nu'}(0)-\int d\boldsymbol{r}\, f_{\nu\nu'}(\boldsymbol{r})h_{\nu\nu'}(\boldsymbol{r})-\frac{1}{2}\int d\boldsymbol{r}\sum_{s=0}^{\infty}[\Delta h_{\nu\nu'}^{(s)}(\boldsymbol{r})]^2\right\}, \tag{23}$$

with

$$\left.\begin{aligned}
&f_{\nu\nu'}(\boldsymbol{r})=\lim_{n\to\infty} f_{\nu\nu'}^{(n)}(\boldsymbol{r}),\quad h_{\nu\nu'}(\boldsymbol{r})=\lim_{n\to\infty} h_{\nu\nu'}^{(n)}(\boldsymbol{r}),\quad \mathcal{F}_{\nu\nu'}(\boldsymbol{k})=\lim_{n\to\infty} F_{\nu\nu'}^{(n)}(\boldsymbol{k})\\
&f_{\nu\nu'}^{(n+1)}(\boldsymbol{r})=f_{\nu\nu'}^{(n)}(\boldsymbol{r})\exp\{\Delta h_{\nu\nu'}^{(n)}(\boldsymbol{r})\}+\exp\{\Delta h_{\nu\nu'}^{(n)}(\boldsymbol{r})\}-1-\Delta h_{\nu\nu'}^{(n)}(\boldsymbol{r})\\
&\Delta h_{\nu\nu'}^{(n)}(\boldsymbol{r})=h_{\nu\nu'}^{(n)}(\boldsymbol{r})-h_{\nu\nu'}^{(n-1)}(\boldsymbol{r})\\
&h_{\nu\nu'}^{(n)}(\boldsymbol{r})=1/V\cdot\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} H_{\nu\nu'}^{(n)}(\boldsymbol{k})\\
&F_{\nu\nu'}^{(n)}(\boldsymbol{k})=\int d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} f_{\nu\nu'}^{(n)}(\boldsymbol{r})\\
&f_{\nu\nu'}^{(0)}(\boldsymbol{r})=\exp\{-\phi_{\nu\nu'}(r)/kT\}-1,\quad f_{\nu\nu'}^{(0)}(-\boldsymbol{r})=f_{\nu\nu'}^{(0)}(\boldsymbol{r})\\
&h_{\nu\nu'}^{(-1)}(\boldsymbol{r})\equiv 0.
\end{aligned}\right\}\qquad (24)^*$$

$H_{\nu\nu'}^{(n)}(\boldsymbol{k})$ in terms of $F_{\nu\nu'}^{(n)}(\boldsymbol{k})$ is given by eqs. (12) and (13). $L(\alpha, \beta, \cdots, \tau; F_{\nu\nu'}^{(n)}(\boldsymbol{k}))$ will be given in terms of $F_{\nu\nu'}^{(n)}(\boldsymbol{k})$ in the next section.

To derive the explicit expressions for $L(\alpha, \beta, \cdots, \tau; F_{\nu\nu'}^{(n)}(\boldsymbol{k}))$ and $h_{\nu\nu'(\alpha,\cdots,\tau)}^{(n)}(\boldsymbol{r})$ in terms of $F_{\nu\nu'}^{(n)}(\boldsymbol{k})$ and to prove eqs. (12) and (13) are the subject of the next section.

§ 3. $L(\alpha, \beta, \cdots, \tau; F_{\nu\nu'}^{(n)}(\boldsymbol{k}))$ and $H_{\nu\nu'(\alpha,\beta,\cdots,\tau)}^{(n)}(\boldsymbol{k})$

The problem of obtaining the expressions for $L(\alpha, \cdots, \tau; F_{\nu\nu'}^{(n)}(\boldsymbol{k}))$ and $H_{\nu\nu'(\alpha,\cdots,\tau)}^{(n)}(\boldsymbol{k})$ in terms of $F_{\nu\nu'}^{(n)}(\boldsymbol{k})$ is solved here by the help of the following lemma, whose proof is given in Appendix I.

LEMMA: If we define $L(\alpha, \cdots, \tau; F_{\nu\nu'}^{(n)}(\boldsymbol{k}))$ by

$$\begin{aligned}
&\frac{1}{2V}\sum_{\boldsymbol{k}}\Big\{-\ln L(\alpha,\cdots,\tau;F_{\nu\nu'}^{(n)}(\boldsymbol{k}))-\sum_{\nu=\alpha}^{\tau}\rho_\nu F_{\nu}^{(n)}(\boldsymbol{k})-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(n)}(\boldsymbol{k})^2\Big\}\\
&=\sum_{n_\alpha=0}^{\infty}\cdots\sum_{n_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu^{n_\nu}}{n_\nu!}\cdot\frac{1}{V}\int\int\cdots\int d\boldsymbol{r}_1\cdots d\boldsymbol{r}_{n_\alpha+\cdots+n_\tau}\sum_{n_\alpha+\cdots+n_\tau\geq i>j\geq 1}\prod f_{ij}^{(n)}\equiv\Big[-\frac{A'}{VkT}\Big]_{R^{(n)*}},\qquad (25)
\end{aligned}$$

Sum over all products which correspond to graphs of ring. See Fig. 6.

then the propagation of the graphs shown in Fig. 7, to be denoted as $K_{\nu(\alpha,\cdots,\tau)}^{(n)}(\boldsymbol{k})$, and the propagation of the graphs in Fig. 8, to be denoted as $K_{\nu\nu'(\alpha,\cdots,\tau)}^{(n)}(\boldsymbol{k})$, are given by

$$\begin{aligned}
K_{\nu(\alpha,\cdots,\tau)}^{(n)}(\boldsymbol{k})&=-\frac{\delta}{\delta F_{\nu}^{(n)}(\boldsymbol{k})}\ln L(\alpha,\cdots,\tau;F_{\nu\nu'}^{(n)}(\boldsymbol{k}))\\
&=-\frac{L_\nu(\alpha,\cdots,\tau;F_{\nu\nu'}^{(n)}(\boldsymbol{k}))}{L(\alpha,\cdots,\tau;F_{\nu\nu'}^{(n)}(\boldsymbol{k}))}\qquad (26)
\end{aligned}$$

* Note that $f_{\nu\nu'}^{(n)}(\boldsymbol{r})$ and $h_{\nu\nu'}^{(n)}(\boldsymbol{r})$ are functions of $r=|\boldsymbol{r}|$ alone and $F_{\nu\nu'}^{(n)}(\boldsymbol{k})$ and $H_{\nu\nu'}^{(n)}(\boldsymbol{k})$ those of $k=|\boldsymbol{k}|$ alone.

and

$$K^{(n)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k}) = -\frac{1}{2}\frac{\delta}{\delta F^{(n)}_{\nu\nu'}(\boldsymbol{k})}\ln L(\alpha,\cdots,\tau;\ F^{(n)}_{\nu\nu'}(\boldsymbol{k}))$$

$$= -\frac{1}{2}\frac{L_{\nu\nu'}(\alpha,\cdots,\tau;\ F^{(n)}_{\nu\nu'}(\boldsymbol{k}))}{L(\alpha,\cdots,\tau;\ F^{(n)}_{\nu\nu'}(\boldsymbol{k}))} \quad \text{for } \nu \leqq \nu', \tag{27}$$

in terms of $L(\alpha,\cdots,\tau;\ F^{(n)}_{\nu\nu'}(\boldsymbol{k}))$, where

$$L_{\nu\nu'}(\alpha,\cdots,\tau;\ F^{(n)}_{\nu\nu'}(\boldsymbol{k})) \equiv \frac{\delta}{\delta F^{(n)}_{\nu\nu'}(\boldsymbol{k})} L(\alpha,\cdots,\tau;\ F^{(n)}_{\nu\nu'}(\boldsymbol{k})).$$

From the definitions of $K^{(n)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k})$ and $H^{(n)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k})$ we have the relations:

$$\rho_\nu{}^2 H^{(n)}_{\nu(\alpha,\cdots,\tau)}(\boldsymbol{k}) = K^{(n)}_{\nu(\alpha,\cdots,\tau)}(\boldsymbol{k}) - \rho_\nu - \rho_\nu{}^2 F^{(n)}_{\nu}(\boldsymbol{k}), \tag{28}$$

$$\rho_\nu\rho_{\nu'} H^{(n)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k}) = K^{(n)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k}) - \rho_\nu\rho_{\nu'} F_{\nu\nu'}(\boldsymbol{k}) \quad \text{for } \nu \leqq \nu'. \tag{29}$$

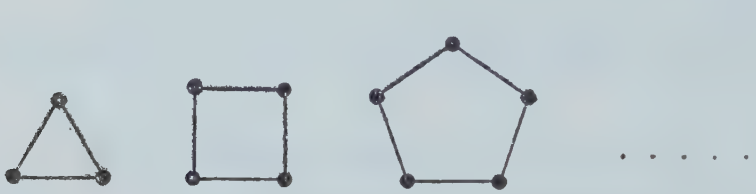

Fig. 6. ● are occupied by one of species $\alpha, \beta, \cdots, \tau$.

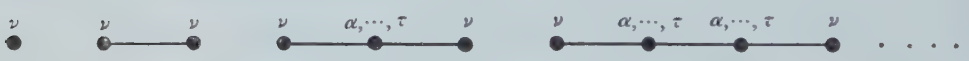

Fig. 7. Graphs corresponding to $K^{(n)}_{\nu(\alpha,\cdots,\tau)}(\boldsymbol{k})$.

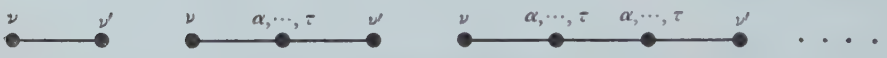

Fig. 8. Graphs corresponding to $K^{(n)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k})$, $\nu \neq \nu'$.

Now we calculate $L(\alpha,\cdots,\tau)$* by starting with $L(\alpha)$ and then proceeding to $L(\alpha,\cdots,\tau)$ with more species in order by the help of the lemma.

For a system with one species α, we have

$$\left[-\frac{A'}{VkT}\right]_{R^{(n)}*} = \frac{1}{2V}\sum_{\boldsymbol{k}}\left\{-\ln(1-\rho_\alpha F^{(n)}_\alpha(\boldsymbol{k})) - \rho_\alpha F^{(n)}_\alpha(\boldsymbol{k}) - \frac{1}{2}\rho_\alpha{}^2 F^{(n)}_\alpha(\boldsymbol{k})^2\right\}. \tag{30}$$

As the consequence,

$$L(\alpha) = 1 - \rho_\alpha F^{(n)}_\alpha(\boldsymbol{k}). \tag{31}$$

* For brevity, we will omit $F^{(n)}_{\nu\nu'}(\boldsymbol{k})$ as the argument of $L(\alpha,\cdots,\tau;\ F^{(n)}_{\nu\nu'}(\boldsymbol{k}))$ in the following in this section.

For a system with two species α and β, we calculate the contribution of the right-hand side of eq. (25) for graphs containing β and those not containing β separately. The latters are graphs of ring of α and have the same contribution as the right-hand side of eq. (30). The former is given by

$$\frac{1}{2V}\sum_{k}\Big\{\sum_{n=1}^{\infty}\frac{1}{n}[\rho_\beta F_\beta^{(n)}(k)+\rho_\beta F_{\alpha\beta}^{(n)}(k)^2K_\alpha^{(n)}(k)]^n$$

$$-\rho_\beta F_\beta^{(n)}(k)-\frac{1}{2}\rho_\beta{}^2F_\beta^{(n)}(k)^2-\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(k)^2\Big\}$$

$$=\frac{1}{2V}\sum_{k}\Big\{-\ln(1-\rho_\beta F_\beta^{(n)}(k)-\rho_\beta F_{\alpha\beta}^{(n)}(k)^2K_\alpha^{(n)}(k))$$

$$-\rho_\beta F_\beta^{(n)}(k)-\frac{1}{2}\rho_\beta{}^2F_\beta^{(n)}(k)^2-\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(k)^2\Big\}$$

where cares have been taken not to include the graphs of the type ⟨⟩ and ◯. Summing these contributions, we have

$$\Big[-\frac{A'}{VkT}\Big]_{R_{\alpha\beta}^{(n)}*}=\frac{1}{2V}\sum_{k}\Big\{-\ln[(1-\rho_\beta F_\beta^{(n)}(k))L(\alpha)-\rho_\beta F_{\alpha\beta}^{(n)}(k)^2K_\alpha^{(n)}(k)L(\alpha)]$$

$$-\rho_\alpha F_\alpha^{(n)}(k)-\rho_\beta F_\beta^{(n)}(k)-\frac{1}{2}\sum_{\nu=\alpha}^{\beta}\sum_{\nu'=\alpha}^{\beta}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(n)}(k)^2\Big\}. \tag{32}$$

As the result, we have

$$L(\alpha,\beta)=L(\alpha)L(\beta)+\rho_\beta F_{\alpha\beta}^{(n)}(k)^2L_\alpha(\alpha)$$

$$=(1-\rho_\alpha F_\alpha^{(n)}(k))(1-\rho_\beta F_\beta^{(n)}(k))-\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(k)^2 \tag{33}$$

by the use of eq. (26) with eq. (31) for $L(\alpha)$ and the definition of $L_\alpha(\alpha)$, eq. (14).

Analogously, $L(\alpha,\beta,\gamma)$ is calculated as the sum of contributions of the graphs which contain γ and those not containing γ as follows:

$$\Big[-\frac{A'}{VkT}\Big]_{R_{\alpha\beta\gamma}^{(n)}*}=\frac{1}{2V}\sum_{k}\Big\{-\ln[1-\rho_\gamma F_\gamma^{(n)}(k)-\rho_\gamma F_{\alpha\gamma}^{(n)}(k)^2K_{\alpha(\alpha,\beta)}(k)$$

$$-\rho_\gamma F_{\gamma\beta}^{(n)}(k)^2K_{\beta(\alpha,\beta)}(k)-2\rho_\gamma F_{\gamma\alpha}^{(n)}(k)F_{\gamma\beta}^{(n)}(k)K_{\alpha\beta(\alpha,\beta)}^{(n)}(k)]$$

$$-\rho_\gamma F_\gamma^{(n)}(k)-\frac{1}{2}\rho_\gamma{}^2F_\gamma^{(n)}(k)^2-\rho_\gamma\rho_\alpha F_{\gamma\alpha}^{(n)}(k)^2-\rho_\gamma\rho_\beta F_{\gamma\beta}^{(n)}(k)^2$$

$$-\ln L(\alpha,\beta)-\rho_\alpha F_\alpha^{(n)}(k)-\rho_\beta F_\beta^{(n)}(k)-\frac{1}{2}\sum_{\nu=\alpha}^{\beta}\sum_{\nu'=\alpha}^{\beta}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(n)}(k)^2\Big\}$$

$$=\frac{1}{2V}\sum_{k}\Big\{-\ln[(1-\rho_\gamma F_\gamma^{(n)}(k))L(\alpha,\beta)+\rho_\gamma F_{\gamma\alpha}^{(n)}(k)^2L_\alpha(\alpha,\beta)$$

$$+\rho_\tau F_{\tau\beta}^{(n)}(\boldsymbol{k})^2 L_\beta(\alpha,\beta)+\rho_\tau F_{\tau\alpha}^{(n)}(\boldsymbol{k}) F_{\tau\beta}^{(n)}(\boldsymbol{k}) L_{\alpha\beta}(\alpha,\beta)]$$
$$-\sum_{\nu=\alpha}^{\tau}\rho_\tau F_{\tau}^{(n)}(\boldsymbol{k})-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(n)}(\boldsymbol{k})^2\Big\} \tag{34}$$

from which we have

$$L(\alpha,\beta,\gamma)=L(\gamma)\,L(\alpha,\beta)+\rho_\tau\sum_{\nu=\alpha}^{\beta}F_{\tau\nu}^{(n)}(\boldsymbol{k})^2 L_\nu(\alpha,\beta)$$
$$+\rho_\tau F_{\tau\alpha}^{(n)}(\boldsymbol{k})F_{\tau\beta}^{(n)}(\boldsymbol{k})L_{\alpha\beta}(\alpha,\beta). \tag{35}$$

In the same way, we get in general

$$L(\alpha,\cdots,\sigma,\tau)=L(\tau)L(\alpha,\cdots,\sigma)+\rho_\tau\sum_{\nu=\alpha}^{\sigma}F_{\tau\nu}^{(n)}(\boldsymbol{k})^2 L_\nu(\alpha,\cdots,\sigma)$$
$$+\rho_\tau\sum_{\nu=\alpha}^{\sigma}\sum_{\substack{\nu'=\alpha\\ \nu\neq\nu'}}^{\sigma}F_{\tau\nu}^{(n)}(\boldsymbol{k})F_{\tau\nu'}^{(n)}(\boldsymbol{k})\frac{1}{2}L_{\nu\nu'}(\alpha,\cdots,\sigma), \tag{36}$$

from which we get

$$L_\tau(\alpha,\cdots,\sigma,\tau)=-\rho_\tau L(\alpha,\cdots,\sigma) \tag{37}$$

and

$$\frac{1}{2}L_{\sigma\tau}(\alpha,\cdots,\rho,\sigma,\tau)=-\rho_\sigma\rho_\tau F_{\sigma\tau}^{(n)}(\boldsymbol{k})L(\alpha,\cdots,\rho)$$
$$+\rho_\tau\sum_{\nu=\alpha}^{\rho}F_{\tau\nu}^{(n)}(\boldsymbol{k})\frac{1}{2}L_{\nu\sigma}(\alpha,\cdots,\rho,\sigma). \tag{38}$$

Thus we have the expression for $L(\alpha,\beta,\cdots,\tau)$ in terms of $L(\nu)=1-\rho_\nu F_\nu^{(n)}(\boldsymbol{k})$, ρ_ν and $F_{\nu\nu'}^{(n)}(\boldsymbol{k})$. From these equations, we find that every term of $L(\alpha,\cdots,\tau)$ corresponds to a graph. If we denote the factor $L(\nu)$ by $\circ_\nu$, factor ρ_ν by $\bullet_\nu$ and the factor $F_{\nu\nu'}^{(n)}(\boldsymbol{k})$ by a line connecting $\bullet_\nu$ and $\bullet_{\nu'}$, the graph corresponding to a term in $L(\alpha,\cdots,\tau)$ consists of some independent $\circ_\nu$ and some independent rings connecting $\bullet_\nu$; all of $\circ_\nu$ and $\bullet_\nu$ in a graph correspond to one-to-one to species in $L(\alpha,\cdots,\tau)$. The number of terms is equal to that of the different possible graphs consisting of points with subscript $\alpha,\beta,\cdots,\tau$. The rings except ◯ are considered to have direction. The sign of a term is positive or negative according as the number of rings are even or odd. A typical term is illustrated in Fig. 9.

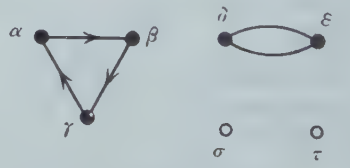

Fig. 9. A graph corresponding to $[-\rho_\alpha\rho_\beta\rho_\tau F_{\alpha\beta}F_{\beta\tau}F_{\tau\alpha}]\times[-\rho_\delta\rho_\varepsilon F_{\delta\varepsilon}{}^2]\,L(\sigma)\,L(\tau)$.

The graphs corresponding to terms of $\frac{1}{2}L_{\nu\nu'}(\alpha,\beta,\cdots,\tau)$ contain a chain connecting $\bullet_\nu$ and $\bullet_{\nu'}$ except for which they are similar to those for $L(\alpha,\beta,\cdots,\tau)$; here rings as well as chains are

Fig. 10

Fig. 11

considered not to have direction. The sign of the term is positive or negative according as the the number of rings is odd or even.

Thus we can obtain the explicit expressions for $L(\alpha, \cdots, \tau)$ and $L_\nu(\alpha, \cdots, \tau)$ and $\frac{1}{2}L_{\nu\nu'}(\alpha, \cdots, \tau)$, and as the consequence, $K^{(n)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k})$, and $H^{(n)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k})$. For example, we now calculate them explicitly for two and three species.

For two species, α and β, the total of the graphs corresponding to $L(\alpha, \beta)$ and $\frac{1}{2}L_{\alpha\beta}(\alpha, \beta)$ are shown in Fig. 10 and Fig. 11, respectively. As the result, we have

$$\left.\begin{aligned}
&L(\alpha,\beta)=(1-\rho_\alpha F_\alpha^{(n)}(\boldsymbol{k}))(1-\rho_\beta F_\beta^{(n)}(\boldsymbol{k}))-\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(\boldsymbol{k})^2\\
&\frac{1}{2}L_{\alpha\beta}(\alpha,\beta)=-\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(\boldsymbol{k})\\
&\rho_\alpha{}^2H^{(n)}_{\alpha(\alpha,\beta)}(\boldsymbol{k})=\frac{\rho_\alpha(1-\rho_\beta F_\beta^{(n)}(\boldsymbol{k}))}{L(\alpha,\beta)}-\rho_\alpha-\rho_\alpha{}^2F_\alpha^{(n)}(\boldsymbol{k})\\
&\rho_\alpha\rho_\beta H^{(n)}_{\alpha\beta(\alpha,\beta)}(\boldsymbol{k})=\frac{\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(\boldsymbol{k})}{L(\alpha,\beta)}-\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(\boldsymbol{k}).
\end{aligned}\right\}\qquad(39)$$

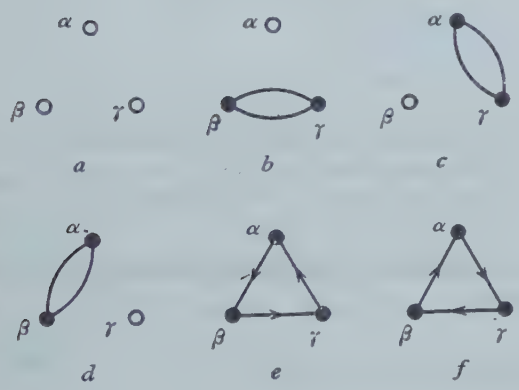

Fig. 12

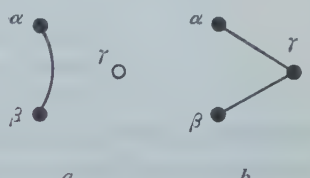

Fig. 13

For three species, the graphs for $L(\alpha, \beta, \gamma)$ and $\frac{1}{2}L_{\alpha\beta}(\alpha, \beta, \gamma)$ are given respectively in Fig. 12 and Fig. 13, and by means of these we have

$$\left.\begin{aligned}
&L(\alpha,\beta,\gamma)=(1-\rho_\alpha F_\alpha^{(n)}(\boldsymbol{k}))(1-\rho_\beta F_\beta^{(n)}(\boldsymbol{k}))(1-\rho_\gamma F_\gamma^{(n)}(\boldsymbol{k}))\\
&\quad-\sum_{\substack{\text{cyclic change}\\ \text{of }(\alpha,\beta,\gamma)}}(1-\rho_\alpha F_\alpha^{(n)}(\boldsymbol{k}))\rho_\beta\rho_\gamma F_{\beta\gamma}^{(n)}(\boldsymbol{k})^2\\
&\quad-2\rho_\alpha\rho_\beta\rho_\gamma F_{\alpha\beta}^{(n)}(\boldsymbol{k})F_{\beta\gamma}^{(n)}(\boldsymbol{k})F_{\gamma\alpha}^{(n)}(\boldsymbol{k})\\
&\frac{1}{2}L_{\alpha\beta}(\alpha,\beta,\gamma)=-\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(\boldsymbol{k})(1-\rho_\gamma F_\gamma^{(n)}(\boldsymbol{k}))-\rho_\alpha\rho_\beta\rho_\gamma F_{\alpha\gamma}^{(n)}(\boldsymbol{k})F_{\beta\gamma}^{(n)}(\boldsymbol{k})\\
&\rho_\alpha{}^2H^{(n)}_{\alpha(\alpha,\beta,\gamma)}(\boldsymbol{k})=\frac{\rho_\alpha L(\beta,\gamma)}{L(\alpha,\beta,\gamma)}-\rho_\alpha-\rho_\alpha{}^2F_\alpha^{(n)}(\boldsymbol{k})\\
&\rho_\alpha\rho_\beta H^{(n)}_{\alpha\beta(\alpha,\beta,\gamma)}(\boldsymbol{k})=-\frac{\frac{1}{2}L_{\alpha\beta}(\alpha,\beta,\gamma)}{L(\alpha,\beta,\gamma)}-\rho_\alpha\rho_\beta F_{\alpha\beta}^{(n)}(\boldsymbol{k}).
\end{aligned}\right\}\qquad(40)$$

§ 4. Radial distribution function

We start with the virial expansion formula for the radial distribution function[3]

$$g_{\nu_1\nu_2}(r_{12})=\exp\{-\phi_{\nu_1\nu_2}(r_{12})/kT\}\cdot\Big\{1+\sum_{m_\alpha=0}^{\infty}\cdots\sum_{m_\tau=0}^{\infty}\cdot\prod_{\nu=0}^{\tau}\frac{{\rho_\nu}^{m_\nu}}{m_\nu!}$$

$$\times\iint\cdots\int d\boldsymbol{r}_3\cdots d\boldsymbol{r}_{m_\alpha+\cdots+m_\tau+2}\sum_{\substack{m_\alpha+\cdots+m_\tau+2\geq i>j\geq 3\\ m_\alpha+\cdots+m_\tau+2\geq k\geq 3\\ 2\geq\varkappa\geq 1}}\prod f_{ij}^{(0)}f_{k\varkappa}^{(0)}\Big\}, \tag{41}$$

Sum over all products of $f_{ij}^{(0)}$ in which each particle of the set $\{\boldsymbol{r}_3,\cdots,\boldsymbol{r}_{m_\alpha+\cdots+m_\tau+2}\}$ is connected to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ by an independent path.

or with that for the potential of average force[3]

$$\ln g_{\nu_1\nu_2}(r_{12})=-\frac{\phi_{\nu_1\nu_2}(r_{12})}{kT}+\sum_{m_\alpha=0}^{\infty}\cdots\sum_{m_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{{\rho_\nu}^{m_\nu}}{m_\nu!}$$

$$\times\iint\cdots\int d\boldsymbol{r}_3\cdots d\boldsymbol{r}_{m_\alpha+\cdots+m_\tau+2}\sum_{\substack{m_\alpha+\cdots+m_\tau+2\geq i>j\geq 3\\ m_\alpha+\cdots+m_\tau+2\geq k\geq 3\\ 2\geq\varkappa\geq 1}}\prod f_{ij}^{(0)}f_{k\varkappa}^{(0)}. \tag{42}$$

Sum over all products of $f_{ij}^{(0)}$ for which each molecule of the set $\{\boldsymbol{r}_3,\cdots,\boldsymbol{r}_{m_\alpha+\cdots+m_\tau+2}\}$ is connected to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ by an independent path and also for which the molecules $\{\boldsymbol{r}_3,\cdots,\boldsymbol{r}_{m_\alpha+\cdots+m_\tau+2}\}$ are connected with each other even when we erase the lines ended at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

The calculation of the contribution of each group can be performed as for the case of a one-component system and we have

$$[g_{\nu_1\nu_2}(r_{12})]_{C^*}=\exp\{-\phi_{\nu_1\nu_2}(r_{12})/kT+h_{\nu_1\nu_2}^{(0)}(r_{12})\} \tag{43C^*}$$

in the approximation C^*,†

$$[g_{\nu_1\nu_2}(r_{12})]_{D^*}=\exp\{-\phi_{\nu_1\nu_2}(r_{12})/kT+h_{\nu_1\nu_2}^{(1)}(r_{12})\} \tag{43D^*}$$

in the approximation D^*. Thus at last we have

$$[g_{\nu_1\nu_2}(r_{12})]_{\mathrm{HNC}}=\exp\{-\phi_{\nu_1\nu_2}(r_{12})/kT+h_{\nu_1\nu_2}(r_{12})\} \tag{43}$$

with $h_{\nu_1\nu_2}(r_{12})$ given in (24).

Rushbrooke[4] has obtained the result in the approximation C^* for the case of binary mixtures, by extending the method of Montroll and Mayer[5] or Born and Green.[6] He[4] has also presented the netted-chain approximation with the object to improve the approximation. It considers some simple graphs among D^* in addition to those in C^*.

† The graphs taken into account in these approximations are Fig. 13 of I with the specification of the species of particles at endpoints as ν_1 and ν_2.

§ 5. Expansion formulae for the free energy and the radial distribution function by means of the hyper-netted chains

The expansion formulae for the free energy and the radial distribution function by means of the hyper-netted chains, having the results in the hyper-netted chain approximation as the leading terms, are obtained as follows:

$$-\frac{A'}{VkT}=\left[-\frac{A'}{VkT}\right]_{\text{HNC, given by eq. (23)}}+\sum_{n_\alpha=0}^{\infty}\cdots\sum_{n_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu^{n_\nu}}{n_\nu!}$$

$$\times\frac{1}{V}\iint\cdots\int d\boldsymbol{r}_1\cdots d\boldsymbol{r}_{n_\alpha+\cdots+n_\tau}\sum_{n_\alpha+\cdots+n_\tau\geq i>j\geq 1}\prod(f_{ij}+h_{ij}), \tag{44}$$

Sum over all products which cosist of junctions alone and are more than singly connected.

$$g_{\nu_1\nu_2}(r_{12})=[g_{\nu_1\nu_2}(r_{12})]_{\text{HNC, given by eq. (43)}}\left\{1+\sum_{m_\alpha=0}^{\infty}\cdots\sum_{m_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu^{m_\nu}}{m_\nu!}\right.$$

$$\times\iint\cdots\int d\boldsymbol{r}_3\cdots d\boldsymbol{r}_{m_\alpha+\cdots+m_\tau+2}\sum_{\substack{m_\alpha+\cdots+m_\tau+2\geq i>j\geq 3\\ m_\alpha+\cdots+m_\tau+2\geq k\geq 3\\ 2\geq\kappa\geq 1}}\prod(f_{ij}+h_{ij})(f_{k\kappa}+h_{k\kappa}), \tag{45}$$

Sum over all products for which each molecule of the set $\{\boldsymbol{r}_3, \cdots, \boldsymbol{r}_{m_\alpha+\cdots+m_\tau+2}\}$ is a junction and is connected to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ by an independent path.

where

$$f_{ij}+h_{ij}=f_{\nu_i\nu_j}(r_{ij})+h_{\nu_i\nu_j}(r_{ij}). \tag{46}$$

The analogous expansion for potential of average force of n particles—the logarithm of the n particle distribution function—will be given in Appendix II.

§ 6. Another expansion scheme

As stated in the introduction, what is to be compared with the theories in the past for the ionic solutions must be the theories for multi-component systems. For this purpose, we translate the formulation in § 5 of I for the case of the multi-component systems.

We divide the potential into two parts as

$$\phi_{\nu\nu'}(r)=\phi_{1\nu\nu'}(r)+\phi_{2\nu\nu'}(r), \tag{47}$$

$\phi_{2\nu\nu'}(r)$ being assumed to have the Fourier transform, define $f_{1\nu\nu'}(r)$ and $j_{\nu\nu'}(r)$ by

$$f_{1\nu\nu'}(r)\equiv\exp\{-\phi_{1\nu\nu'}(r)/kT\}-1,\quad j_{\nu\nu'}(r)=-\phi_{2\nu\nu'}(r)/kT, \tag{48}$$

and expand as

$$\exp\{-\phi_{\nu\nu'}(r)/kT\}-1=f_{1\nu\nu'}(r)+\sum_{n=1}^{\infty}\frac{1}{n!}j_{\nu\nu'}(r)^n+f_{1\nu\nu'}\sum_{n=1}^{\infty}\frac{1}{n!}j_{\nu\nu'}(r)^n. \tag{49}$$

What to be calculated here are eq. (2′) and eq. (41) with $f_{\nu\nu'}^{(0)}(r)$ replaced by the right-hand side of eq. (49). We represent the terms by graphs by corresponding a line, denoted as b, to the factors $f_{1\nu\nu'}(r)$ and $j_{\nu\nu'}(r)$ and group the graphs by the times of identification needed to reduce them to a line or a ring. The groups of graphs thus grouped are shown in Fig. 16 of I if we put a particle of one of species $\alpha, \cdots, \tau$ in any of ●'s. Then we obtain eqs. (18)–(23), if we make the correspondence $B^{(0)}=b$, $B^{(1)}=b'+c''$, $B^{(2)}=d''$, $\cdots$, and $R^{(0)}=R_b$, $R^{1)}=R_c$, $\cdots$, as in I, with the recurrence equations

$$\left.\begin{aligned}
&f_{\nu\nu'}(r)=\lim_{n\to\infty} f_{\nu\nu'}^{(n)}(r),\quad h_{\nu\nu'}(r)=\lim_{n\to\infty} h_{\nu\nu'}^{(n)}(r),\quad \mathcal{F}_{\nu\nu'}(k)=\lim_{n\to\infty} F_{\nu\nu'}^{(n)}(k)\\
&f_{\nu\nu'}^{(n+1)}(r)=f_{\nu\nu'}^{(n)}(r)\exp\{\Delta h_{\nu\nu'}^{(n)}(r)\}+\exp\{\Delta h_{\nu\nu'}^{(n)}(r)\}-1-\Delta h_{\nu\nu'}^{(n)}(r),\quad n\geqq 1,\\
&f_{\nu\nu'}^{(1)}(r)=f_{1\nu\nu'}(r)\exp\{h_{\nu\nu'}^{(0)}(r)+j_{\nu\nu'}(r)\}+\exp\{h_{\nu\nu'}^{(0)}(r)+j_{\nu\nu'}(r)\}-1-h_{\nu\nu'}^{(0)}(r)\\
&f_{\nu\nu'}^{(0)}(r)=f_{1\nu\nu'}(r)+j_{\nu\nu'}(r)
\end{aligned}\right\}\quad (50)$$

and the equations to obtain $h_{\nu\nu'}^{(n)}(r)$ from $f_{\nu\nu'}^{(n)}(r)$, which are identical as in the previous sections: eqs. (26)–(29) with (36)–(39).

6a. For the purpose of comparisons with the results in the past for the ionic solutions, we write the formulae for the free energy in the ring and watermelon approximation:

In the ring approximation:

$$\begin{aligned}
\left[-\frac{A'}{VkT}\right]_{b+b'+R_b}=&\frac{1}{2V}\sum_{k}\Big\{-\ln L(\alpha,\cdots,\tau;\,F_{\nu\nu'}^{(0)}(k))\\
&-\sum_{\nu=\alpha}^{\tau}\rho_\nu F_\nu^{(0)}(k)-\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(0)}(k)^2\Big\}\\
&+\frac{1}{2}\sum_{\nu=\alpha}^{\tau}\sum_{\nu'=\alpha}^{\tau}\Big\{F_{\nu\nu'}^{(0)}(0)+\int d\boldsymbol{r}\,f_{1\nu\nu'}(r)\,j_{\nu\nu'}(r)+\frac{1}{2}\int d\boldsymbol{r}\,j_{\nu\nu'}(r)^2\Big\}.
\end{aligned}\quad (51)$$

In the watermelon approximation:

$$\begin{aligned}
\left[-\frac{A'}{VkT}\right]_{b+b'+R_b+c''}=&\left[-\frac{A'}{VkT}\right]_{b+b'+R_b}+\frac{1}{2}\sum_{\nu}\sum_{\nu'}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}\\
&\times\Big\{f_{1\nu\nu'}(r)[\exp\{h_{\nu\nu'}^{(0)}(r)+j_{\nu\nu'}(r)\}-1-h_{\nu\nu'}^{(0)}(r)-j_{\nu\nu'}(r)]\\
&+\exp\{h_{\nu\nu'}^{(0)}(r)+j_{\nu\nu'}(r)\}-1-h_{\nu\nu'}^{(0)}(r)-j_{\nu\nu'}(r)-\frac{1}{2}\,(h_{\nu\nu'}^{(0)}(r)+j_{\nu\nu'}(r))^2\Big\}
\end{aligned}\quad (52)$$

$$\begin{aligned}
=&\frac{1}{2V}\sum_{k}\Big\{-\ln L(\alpha,\cdots,\tau;\,F_{\nu\nu'}^{(0)}(k))-\sum_{\nu}\rho_\nu F_\nu^{(0)}(k)-\frac{1}{2}\sum_{\nu}\sum_{\nu'}\rho_\nu\rho_{\nu'}F_{\nu\nu'}^{(0)}(k)^2\Big\}\\
&+\frac{1}{2}\sum_{\nu}\sum_{\nu'}\rho_\nu\rho_{\nu'}\Big\{F_{\nu\nu'}^{(1)}(0)-\int d\boldsymbol{r}\,f_{\nu\nu'}^{(0)}(r)\,h_{\nu\nu'}^{(0)}(r)-\frac{1}{2}\int d\boldsymbol{r}\,h_{\nu\nu'}^{(0)}(r)^2\Big\}.
\end{aligned}\quad (52')$$

6b. If the potential between particles is of the form:

$$\phi_{\nu\nu'}(r)=e_\nu e_{\nu'}\phi(r) \quad (\nu, \nu'=\alpha, \cdots, \tau) \tag{53}$$

and $\phi(r)$ has the Fourier transform, then we may take

$$f_{1\nu\nu'}(r)\equiv 0 \quad \text{and} \quad j_{\nu\nu'}(r)=-\frac{1}{kT}\phi_{\nu\nu'}(r) \tag{54}$$

and hence

$$\begin{aligned} &f^{(0)}_{\nu\nu'}(r)=e_\nu e_{\nu'} f^{(0)}(r), \quad f^{(0)}(r)=-\frac{1}{kT}\phi(r), \\ &F^{(0)}_{\nu\nu'}(k)=e_\nu e_{\nu'} F^{(0)}(k). \end{aligned} \tag{55}$$

For this case, it can be easily verified that

$$L(\alpha, \cdots, \tau; F^{(0)}(k))=1-\epsilon F^{(0)}(k) \tag{56}$$

$$\frac{1}{2}L_{\nu\nu'}(\alpha, \cdots, \tau; F^{(0)}_{\nu\nu'}(k))=-\rho_\nu\rho_{\nu'}e_\nu e_{\nu'}F^{(0)}(k) \quad \text{for} \quad \nu\neq\nu', \tag{57}$$

where

$$\epsilon\equiv\sum_{\nu=\alpha}^{\tau}\rho_\nu e_\nu^2, \tag{58}$$

by the mathematical induction. For a one-component system, the former of these is eq. (31) itself and for a two-component system, these are obtained from eq. (39) when (55) is substituted there, thence we verify (56) and (57) by confirming that the substitution of "(56) and (57) for n species" in the right-hand side of (36) and (38) gives us "(56) and (57) for $n+1$ species".

Substituting these in eq. (29) with (27) and eq. (28) with (26) and (27), we get

$$\left.\begin{aligned} &H^{(0)}_{\nu\nu'(\alpha,\cdots,\tau)}(k)=e_\nu e_{\nu'}H^{(0)}(k) \quad (\nu, \nu'=\alpha, \cdots, \tau), \\ &H^{(0)}(k)=\frac{F^{(0)}(k)}{1-\epsilon F^{(0)}(k)}-F^{(0)}(k), \\ &h^{(0)}_{\nu\nu'}(r)=e_\nu e_{\nu'}h^{(0)}(r). \end{aligned}\right\} \tag{59}$$

Then, (51) and (52) can be written as

$$\left[-\frac{A'}{VkT}\right]_{b+b'+R_b}=\frac{1}{2V}\sum_k\left\{-\ln(1-\epsilon F^{(0)}(k))-\epsilon F^{(0)}(k)\right\}+\frac{1}{2}F^{(0)}(0)\left(\sum_\nu\rho_\nu e_\nu\right)^2 \tag{60}$$

and

$$\begin{aligned} \left[-\frac{A'}{VkT}\right]_{b+b'+R_b+c''}=&\frac{1}{2V}\sum_k\left\{-\ln(1-\epsilon F^{(0)}(k))-\epsilon F^{(0)}(k)\right\} \\ &+\frac{1}{2}F^{(0)}(0)\left(\sum_\nu\rho_\nu e_\nu\right)^2+\frac{1}{2}\sum_\nu\sum_{\nu'}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}\left\{\exp(e_\nu e_{\nu'}[h^{(0)}(r)+f^{(0)}(r)])-1\right. \end{aligned}$$

$$-e_\nu e_{\nu'}[h^{(0)}(r)+f^{(0)}(r)]-\frac{1}{2}e_\nu^2 e_{\nu'}^2[h^{(0)}(r)+f^{(0)}(r)]^2\Big\} \tag{61}$$

$$= \frac{1}{2V}\sum_k\Big\{-\ln(1-\epsilon F^{(0)}(k))-\epsilon F^{(0)}(k)\Big\}$$

$$+\frac{1}{2}\sum_\nu{}'\sum_{\nu'}{}'\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}\Big\{\exp(e_\nu e_{\nu'}[h^{(0)}(r)+f^{(0)}(r)])-1\Big\}$$

$$-\frac{\epsilon^2}{4}\int d\boldsymbol{r}[h^{(0)}(r)+f^{(0)}(r)]^2-\frac{1}{2}H^{(0)}(0)\Big(\sum_\nu \rho_\nu e_\nu\Big)^2. \tag{61'}$$

For the case of Coulombic potential, $\phi(r)=1/r$,

$$\left.\begin{aligned} f^{(0)}(r)&=-\frac{1}{kT}\cdot\frac{1}{r}, \quad F^{(0)}(k)=-\frac{1}{kT}\cdot\frac{4\pi}{k^2}\\ H^{(0)}(k)+F^{(0)}(k)&=-\frac{1}{kT}\cdot\frac{4\pi}{k^2+\kappa^2}\\ h^{(0)}(r)+f^{(0)}(r)&=-\frac{1}{kT}\frac{1}{r}e^{-\kappa r}\end{aligned}\right\} \tag{62}$$

where

$$\kappa^2=\frac{4\pi\epsilon}{kT}=\frac{4\pi}{kT}\sum_{\nu=\alpha}^{\tau}{}'\rho_\nu e_\nu^2.$$

The integrations in (60) and (61′) are easily performed except the second term of (61′):

$$\Big[-\frac{A'}{VkT}\Big]_{b+b'+R_b}=\frac{\kappa^3}{12\pi}, \tag{63}$$

$$\Big[-\frac{A'}{VkT}\Big]_{b+b'+R_b+c''}=\frac{\kappa^3}{96\pi}+\frac{1}{2}\sum_\nu\sum_{\nu'}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}\Big\{\exp(e_\nu e_{\nu'}[h^{(0)}(r)+f^{(0)}(r)])-1\Big\}, \tag{64}$$

where the last terms of Formulae (60) and (61′) have been dropped because $\sum_\nu \rho_\nu e_\nu$ vanishes for neutral systems. Formula (63) is the result of Debye and Hückel[7),8)] and $-e_\nu e_{\nu'}[h^{(0)}(r)+f^{(0)}(r)]kT=-e_\nu e_{\nu'}\exp(-\kappa r)/r$ is the selfconsistent potential of Debye.[7)] Formula (61) with terms of triangles of watermelons has been given by Yukhnovsky.[9)]

6c. Even when the potential has core part, if the long range part of the potential is of the form:

$$e_\nu e_{\nu'}\phi(r) \quad (\nu,\nu'=\alpha,\cdots,\tau), \tag{65}$$

then we can perform the integrations by taking the division of the potential in such a way as

$$f^{(0)}_{\nu\nu'}(r)=e_\nu e_{\nu'} f^{(0)}(r) \quad \text{with} \quad f^{(0)}(r)=-\frac{1}{kT}\phi(r) \tag{66}$$

is valid. A substitution of this in eqs. (47), (48) and (50) gives for $f_{1\nu\nu'}(r)$ and $j_{\nu\nu'}(r)$:

$$\exp\{-f_{1\nu\nu'}(r)\}[1+f_{1\nu\nu'}(r)]=\exp\{-\frac{1}{kT}\phi_{\nu\nu'}(r)-e_\nu e_{\nu'} f^{(0)}(r)\},$$

$$j_{\nu\nu'}(r)=e_\nu e_{\nu'} f^{(0)}(r)-f_{1\nu\nu'}(r). \tag{67}$$

For instance, in the case of Coulombic potential with hard core:

$$\phi_{\nu\nu'}(r)=\begin{cases}\infty & r<a_{\nu\nu'}\\ e_\nu e_{\nu'}\dfrac{1}{r} & r>a_{\nu\nu'},\end{cases} \tag{68}$$

we will adopt

$$\left.\begin{aligned} &f^{(0)}_{\nu\nu'}(r)=e_\nu e_{\nu'} f^{(0)}(r), \quad f^{(0)}(r)=-\frac{1}{kT}\frac{1}{r}\\ &f_{1\nu\nu'}(r)=\begin{cases}-1 & r<a_{\nu\nu'}\\ 0 & r>a_{\nu\nu'}\end{cases}\\ &j_{\nu\nu'}(r)=\begin{cases}e_\nu e_{\nu'} f^{(0)}(r)+1 & r<a_{\nu\nu'}\\ e_\nu e_{\nu'} f^{(0)}(r) & r>a_{\nu\nu'}.\end{cases}\end{aligned}\right\} \tag{69}$$

Substitutions of (67) into (51) and (52) give

$$\left[-\frac{A'}{VkT}\right]_{b+b'+R_b}=-\frac{1}{2V}\sum_k\left\{-\ln(1-\epsilon F^{(0)}(k))-\epsilon F^{(0)}(k)\right\}+\frac{1}{2}F^{(0)}(0)\left(\sum_\nu \rho_\nu e_\nu\right)^2$$

$$-\frac{1}{4}\sum_\nu\sum_{\nu'}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}\, f_{1\nu\nu'}(r)^2, \tag{70}$$

and

$$\left[-\frac{A'}{VkT}\right]_{b+b'+R_b+c''}=-\frac{1}{2V}\sum_k\left\{-\ln(1-\epsilon F^{(0)}(k))-\epsilon F^{(0)}(k)\right\}-\frac{1}{2}H^{(0)}(0)\left(\sum_\nu \rho_\nu e_\nu\right)^2$$

$$+\frac{1}{2}\sum_\nu\sum_{\nu'}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}\,(1+f_{1\nu\nu'}(r))\left\{\exp\{e_\nu e_{\nu'}[h^{(0)}(r)+f^{(0)}(r)]-f_{1\nu\nu'}(r)\}-1\right\}$$

$$+\frac{1}{2}\sum_\nu\sum_{\nu'}\rho_\nu\rho_{\nu'}\int d\boldsymbol{r}\, f_{1\nu\nu'}(r)-\frac{\epsilon^2}{4}\int d\boldsymbol{r}[h^{(0)}(r)+f^{(0)}(r)]^2. \tag{71}$$

For the Coulombic potential with hard core, these become

$$\left[-\frac{A'}{VkT}\right]_{b+b'+R_b}=\frac{\kappa^3}{12\pi}-\frac{1}{4}\sum_\nu\sum_{\nu'}\rho_\nu\rho_{\nu'}\frac{4\pi}{3}a^3_{\nu\nu'} \tag{72}$$

and

$$\left[-\frac{A'}{VkT}\right]_{b+b'+R_b+c''}=\frac{\kappa^3}{96\pi}+\frac{1}{2}\sum_{\nu}\sum_{\nu'}\rho_\nu\rho_{\nu'}$$
$$\times\left\{\int_{a_{\nu\nu'}}^{\infty}4\pi r^2 dr\left[\exp\left\{-\frac{1}{kT}e_\nu e_{\nu'}\frac{1}{r}e^{-\kappa r}\right\}-1\right]-\frac{4\pi}{3}a_{\nu\nu'}^3\right\}. \tag{73}$$

The osmotic pressure obtained by

$$\frac{p}{kT}=\sum_{\nu=\alpha}^{\tau}\rho_\nu\frac{\partial}{\partial\rho_\nu}\frac{A}{VkT}-\frac{A}{VkT}=\rho-\frac{A'}{VkT}+\sum_{\nu}\rho_\nu\frac{\partial}{\partial\rho_\nu}\frac{A'}{VkT} \tag{74}$$

with the use of (73) as the approximation for A'/VkT is identical to that obtained by Meeron[8),10)].

The formulae of the radial distribution function in the approximations of c^* and $d^{*\dagger}$ have the form of (43C^*) and (43D^*) and that in the hyper-netted chain approximation the form of (43), again. Meeron[11)] has obtained the expansion of $g(r)$ which has $[g(r)]_{c^*}$ as the leading term.

§ 7. Conclusion

We have obtained the formulae for the free energy and the radial distribution function in the hyper-netted chain approximation, taking into account of far more graphs than considered in the approximations in the past for the case of the multi-component system: Final formulae are given by (23) and (43) with (24), (12)–(14) and (36)–(38). When we are to apply these approximation to practical problems, we have to solve the recurrence equation containing Fourier transformations.

Numerical computations for strong electrolytes will be the subject of a subsequent paper in this series.

Appendix I: A proof of the lemma

Here, a proof of the lemma will be given, based on a known relation between the free energy and the radial distribution function.

We use the relation by Bogolyubov[12)] and Hiroike,[13)] generalized to the multi-component system:

$$\rho_\nu\rho_{\nu'}g_{\nu\nu'}(r)=\frac{1}{V}\int d\boldsymbol{r}^N\sum_{k=1}^{n_\nu}\sum_{l=1}^{n_{\nu'}}{}'\,\delta(\boldsymbol{r}-\boldsymbol{r}_{kl})$$
$$\times\exp\{-\sum_{i>j}\phi_{\nu_i\nu_j}(r_{ij})/kT\}/\int d\boldsymbol{r}^N\exp\{-\sum_{i>j}\phi_{\nu_i\nu_j}(r_{ij})/kT\}$$

(Σ′: $k\neq l$ if $\nu=\nu'$)

† The graphs considered in these approximations will be represented by Fig. 13 of I with the specification of the species of particles at end points as ν and ν', if we are to replace the large letters B, C, D, ⋯ by the corresponding small letters b, c, d, ⋯, there.

$$= \frac{1}{V}(1+\delta_{\nu,\nu'}) \int d\boldsymbol{r}^N \sum_{N\geq k>l\geq 1} \sum \frac{\delta \phi_{\nu_k \nu_l}(\boldsymbol{r}_{kl})}{\delta \phi_{\nu\nu'}(\boldsymbol{r})}$$

$$\times \exp\{-\sum_{i>j} \phi_{\nu_i \nu_j}(r_{ij})/kT\} / \int d\boldsymbol{r}^N \exp\{-\sum_{i>j} \phi_{\nu_i \nu_j}(r_{ij})/kT\}$$

$$= \frac{1}{V}(1+\delta_{\nu,\nu'}) \frac{\delta A}{\delta \phi_{\nu\nu'}(\boldsymbol{r})} = (1+\delta_{\nu,\nu'}) \exp\{-\phi_{\nu\nu'}(r)/kT\} \frac{\delta(-A/VkT)}{\delta f_{\nu\nu'}^{(0)}(\boldsymbol{r})}, \tag{A·1}$$

with the definition of the functional derivative by

$$\delta\phi(\boldsymbol{r})/\delta\phi(\boldsymbol{x}) = \delta(\boldsymbol{r}-\boldsymbol{x}), \quad \delta\psi(\boldsymbol{r})/\delta\phi(\boldsymbol{r}) = 0, \tag{A·2}$$

which implies that

$$\text{if } g(\boldsymbol{r}) = \delta A/\delta f(\boldsymbol{r}), \quad \text{then } G(\boldsymbol{k}) = \delta A/\delta F(\boldsymbol{k}) \tag{A·3}$$

with

$$\delta f(r)/\delta F(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \quad \text{and} \quad \delta F(\boldsymbol{k}')/\delta F(\boldsymbol{k}) = V\delta_{\boldsymbol{k},\boldsymbol{k}'},$$

where $F(\boldsymbol{k})$ and $G(\boldsymbol{k})$ are the Fourier transforms of $f(\boldsymbol{r})$ and $g(\boldsymbol{r})$, respectively.

We substitute eqs. (2) and (41) respectively into both sides of this equation (A·1) and have, by picking up the terms in a set of powers of ρ_ν's,

$$\rho_\nu \rho_{\nu'} \prod_{i=\alpha}^{\tau} \frac{\rho_i^{m_i}}{m_i!} \int\int \cdots \int d\boldsymbol{r}_3 \cdots d\boldsymbol{r}_{m_\alpha + \cdots + m_\tau + 2} \sum \Pi f_{ij}^{(0)} f_{k\kappa}^{(0)}$$

$$= (1+\delta_{\nu,\nu'}) \frac{\delta}{\delta f_{\nu\nu'}^{(0)}(\boldsymbol{r})} \prod_{i=\alpha}^{\tau} \frac{\rho_i^{n_i}}{n_i!} \int\int \cdots \int d\boldsymbol{r}_1 \cdots d\boldsymbol{r}_{n_\alpha + \cdots + n_\tau} \sum \Pi f_{ij}^{(0)} \tag{A·4}$$

where $m_i = n_i - \delta_{i,\nu} - \delta_{i,\nu'}$. In this equation, we can equate the terms in the same powers of $f_{ij}^{(0)}$. Then if we pick up the terms of ring in the right-hand side, we have for the right-hand side

$$(1+\delta_{\nu,\nu'}) \frac{\delta}{\delta f_{\nu\nu'}^{(0)}(\boldsymbol{r})} \sum_{n_\alpha=0}^{\infty} \cdots \sum_{n_\tau=0}^{\infty} \cdot \prod_{i=\alpha}^{\tau} \frac{\rho_i^{n_i}}{n_i!} \int\int \cdots \int d\boldsymbol{r}_1 \cdots d\boldsymbol{r}_{n_\alpha + \cdots + n_\tau} \sum \Pi f_{ij}^{(0)}$$

Rings shown in Fig. 6.

$$= (1+\delta_{\nu,\nu'}) \frac{\delta}{\delta f_{\nu\nu'}^{(0)}(\boldsymbol{r})} \frac{1}{2V} \sum_{\boldsymbol{k}} \Big\{ -\ln L(\alpha, \cdots, \tau;\ F_{\nu\nu'}^{(0)}(\boldsymbol{k}))$$

$$- \sum_{\nu} \rho_\nu F_\nu^{(0)}(\boldsymbol{k}) - \frac{1}{2} \sum_{\nu} \sum_{\nu'} \rho_\nu \rho_{\nu'} F_{\nu\nu'}^{(0)}(\boldsymbol{k})^2 \Big\} \tag{A·5R}$$

and the corresponding left-hand side becomes

$$\rho_\nu \rho_{\nu'} \sum_{m_\alpha=0}^{\infty} \cdots \sum_{m_\tau=0}^{\infty} \cdot \prod_{i=\alpha}^{\tau} \frac{\rho_i^{m_i}}{m_i!} \int\int \cdots \int d\boldsymbol{r}_3 \cdots d\boldsymbol{r}_{m_\alpha + \cdots + m_\tau + 2} \sum \Pi f_{ij}^{(0)} f_{k\kappa}^{(0)}$$

Chains between $\nu_1 = \nu$ and $\nu_2 = \nu'$ shown in Fig. 8.

$$= \rho_\nu \rho_{\nu'} h_{\nu\nu'(\alpha,\cdots,\tau)}^{(0)}(\boldsymbol{r}_{12}). \tag{A·5L}$$

An application of the relation (A·3) to this equation gives

$$\rho_\nu{}^2 H^{(0)}_{\nu(\alpha,\cdots,\tau)}(\boldsymbol{k}) = -\frac{\partial}{\delta F_\nu{}^{(0)}(\boldsymbol{k})} \ln L(\alpha, \cdots, \tau;\ F^{(0)}_{\nu\nu'}(\boldsymbol{k})) - \rho_\nu - \rho_\nu{}^2 F_\nu{}^{(0)}(\boldsymbol{k}),$$

$$\rho_\nu \rho_{\nu'} H^{(0)}_{\nu\nu'(\alpha,\cdots,\tau)}(\boldsymbol{k}) = -\frac{1}{2}\frac{\partial}{\delta F^{(0)}_{\nu\nu'}(\boldsymbol{k})} \ln L(\alpha, \cdots, \tau;\ F^{(0)}_{\nu\nu'}(\boldsymbol{k})) - \rho_\nu \rho_{\nu'} F^{(0)}_{\nu\nu'}(\boldsymbol{k}). \tag{A·6}$$

The identity (A·4) holds for any function $f^{(0)}_{\nu\nu'}(\boldsymbol{r})$, especially even when we replace $f^{(0)}_{\nu\nu'}(\boldsymbol{r})$ by $f^{(n)}_{\nu\nu'}(\boldsymbol{r})$. As the consequence, (A·5R) and (A·5L) in which (0) is replaced by (n) are equal and hence we can obtain equations (A·6) with (0) relaced by (n), which are to be proved.

Appendix II: Expansion formulae for the potentials of average force of n particles by means of the hyper-netted chains*

For the sake of completeness, we present here the expansion formulae for the potentials of average force of n particles by means of the hyper-netted chains.

The potential of average force of n particles, denoted as $W^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$, will be defined by[14]

$$W^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = -kT \ln[\rho^{(n)}_{\nu_1,\cdots,\nu_n}(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n)/\rho^n] \tag{A·7}$$

where $\boldsymbol{x}_i$ denotes $(\boldsymbol{r}_i, \nu_i)$, coordinate and species of i-th particle, and $\rho^{(n)}_{\nu_1\cdots\nu_n}(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_n)$ is the distribution function of n particles of species $\nu_1, \cdots, \nu_n$; $W^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = -kT \ln g_{\nu_1\nu_2}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$. Following Mayer,[15] we divide $W^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ into component potentials of average force, $w^{(m)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m)$, as

$$W^{(1)}(\boldsymbol{x}) = 0$$

$$W^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = w^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$$

$$W^{(3)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = w^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) + w^{(2)}(\boldsymbol{x}_2, \boldsymbol{x}_3) + w^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_3) + w^{(3)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$$

$$\cdots\cdots\cdots\cdots$$

$$W^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \sum_{m=2}^{n} \sum_{n \geq i_m > \cdots > i_2 > i_1 \geq 1} \sum \cdots \sum w^{(m)}(\boldsymbol{x}_{i_1}, \cdots, \boldsymbol{x}_{i_m}). \tag{A·8}$$

The virial expansion formulae for the potentials of average force given by Meeron[3] are taken as these equations with

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* After the completion of this manuscript, I received a reprint of a paper[16] from Dr. E. Meeron, in which he has obtained the expansion formula for the potential of average force of n particles by means of the hyper-netted chains for the case of one-component system. It can be easily verified that our results in this Appendix contain his result as a special one. I express here my sincere thanks to Dr. E. Meeron for sending me reprints of his papers published. (Oct. 13, 1958)

$$w^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \phi_{\nu_1\nu_2}(r_{12}) - kT \sum_{s_\alpha=0}^{\infty} \cdots \sum_{s_\tau=0}^{\infty} \cdot \prod_{\nu=\alpha}^{\tau} \frac{\rho_\nu^{s_\nu}}{s_\nu!}$$

$$\times \iint \cdots \int d\boldsymbol{r}_3 \cdots d\boldsymbol{r}_{s_\alpha+\cdots+s_\tau+2} \sum_{\substack{s_\alpha+\cdots+s_\tau+2 \geq i > j \geq 3 \\ s_\alpha+\cdots+s_\tau+2 \geq k \geq 3 \\ 2 \geq \kappa \geq 1}} \prod f_{ij}^{(0)} f_{k\kappa}^{(0)}$$

$$w^{(m)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m) = -kT \sum_{s_\alpha=0}^{\infty} \cdots \sum_{s_\tau=0}^{\infty} \cdot \prod_{\nu=\alpha}^{\tau} \frac{\rho_\nu^{s_\nu}}{s_\nu!}$$

$$\times \iint \cdots \int d\boldsymbol{r}_{m+1} \cdots d\boldsymbol{r}_{m+s_\alpha+\cdots+s_\tau} \sum_{\substack{m+s_\alpha+\cdots+s_\tau \geq i > j \geq m+1 \\ m+s_\alpha+\cdots+s_\tau \geq k \geq m+1 \\ m \geq \kappa \geq 1}} \prod f_{ij}^{(0)} f_{k\kappa}^{(0)}, \quad m > 2 \qquad (A \cdot 9)$$

where sums in the right-hand side should be over all products for which each molecule of the set $\{\boldsymbol{x}_{m+1}, \cdots, \boldsymbol{x}_{m+s_\alpha+\cdots+s_\tau}\}$ is connected to all of the molecules $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m\}$ directly or indirectly and moreover at least two of $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m\}$ by an independent path and also for which the molecules $\{\boldsymbol{x}_{m+1}, \cdots, \boldsymbol{x}_{m+s_\alpha+\cdots+s_\tau}\}$ are connected with each other even when we erase the lines ended at the molecules $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m\}$.

Just as in § 5, these expansions can be rewritten as

$$w^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \phi_{\nu_1\nu_2}(r_{12}) - kT\, h_{\nu_1\nu_2}(r_{12}) - kT \sum_{s_\alpha=0}^{\infty} \cdots \sum_{s_\tau=0}^{\infty} \cdot \prod_{\nu=\alpha}^{\tau} \frac{\rho_\nu^{s_\nu}}{s_\nu!}$$

$$\times \iint \cdots \int d\boldsymbol{r}_3 \cdots d\boldsymbol{r}_{s_\alpha+\cdots+s_\tau+2} \sum_{\substack{s_\alpha+\cdots+s_\tau+2 \geq i > j \geq 3 \\ s_\alpha+\cdots+s_\tau+2 \geq k \geq 3 \\ 2 \geq \kappa \geq 1}} \prod (f_{ij}+h_{ij})(f_{k\kappa}+h_{k\kappa})$$

$$w^{(m)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m) = -kT \sum_{s_\alpha=0}^{\infty} \cdots \sum_{s_\tau=0}^{\infty} \cdot \prod_{\nu=\alpha}^{\tau} \frac{\rho_\nu^{s_\nu}}{s_\nu!}$$

$$\times \iint \cdots \int d\boldsymbol{r}_{m-1} \cdots d\boldsymbol{r}_{m-s_\alpha+\cdots+s_\tau} \sum_{\substack{m+s_\alpha+\cdots+s_\tau \geq i > j \geq m+1 \\ m+s_\alpha+\cdots+s_\tau \geq k \geq m+1 \\ m \geq \kappa \geq 1}} \prod (f_{ij}+h_{ij})(f_{k\kappa}+h_{k\kappa}), \quad m > 2, \qquad (A \cdot 10)$$

where sums in the right-hand sides should be over all products for which each molecule of $\{\boldsymbol{x}_{m+1}, \cdots, \boldsymbol{x}_{m+s_\alpha+\cdots+s_\tau}\}$ is a junction and is connected to all of the molecules $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m\}$ directly or indirectly and moreover at least two of $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m\}$ by an independent path and also for which the molecules $\{\boldsymbol{x}_{m-1}, \cdots, \boldsymbol{x}_{m+s_\alpha+\cdots+s_\tau}\}$ are connected with each other even when we erase the lines ended at the molecules $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m\}$.

In these equations, the factors in $w^{(m)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m)$ for $m \geq 3$ correspond to graphs which do not reduce to a line, so that if we adopt the hyper-netted chain approximation for the calculations of $W^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ for $n \geq 3$ then we will drop all the terms $w^{(m)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m)$ for $m \geq 3$ and retain the terms which reduce to a line by identifications in $w^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$; as the result we are only to calculate $W^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and use the generalized superposition approximation

$$W^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \sum_{n \geq i > j \geq 1} \sum W^{(2)}(\boldsymbol{x}_i, \boldsymbol{x}_j) \qquad (A \cdot 11)$$

in order to get $W^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ in the hyper-netted chain approximation, which gives simply

$$W^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \sum_{n \geq i > j \geq 1} \sum [\phi_{\nu_i \nu_j}(r_{ij}) - kT\, h_{\nu_i \nu_j}(r_{ij})]. \qquad (A \cdot 12)$$

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Spin-Orbit Potential with Pion Scattering Correction

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Dispersion relation is applied to the pion theory of spin-orbit potential.

§ 1. Introduction

It is an interesting problem in pion theory of nuclear potential that the large two body spin-orbit potential does exist or not. If we assumed the phenomenological two body spin-orbit potential with the appropriate sign and good order of magnitude, we would have an easiest explanation for the polarization in nucleon-nucleon scattering and for the nuclear shell structure The matter is, however, not so simple. In fact there is still a possibility of having an explanation of polarization or nuclear level splitting without assuming the large two body spin-orbit potential.[1] In this aspect we shall here calculate the two body spin-orbit potential from pion theory in a better approximation that the old one.

Hitherto the spin-orbit potential has been calculated in several approximations. For instance, it has been shown that the fourth order T. M. O. potential has a rather small spin-orbit potential of the order (μ/M).[2] On the other hand, the technique of dispersion relation paved a way for taking the effects of the multiple pion-nucleon scattering into account.[3] Here the spin-orbit potential with scattering correction is derived by making use of dispersion relation of pion-nucleon scattering in expectation that the dominant resonance in the scattering data (3/2, 3/2) may give a rather large contribution to the spin-orbit potential. The spin-orbit potential is obviously non-static, so we may think that the static dispersion relation cannot be used. However, it is not necessarily so, as we shall see in the next section where it will be seen that the spin-orbit potential in the approximation of order (μ/M) can be calculated by the static dispersion relation.

Two approximations are made: (a) the number of exchanged pions is limited to two, (b) non-static effect is treated in the first order approximation, that is, the velocity dependent potential of order (μ/M) is retained.

It is to be noted that the number of pions emitted and absorbed by the same nucleon is not limited, and that all the velocity dependence is of the spin-orbit type in our order of approximation.[4] The recoil corrections independent of velocity are contained partly in the experimental pion-nucleon cross section.

§ 2. Recoil correction to static dispersion relations

The velocity dependent interaction between two nucleons has a close analogy with the retarded interaction between two electrons. The retarded electromagnetic field of uniformly moving electron can be obtained from the coulomb field of rest electron by the Lorentz transformation. This situation is quite the same in the case of pion field of uniformly moving nucleon. As we shall see in the following, it is only this retarded effect that contributes to velocity dependent correction to the static potential in the order of approximation (μ/M), and the deviation from uniform motion contributes to the potential only in the order higher than (μ/M).

The S-matrix of pion scattering with fixed nucleon $(j, q/S, i, k)$ is obtained from the dispersion relation as a functional of experimental cross section of pion-nucleon scattering.

Then the S-matrix in the case of uniformly moving nucleon with velocity $\boldsymbol{V}=\boldsymbol{P}/M$ can be given by the following substitutions which represent the Galilei transformation,

$$\boldsymbol{k}\rightarrow\boldsymbol{k}-(k_0/M)\boldsymbol{P}, \tag{1}$$

$$k_0\rightarrow k_0-(\boldsymbol{P}\cdot\boldsymbol{k})/M. \tag{1'}$$

Though the above consideration might appear too crude, the result represented by the formulae (1) and (1′) are just the same as those which are derived by the more standard calculation in the following.

The total Hamiltonian for P-wave pion is

$$H=(\boldsymbol{P}^2/2M)+\sum_k \omega_k \boldsymbol{a}_k^*\boldsymbol{a}_k+(f_0/\mu)\sum_k (i/\sqrt{2\omega_k})$$
$$\times[(\boldsymbol{\sigma}\cdot\boldsymbol{k})-\omega_k(\boldsymbol{\sigma}\cdot\boldsymbol{P})/M]\times[\boldsymbol{a}_k\exp(i\boldsymbol{k}\cdot\boldsymbol{r})-\boldsymbol{a}_k^*\exp(-i\boldsymbol{k}\cdot\boldsymbol{r})]\boldsymbol{\tau}, \tag{2}$$

in which one nucleon interacting with pion field is considered, and as we are interested only in the velocity dependence of the order (μ/M), the order of $(\boldsymbol{\sigma}\cdot\boldsymbol{P})$ and $\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$ in eq. (2) is unimportant.

Jost transformation

$$U=\exp[i\sum_k (\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{a}_k^*\boldsymbol{a}_k]$$

transforms H to H':

$$H'=UHU^{-1}$$
$$=(\boldsymbol{P}^2/2M)+\sum_k[\omega_k-(\boldsymbol{P}\cdot\boldsymbol{k})/M]\boldsymbol{a}_k^*\boldsymbol{a}_k+(f_0/\mu)$$
$$\times\sum_k(i/\sqrt{2\omega_k})[(\boldsymbol{\sigma}\cdot\boldsymbol{P})-\omega_k(\boldsymbol{\sigma}\cdot\boldsymbol{P})/M]\times(\boldsymbol{a}_k-\boldsymbol{a}_k^*)\boldsymbol{\tau}+\Delta H, \tag{3}$$

where ΔH is the term of order (μ/M) and does not contain the nucleon momentum $\boldsymbol{P}$. ΔH gives no contribution to the spin-orbit potential in the approximation (b) of § 1, hence it can be neglected. Then H' commutes with $\boldsymbol{P}$, that is, $\boldsymbol{P}$ is a

constant of motion. The nucleon can be regarded as moving uniformly. Comparing the Hamiltonian H' with static Hamiltonian, we find that the difference between them is just the substitution (1) and (2) in the four dimensional language. For the case of two nucleons, the situation is the same.

§ 3. Derivation of potential

The S-matrix of nucleon-nucleon scattering is

$$S=-\frac{1}{2(2\pi)^3}\iint\frac{(j,\,q,\,\boldsymbol{P}'|S^{(1)}|i,\,k,\,\boldsymbol{P})}{(k_0{}^2-\omega_k{}^2+i\varepsilon)}\times\frac{(j,\,-q,\,-\boldsymbol{P}'|S^{(2)}|i,\,-k,\,-\boldsymbol{P})}{(q_0{}^2-\omega_q{}^2+i\varepsilon)}d^4k\,d^4q, \tag{4}$$

where notations are borrowed from Miyazawa's.[3] $S^{(1)}$ and $S^{(2)}$ are obtained by the method discussed in the preceding section. S is composed of two terms,

$$S=S\text{ (static)}+S\text{ (recoil)}, \tag{5}$$

where S (static) is the one discussed by Miyazawa and Konuma.[5] S (recoil) is composed of two terms,

$$S\text{ (recoil)}=S(f^4)+S\text{ (scattering correction)}, \tag{6}$$

where $S(f^4)$ is the contribution from the fourth order perturbation theory which was discussed by one of the authors.[21] By using the method similar to Miyazawa's,[5] we find that the final expression is as follows:

$$V\text{ (recoil)}=V(f^4)+V\text{ (scattering correction)}, \tag{7}$$

$$V(f^4)=-(2\mu)(\mu/M)(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(f^2/4\pi)^2\times(1+2x+2x^2+x^3)(e^{-2x}/x^6)(\boldsymbol{L}\cdot\boldsymbol{S}), \tag{8}$$

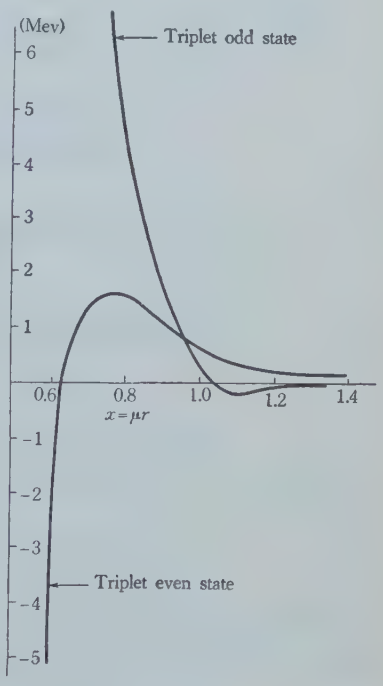

Fig. 1. $V(x,\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$.

V (scattering correction)

$$=(2\mu)(\mu/M)(6+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{x}\frac{d}{dx}+\frac{2}{x^2}\right)\frac{1}{x^2}\times\left(\frac{d^2}{dx^2}-\frac{4}{x}\frac{d}{dx}+\frac{4}{x^2}\right)\left\{\frac{2}{9(2\pi)^3}\left(\frac{f^2}{4\pi}\right)F_1(x)-\frac{2}{81(2\pi)^5}F_2(x)\right\}(\boldsymbol{L}\cdot\boldsymbol{S})$$

$$-(2\mu)(\mu/M)\frac{1}{4(2\pi)^3}\frac{1}{x^4}\left(\frac{d^4}{dx^4}-\frac{12}{x}\frac{d^3}{dx^3}+\frac{60}{x^2}\frac{d^2}{dx^2}\right.$$

$$\left.-\frac{144}{x^3}\frac{d}{dx}+\frac{144}{x^4}\right)\left\{\frac{4}{9}(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f^2}{4\pi}\right)F_5(x)\right.$$

$$\left.-\frac{2}{9}(6-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f^2}{4\pi}\right)F_3(x)+\frac{1}{81\pi^2}(6-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)F_4(x)\right\}(\boldsymbol{L}\cdot\boldsymbol{S}), \tag{9}$$

where $x=\mu r$ and

$$\frac{1}{\mu^2}F_1(x)=\int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\,\frac{\sigma_{33}(p)}{\omega_p\omega_k(\omega_p+\omega_k)}, \tag{10}$$

$$\frac{1}{\mu^4}F_2(x)=\int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\int_0^\infty dq\,\frac{\sigma_{33}(p)\sigma_{33}(q)}{\omega_p\omega_q(\omega_k+\omega_p)(\omega_k+\omega_q)}, \tag{11}$$

$$\frac{1}{\mu^4}F_3(x)=\int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\,\frac{\sigma_{33}(p)}{\omega_p}$$

$$\times\left[\frac{2\omega_k^2+\omega_k\omega_p+\omega_p^2}{\omega_p^2\omega_k^3(\omega_p+\omega_k)}+\frac{1}{\omega_p\omega_k(\omega_p+\omega_k)^2}\right], \tag{12}$$

$$\frac{1}{\mu^6}F_4(x)=\int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\int_0^\infty dq\,\frac{\sigma_{33}(p)\omega_{33}(q)}{\omega_p\omega_q}$$

$$\times\frac{1}{(\omega_k+\omega_p)(\omega_k+\omega_q)(\omega_p+\omega_q)}\left[\frac{\omega_p+\omega_q}{\omega_k(\omega_k+\omega_q)}+\frac{2}{\omega_k+\omega_p}+\frac{2}{\omega_p+\omega_q}\right], \tag{13}$$

$$\frac{1}{\mu^4}F_5(x)=\int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\,\frac{\sigma_{33}(p)}{\omega_p^3\omega_k^2}$$

$$=\frac{\pi}{2}\exp(-2x)\int_0^\infty dp\,\frac{\sigma_{33}(p)}{\omega_p^3}. \tag{14}$$

We have assumed $\sigma_{33}\gg\sigma_{13}$, σ_{31}, σ_{11} in the above expressions. The numerical results are shown in Fig. 1, where

$$V\ (\text{recoil})=V(x,\ \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{L}\cdot\boldsymbol{S}).$$

§ 4. Discussions

Unfortunately, our numerical results are not satisfactory as to the magnitude and sign that are required from the shell model.

If there should be any corrections which have been discarded in our approximations but would modify the quantitative behavior of our potential, it would be

he contribution of the second or even higher orders in (μ/M). A more satisfactory olution would be to find a mathematical technique which does not resort to the xpansion in the power of (μ/M).

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Appendix. *S*-**wave contribution to spin-orbit potential**

To give completeness we have derived the *S*-wave contribution in the approximation that the *S*-wave interactions were taken into account in the Born approximation.

The additional potential due to *S*-wave scattering is as follows,

$$
\begin{aligned}
V' = {} & (2\mu)(\mu/M)\frac{1}{x}\frac{d}{dx}\left\{-\frac{8}{9(2\pi)^3}\mu(a_1+2a_3)\left(\frac{1}{x^2}\right)F_6(x)\right. \\
& \left.+(\tau_1\cdot\tau_2)\frac{4}{27(2\pi)^3}\mu(a_1-a_3)\left(\frac{1}{x^2}\right)F_7(x)\right\}(\boldsymbol{L}\cdot\boldsymbol{S}) \\
& -(2\mu)(\mu/M)(\tau_1\cdot\tau_2)\left(\frac{f^2}{4\pi}\right)\left(\frac{1}{3}\right)\mu(a_1-a_3)\frac{(1+x)^2}{x^6}\exp(-2x)(\boldsymbol{L}\cdot\boldsymbol{S}) \\
& -(2\mu)(\mu/M)\frac{2}{9(2\pi)^3}\left(\frac{1}{x^4}\right)\left(\frac{d^2}{dx^2}-\frac{4}{x}\frac{d}{dx}+\frac{4}{x^2}\right) \\
& \times\left\{2\mu(a_1+2a_3)F_8(x)+(\tau_1\cdot\tau_2)\left(\frac{1}{3}\right)\mu(a_1-a_3)F_9(x)\right\}(\boldsymbol{L}\cdot\boldsymbol{S}), \qquad \text{(A·1)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\mu}F_6(x) &= \int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\,\frac{\sigma_{33}(p)}{\omega_p(\omega_k+\omega_p)}, \\
F_7(x) &= \int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\,\frac{\sigma_{33}(p)}{\omega_k+\omega_p}, \\
\frac{1}{\mu^3}F_8(x) &= \int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\,\frac{\sigma_{33}(p)}{\omega_p\omega_k(\omega_p+\omega_k)^2}, \\
\frac{1}{\mu^2}F_9(x) &= \int_0^\infty dk\,k\sin(2kr)\int_0^\infty dp\,\frac{\sigma_{33}(p)}{\omega_p(\omega_p+\omega_k)}\left(\frac{1}{\omega_k}+\frac{1}{\omega_p+\omega_k}\right). \qquad \text{(A·2)}
\end{aligned}
$$

This potential is very small in its magnitude compared with the one given by eq. (9).

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Electrostatic Interactions in an Electron-Ion Gas at High Density

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First, consideration is given to the influence of the Coulomb interaction of electrons upon the electron-ion interaction and the ionic oscillation of an electron-ion gas at high density by utilizing a method of normal modes on the one hand and a collective description on the other hand. Following these methods, we derive the so-called Froelich Hamiltonian for the electron-phonon interaction by eliminating the electronic plasmon from the total Hamiltonian completely. As the interactions between electron pairs with opposite momenta and with different spins which are essential to the superconductivity are discarded in the method of normal modes, the results obtained by this method are considered as valid only for the normal state. These important interactions of electron pairs can be taken into account in the collective description, by means of which the electron-phonon Hamiltonian with the screened interaction of electrons is obtained.

Second, a dispersion equation for the system under consideration is obtained by the sound approximation proposed in a previous paper, which proves to be equivalent to Sawada's pair theory for small momentum transfers. The dispersion equation obtained exhibits a "metastable" state corresponding to the phonon excitation. The eigenfrequency and the decay time are estimated for a certain region of wave numbers. By the perturbation expansion of the density matrix in the sound approximation a relationship between the correlation energy and the dispersion equation is pointed out and the role of the Coulomb interaction of electrons is illustrated graphically.

§ 1. Introduction

In the previous paper referred to as I,[1] the so-called sound approximation has been introduced by linearizing the equations of the density matrices to obtain the excited levels and the correlation energy of an electron gas immersed in a uniform positive charge distribution. In another paper referred to as II[2], the dispersion equations and the correlation energy of an electron gas with the Bloch elecron-phonon interaction have been studied. There the Coulomb interaction of electrons has been considered separately and the plasma mode has been eliminated from the Hamiltonian by a method of the collective description, in which the problem of the subsidiary condition unsettled in Bohm's theory[3] has been solved. Although the sound approximation is exact in the high density limit and capable of yielding the correlation energy identical with the result of Gell-Mann and Brueckner,[4] we have been satisfied with the result identical with Macke's[5] of less accuracy in the high density limit by discarding a number of perturbation terms beyond the second order. Further we have found in dealing with the electron-phonon interaction that the

sound approximation or the linearization method becomes so poor for this interaction that it leads to a divergent result. In this approximation only the interaction between electrons and holes are considered, the other important interactions being neglected, especially the interactions between electron pairs with opposite momenta and with different spins which are essential to the superconductivity.[6]

As pointed out by Bardeen[7] and Nakajima,[8] if the Bloch interaction between electrons and phonons is employed the Coulomb interaction does not appear explicitly. With the use of the collective description of Bohm and Pines,[3] Bardeen and Pines[9] refined Nakajima's theory to derive the Froelich interaction appearing in the theory of superconductivity,[10] starting with the Hamiltonian having the attractive Coulomb interaction between electrons and ions and the energy of the ionic oscillation with frequency $\Omega_0=(4\pi nze^2/M)^{1/2}$, where n, z and M are the electronic density, the ionic charge and mass. Following their standpoint we shall consider the electron ion interaction by means of three methods; the method of normal modes similar to Sawada's,[11] the sound approximation and the collective description.

In section 2, considering the commutation relations of the density matrices on the linearization assumption, we shall show that the sound approximation is equivalent to the method of Gell-Mann and Brueckner[4] in obtaining the correlation energy, the energy shift for the fully degenerate state. In sections 3 and 4, dealing with a method of normal modes equivalent to Sawada's pair theory,[11] it will be shown that the interaction between electrons and phonons corresponding to the Bloch interaction for metals is derived by eliminating the interaction between the electronic plasmon and the ionic plasmon. It can be pointed out that the matrix elements for transitions of the individual electrons almost perpendicular to the Fermi surface become negligibly small compared with those almost parallel to the Fermi surface.

Recently, Wentzel[12] has proposed a method of "equivalent Hamiltonian" to obtain the dispersion equation, the correlation energy and the diamagnetism of an electron gas at high density. In this theory the Bloch interaction and the Coulomb interaction have been introduced simultaneously. As above-mentioned, if the Coulomb interaction of electrons is introduced in the explicit form the attractive Coulomb interaction between electrons and ions should be employed. Taking account of this point, we can obtain the dispersion equation compatible with the picture of Bardeen[7] and Nakajima.[8] In sections 5 and 6, the dispersion equation will be obtained by the sound approximation. From this dispersion equation the phonon frequency and its decay time are estimated. In the appendix the role of the Coulomb interaction of electrons will be illustrated graphically by expanding the density matrix in a sum of linked clusters.

In the method of normal modes the interactions of excited electrons and of holes between themselves are completely neglected. As pointed out by Bardeen,

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Preliminary reports have appeared in: Busseiron Kenkyu **3**(1958), 83; 354; **4**(1958), 1; Prog. Theor. Phys. **20**(1958), 245.

Cooper and Schrieffer[6] the interactions between excited electrons or between holes with antiparallel spins are essential to the superconducting state. Therefore we can expect no superconducting state from the method of normal modes. Recently Bogolubov and his co-workers[13] have made a distinguished contribution to the theory of the superconductivity. As long as the Coulomb interaction is concerned, inspite of their exact formulation, its complexity seems to prevent us from arriving at a quantitatively definite result.

In section 7 we shall separate the plasma mode by means of the collective description proposed in II in order to consider the same problem as dealt in section 4. Following this method, we can take the interactions between holes and holes and those between excited electrons and excited electrons into account. The total Hamiltonian splits up into the plasmon energy, the phonon energy, the energy of screened electrons and the interaction energy between screen screened electrons and phonons which becomes of importance in the theory of superconductivity.

§ 2. Commutation relations of density matrices

On the linearization assumption proposed in II, we shall consider a method of normal modes which proves to be equivalent to Sawada's pair theory, in which the interactions of holes and of excited electrons between themselves are solely discarded. It can be shown that these interactions disappear in obtaining the energy shift for the lowest state. This is considered to be a reason why these two methods of approach lead to the identical expression for the correlation energy of an electron gas at high density first given by Gell-Mann and Brueckner.[4]

Now, we consider the commutation relations of density matrices expressed in the quantized form

$$g(\boldsymbol{P},\boldsymbol{q})=a^*(\boldsymbol{P}-\tfrac{1}{2}\boldsymbol{q})a(\boldsymbol{P}+\tfrac{1}{2}\boldsymbol{q}), \tag{2·1}$$

where a^* and a are the creation and annihilation operator anticommutative with one another except that $[a(\boldsymbol{P}),\ a^*(\boldsymbol{P})]_+=1$. Here we denote $g(\boldsymbol{P},\ \boldsymbol{q})$ by $B_P(\boldsymbol{q})$ if $(\boldsymbol{P}\cdot\boldsymbol{q})$ is positive and by $B_P^*(-\boldsymbol{q})$ if $(\boldsymbol{P}\cdot\boldsymbol{q})$ is negative. As will be shown readily, $-(\boldsymbol{P}\cdot\boldsymbol{q})$ implies the frequency of $g(\boldsymbol{P}\cdot\boldsymbol{q})$. From the anticommutation relations of a^* and a we find

$$\begin{aligned}[g(\boldsymbol{P},\ \boldsymbol{q}),\ g(\boldsymbol{P}',\ \boldsymbol{q}')]=&a^*(\boldsymbol{P}-\tfrac{1}{2}\boldsymbol{q})a(\boldsymbol{P}'+\tfrac{1}{2}\boldsymbol{q}')\delta_{P+(q/2)P'-(q'/2)}\\&-a^*(\boldsymbol{P}'-\tfrac{1}{2}\boldsymbol{q}')a(\boldsymbol{P}+\tfrac{1}{2}\boldsymbol{q})\delta_{P'+(q'/2)P-(q/2)}.\end{aligned} \tag{2·2}$$

According to the linearization approximation we ignore the correlation between different momenta $\boldsymbol{q}$ and the above expression is replaced by

$$[g(\boldsymbol{P},\ \boldsymbol{q}),\ g(\boldsymbol{P}',-\boldsymbol{q}')]=[n(\boldsymbol{P}-\tfrac{1}{2}\boldsymbol{q})-n(\boldsymbol{P}+\tfrac{1}{2}\boldsymbol{q})]\,\delta_{PP'}\,\delta_{qq'}, \tag{2·3}$$

$$= \begin{cases} [B_P(\boldsymbol{q}), B^*_{P'}(\boldsymbol{q}')], & (\boldsymbol{P}\cdot\boldsymbol{q})>0,\ (\boldsymbol{P}'\cdot\boldsymbol{q}')>0, \\ [B^*_P(-\boldsymbol{q}), B_{P'}(-\boldsymbol{q}')], & (\boldsymbol{P}\cdot\boldsymbol{q})<0,\ (\boldsymbol{P}'\cdot\boldsymbol{q}')<0, \\ \text{etc.}, & \end{cases} \quad (2\cdot3)'$$

$$n(\boldsymbol{P}) = a^*(\boldsymbol{P})a(\boldsymbol{P}) .$$

Further, if the state $\boldsymbol{P}-\frac{1}{2}\boldsymbol{q}$ is occupied and state $\boldsymbol{P}+\frac{1}{2}\boldsymbol{q}$ is vacant, (2·3) may be written as

$$[B_P(\boldsymbol{q}), B^*_{P'}(\boldsymbol{q}')] = \delta_{PP'}\,\delta_{qq'}, \qquad (\boldsymbol{P}\cdot\boldsymbol{q})>0 . \quad (2\cdot4)$$

Among the quantities $g(\boldsymbol{P}, \boldsymbol{q})$ those specified by two momenta, $\boldsymbol{P}-\frac{1}{2}\boldsymbol{q}$ and $\boldsymbol{P}+\frac{1}{2}\boldsymbol{q}$ lying within the Fermi sphere, can be considered as creating a hole at $\boldsymbol{P}+\frac{1}{2}\boldsymbol{q}$ and annihilating another hole at $\boldsymbol{P}-\frac{1}{2}\boldsymbol{q}$. Therefore these quantities give rise to interactions of holes between themselves. In the same way, the quantities specified by two momenta lying out of the Fermi sphere give rise to interactions of excited electrons between themselves. If we are permitted to exclude these interactions from the Hamiltonian, we have only to deal with those operators $B^*_P(\boldsymbol{q})$ and $B_P(\boldsymbol{q})$ representing a creation and an annihilation operator of a pair of electron and hole creating a pair out of and annihilating a pair into the Fermi sea. In place of (2·4) we can put

$$[B_P(\boldsymbol{q}), B^*_{P'}(\boldsymbol{q}')] = \delta_{PP'}\,\delta_{qq'}, \qquad |\boldsymbol{P}-\tfrac{1}{2}\boldsymbol{q}|\leq P_0,\ |\boldsymbol{P}+\tfrac{1}{2}\boldsymbol{q}|>P_0, \quad (2\cdot5)$$

for these operators, where $\boldsymbol{P}-\frac{1}{2}\boldsymbol{q}$ lies within the Fermi surface and $\boldsymbol{P}+\frac{1}{2}\boldsymbol{q}$ lies out of it. These operators are precisely equivalent to Sawada's operator composed on the analogy with the positron theory. As these boson operators are composed referring to the Fermi sphere, their objects of application are inevitably confined to cases of weak interactions. In cases of considerably strong interations for which non-degenerate states play an important role, the commutation relations (2·5) become physically less meaningful. For most cases, however, (2·5) may be considered as valid with sufficient accuracy.

In I, on the analogy with Tomonaga's theory of sound waves,[14] we have introduced creation operators b^* and annihilation operators b in place of B^* and B by

$$\boldsymbol{b}^*_s(\boldsymbol{q}) = (2\pi/q)^{1/2}\sum_{\boldsymbol{P}}{}^{(\boldsymbol{P}\cdot\boldsymbol{s})=s^2}_{(\boldsymbol{P}\cdot\boldsymbol{q})>0} B^*_P(\boldsymbol{q}) , \quad (2\cdot6)$$

$$\boldsymbol{b}_s(\boldsymbol{q}) = (2\pi/q)^{1/2}\sum_{\boldsymbol{P}}{}^{(\boldsymbol{P}\cdot\boldsymbol{s})=s^2}_{(\boldsymbol{P}\cdot\boldsymbol{q})>0} B_P(\boldsymbol{q}) , \quad (2\cdot7)$$

where $\boldsymbol{s}$ stands for a radial vector in the (x, y)-plane perpendicular to $\boldsymbol{q}$ and the summations are carried out along the z-axis under the condition $(\boldsymbol{P}\cdot\boldsymbol{s})=s^2$. For these operators we can obtain the commutation relations

$$[\boldsymbol{b}_s(\boldsymbol{q}), \boldsymbol{b}^*_{s'}(\boldsymbol{q}')] = (2\pi/q)\sum_{\boldsymbol{P}}{}^{(\boldsymbol{P}\cdot\boldsymbol{s})=p^2}_{(\boldsymbol{P}\cdot\boldsymbol{q})>0}[n(\boldsymbol{P}-\boldsymbol{q}/2)-n(\boldsymbol{P}+\boldsymbol{q}/2)]\,\delta_{ss'}\,\delta_{qq'}, \quad (2\cdot8)$$

from (2·3). Contrasted to (2·5), even if a number of holes and excited electrons exist in the neighborhood of the Fermi surface, (2·8) can be written as

$$[\boldsymbol{b}_s(\boldsymbol{q}), \boldsymbol{b}^*_{s'}(\boldsymbol{q}')]=\delta_{ss'}\,\delta_{qq'}\,, \tag{2·9}$$

for small values of s and s' as well as for small values of q and q'. For operators with large values of s and s' (2·9) is not valid any more so that it should be replaced by (2·8). As will be shown in section 5 and Appendix 2, the correlation energy obtained by using (2·9) is identical with the result of Gell-Mann and Brueckner in the absence of the ionic motion.

Next we shall inquire about the fundamental reason why these three different methods, Gell-Mann and Bruekner's, Sawada's, and the method of sound approximation, lead to the identical result. As is well-known, the energy shift for the lowest state, familiar as the "vacuum" polarization in the quantum electrodynamics, is given by the contribution from the closed loops of linked clusters in the expansion of the S-matrix.[19),20),26)] The terms corresponding to these closed loops are composed of the linked products of contractions introduced by Wick.[21)] According to the linearization procedure the contraction of the density operators $\rho(\boldsymbol{q}, t)$ and $\rho(\boldsymbol{q}', t')$ becomes

$$(\rho(\boldsymbol{q}, t), \rho(\boldsymbol{q}'t'))_c=$$
$$\begin{cases}\sum_{P}[n(\boldsymbol{P}-(\boldsymbol{q}/2))-n(\boldsymbol{P}+(\boldsymbol{q}/2))]\exp[-i\omega_P(\boldsymbol{q})(t-t')]\,\delta_{q-q'}\,, & t>t' \\ \sum_{P}[n(\boldsymbol{P}+(\boldsymbol{q}/2))-n(\boldsymbol{P}-(\boldsymbol{q}/2))]\exp[-i\omega_P(\boldsymbol{q})(t'-t)]\delta_{q-q'}\,, & t'>t\,,\end{cases}$$
$$m=\hbar=1 \tag{2·10}$$

where $\omega_p(\boldsymbol{q})=|(\boldsymbol{P}\cdot\boldsymbol{q})|$ implies the frequency and we have used (2·3) assuming

$$\rho(\boldsymbol{q}, t)=\sum_{P} g(\boldsymbol{P}, \boldsymbol{q})\exp[-i(\boldsymbol{P}\cdot\boldsymbol{q})t]$$
$$=\sum_{P}\{B^*_{-P}(-\boldsymbol{q})\exp[i\omega_P(\boldsymbol{q})t]+B_P(\boldsymbol{q})\exp[-i\omega_P(\boldsymbol{q})t]\},$$

If the interaction energy is a functional of the density operator the correlation energy is given by a sum of the averages of the linked products of contractions with respect to the fully degenerate state, the "vacuum" state satisfying

$$B_I(\boldsymbol{q})|0>=0\,.$$

These averages are expressed by

$$\Delta_q(t-t')\delta_{q-q'}=\langle T|\rho(\boldsymbol{q}, t),\ \rho(\boldsymbol{q}', t')|\rangle_0=\sum_{P\ |P+(1/2)q|>P_0}^{|P-(1/2)q|\leqq P_0}\exp[-i\omega_P(\boldsymbol{q})|t-t'|]\,,\delta_{q-q'} \tag{2·11}$$

where $T[\cdots]$ means the chronological operator and $\langle\cdots\rangle_0$ the average for the fully degenerate state. For a small momentum transfer q we integrate over the component of P parallel to q obtaining

$$\Delta_q(t-t')\,\delta_{q-q'}=(q/2\pi)\sum_{s}\exp[-i\omega_s(\boldsymbol{q})|t-t'|]\,,\,\delta_{q-q'}\,, \tag{2·12}$$

which is the contraction in the sound approximation, where s stands for a two

dimensional vector in a plane perpendicular to $\boldsymbol{q}$. The expression (2·11) can also be derived from Sawada's theory which neglects the interaction of holes and of excited electrons between themselves and by assuming the commutation relations in the form given by (2·5).

From these facts we can see the equivalence in the above-mentioned three methods. The same conclusion can be derived from the expansion of the density matrix in the low temperature limit (Appendix 3). In section 5 we shall evaluate the correlation energy by expanding the density matrix in terms of the products of contractions replacing the time variable t by the imaginary time $\tau=it$ proportional to the inverse temperature.

§ 3. Ionic oscillation

In the following we shall consider the transformation of the interaction between electrons and ionic plasmons due to the mutual interaction of electrons. This problem has been ivestigated by Bohm and Staver[15] and by Overhauser[16] by applying the Thomas-Fermi approximation, and more recently by Bardeen and Pines[9] by utilizing the collective description. In these theories the energy of ions has been assumed as

$$H^{(i)}=\tfrac{1}{2}\sum_{q}H^{(i)}(\boldsymbol{q}),\quad H^{(i)}(\boldsymbol{q})=P(\boldsymbol{q})P(-\boldsymbol{q})+\Omega_0^2Q(\boldsymbol{q})Q(-\boldsymbol{q})-\Omega_0\,, \tag{3·1}$$

where Ω_0 is the frequency of the ionic plasmon mentioned in section 1. The attractive Coulomb interaction denoted by $V^{(e\cdot i)}$ is expressed in terms of the coordinate of the ionic plasmon $Q(\boldsymbol{q})$ in the from

$$V^{(e\cdot i)}=\tfrac{1}{2}\sum V^{(e\cdot i)}(\boldsymbol{q}),\quad V^{(e\cdot i)}(\boldsymbol{q})=-(4\pi e^2/q^2)^{1/2}\Omega_0[\rho(\boldsymbol{q})Q(-\boldsymbol{q})+\rho(-\boldsymbol{q})Q(\boldsymbol{q})], \tag{3·2}$$

$$Q(\boldsymbol{q})=q^{-1}(M/N_0)^{1/2}\rho^{(i)}(\boldsymbol{q})\,, \tag{3·3}$$

where $\rho^{(i)}(\boldsymbol{q})$ stands for a fourier coefficient of the density of ions (unit volume).

As shown by Bohm and Staver, the frequency of the ionic oscillation modified by the mutual Coulomb interaction of electrons can be obtained from the linearized equations of the Thomas-Fermi approximation. Remembering that such a linearization procedure has been initiated by Bloch[17] in his theory of sound waves, it can be shown that the same result as Bohm and Staver is rederived from Bloch's theory. Here it deserves to notice that our method of normal modes as well as Sawada's theory may be considered as a generalization of Bloch's theory of sound waves. On the other hand, the theory of Bardeen and Pines is based upon Bohm's collective description, which, however, involves something dubious concerning the subsidiary condition and the cut-off wave number leading to the overestimation of the correlation energy as pointed out by Gell-Mann and Brueckner.[18] Recently, Bogolubov and his collaborators have applied Sawada's theory to the theory of superconductivity to investigate the influence of the Coulomb interaction of electrons. Here we should make a contribution to the problem of electron-ion interaction obtaining the Bloch type interaction between electrons and phonons, the dispersion equ-

ation furnishing the phonon frequency and the correlation energy with the aid of the method of normal modes on the one hand and of the collective description on the other hand.

Now the total Hamiltonian in the quantized form is expressed by

$$H=T+V^{(e)}+V^{(e\text{-}i)}+H^{(i)}-2\pi ne^2\sum_q q^{-2}, \tag{3·4}$$

where T is the kinetic energy of electrons and $V^{(e)}$ is the Coulomb interaction energy and

$$T=\tfrac{1}{2}\sum_{\boldsymbol{P}}\boldsymbol{P}^2 n(P), \quad n(\boldsymbol{P})=a^*(\boldsymbol{P})a(\boldsymbol{P}), \tag{3·5}$$

$$V^{(e)}=\tfrac{1}{2}\sum V^{(e)}(\boldsymbol{q}), \quad V^{(e)}(\boldsymbol{q})=(4\pi e^2\cdot q^2)\rho(\boldsymbol{q})\rho(-\boldsymbol{q}), \quad \rho(\boldsymbol{q})=\sum_{\boldsymbol{P}} g(\boldsymbol{P},\boldsymbol{q}). \tag{3·6}$$

Following the method of normal modes as well as the collective description we can separate out the plasma mode of electrons from the Hamiltonian and the interaction energy $V^{(e\text{-}i)}$ splits up into the interaction between the electronic plasmon and the ionic plasmon and the interaction between the individual modes of electrons and the ionic plasmon $V^{(i,i)}$. After eliminating the former interaction, we find an interaction between the individual screened electrons and the plasma phonon, the modified ionic plasmon $V^{(i,p)}$. The coupling parameter of this interaction may be considered as corresponding to Bloch's parameter in the theory of electric conductivity. The phonon frequency involved in this parameter is real contrasted to the imaginary frequency given by Wentzel which will be discussed in section 5.

§ 4. Method of normal modes

The essential point of the method of normal modes lies in replacing the kinetic energy of electrons T by

$$T=\tfrac{1}{2}\sum_{\boldsymbol{q}} T(\boldsymbol{q}), \quad \sum_{\boldsymbol{P}}\cdots=\left(\frac{1}{2\pi}\right)^3\int d\boldsymbol{P} \tag{4·1}$$

$$T(\boldsymbol{q})=\sum_P \omega_P(\boldsymbol{q})[B_P^*(\boldsymbol{q})B_P^*(\boldsymbol{q})+B_{-P}^*(-\boldsymbol{q})B_{-P}(-\boldsymbol{q})]. \tag{4·2}$$

Then the total Hamiltonian becomes separable with respect to the momentum transfers $\boldsymbol{q}$ so that it can be easily diagonalized. Instead of diagonalizing the total Hamiltonian directly as done by Wentzel, we here diagonalize the electron energy $H^{(e)}=T+V^{(e)}$ with the aid of the normal coordinates given in Appendix 1. From (A·1·8·9) the density operator $\rho(\boldsymbol{q})$ is expanded in terms of the normal coordinates as

After this paper had been written, we read the recent paper of Pines and Nozières[18] in which they show that their result for not so extremely high density becomes identical to the value extrapolated from the expression given by Gell-Mann and Brueckner. As will be pointed out in Appendix 1, the collective description given here proves to be exact in the high density limit.

$$\rho(\boldsymbol{q})=\sum_{P}[\nu_{-P}(-\boldsymbol{q})\eta^{*}_{-P}(-\boldsymbol{q})+\nu^{+}_{P}(\boldsymbol{q})\eta_{P}(\boldsymbol{q})]+\nu_{pl}\hat{\xi}(\boldsymbol{q})\,, \tag{4·3}$$

where η_i^* and η_i are the creation and the annihilation operators of the individual modes, $\hat{\xi}$ the coordinate of the electronic plasmon and $\nu^{\pm}$ the normalization factors. Accoordingly, the total Hamiltonian splits up into the energy of the individual modes $H^{(i,d.)}$, the interaction energy between the individual modes and the ionic plasmon $V^{(i,i)}$and the energy of the two plasmons $H^{(e)}$. Explicitly, these energies are

$$H=H^{(i,d.)}+V^{(i,i)}+H^{(e)}\,, \tag{4·4}$$

$$H^{(i,d.)}=\sum_{\boldsymbol{q}}\sum_{\boldsymbol{P}}[\Omega_{\boldsymbol{P}}(\boldsymbol{q})\{\eta^{*}_{\boldsymbol{P}}(\boldsymbol{q})\eta_{\boldsymbol{P}}(\boldsymbol{q})+\tfrac{1}{2}\}-\omega_{\boldsymbol{P}}(\boldsymbol{q})/2]\,, \tag{4·5}$$

$$V^{(i,i)}=-\sum_{\boldsymbol{q}}\sum_{\boldsymbol{P}}(4\pi e^2/q^2)^{1/2}\Omega_0[\nu^{-}_{\boldsymbol{P}}(\boldsymbol{q})\eta^{*}_{-\boldsymbol{P}}(-\boldsymbol{q})+\nu^{+}_{\boldsymbol{P}}(\boldsymbol{q})\eta_{\boldsymbol{P}}(\boldsymbol{q})]Q(-\boldsymbol{q}), \tag{4·6}$$

$$H^{(e)}=\tfrac{1}{2}\sum_{\boldsymbol{q}}H^{(e)}(\boldsymbol{q}),\quad H^{(e)}(\boldsymbol{q})=\sigma(\boldsymbol{q})\sigma(-\boldsymbol{q})+\omega_{pl}^2\hat{\xi}(\boldsymbol{q})\hat{\xi}(-\boldsymbol{q})$$

$$-\omega_c\Omega_0[\hat{\xi}(\boldsymbol{q})Q(-\boldsymbol{q})+\hat{\xi}(-\boldsymbol{q})Q(\boldsymbol{q})]+H^{(i)}(q), \tag{4·7}$$

where $\Omega_{\boldsymbol{P}}(\boldsymbol{q})$ stand for the zero-points of $\varphi^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})$ (A·1·5) except for the plasma mode. First, the Hamiltonian $H^{(e)}$ is readily diagonalized by an orthogonal transformation

$$\left.\begin{aligned}\hat{\xi}(\boldsymbol{q})&=O_1\theta_1(\boldsymbol{q})+O_2\theta_2(\boldsymbol{q})\\ Q(\boldsymbol{q})&=O_1\theta_2(\boldsymbol{q})-O_2\theta_1(\boldsymbol{q})\end{aligned}\right\} \tag{4·8}$$

$$\left.\begin{aligned}O_1&=(1/2)^{1/2}[1-(\omega_{pl}^2-\Omega_0^2)/(\Omega_P^2-\Omega_L^2)]^{1/2}\\ O_2&=(1/2)^{1/2}[1+(\omega_{pl}^2-\Omega_0^2)/(\Omega_P^2-\Omega_L^2)]^{1/2}\end{aligned}\right\} \tag{4·9}$$

where θ_1 and θ_2 represent the plasma phonon and the plasmon with frequencies Ω_L and Ω_P respectively. These two modes resemble the acoustical and the optical branches of an ionic crystal. The eigenfrequencies are given as the two roots of the secular equation

$$(\omega_{pl}^2-\Omega^2)(\Omega_0^2-\Omega^2)-\Omega_0^2\omega_c^2=0\,, \tag{4·10}$$

whose smaller value becomes

$$\Omega_L^2=\tfrac{3}{5}\left(\frac{\Omega_0^2}{\omega_c^2}\right)\omega_0(\boldsymbol{q})^2+0(q^4)+\cdots \tag{4·11}$$

The corresponding result given by the Thomas-Fermi approximation is

$$\Omega^{(TF)}(q)^2=\tfrac{1}{3}(z/M)\omega_0(\boldsymbol{q})^2+0(q^4)+\cdots,\quad(\Omega_0/\omega_c)^2=z/M\,, \tag{4·12}$$

being somewhat smaller than (4·11). The larger frequency becames slightly larger than the electronic plasmon frequency (see (6·2)). After all $H^{(e)}$ turns out into

$$H^{(e)}=\tfrac{1}{2}\sum_{\boldsymbol{q}}[\pi_1(\boldsymbol{q})\pi_1(-\boldsymbol{q})+\pi_2(\boldsymbol{q})\pi_2(-\boldsymbol{q})+\Omega_L^2\theta_1(\boldsymbol{q})\theta_1(-\boldsymbol{q})$$

$$+\Omega_P^2\theta_2(\boldsymbol{q})\theta_2(-\boldsymbol{q})-\Omega_0], \tag{4·13}$$

$$[\theta_j(\boldsymbol{q}), \pi_j(\boldsymbol{q}')]=i\delta_{\boldsymbol{q}\boldsymbol{q}'}, \quad j=1, 2.$$

We neglect the interaction between the plasmon and the individual modes, then the interaction between the plasma phonon and the individual modes takes the from

$$V^{(i,p)}=-\sum_{\boldsymbol{q}}\sum_{\boldsymbol{P}}(2\Omega_I)^{1/2}[M^{-}_{\boldsymbol{P}}(-\boldsymbol{q})\gamma^{*}_{-\boldsymbol{P}}(-\boldsymbol{q})+M^{+}_{\boldsymbol{P}}(\boldsymbol{q})\gamma_{\boldsymbol{P}}(\boldsymbol{q})]\theta_1(-\boldsymbol{q}), \quad (4\cdot14)$$

$$M^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})=(2\pi e^2/\Omega_I q^2)^{1/2}\Omega_0\nu^{\pm}_{\boldsymbol{P}}(\boldsymbol{q}), \quad (4\cdot15)$$

where the coupling parameter $M^{(\pm)}_{\boldsymbol{P}}(\boldsymbol{q})$ corresponds to Bloch's coefficient. For transitions of the individual electrons almost perpendicular to the Fermi surface, (4·15) turns out into

$$|M^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})|=\pi q(2\pi e^2/\Omega_I)^{1/2}\Omega_0/[2e^2P_0\ln|\omega_0-\omega_{\boldsymbol{P}}(\boldsymbol{q})|], \ \omega_{\boldsymbol{P}}(\boldsymbol{q})\simeq\omega_0 \quad (4\cdot16)$$

which is negligibly small. On the other hand, for the transitions almost parallel to the Fermi surface, namely for $\omega_{\boldsymbol{P}}(\boldsymbol{q})\simeq0$, we get

$$|M^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})|=\pi q(2\pi e^2/\Omega_I)^{1/2}\Omega_0/|\pi q^2+2e^2P_0\pm ie^2q^{-1}\omega_{\boldsymbol{P}}(\boldsymbol{q})|\simeq\pi(5/18)^{1/2}(\Omega_I/P_0)^{1/2} \quad (4\cdot17)$$

using the expression (A·1·6). As long as we are dealing with the method of normal modes, we are obliged to discard the interactions between electron pairs with opposite momenta and with different spins essential to the superconductivity. These interactions discarded here will be restored in the collective description (section 7) and illustrated in Fig. 1 by the graph of Hugenholtz.[19] In this section we have tried to diagonalize the electron Hamiltonian in the first step and considered the electron-ion interaction in the second step so that we may inquire about the relationship to the collective description.

The phonon frequency (4·11) is not the observed one. It is shifted by the Bloch type interaction (4·14) to the value given in section 6. Instead of treading into this problem, here we have considered the coefficient of the Bloch interaction which will be useful for the theory of electron-ion relaxation, especially electric conductivity.

§ 5. Dispersion equation

In order to obtain the dispersion equation we should like to emphasize that the method of normal modes is equivalent to Wentzel's method of "equivalent Hamiltonian." From the dispersion equation it is shown that the Fourier transform of the propagator for electrons F_q is changed into $F_q/1+F_q$ under the influence of the Coulomb interaction of electrons. As far as the small momentum transfers are concerned, we are allowed to deal with the problem in the sound approximation which is exact in the high density limit. The Hamiltonian in the sound approximation corresponding to (4·1) and (4·4) will be given in Appendix 2. From this Hamiltonian the dispersion equation yielding the eigenvalues is readily obtained as

$$\varphi_q(\omega)=1+F_q(\omega)-F_q(\omega)G_q(\omega)=0\,, \tag{5·1}$$

$$F_q(\omega)=(8e^2/q)\sum\omega_s[\omega_s^2-\omega^2]^{-1}\,,\quad G_q(\omega)=\Omega_0^2[\Omega_0^2-\omega^2]^{-1}, \tag{5·2}$$

where $F_q(\omega)$ and $G_q(\omega)$ are the fourier transforms of the propagators for electrons and for the ionic plasmon, respectively. The value of F_q has been doubled due to the spin degeneracy. In the absence of the Coulomb interaction the dispersion equation turns into

$$\varphi_q^{(0)}(\omega)=1-F_q(\omega)G_q(\omega)=0\,, \tag{5·3}$$

which is in accordance with the result given in II. Comparing (5·2) with (5·3) we can see that the Fourier transform F_q is transformed into $F_q'=F_q/1+F_q$. Inspite of such a formal similarity between (5·2) and (5·3) there is an essential difference between them such that the zero points of $\varphi_q(\omega)$ are confined in the right half space exclusive of the imaginary axis, while $\varphi_q^{(0)}(\omega)$ has a zero point on the imaginary axis. This is the reason why the correlation energy obtained from (5·2) is convergent but that obtained from (5.3) becomes divergent. As pointed out by Wentzel, the correlation energy is furnished by the dispersion function $\varphi_q(\omega)$. Therefore we can find the dispersion equation from the expression for the correlation energy given by the sum of the contributions from closed loops, namely from linked clusters for the "vacuum" poralization in the expansion of the density matrix (Appendix 3). Now the energy shift for the Hamiltonian $H_s(q)$ given by (A·2·4) becomes[21),26)]

$$\epsilon_0(q)=\sum_{n=2}^{\infty}(-1)^n/n!\sum_{m=0}^{[n/2]}\binom{n}{2m}\int_{-\infty}^{\infty}\cdots\int d\tau_1\cdots d\tau_n$$

$$\langle T[V^{(e\cdot i)}(q,\tau_1)\cdots V^{(e\cdot i)}(q,\tau_{2m})\,V^{(e)}(q,\tau_{2m+1})\cdots V^{(e)}(q,\tau_n)]\rangle_0\delta(\tau_1)\,, \tag{5·4}$$

where $[n/2]$ implies the largest integer not exceeding $n/2$. The calculation will be given in Appendix 2. Inspite of the fact that the phonon excitation becomes metastable exhibiting an imaginary frequency the correlation energy remain real by taking the low temperature limit. It will not be right if one deals with the expansion of the S-matrix on the usual adiabatic assumption[20)]. Finally the energy shift is given by

$$\epsilon_0(q)=\frac{1}{2\pi}\int_{-\infty}^{\infty}du[\ln\{1+F_q(iu)-F_q(iu)G_q(iu)\}-F_q(iu)]\,, \tag{5·5}$$

which can be transformed into

$$\epsilon_0(q)=\frac{1}{2\pi i}\oint wdw^2\frac{d\ln\phi_q(w)}{dw^2}-\frac{1}{2\pi}\int_{-\infty}^{\infty}duF_q(iu)\,, \tag{5·6}$$

by adding a large hemicycle on the right-hand side of the imaginary axis and integrating by parts, where we have put $w=iu$. From the first term in the right-hand side of (5·6) we obtain the dispersion function $\varphi_q(w)$. The Cauchy integral in (5·6) is nothing but the zero-point energy of $H_s(q)$, hence the former method pro-

posed in II proves to be another way of estimating this integral according to the Wigner-Brillouin perturbation expansion in the absence of the ionic motion.

§ 6. Phonon frequency

The frequency of the plasma phonon given in section 4 is not considered as the observed one, because it is influenced considerably by the Bloch type interaction In the scheme of the sound approximation the observed phonon frequency is obtained from the dispersion equation which may be written as

$$\Omega^2=\Omega_0^2/1-F_q(\Omega)\ . \tag{6·1}$$

An eigenfrequncy corresponding to the observed plasma frequency becomes slightly larger than the electronic plasma frequency such as

$$\Omega_{pl}^2=\omega_{pl}^2+\Omega_0^2+0(q^2)+\cdots\ . \tag{6·2}$$

For a momentum larger than Ω_0/P_0 we can find an unstable state. The eigenfrequency can be estimated by replacing Ω in $f(\Omega)$ by Ω_0 to the effect

$$|\Omega_{pl}^2|=\Omega_0^2/1-F_q(\Omega_0)|,\quad (\mathrm{Im}\Omega_{ph}^2/\mathrm{Re}\Omega_{ph}^2)=\mathrm{Im}F_q(\Omega_0)/1-\mathrm{Re}F_q(\Omega_0), \tag{6·3}$$

$$\mathrm{Re}F_q(\Omega)=-(4e^2P_0/\pi q^2)\left[1-(\Omega/2\omega_0)\ln\frac{\omega_0+\Omega}{\omega_0-\Omega}\right],\ \mathrm{Im}F_q(\Omega)=2e^2\Omega/q^3. \tag{6·4}$$

In this approximation the frequency becomes pure imaginary, if $q<\Omega_0/P_0$ since $F_q(\Omega_0)$ becomes larger than ω_c^2/Ω_0^2. For the momenta q, $\Omega_0/P_0<q\ll\omega_c/P_0$, the ratio of the imaginary part to the real part becomes $\pi\Omega_0/2\omega_0$ and the absolute value is $\Omega_0^2\omega_0^2/3\omega_c^2$ which is equal to the value given by the Thomas-Fermi approximation. Hence the inverse decay time reads

$$\tau^{-1}=\mathrm{Im}\,\Omega_{ph}=(\Omega_0\omega_0/3^{1/2}\omega_c)\sin(\pi\Omega_0/4\omega_0)\ . \tag{6·5}$$

This result can be improved by inserting the Thomas-Fermi value for the Ω in the right-hand side of (6·1) to the effect:

$$\tau^{-1}=(\Omega_0\omega_0/3^{1/2}\omega_c)\sin(\pi\Omega_0/4\cdot3^{1/2}\omega_c)\simeq\frac{\pi}{12}\left(\frac{\Omega_0}{\omega_c}\right)^2\omega_0\,, \tag{6·6}$$

which is considerably smaller than $\mathrm{Re}\,\Omega_{ph}$ by the ratio $\sim\Omega_0/\omega_c$. In section 4 we have divided the diagonalization process into two steps to obtain the familiar electron phonon Hamiltonian. It can be shown that the more or less similar result to the above is obtained from (4·14) assuming the dispersion equation in the type of (5·3).

Here we should point out that the appearance of the imaginary frequency resembles the fact that the excitation energy of a Bose gas with an attractive interaction force becomes pure imaginary for small wave numbers.[22] Noting that the improved result (6·6) is valid for small momenta $q<\omega_c/P_0$, we can see that the real part of the frequency predominates over the imaginary part for the small momenta. These excited states of imaginary frequencies with the small imaginary

parts are called "metastable," while those of pure imaginary frequencies are called "essentially unstable."

§ 7. Collective description

In this section we shall derive the Bloch interaction by the collective description. This problem has been considered by Bardeen and Pines, but the collective description used by them involves at least two dubious points about the subsidiary conditions and the cut-off momentum. These points have been criticized by several authors.[18] The difficulties concerning these points have been overcome by a unitary transformation proposed in II. According to it, the collective mode is separated from the Hamiltonian so completely that the transformed Hamiltonian becomes commutative with both the collective coordinate and its canonical conjugate.[23] In terms of the collective coordinate $\xi(\boldsymbol{q})$ and the canonical conjugate $\sigma(\boldsymbol{q})$ given in (A·1·11) and (A·1·12) the transformation function becomes

$$\left.\begin{aligned} &U=U_1U_2U_1\,,\\ &U_1=\prod_{\boldsymbol{q}}\exp[i\xi(\boldsymbol{q})\beta(\boldsymbol{q})]\,,\quad U_2=\prod_{\boldsymbol{q}}\exp[-i\sigma(\boldsymbol{q})\alpha(\boldsymbol{q})]\,,\\ &[\alpha(\boldsymbol{q}),\ \beta(\boldsymbol{q}')]=i\delta_{\boldsymbol{q}\boldsymbol{q}'}, \end{aligned}\right\} \tag{7·1}$$

where $\alpha(\boldsymbol{q})$ is the auxiliary variable and $\beta(\boldsymbol{q})$ is its canonical conjuagte to represent the collective motion. As long as no confusion occurs, we shall be allowed to employ the same notations, such as a^*, a, B^*, B and $g(\boldsymbol{P},\boldsymbol{q})$ as used in the above, still in the collective representation to describe the individual motion of electrons. By the transformation given by (7·1) we can go over from the original representation imposed by a set of subsidiary conditions

$$\mathfrak{B}(\boldsymbol{q})\Phi_1=0,\ \mathfrak{B}(\boldsymbol{q})=(\omega_{pl}/2)^{1/2}\alpha(\boldsymbol{q})+i(2\omega_{pl})^{-1/2}\beta(-\boldsymbol{q}), \tag{7·2}$$

to the collective representation imposed by

$$\mathfrak{A}(\boldsymbol{q})\Phi_c=0,\ \mathfrak{A}(\boldsymbol{q})=(\omega_{pl}/2)^{1/2}\xi(\boldsymbol{q})+i(2\omega_{pl})^{-1/2}\sigma(-\boldsymbol{q}). \tag{7·3}$$

A quantity in the old representation A_0 can be expressed in the collective representation by

$$A_0=U^{-1}A_cU\,. \tag{7·4}$$

First we consider the density operator $\rho(\boldsymbol{q})$. Corresponding to (4·3) it is transformed as

$$\rho(\boldsymbol{q})=U^{-1}\rho_c(\boldsymbol{q})U=\sum_{\boldsymbol{P}}\lambda(\boldsymbol{P},\boldsymbol{q})g(\boldsymbol{P},\boldsymbol{q})+\nu_{pl}\alpha(\boldsymbol{q}), \tag{7·5}$$

$$\lambda(\boldsymbol{P},\boldsymbol{q})=[\omega_{pl}^2-\omega_c^2-\omega_{\boldsymbol{P}}(\boldsymbol{q})^2]/[\omega_{pl}^2-\omega_{\boldsymbol{P}}(\boldsymbol{q})^2]\,. \tag{7·6}$$

In the same way the total Hamiltonian is transformed into

$$H=H^{(e)}+V^{(s,i)}+H^{(F)} \tag{7·7}$$

$$H^{(e)}=T+V^{(e)}-\tfrac{1}{2}\sum_{q}[\sigma(\boldsymbol{q})\sigma(-\boldsymbol{q})+\omega_{pl}^{2}\xi(\boldsymbol{q})\xi(-\boldsymbol{q})]\,, \tag{7·8}$$

$$V^{(s,i)}=-\Omega_{0}\sum_{q}(4\pi e^{2}/q^{2})^{1/2}\sum_{P}\lambda(\boldsymbol{P},\boldsymbol{q})g(\boldsymbol{P},\boldsymbol{q})Q(-\boldsymbol{q})\,, \tag{7·9}$$

$$H^{(P)}=\tfrac{1}{2}\sum_{q}[\beta(\boldsymbol{q})\beta(-\boldsymbol{q})+\omega_{pl}^{2}\alpha(\boldsymbol{q})\alpha(-\boldsymbol{q})]$$

$$-\omega_{c}\Omega_{0}\sum_{q}\alpha(\boldsymbol{q})Q(-\boldsymbol{q})+H^{(i)}, \tag{7·10}$$

where $H^{(e)}$ represents the energy of screened electrons, $V^{(s,i)}$ the interaction energy between the screened electrons and the ionic plasmon and $H^{(P)}$ the energy of two plasmons. Using the quite similar method as that used in section 4, we can diagonalize $H^{(P)}$ into the form

$$H^{(P)}=\tfrac{1}{2}\sum_{q}[\beta_{1}(\boldsymbol{q})\beta_{1}(-\boldsymbol{q})+\beta_{2}(\boldsymbol{q})\beta_{2}(-\boldsymbol{q})+\Omega_{L}^{2}\alpha_{1}(\boldsymbol{q})\alpha_{1}(-\boldsymbol{q})$$

$$+\Omega_{P}^{2}\alpha_{2}(\boldsymbol{q})\alpha_{2}(-\boldsymbol{q})-\Omega_{0}]\,, \tag{7·11}$$

where

$$\alpha_{1}(\boldsymbol{q})=O_{1}\alpha(\boldsymbol{q})-O_{2}Q(\boldsymbol{q})\,,\quad \alpha_{2}(\boldsymbol{q})=O_{2}\alpha(\boldsymbol{q})+O_{1}Q(\boldsymbol{q})\,,$$

$$[\alpha_{1}(\boldsymbol{q}),\beta_{1}(\boldsymbol{q}')]=[\alpha_{2}(\boldsymbol{q}),\beta_{2}(\boldsymbol{q}')]=i\delta_{qq'}\,. \tag{7·12}$$

Corresponding to (4·13), the interaction energy between the screened electrons and the plasma phonon becomes

$$V^{(s\cdot p)}=\sum_{q}\sum_{P}(2\Omega_{L})^{1/2}M_{P}^{(c)}(\boldsymbol{q})g(\boldsymbol{P},\boldsymbol{q})\alpha_{1}(-\boldsymbol{q})\,, \tag{7·13}$$

$$|M_{P}^{(c)}(\boldsymbol{q})|=\Omega_{0}(2\pi e^{2}/\Omega_{L}q^{2})^{1/2}\lambda(\boldsymbol{P},\boldsymbol{q}) \tag{7·14}$$

$$\simeq(5\Omega_{L}/6n)^{1/2}|(3P_{0}^{2}/5)-P^{2}\cos^{2}\theta|,\ \cos\theta=(\boldsymbol{P}\cdot\boldsymbol{q})/Pq \tag{7·15}$$

where we have again neglected the interaction between the screened electrons and the plasmon. For transitions of the screened electrons parallel to the Fermi surface, namely for $P\simeq P_{0}$, $\theta\simeq\pi/2$, (7·15) turns out to be $\pi(9/10)^{1/2}(\Omega_{L}/P_{0})^{1/2}$. For transitions perpendicular to the Fermi surface it becomes a little smaller as $\pi(2/5)^{1/2}(\Omega_{L}/P_{0})^{1/2}$ while (4·16) become negligibly small. The screened interaction is responsible for this difference which disappears by restricting ourselves to the interactions between excited electrons and holes, since the whole quantities of the individual electrons inclusive of the screened interaction are expressed by the normal coordinates $\eta_{P}^{*}(\boldsymbol{q})$ and $\eta_{P}(\boldsymbol{q})$ as in the method of normal modes and the whole results given in the above are reproduced in the collective description. It is interesting to compare this interaction of electrons with Bohm's and Pines' which reads

$$V_{sh.}^{(P)}=2\pi e^{2}\sum_{q>q_{c}}q^{-2}\rho(\boldsymbol{q})\rho(-\boldsymbol{q}), \tag{7·16}$$

where q_{c} stands for the cut-off momentum, while the screened interaction given here becomes

$$V_{sh.}^{(c)}=2\pi e^2\sum_q q^{-2}\rho(\boldsymbol{q})\rho(-\boldsymbol{q})-\tfrac{1}{2}\sum_q \omega_{pl}(\boldsymbol{q})^2\xi(\boldsymbol{q})\xi(-\boldsymbol{q}). \tag{7·17}$$

The energy of electrons $H^{(e)}$ inclusive of this interaction contributes to the correlation energy as the energy for the scattering states of Sawada. This difference in the expression of the screened interaction will be responsible for the discrepancy between Sawada's result and Pines' result.[18] In the present collective description the problem of the subsidiary condition has been solved. On performing the canonical transformation in this section it is not necessary to use the Bose commutation relations (2·5) but we may use the more exact ones (2·3). Dealing with these commutation relations, we can find the dispersion equation corresponding to (5·1):

$$\varphi_q(\omega)=1-\frac{4\pi e^2\omega^2}{\omega^2-\Omega_0^2}\sum_{\boldsymbol{P}}\frac{n(\boldsymbol{P})}{\{\omega-(\boldsymbol{P}\cdot\boldsymbol{q})\}^2-(q^2/2)^2}=0, \tag{7·18}$$

which leads to the Bohm-Pines dispersion equation by putting $\Omega_0=0$.

A merit of the collective description lies in the fact that we can take account of the interactions between excited electrons and those between holes with differents spins. In Appendix 2, we shall illustrate those graphs discarded in the method of normal modes by the graphical representation of Hugenholtz.[19]

§ 8. Summary

In section 2 we have investigated the method of normal modes and the sound approximation from the linearization procedure. It has been pointed out that the two methods lead to the identical result for the lowest state of an electron gas at high density. In sections 4 and 7 the electronic plasmon has been separated from the total Hamiltonian of an electron-ion gas by the method of normal modes and by the collective description respectively. In the method of normal modes the electron energy has been diagonalized, the interactions between electron pairs with different spins being discarded. In the collective description the screened interaction of electrons is left yet not diagonalized giving rise to the short-range correlation between the individual electrons. By this method we can take account of the interactions of electron pairs discarded in the method of normal modes. The descriptions given in these sections cover a similar ground to the Thomas-Fermi approximation and these two approaches are considered as affording the more extended and exact treatments than the Thomas-Fermi approximation. In sections 5 and 7 we have obtained the dispersion equation and the phonon frequency, using the sound approximation. It has been shown that the phonon excitation becomes metastable. More or less similar results will be obtained by the method of normal modes and also by the collective description restricting ourselves to the electron-hole interaction and eliminating the real plasmon corresponding to the solution with the largest eigenfrequency of the dispersion equation (7·18). The method of normal modes will be useful for obtaining the normal electric conductivity, while it is expected that the collective description given here is applicable to the theory of superconductivity with

little modification of the present prescription.

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Appendix 1. Normal coordinates

The normal modes for the Hamiltonian of electrons $H^{(e)}=T+V^{(e)}$ have been given by Brout[24]. These coordinates are expressed in the present notation in the form

$$[\eta_{\boldsymbol{P}}(\boldsymbol{q}),\eta^*_{\boldsymbol{P}'}(\boldsymbol{q}')]=\delta(\boldsymbol{P}-\boldsymbol{P}')\delta(\boldsymbol{q}-\boldsymbol{q}')\,, \tag{A·1·1}$$

$$\eta^*_{\boldsymbol{P}}(\boldsymbol{q})=B^*_{\boldsymbol{P}}(\boldsymbol{q})+\frac{4\pi e^2}{q^2}\nu^-_{\boldsymbol{P}}(\boldsymbol{q})\int\frac{d\boldsymbol{P}'}{(2\pi)^3}[c^+_{\boldsymbol{q}}(\boldsymbol{P},\boldsymbol{P}')B^*_{\boldsymbol{P}'}(\boldsymbol{q})+d_{\boldsymbol{q}}(\boldsymbol{P},\boldsymbol{P}')B_{-\boldsymbol{P}'}(-\boldsymbol{q})]\,, \tag{A·1·2}$$

$$\eta_{\boldsymbol{P}}(\boldsymbol{q})=B_{\boldsymbol{P}}(\boldsymbol{q})+\frac{4\pi e^2}{q^2}\nu^-_{\boldsymbol{P}}(\boldsymbol{q})\int\frac{d\boldsymbol{P}'}{(2\pi)^3}[c^-_{\boldsymbol{q}}(\boldsymbol{P},\boldsymbol{P}')B_{\boldsymbol{P}'}(\boldsymbol{q})+d_{\boldsymbol{q}}(\boldsymbol{P},\boldsymbol{P}')B^*_{-\boldsymbol{P}'}(-\boldsymbol{q})], \tag{A·1·3}$$

$$c^{\pm}_{\boldsymbol{q}}(\boldsymbol{P},\boldsymbol{P}')=[\omega_{\boldsymbol{P}}(\boldsymbol{q})-\omega_{\boldsymbol{P}'}(\boldsymbol{q})\pm i\alpha]^{-1},\ d_{\boldsymbol{q}}(\boldsymbol{P},\boldsymbol{P}')=[\omega_{\boldsymbol{P}}(\boldsymbol{q})+\omega_{\boldsymbol{P}'}(\boldsymbol{q})]^{-1},$$
$$|\boldsymbol{P}'+\frac{\boldsymbol{q}}{2}|>P_0\,,\ |\boldsymbol{P}'-\frac{\boldsymbol{q}}{2}|\leq P_0\,, \tag{A·1·4}$$

where the normalization factor $\nu^{\pm}$ takes the form

$$\nu^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})=\varphi^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})^{-1}\,, \tag{A·1·5}$$

$$\varphi^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})=1-\frac{4\pi e^2}{q^2}\int\frac{d\boldsymbol{P}'}{(2\pi)^3}[c^{\pm}_{\boldsymbol{q}}(\boldsymbol{P},\boldsymbol{P}')-d_{\boldsymbol{q}}(\boldsymbol{P},\boldsymbol{P}')]\,, \tag{A·1·6}$$

$$\mathrm{Re}\varphi^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})=1-\frac{4e^2\omega_0}{\pi q^3}\left[1-\frac{\omega_{\boldsymbol{P}}(\boldsymbol{q})}{2\omega_0}\ln\frac{\omega_0+\omega_{\boldsymbol{P}}(\boldsymbol{q})}{\omega_0-\omega_{\boldsymbol{P}}(\boldsymbol{q})}\right],$$
$$\mathrm{Im}\varphi^{\pm}_{\boldsymbol{P}}(\boldsymbol{q})=\mp 2ie^2\omega_{\boldsymbol{P}}(\boldsymbol{q})/q^3\,. \tag{A·1·7}$$

Here the spin degeneracy has been taken into account in obtaining (A·1·7). Inversely, $B^*_{\boldsymbol{P}}(\boldsymbol{q})$ and $B_{\boldsymbol{P}}(\boldsymbol{q})$ are expressed in terms of these normal modes as follows:

$$B^*_{\boldsymbol{P}}(\boldsymbol{q})=\eta^*_{\boldsymbol{P}}(\boldsymbol{q})+\frac{4\pi e^2}{q^2}\int\frac{d^3\boldsymbol{P}'}{(2\pi)^3}[\nu^-_{\boldsymbol{P}'}(\boldsymbol{q})c^-_{\boldsymbol{q}}(\boldsymbol{P}',\boldsymbol{P})\eta^*_{\boldsymbol{P}'}(\boldsymbol{q})$$
$$-\nu^+_{-\boldsymbol{P}'}(-\boldsymbol{q})d_{\boldsymbol{q}}(\boldsymbol{P}',\boldsymbol{P})\eta_{-\boldsymbol{P}'}(-\boldsymbol{q})]+(4\pi e^2/q^2)\nu_{pl}(\boldsymbol{q})[\{\omega_{pl}-\omega_{\boldsymbol{P}}(\boldsymbol{q})\}^{-1}\mathfrak{N}^*(\boldsymbol{q})$$
$$-\{\omega_{pl}-\omega_{\boldsymbol{P}}(\boldsymbol{q})\}^{-1}\mathfrak{N}(-\boldsymbol{q})]\,, \tag{A·1·8}$$

$$B_{\boldsymbol{P}}(\boldsymbol{q})=\eta_{\boldsymbol{P}}(\boldsymbol{q})+\frac{4\pi e^2}{q^2}\int\frac{d^3\boldsymbol{P}'}{(2\pi)^3}[\nu^+_{\boldsymbol{P}'}(\boldsymbol{q})c^+_{\boldsymbol{q}}(\boldsymbol{P}',\boldsymbol{P})\eta_{\boldsymbol{P}'}(\boldsymbol{q})$$
$$-\nu^-_{-\boldsymbol{P}'}(-\boldsymbol{q})d_{\boldsymbol{q}}(\boldsymbol{P}',\boldsymbol{P})\eta^*_{-\boldsymbol{P}'}(-\boldsymbol{q})] \tag{A·1·9}$$

$$+(4\pi e^2/q^2)\nu_{pl}(\boldsymbol{q})[\{\omega_{pl}-\omega_P(\boldsymbol{q})\}^{-1}\mathfrak{A}(\boldsymbol{q})-\{\omega_{pl}-\omega_P(\boldsymbol{q})\}^{-1}\mathfrak{A}^*(-\boldsymbol{q})],$$

$\nu_{pl}=n^{1/2}q$,

where $\mathfrak{A}^*(\boldsymbol{q})$ and $\mathfrak{A}(\boldsymbol{q})$ are the creation and the annihilation operator of the electronic plasmon and

$$\mathfrak{A}^*(\boldsymbol{q})=|q|^{-1}\omega_c^2(2n\omega_{pl})^{-1/2}\int\frac{d\boldsymbol{P}}{(2\pi)^3}\left[\frac{B_P^*(\boldsymbol{q})}{\omega_{pl}-\omega_P(\boldsymbol{q})}+\frac{B_{-P}(-\boldsymbol{q})}{\omega_{pl}+\omega_P(\boldsymbol{q})}\right],$$

$$\omega_c^2=4\pi ne^2,\ \omega_{pl}^2=\omega_c^2+(3/5)\omega_0^2+\cdots. \qquad (A\cdot1\cdot10)$$

The coordinate of the plasmon is given by

$$\tilde{\xi}(\boldsymbol{q})=[2\omega_{pl}(\boldsymbol{q})]^{-1/2}[\mathfrak{A}^*(-\boldsymbol{q})+\mathfrak{A}(\boldsymbol{q})], \qquad (A\cdot1\cdot11)$$

$$\sigma(\boldsymbol{q})=i[\omega_{pl}(\boldsymbol{q})/2]^{1/2}[\mathfrak{A}^*(\boldsymbol{q})-\mathfrak{A}(-\boldsymbol{q})]. \qquad (A\cdot1\cdot12)$$

Further, it deserves to notice that these normal coordinates (A·1·2) and (A·1·3) become the normal coordinates for the energy of the screened electrons in the collective description, too, and that the energy for the scattering states of Sawada is given as the expectation value of $H^{(e)}$ for the lowest state satisfying

$$\eta_P(\boldsymbol{q})\Phi_c=0, \qquad (A\cdot1\cdot13)$$

together with the subsidiary conditions (7·3). To prove this we have only to point out that

$$[H^{(e)},\eta_P^*(\boldsymbol{q})]=\omega_P(\boldsymbol{q})\eta_P^*(\boldsymbol{q}),\ [H^{(e)},\eta_P(\boldsymbol{q})]=-\omega_P(\boldsymbol{q})\eta_P(\boldsymbol{q}),$$
$$[H^{(e)},\mathfrak{A}^*(\boldsymbol{q})]=[H^{(e)},\mathfrak{A}(\boldsymbol{q})]=0, \qquad (A\cdot1\cdot14)$$

which imply that there is no matrix element combining these normal modes with the plasma mode, the unique independent mode to complete the orthonormal set. As the plasma zero-point energy is absorbed in the auxiliary field, it does not appear in this expectation value, which is shown to be in accordance with Sawada's former result discarding the plasma energy.

Appendix 2. Sound approximation and correlation energy

In the sound approximation there appear the operators b^* and b defined by (2·6) and (2·7) instead of B^* and B. The Hamiltonian corresponding to (4·4) can be obtained by insterting $(2\pi/q)^{1/2}b_s(\boldsymbol{q})$ and $(2\pi/q)^{1/2}b_s^*(\boldsymbol{q})$ for $B_P(\boldsymbol{q})$ and $B_P^*(\boldsymbol{q})$ respectively and the two dimensional sum $(q/2\pi)\sum_s$ for the three dimensional sum $\sum_P$ in the expressions (3·6) and (4·2). In place of $\rho(\boldsymbol{q})$ and $T(\boldsymbol{q})$ we get

$$\rho(\boldsymbol{q})=\sum_P g(\boldsymbol{P},\boldsymbol{q})=(q/2\pi)^{1/2}\sum_r(2\omega_r)^{1/2}q_r(\boldsymbol{q}), \qquad (A\cdot2\cdot1)$$

$$T_S(\boldsymbol{q})=\sum_r[p_r(\boldsymbol{q})p_r(-\boldsymbol{q})+\omega_r(\boldsymbol{q})^2q_r(\boldsymbol{q})q_r(-\boldsymbol{q})-\omega_r(\boldsymbol{q})], \qquad (A\cdot2\cdot2)$$

where $\omega_s(\boldsymbol{q})=(P_0^2-s^2)^{1/2}q$ and the coordinates q_r and p_r are

$$\left.\begin{aligned} q_r(\boldsymbol{q}) &= (2\omega_r)^{-1/2}[b_r^*(-\boldsymbol{q}) + b_r(\boldsymbol{q})]\,, \\ p_r(\boldsymbol{q}) &= i(\omega_r/2)^{1/2}[b_r^*(\boldsymbol{q}) - b_r(-\boldsymbol{q})]\,. \end{aligned}\right\} \qquad (A\cdot 2\cdot 3)$$

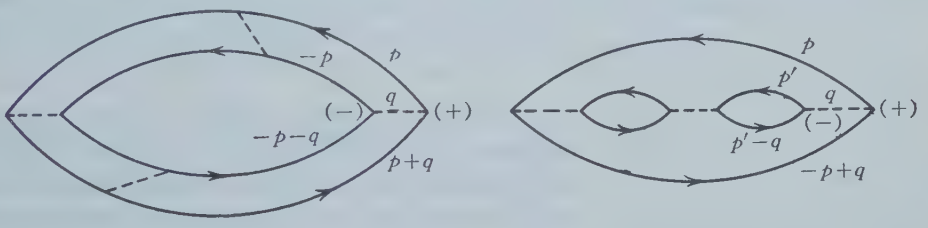
Accordingly, the Hamiltonian is expressed by

$$H_S(\boldsymbol{q}) = T_S(\boldsymbol{q}) + V_S^{(e)}(\boldsymbol{q}) + V_S^{(e\cdot i)}(\boldsymbol{q}) + H^{(i)}(\boldsymbol{q})\,, \qquad (A\cdot 2\cdot 4)$$

$$V_S^{(e)}(\boldsymbol{q}) = (4e^2/q) \sum_{r,s}{}' (\omega_r\omega_s)^{1/2} q_r(\boldsymbol{q}) q_s(-\boldsymbol{q})\,, \qquad (A\cdot 2\cdot 5)$$

$$V_S^{(e\cdot i)}(\boldsymbol{q}) = -(4e^2/q)^{1/2} \Omega_0 \sum_r{}' \omega_r^{1/2}[q_r(\boldsymbol{q}) Q(-\boldsymbol{q}) + q_r(-\boldsymbol{q}) Q(\boldsymbol{q})]\,. \qquad (A\cdot 2\cdot 6)$$

In the sound approximation as well as in the method of normal modes we are obliged to discard those graphs representing the interactions between electron pairs with different spins essential to the superconductivity as illustrated in Fig. 1 (a) by the Hugenholtz representation.



(a) (b)

Fig. 1. Hugenholtz Graphs for $V^{(e\cdot i)}$; ⟵ electron line, ⟶ hole line, ····· ionic plasmon line. (±); +spin and −spin.
(a) A graph discarded in the method of normal modes representing the interaction betwen different spins with opposite momenta ±$\boldsymbol{p}$.
(b) A graph representing the interaction between electrons and holes.

Now from the expansion of the density matrix given by (5·4) we can express the energy shift $\epsilon_0(\boldsymbol{q})$ by a sum of the following products of the propagators

$$\epsilon_0(\boldsymbol{q}) = -\sum_{n=2}^{\infty}\Big[(-1)^n/n!\Big]\sum_{m=0}^{[n/2]}{}'\binom{n}{2m} N_c(m, n) \int_{-\infty}^{\infty}\cdots\int d\tau_1\cdots d\tau_n D_q(\tau_1-\tau_2) S_q(\tau_2-\tau_3)\cdots$$

$$D_q(\tau_{2m-1}-\tau_{2m}) S_q(\tau_{2m}-\tau_{2m-1})\cdots S_q(\tau_n-\tau_1)\delta(\tau_1), \qquad (A\cdot 2\cdot 7)$$

where $D_q(\tau-\tau')$ stands for the propagator of the ionic plasmon and $S_q(\tau-\tau')$ the propagator of the electron pair shown by a dotted line and a solid line, respectively, in Fig. 2. These quantities are defined by the contraction given by the averages of the T-products with respect to the fully degenerate state as follows:

$$S_q(\tau-\tau') = (8\pi e^2/q^2)\langle T[\rho(\boldsymbol{q}, \tau), \rho(-\boldsymbol{q}, \tau')]\rangle_0 = (4e^2/q)\sum_s{}' \exp[-\omega_s(\boldsymbol{q})|\tau-\tau'|], \qquad (A\cdot 2\cdot 8)$$

$$D_q(\tau-\tau') = \Omega_0^2\langle T[Q(\boldsymbol{q}, \tau), Q(-\boldsymbol{q}, \tau')]\rangle_0 = (\Omega_0/2)\exp[-\Omega_0|\tau-\tau'|], \qquad (A\cdot 2\cdot 9)$$

where $i|t|$ in (2·12) has been replaced by $|\tau|$. The Fourier transforms of these propagators F_q and G_q appearing in the dispersion eqution (5·1) are defined by

$$F_q(iu)=\int_{-\infty}^{\infty}d\tau S_q(\tau)\exp[iu\tau]=(4e^2P_0/\pi q^2)R(u/P_0q)\,, \qquad (A\cdot2\cdot10)$$

$$R(x)=1-x\tan^{-1}x^{-1}\,,$$

$$G_q(iu)=\int_{-\infty}^{\infty}d\tau D_q(\tau)\exp[iu\tau]=\Omega_0^2/(u^2+\Omega_0^2). \qquad (A\cdot2\cdot11)$$

$N_c(m,n)$ in $(A\cdot2\cdot7)$ implies the number of identical sets of the contraction products appearing in the expansion of the density matrix. To clarify the role of the Coulomb interaction of electrons we illustrate this number in Fig. 2 for $m=1$, $n=4$. If $n=2m$ there appear $2(2m-1)!$ idential products for the interaction $V_q^{(e\cdot i)}$. These products are represented by a closed alternating chain of the m electron lines and the m plasmon lines. $N_c(m,n)$ can be counted by adding $n-2m$ points corresponding to the mutual interaction of electrons $V_q^{(e)}$ to the m electron lines in each one of the $2(2m-1)!$ closed chains. In this way we obtain $N_c(m,n)=2(2m-1)!\,(n-m-1)!/(m-1)!$.

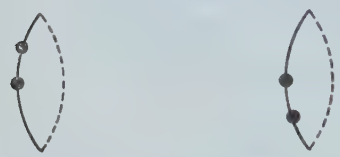


Fig. 2. Graphs for $n=4$, $m=1$; $N_c(1,4)=4$.

The number of sets of the contraction products $N_c(m,n)$ can be counted as the number of graphs indicating the possible ways of adding the n-$2m$ vertices for the Coulomb interaction $V^{(e)}(q)$ to the lines of electron pairs in the graph of the $2m$-th T-product for the free electron pairs and the ionic plasmons represented by an alternating closed chain of m solid lines and m dotted lines.

Without any fuller analytical computations, by replacing $1-G_q(iu)$ by $\omega_{pl}^2/\omega_{pl}^2+\Omega_0^2$; $u=P_0q_c$, $q_c=\omega_{pl}/P_0$ (cut-off wave number of Sawada) it can be seen that the absolute value of the correlation energy is reduced slightly by the ratio ω_{pl}^2/Ω_p^2 corresponding to the result of Herring.[25] One might suspect that the correlation energy is not imaginary inspite of the appearance of the metastable phonon state. Here, instead of treading into details of this problem we would point out that the lowest energy may remain real even if the exited states are metastable.[19]

Appendix 3. Energy shift for the lowest state

By expanding the S-matrix on the usual adiabatic assumption we arrive at an expression for the correlation energy identical to $(5\cdot7)$ except that the contour integral is performed in the upper half plane.[20] If an imaginary pole is enclosed by the contour the imaginary correlation energy would result. Even in such a case, the correlation energy is not always imaginary if the integration is performed enclosing the whole right half plane. This contour integral is given by the expansion of the density matrix in the low temperature limit. In general, for a Hamiltonian $H=H_0+V$ we define D_l and D_r by

$$\exp(\tau H)=D_l(\tau)\exp(\tau H_0),\quad \exp(-\tau H)=\exp(-\tau H_0)D_r(\tau)\,, \qquad (A\cdot3\cdot1)$$

$$\left.\begin{aligned} D_l(\tau) &= 1+\sum_{n=1}^{\infty}(-1)^n\int_{\tau}^{0}d\tau_1\int_{\tau}^{\tau_1}d\tau_2\cdots\int_{\tau}^{\tau_{n-1}}d\tau_n\, V(\tau_1)\cdots V(\tau_n)\,, \\ D_r(\tau) &= 1+\sum_{n=1}^{\infty}(-1)^n\int_{0}^{\tau}d\tau_1\int_{0}^{\tau_1}d\tau_2\cdots\int_{0}^{\tau_{n-1}}d\tau_n V(\tau_1)\cdots V(\tau_n)\,, \\ V(\tau) &= \exp(\tau H_0)\, V \exp(-\tau H_0)\,. \end{aligned}\right\} \qquad (A\cdot 3\cdot 2)$$

Then we find that

$$\begin{aligned} D(\infty,-\infty) &= \lim_{\tau=\infty} D_r(\tau) D_l(-\tau) \\ &= 1+\sum_{n=1}^{\infty}(-1)^n\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2\cdots\int_{-\infty}^{\tau_{n-1}}d\tau_n V(\tau_1)\cdots V(\tau_n), \end{aligned} \qquad (A\cdot 3\cdot 3)$$

which is identical to the expansion of the S-matrix except that the coefficient $(-i)^n$ has been replaced by $(-1)^n$.[26] Now the energy shift for the lowest state $H_0|0\rangle = E_0^{(0)}|0\rangle$ is furnished by

$$\langle 0|D(\infty,-\infty)|0\rangle = \exp M_0\,, \quad M_0 = -2\pi\delta(0)\,\Delta E_0 \qquad (A\cdot 3\cdot 4)$$

where $\delta(0) = (2\pi)^{-1}\lim_{\alpha\to 0}\int_{-\infty}^{\infty}d\tau\exp(i\alpha\tau)$. The expression (5·4) given by the sum of the linked clusters is identified to ΔE_0 given here. We have not trodden into the details of the proof of this result as it will readily be found in the same way as shown by Gell-Mann and Low,[20] and more recently by Hubbard[26] and by Sucher,[20] noting that

$$\lim_{\tau\to-\infty} HD_l(\tau)|0> = \lim_{\tau\to-\infty} E_0 D_l(\tau)|0>, \quad \lim_{\tau\to\infty} <0|D_r(\tau)H = \lim_{\tau\to\infty} <0|D_r(\tau)E_0\,,$$

$$\Delta E_0 = E_0 - E_0^{(0)},$$

and introducing the adiabatic parameter α as

$$V_\alpha(\tau) = V(\tau)\exp(-\alpha|\tau|).$$

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Interactions of Hydromagnetic Waves

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A generalized Rankine-Hugoniot relation for hydromagnetic shock waves propagating perpendicularly to the magnetic field is discussed. Then the interaction of the shock or rarefaction waves with the contact surface on which the magnetic field has a discontinuity is analyzed by a graphical method.

The motion of the high temperature part of a plasma surrounded by the strong magnetic field and the electric current is investigated applying the result of the above analysis. It is shown that under some initial condition the contracting motion of the plasma occurs succeeding its expansion.

§ 1. Introduction

The hydromagnetic shock waves have been investigated by several authors.[1)–5)] The results obtained by them show that the propagation of shocks is generally complicated for the arbitrary configuration of the magnetic field. However, it was shown by Taniuti[6)] that the analysis of the fluid with infinite conductivity can be comparatively simplified, and that the propagation of hydromagnetic waves may be described in the way analogous to that of the ordinary hydrodynamics so far as the one-dimensional propagation of the hydromagnetic wave perpendicular to the magnetic field is concerned.

In the present article, we deal with these perpendicular shock and rarefaction waves and their interaction. In § 2, the various shock relations, involving an equation which corresponds to the Rankine-Hugoniot equation in the hydrodynamics, will be presented.[4)] The analysis of the interaction between the shock, rarefaction wave and the contact discontinuity will be given in § 3. In § 4 we shall discuss, on the basis of these results, an initial value problem in which the fluid is initially at rest and the high density part of the fluid associated with a weak magnetic field is surrounded by the low density part with a strong magnetic field. The subsequent behaviour of the fluid and the magnetic field will be calculated in detail and it will be pointed out that under certain conditions the contact surface can oscillate due to its interaction with the rarefaction and shock waves.

We hope that this result concerning the motion of the contact surface may be of some use for the qualitative explanations of some experimental facts, resulting from the pinched discharge in the cylindrical tube, although it may be difficult to find any actual phenomenon exactly corresponding with this kind of initial value problem.

§ 2. Shock relations

In the following considerations we confine ourselves to the one-dimensional case in which the motion of the fluid is restricted to the x-direction, the magnetic field lies in the z-direction and all the quantities are independent of y and z.

The two constant states (1) and (2) separated by the shock front are specified by the pressure p_1, p_2, the density ρ_1, ρ_2 (the specific volume τ_1, τ_2), the flow speed u_1, u_2 and the strength of the magnetic field H_1, H_2. The indices 1 and 2 will apply to the state of the fluid ahead and behind the shock wave, respectively.

Then the conservation laws of mass, momentum and energy are expressed as follows: (cf. references 2) and 3))

$$\rho_1 v_1 = \rho_2 v_2 = m\,, \tag{2·1}$$

$$p_1 + \rho_1 v_1^2 + H_1^2/(8\pi) = p_2 + \rho_2 v_2^2 + H_2^2/(8\pi)\,, \tag{2·2}$$

$$\rho_1 v_1 (v_1^2/2 + e_1 + p_1\tau_1) + v_1 H_1^2/(4\pi) = \rho_2 v_2 (v_2^2/2 + e_2 + p_2\tau_2) + v_2 H_2^2/(4\pi)\,, \tag{2·3}$$

where $v = u - U$, and U is the speed of the shock front. The internal energy e is a function of p and τ. In our consideration, it is assumed that the electric conductivity of the fluid is infinite so that the magnetic flux is "*frozen*" in the material throughout the process, and that the following relation holds:

$$H_1/\rho_1 = H_2/\rho_2 = \kappa\,, \tag{2·4}$$

where κ is a constant.

1) *Hugoniot curve*

Following the way familiar in the fluid dynamics[7] the Hugoniot relation is obtained from (2·1), (2·2), (2·3) and (2·4):

$$\begin{aligned}\mathcal{H}(p, \tau) \equiv{}& e(p, \tau) + H^2\tau/(8\pi) - e(p_1, \tau_1) \\ & - H_1^2\tau_1/(8\pi) + 1/2(\tau - \tau_1)(p_1^* + p^*) = 0\,,\end{aligned} \tag{2·5}$$

where p^* is the sum of the gas pressure and the magnetic pressure $p_m = H^2/(8\pi)$,[3] i.e.

$$p^* = p + H^2/(8\pi)\,. \tag{2·6}$$

$\mathcal{H}(p, \tau)$ is the generalized form of the Hugoniot function in the fluid dynamics. For polytropic gases with

$$e(p, \tau) = 1/(\gamma - 1)\, p\tau = (1 - \mu^2)/(2\mu^2)\, p\tau\,, \tag{2·7}$$

the Hugoniot function is given by

$$\begin{aligned}\mathcal{H}(p, \tau) \equiv{}& \kappa^2/(16\pi)\,(3/\tau - 3/\tau_1 - \tau_1/\tau^2 + \tau/\tau_1^2) \\ & + 1/(2\mu^2)\,(p\tau - p_1\tau_1 + \mu^2 p_1\tau - \mu^2 p\tau_1) = 0\,,\end{aligned} \tag{2·8}$$

where γ is the ratio of the specific heats and μ is defined by $\mu^2 = (\gamma - 1)/(\gamma + 1)$. The equation (2·8) gives all pairs of values (τ, p) for the state on one side of the shock front in terms of the values τ_1, p_1 and H_1 (or κ) on the other side.

In the following we consider the (p^*, τ)-relation instead of the (p, τ)-relation. Equation (2·8), then, becomes

$$p^*\tau - p_1^*\tau_1 + \mu^2(p_1^*\tau - p^*\tau_1) + \kappa^2(3\mu^2-1)/(8\pi)\,(1/\tau - 1/\tau_1) = 0\,. \qquad (2\cdot9)$$

Here it should be noted that from equation (2·6) the total pressure p^* is restricted by the inequality:

$$p^* > H_1^2/(8\pi)\,(\tau_1/\tau_{\max})^2. \qquad (2\cdot10)$$

ii) *Properties of the Hugoniot curves (for polytropic gases)*[4]

It can be easily shown that in the (τ, p^*)-plane the Hugoniot relation is represented as a continuous curve, the main features of which are not different from those of the fluid dynamics.[7] Here, we summarize simply the results of the calculations: (see Fig. 1)

A. $\tau \to \tau_{\min} = \mu^2\tau_1$, for $p^* \to \infty$,

$\tau = \tau_{\max}$, for $p^* = H_1^2/(8\pi)\,(\tau_1/\tau_{\max})^2$,

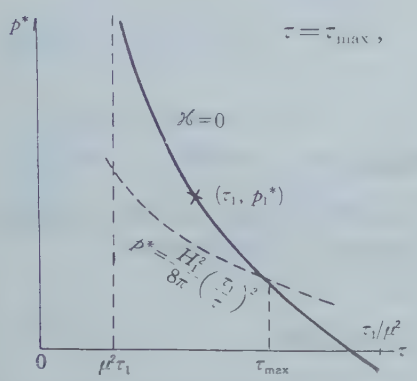

Fig. 1. Hugoniot curve, $\mathcal{H}=0\,(\mu^2\tau_1 < \tau < \tau_{max})$, characterizes all pairs of values (τ, p^*) for the state on one side of the shock front that are compatible with the shock relations when the values (τ_1, p_1^*) and H_1 (or κ) on the other side are given.

where $\tau_{\max}$ is given by the cross point of two curves, the Hugoniot curve (2·9) and curve $p^* = H_1^2/(8\pi)\,(\tau_1/\tau)^2$, and is the root of the third order algebraic equation with respect to τ. One can easily see that $\tau_{\max}$ lies in the following region:

$$\tau_1 < \tau_{\max} < \tau_1/\mu^2. \qquad (2\cdot11)$$

We do not present the actual value of $\tau_{\max}$ explicitly, since it is unnecessary for the following discussions.

B. Along the Hugoniot curve,

$$dp^*/d\tau < 0, \quad d^2p^*/d\tau^2 > 0.$$

C. If the strength of the magnetic field increases, $|dp^*/d\tau|$ increases for the fixed value of p^* or τ. Hence for the same value of $|\tau - \tau_1|$, $|p^* - p_1^*|$ has the larger value for the larger H_1.

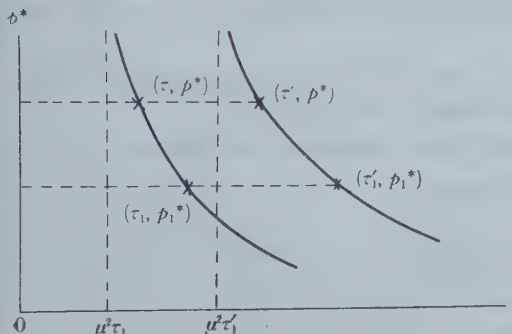

Fig. 2. Two Hugoniot curves, $\mathcal{H}$ and $\mathcal{H}'$, passing through the points (τ_1, p_1^*) (τ_1', p_1^*) respectively.

D. Consider the two Hugoniot curves, $\mathcal{H}$ and $\mathcal{H}'$, passing through the points (τ_1, p_1^*) and (τ_1', p_1^*) respectively $(\tau' > \tau_1)$. For a fixed p^* we have the points

(τ, p^*) on $\mathcal{H}$ and (τ', p^*) on $\mathcal{H}'$ (cf. Fig. 2). Then there exist the following relations between the volume changes, $|\tau-\tau_1|$ and $|\tau'-\tau_1'|$, (cf. Appendix A).

(1) If the shock is weak, i.e. $(p^*-p_1^*)/p^*{}_1 \ll 1$,

$$|\tau-\tau_1| \gtrless |\tau'-\tau_1'|, \quad \text{for} \quad \bar{c}_1\rho_1 \lesseqgtr \bar{c}_1'\rho_1',$$

where $\bar{c}$ is defined by (2·12).

(2) If the shock is strong, i.e. $(p^*-p_1^*)/p_1^* \gg 1$,

$$|\tau-\tau_1| \gtrless |\tau'-\tau_1'|, \quad \text{for} \quad \rho_1 \lesseqgtr \rho_1'.$$

iii) *Properties of the shock transition*

In comparison with the shock transition in the fluid dynamics no other remarkable fact is obtainable under our present assumptions and configurations, except that the shock velocity exceeds the *effective* sound speeds $\bar{c}$[3),4),6)] which is defined by

$$\left.\begin{aligned} &\bar{c}^2 = c_s^2 + c_a^2, \\ &c_s^2 = \partial p/\partial\rho, \\ &c_a^2 = H^2/(4\pi\tau). \end{aligned}\right\} \qquad (2\cdot12)$$

§ 3. (u, p^*)-relation

As in the ordinary hydrodynamics,[7)] the (u, p^*)-relation is more suitable to the following study of the wave interactions than the $(\tau-p^*)$-relation.

i) *Shock waves* ($\boldsymbol{S}$)

From the shock relations (2·1) and (2·2), one gets

$$-m = -\rho_1 v_1 = -\rho_2 v_2 = (p_2^*-p_1^*)/(u_2-u_1) = \pm\sqrt{(p_2^*-p_1^*)/(\tau_1-\tau_2)}, \qquad (3\cdot1)$$

$$\left.\begin{aligned} &u_2 = u_1 \pm \phi_1(p_2^*), \\ &\phi_1(p^*) = (p^*-p_1^*)\sqrt{(\tau_1-\tau)/(p^*-p_1^*)}, \end{aligned}\right\} \qquad (3\cdot2)$$

where according to the Hugoniot relation (2·9) τ is a function of p_1^*, τ_1, H_1 (or κ) and p^*.

In eqs. (3·1) and (3·2), the upper sign must be taken for the forward-facing wave $(\boldsymbol{S}_{\rightarrow})$* and the lower for the backward-facing wave $(\boldsymbol{S}_{\leftarrow})$.

The main properties of the function $\phi_1(p^*)$ are expressed as follows,

A. $$\phi_1(p_2^*) = -\phi_2(p_1^*).$$

B. $$\left.\begin{aligned} &\phi_1(p^*) \to \infty, \\ &\phi_1'(p^*) \to 0, \end{aligned}\right\} \quad \text{for} \quad p^* \to \infty.$$

C. $$\phi_1'(p^*) > 0.$$

* The notations $\boldsymbol{S}_{\rightleftarrows}$ and $\boldsymbol{R}_{\leftrightarrows}$, etc., are used in the same meaning as in 7).

D. Corrseponding to the property D of the Hugoniot curve, i.e. the relations between the volume changes, we have the following relations between the velocity changes,

(1) if the shock is weak, i.e. $(p^*-p_1^*)/p_1^* \ll 1$,

$$|u-u_1| \gtrless |u'-u_1'|, \quad \text{for} \quad \bar{c}_1\rho_1 \lesseqgtr \bar{c}_1'\rho_1',$$

(2) if the shock is strong, i.e. $(p^*-p_1^*)/p_1^* \gg 1$,

$$|u-u_1| \gtrless |u'-u_1'|, \quad \text{for} \quad \rho_1 \lesseqgtr \rho_1'.$$

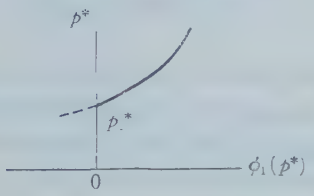

Fig. 3 (a). The pressure behind a shock front as a function of a velocity change.

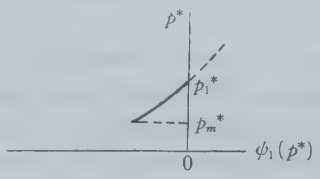

Fig. 3 (b). The pressure behind a rarefaction wave as a function of a velocity change, $p^* > \frac{H_1^2}{8\pi}\left(\frac{\tau_1}{\tau_{\max}}\right)^2 = p^*_m$.

ii) *Rarefaction waves* (***R***)

Across a rarefaction wave $\boldsymbol{R}_{\rightleftarrows}$ we have[7]

$$\left.\begin{aligned} &u \mp l(\rho) = \text{constant}, \\ &l(\rho) = \int^{\rho} \frac{\bar{c}}{\rho}\, d\rho = \int^{p^*} \frac{dp^*}{\rho\bar{c}}, \end{aligned}\right\} \tag{3·3}$$

respectively, or

$$\left.\begin{aligned} &u_2 - u_1 = \pm\psi_1(p_2^*), \\ &\psi_1(p^*) = \int_{\rho_1}^{\rho} \frac{\bar{c}}{\rho}\, d\rho = \int_{p_1^*}^{p^*} \frac{dp^*}{\rho\bar{c}}. \end{aligned}\right\} \tag{3·4}$$

The function $\psi_1(p^*)$ has the following properties:

A. $$\psi_1(p_2^*) = -\psi_2(p_1^*).$$

B. $$\left.\begin{aligned} &\psi_1(p^*) \to \infty, \\ &\psi_1'(p^*) \to 0, \end{aligned}\right\} \quad \text{for} \quad p^* \to \infty.$$

C. $$\psi_1'(p^*) > 0.$$

D. $$\phi_1'(p_1^*) = \psi_1'(p_1^*).$$

E. $\psi_1(p^*)$ is defined in the domain $p^* > p_m^* = H_1^2/(8\pi)\,(\tau_1/\tau_{\max})^2$ (cf. (2·10)). The two functions $\phi_1(p^*)$ and $\psi_1(p^*)$ are shown graphically in Fig. 3.

§ 4. Interactions of elementary waves

In this section the interactions between the elementary waves (the shock and

the rarefaction waves) and the contact discontinuity are analyzed. Although the intermediate states of the interactions are too much complicated to be treated exactly, one can determine the elementary waves and the constant states which appear eventually after the interactions, making use of the results of the preceding sections, in particular, the (u, p^*)-relations.

The main characteristics of the interaction of the elementary waves are not so different from those in the fluid dynamics,[7] if the contact surface does not exist initially.

Hence we first investigate the interaction of a shock wave ($\boldsymbol{S}$) and a contact discontinuity ($\boldsymbol{T}$).

Under the present assumptions, the magnetic field $\boldsymbol{H}$ is always perpendicular to the motion of the shock front and has the z-component alone; therefore, it can be easily seen that an electric current proportional to $|H_1-H_2|$ exists on the plane of the contact discontinuity, because we have the relation

$$\operatorname{rot} \boldsymbol{H}=(4\pi/c)\boldsymbol{j} \quad \begin{pmatrix} \boldsymbol{j}: \text{ electric current} \\ c: \text{ light velocity} \end{pmatrix}, \tag{4·1}$$

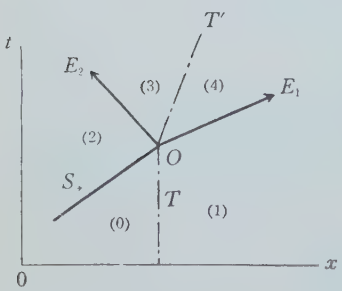

Fig. 4. The various zones of a fluid before and after the collision between a shock front and a contact surface.

The whole space is divided into the five regions (0)~(4) by the elementary waves ($\boldsymbol{S}_{\rightarrow}$, OE_1 and OE_2) and the contact surfaces ($\boldsymbol{T}$ and $\boldsymbol{T}'$). All the quantities in respective regions are specified by the respective subscripts.

Suppose that the fluid is initially at rest, the shock $\boldsymbol{S}_{\rightarrow}$ invades the fluid in the state (0), and hits the fluid in the state (1), at a discontinuity surface $\boldsymbol{T}$. (c.f. Fig. 4). Then the reflection and the refraction of the shock wave occur according to the two formulae;

(a) $\boldsymbol{S}_{\rightarrow}\boldsymbol{T} \rightarrow \boldsymbol{S}_{\leftarrow}\boldsymbol{T}\boldsymbol{S}_{\rightarrow}$,

if shock is weak and $\bar{c}_0\rho_0$ is smaller than $\bar{c}_1\rho_1$
or if shock is strong and ρ_0 is smaller than ρ_1,

(b) $\boldsymbol{S}_{\rightarrow}\boldsymbol{T} \rightarrow \boldsymbol{R}_{\leftarrow}\boldsymbol{T}\boldsymbol{S}_{\rightarrow}$,

if shock is weak and $\bar{c}_0\rho_0$ is larger than $\bar{c}_1\rho_1$
or if shock is strong and ρ_0 is larger than ρ_1.

These formulae represent the fact that a reflected (OE_2) and a transmitted (OE_1) shock waves result in the case (a) whereas a rarefaction wave (OE_2) is reflected but a shock wave (OE_1) is still transmitted in the case (b).

The proof of the above formulae is given as follows.

On the basis of the results obtained in § 3, the constant states which can be connected with the fixed states (0) and (1) by the shock waves $\boldsymbol{S}_{\rightarrow}$ are represented by the curves g_0 and g_1 (Fig. 5) respectively.

Figs. 5 (a) and (b) correspond to the cases (a) and (b) respectively. Then, in the way quite similar to that used in the ordinary fluid dynamics, we can determine the remaining states (2), (3) and (4) as indicated in Fig. 5. It can also be proved that OE_1 cannot be a rarefaction wave.

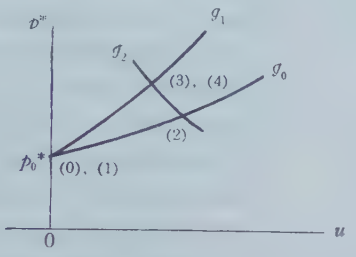

Fig. 5 (a). Collision between a shock front and a contact surface. Case (a) $S_{\rightarrow}T \rightarrow S_{\leftarrow}TS_{\rightarrow}$.

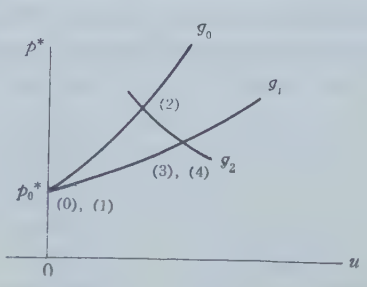

Fig. 5 (b). Collision between a shock front and a contact surface. Case (b) $S_{\rightarrow}T \rightarrow R_{\leftarrow}TS_{\rightarrow}$.

Next we study the interaction of a rarefaction wave ($\boldsymbol{R}$) and a contact discontinuity ($\boldsymbol{T}$). In this case the interaction leads at first to a period of penetration during which the flow cannot be described as made up of simple waves. If this penetration is completed in finite time the regions separated by two emerging elementary waves and a contact discontinuity are in constant terminal states and the effect of the interaction can be described symbolically as follows;

(a) $\boldsymbol{R}_{\rightarrow}\boldsymbol{T} \rightarrow \boldsymbol{R}_{\leftarrow}\boldsymbol{T}\boldsymbol{R}_{\rightarrow}$, for $\bar{c}_0\rho_0 < \bar{c}_1\rho_1$,

(b) $\boldsymbol{R}_{\rightarrow}\boldsymbol{T} \rightarrow \boldsymbol{S}_{\leftarrow}\boldsymbol{T}\boldsymbol{R}_{\rightarrow}$, for $\bar{c}_0\rho_0 > \bar{c}_1\rho_1$,

where the reflected compression wave in which a shock wave is eventually formed is simply denoted as $\boldsymbol{S}_{\leftarrow}$. In the present case a shock or compression wave cannot be transmitted. If the period of the penetration is not finite and the curves g_0 and g_2 has no cross point within the region $(2\cdot10)$, the analysis is much complicated and is not presented in this article.

§ 5. Application

We first try to find the flow and the field resulting from an initial state in which the plasma on the right-hand side, $x>0$, is in a constant state (1) given by u_1, p_1, τ_1 and H_1, and that on the left-hand side, $x<0$, is in a constant state (0): u_0, p_0, τ_0 and H_0. The analysis of the interaction process discussed in the previous section likewise applies to this problem.

We are particularly interested in such a special case that in the initial state the plasma is at rest ($u_0=u_1=0$) and the inequality, $p_0^* > p_1^*$ holds.

If the magnetic field is put equal to zero, this problem reduces to that of the ordinary hydrodynamics which is well-known in the experiment of shock tubes.[7] In the hydromagnetic configuration described above, the respective values of $\kappa(=H/\rho)$

are constant everywhere in the two regions, $\mathcal{R}$ and $\mathcal{L}$, covered by the particle paths issuing out of the right-hand side, $x>0$ and the left-hand side, $x<0$ of the x-axis, respectively (c.f. Fig. 6), since the initial values of κ are constant, equal to H_0/ρ_0 and H_1/ρ_1 in the two regions, $x>0$ and $x<0$ respectively and κ is constant along the particle path. That is to say, the whole (x, t)-space can be divided into the two domains of semi-simple wave, $\mathcal{R}$ and $\mathcal{L}$.[6] Hence in closer analogy with the hydrodynamical case we can apply the theory of interaction discussed in § 4. The result is indicated in Fig. 6, in which the region $\mathcal{R}$ is composed of the two constant states (1), (3) and the shock wave $\boldsymbol{S}_+$, the region $\mathcal{L}$ is divided into the three domains, constant states, (0) and (2), and the rarefaction wave $\boldsymbol{R}_-$. The both regions $\mathcal{R}$ and $\mathcal{L}$ are in contact with each other at the discontinuity surface $\boldsymbol{T}$. The pressure p_2^* $(=p_3^*)$ and the velocity $u_2(=u_3)$ in the domains (2) and (3) can be calculated on the basis of the (u, p^*)-relation in § 3 and are given in Fig. 6 (b).

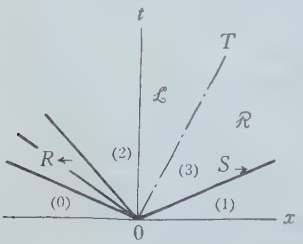

Fig. 6 (a). Wave motion if a pressure difference exists initially

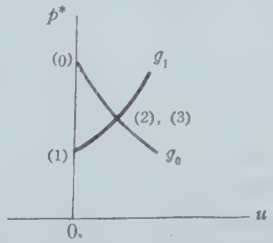

Fig. 6 (b). (u, p^*)-diagram for the wave motion described in Fig. 7 (a).

Let us now proceed to investigate the initial value problem involving the interactions between rarefaction waves, shocks and contact discontinuities. The initial conditions are prescribed as follows: at $t=0$ the plasma is at rest, i.e. $u=0$, the pressure p and the magnetic field H are given by

$$\left.\begin{aligned} p &= \begin{cases} p_0 & \text{in} \quad D_0 \\ p_1 & \text{in} \quad D_1, \end{cases} \\ H &= \begin{cases} H_0 & \text{in} \quad D_0 \\ H_1 & \text{in} \quad D_1; \end{cases} \end{aligned}\right\} \qquad (5\cdot1)$$

in which p_0's and H_0's are constant, the domains D_0 and D_1 are defined by $x\leqq a$ and $|x|>a$ respectively, moreover, p_0 is much greater than p_1 and H_0 is much smaller than H_1† so that the total pressure p_0^* is slightly larger than p_1^*, i.e. $p_0 \gg p_1$, $H_0 \ll H_1$ and $p_0^* \gtrsim p_1^*$. In this case we can safely assume that $\tau_0 \ll \tau_1$ and $\tau_0/\bar{c}_0 \ll \tau_1/\bar{c}_1$.

This case seems to correspond to a condition which is realized experimentally in the process of the pinched discharge.

† Under this condition it may occur that in D_1 the cyclotron radius r is smaller than the collision mean free path λ. It should, however, be noted that in the case $\lambda \gg r$, the hydromagnetic equation is also valid in which the velocity u should be considered as the drift velocity if the magnetic field is constant.[8]

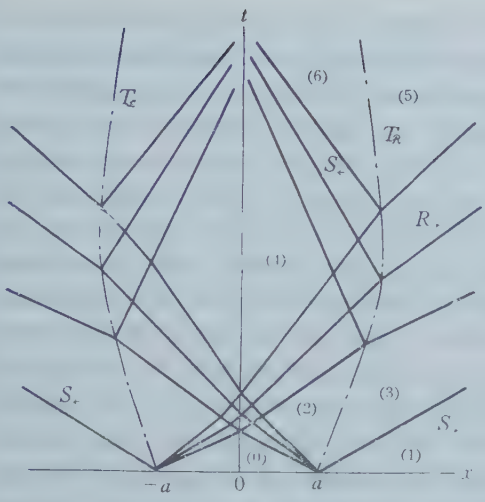

Fig. 7. The domain (0) is circumscribed by the rarefaction waves $\boldsymbol{R}_{\leftarrow}$, $\boldsymbol{R}_{\rightarrow}$ and the x-axis. The region (1) is bounded by the shock $\boldsymbol{S}_{\rightarrow}$ or $\boldsymbol{S}_{\leftarrow}$ and the x-axis. The domain (3) is adjacent to the state (1) and the rarefaction zone and is in contact with $\boldsymbol{T}$. The domain (2) is adjacent to (3) and is circumscribed by $\boldsymbol{R}_{\leftarrow}$ and $\boldsymbol{R}_{\rightarrow}$. All the quantities in respective regions are specified by the respective subscripts.

Suppose that the plasma in D_0 is of much higher pressure and density, or much higher temperature than that in D_1 and is confined to be at rest by the magnetic field H_1, and then the magnetic field H_1 is suddenly, but slightly, reduced at the time $t=0$. The motion which follows this initial stage is illustrated in Fig. 7.

Since the pressure $p_0{}^*$ is nearly equal to $p_1{}^*$, the shocks, $\boldsymbol{S}_{\rightarrow}$ and $\boldsymbol{S}_{\leftarrow}$, are quite weak and may be considered as the magnetosonic disturbances with the effective sound speed $\bar{c}$. Hence, across the shock fronts, the temperature rise is quite weak and the plasma velocities in (2) and (3) are small, but the pressure (temperature) difference across $\boldsymbol{T}$ is sufficiently large. All these quantities can be calculated on the basis of Fig. 6.

Let t_c be the critical instance at which the first interaction between the contact discontinuity $\boldsymbol{T}_{\mathcal{R}}$ (or $\boldsymbol{T}_{\mathcal{L}}$) and the rarefaction wave $\boldsymbol{R}_{\rightarrow}$ (or $\boldsymbol{R}_{\leftarrow}$) occurs.

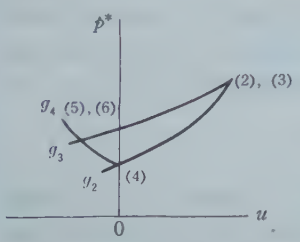

Fig. 8. (u, p^*)-diagram for the case described in Fig. 7.

Until this instance, t_c, the plasma of high temperature expands slowly, the surface of which moves with the small velocity u_2, keeping contact with the plasma of the low pressure, p_3. Then, as can be seen from Figs. 7 and 8, the contact surface begins to move backwards due to the interaction with the rarefaction wave $\boldsymbol{R}_{\rightarrow}$ (or $\boldsymbol{R}_{\leftarrow}$). This backward motion of the contact surfaces leads to the compression of the inner plasma bounded by them. The slow plasma expansion and the subsequent compression established above arise, of course, from the large difference of the magnetic field across $\boldsymbol{T}$.

In the process of the high current discharge, we can find the circumstances analogous to that described above. Suppose that the plasma accelerated towards the discharge axis by the self-magnetic force is pinched and confined within a column of small diameter during a short time interval. Then follows the expansion of this plasma column. In this last process occurring after the first pinch of the plasma, the motion may be considered as if it resulted from an initial condition such as corresponding to (5·1). Therefore we can apply the result of the above initial value problem to the description of this motion.

The results obtained by such analogy are stated as follows. A weak divergent hydromagnetic shock precedes with a velocity nearly equal to the effective sound speed. Since it may be considered as a magnetosonic disturbance, the temperature rise resulting from this shock wave is negligible, though the effective sound speed is large if the magnetic field is strong. On the other hand, the inner plasma column of high temperature bounded by a contact surface begins to expand slowly, and is decelerated immediately by the magnetic field due to the current flowing on the contact surface, then tends to the backward motion which leads to the re-compression of the plasma. Of course, the practical experimental configuration is cylindrical whereas a plane geometry has been considered here, and the actual phenomenon seems to be more complicated.* Further it is doubtful whether such an initial condition as mentioned above is realizable in practice or not; moreover, in the case of the actual pinch, the density ρ_1 is so small that the applicability of the hydrodynamical description is questionable in D_1. Nevertheless, one can expect that in crude approximation the model under our consideration is of interest so far as qualitative discussions are concerned, since it provides an explanation of re-compression of the plasma of high density which was already suggested by Kurchatov[9] in connection with the time lag of neutron emission. The present theory suggests only the possibility of such an oscillation of the plasma column. The connection with other theories[10] is not so clear because the theory described in this paper is constructed on the basis of the interactions of elementary waves and contact discontinuities** and it seems to be different from that obtained by Leontovich and Osovetz.[10]

The discussion along the line of the present thought, cannot be expected to be valid throughout the process of the pinched discharge. To understand the mechanism of this phenomenon we have to wait for more detailed investigation.

Acknowledgements

In conclusion, the author expresses his sincere thanks to Professor T. Taniuti for suggesting this problem and for his continual discussions. He is also indebted to Professor I. Imai, Mr. S. Kuwabara and Mr. N. Mugibayashi for valuable discussions and to Professor Y. Tanikawa for his kind interest.

Appendix A

The volume change $|\tau_2 \sim \tau_1|$ induced by the shock transitions from the state (1) to the state (2) can be derived from eqs. (2·4) and (2·9); i.e.

$$\frac{\tau_2 - \tau_1}{p_1^* - p_2^*} = \frac{(1-\mu^2)\tau_1}{p_2^* + \mu^2 p_1^* + (1-3\mu^2) H_1 H_2 / (8\pi)} . \qquad (A\cdot 1)$$

* Some qualitative considerations on this point are given in Appendix B.

** With this respect, our theory may have some connection with that given by Allen.[11]

Approximations to this equation for the two extreme cases are presented below and imply the property D of the Hugoniot curves described in § 2, ii);

(1) for a sufficiently weak shock ($p_2^* \approx p_1^*$),

$$\frac{\tau_2-\tau_1}{p_1^*-p_2^*} \sim \left(-\frac{d\tau}{dp^*}\right)_{p^*=p_1^*} = \frac{\tau_1^2}{\bar{c}_1^2}, \tag{A·2}$$

(2) for a sufficiently strong shock ($p_2^* \to \infty$),

$$\frac{\tau_2-\tau_1}{p_1^*-p_2^*} \sim \frac{(1-\mu^2)\tau_1}{p_2^*+\mu^2 p_1^*+(1-3\mu^2)\kappa^2/(8\pi\mu^2\tau_1)} \sim \frac{(1-\mu^2)\tau_1}{p_2^*}, \tag{A·3}$$

since $\tau_2 \sim \mu^2\tau_1$, for $p_2^* \to \infty$ (cf. Fig. 1).

Appendix B

In the latter part of § 5, we have discussed the two-dimensional motion of the plasma with cylindrical symmetry, applying the result of the analysis in the case of the one-dimensional motion. It was supposed there that the qualitative pattern of the flow would not be much altered by this change of the configuration. The validity of this supposition, however, should be examined, in particular near the axis of symmetry. (cf. Fig. 8) In this appendix some consideration on this point will be given, though it is not rigorous from the mathematical point of view.

First we present the sets of the equations of the isentropic flow for the following four cases.

1) Unsteady one-dimensional flow.

$$\partial\rho/\partial t+\partial\rho u/\partial x=0, \tag{B·1}$$

$$\partial u/\partial t+u\partial u/\partial x+(1/\rho)(\partial p/\partial x)=0. \tag{B·2}$$

2) Unsteady two-dimensional flow.

$$\partial\rho/\partial t+\partial\rho U/\partial r+\rho U/r=0, \tag{B·3}$$

$$\partial U/\partial t+U\partial U/\partial r+(1/\rho)\partial p/\partial r=0, \tag{B·4}$$

where r is the distance from the axis of symmetry and U is the radial velocity.

3) Steady two-dimensional irrotational flow in the (x, z)-plane.

$$\partial\rho w/\partial z+\partial\rho u/\partial x=0, \tag{B·5}$$

$$w\partial u/\partial z+u\partial u/\partial x+(1/\rho)\partial p/\partial x=0, \tag{B·6}$$

$$\tfrac{1}{2}(u^2+w^2)+\int\frac{dp}{\rho}=\text{constant}. \tag{B·7}$$

4) Steady three-dimensional irrotational flow with cylindrical symmetry.

$$\partial\rho w/\partial z+\partial\rho U/\partial r+\rho U/r=0, \tag{B·8}$$

$$w\partial U/\partial z+U\partial U/\partial r+(1/\rho)\partial p/\partial r=0, \tag{B·9}$$

$$\tfrac{1}{2}(U^2+w^2)+\int\frac{dp}{\rho}=\text{constant}. \tag{B·10}$$

Now we consider the steady flow. It has been shown experimentally[12] that if the flow is supersonic in the case 4) with eqs. (B. 8–10) the propagation of the weak wave including its reflection at the axis of symmetry is qualitatively similar to the corresponding one in the case 3), where the exact analysis is possible (cf. reference 7), §§ 114, 115 and 148). This suggests that the appearance of the term $\rho U/r$ in the equation of continuity (B·8) does not affect the qualitative feature of the propagation of the weak wave. (This supposition, however, has not yet been justified by the mathematical analysis; for example, the calculation with the linearized theory gives a singularity of the velocity field along the axis $r=0$,[13),14)] and the exact treatment near the axis does not seem to have been published.)

Next, we consider the unsteady flow 1) and the steady flow 3), then the exact analysis with the method of characteristics shows that the qualitative feature of the propagation and the interaction of the weak wave which are symmetrical with respect to the axis $x=0$ is not so different between these two cases.

Thus as a conclusion from the discussions given above, we can suppose that, analogously to the cases of the steady flow, the time dependent propagation of the weak wave with cylindrical symmetry is similar to the corresponding one in the one-dimensional case. The situation may not be different in the magnetohydrodynamics if the magnetic field is perpendicular to the motion of the plasma where we have only to replace p by p^* in eqs. (B·1–4).

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Addendum

The result of the analysis presented in § 4 of this paper is contradictory to the description in reference 7), p. 179, in the limit $H \to 0$. Recently, however, the author has received a letter from Profs. R. Courant and K. O. Friedrichs in which they have mentioned the correction to the error in their book. The author thanks them for their kind communication.

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On the Kirkwood Superposition Approximation

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The Kirkwood superposition approximation is investigated by making use of the expansion theorem for the potential of average force and its correction term is found in a form of the expansion in powers of particle number density ρ. The lowest order term of this correction is calculated for a special configuration of three particles in a fluid consisting of hard spheres and the Kirkwood approximation is shown to overestimate the distribution function of triplets in this case. The Kirkwood integral equation with correction is solved by expanding the radial distribution function in powers of ρ. The solution thus obtained is shown to be exact up to the order of ρ^2, while the Kirkwood approximation itself does not yield the correct term of the order of ρ^2. It is shown that the distribution function of triplets can, in principle, be expanded in terms of radial distribtution functions and a few terms of this expansion are calculated explicitly, of which the first term just corresponds to the Kirkwood superposition approximation.

§ 1. Introduction

The integral equation method for evaluating distribution functions in fluid systems has provided a useful means, for instance, to determine the radial distribution function itself, to calculate the equation of state, to study the condensation phenomena, etc. Since both the Kirkwood[1)2)3)] and the Yvon-Born-Green[4)5)6)] integral equations involve the distribution functions in sets of three particles in addition to the radial distribution functions, it is necessary to introduce the assumption of a suitable closure, that is, a relation between these distribution functions, in order to solve the integral equations. The closure which has been widely used is the Kirkwood superposition approximation in which the potential of average force in a set of three particles is assumed to be the sum of the three pair potentials of average force. This is equivalent to assuming that the triplet distribution function is a product of the three radial distribution functions.

So far, there have been several methods for estimating the extent of validity of the superposition approximation. The most rigorous method is to compare the exact expansions of the radial distribution function and the pressure in powers of the number density ρ with the corresponding one obtained with the aid of the superposition approximation. This has been done for the case of the third and fourth virial coefficients in a system of hard spheres, by Hart, Wallis and Pode[7)], Rushbrooke and Scoins[8)], Nijboer and van Hove[9)], and Hiroike[10)]. They found that

the superposition approximation yields an exact third virial coefficient, independently of the potential used. (Meeron[11] has shown that this comparison is less significant than it seems since the third virial coefficient is given correctly by any closure whatsoever.) However, the fourth virial coefficient is no longer exact and, in the case of hard spheres, the discrepancy amounts to about 21.5%. Also, as was shown by Nijboer and van Hove[9], the fourth virial coefficient for hard spheres shows a wide discrepancy when calculated via the virial theorem, using the superposition approximation, and via the compressibility integral. Thus the superposition aproximation seems to be inaccurate even at low densities and gives inconsistent results when substituted in different equations.

On the other hand, the direct comparison[12,13] of the equation of state and radial distribution function obtained via the superposition approximation with the experimental data shows a surprisingly good agreement up to quite high densities. Likewise, a comparison with the results calculated by the Monte Carlo method[14,15,16] shows good agreement up to moderately high densities. Thus the superposition approximation seems to be more accurate than could be expected from the comparison with the fourth virial coefficient.

Confronting with these facts, several authors have attempted to justify[11,17] or to improve[18] the superposition approximation or to use[19,20] it so that it will lead to less inaccuracies than those associated with the previous equations. However, it seems that the statistical mechanical foundation of the superposition approximation has not yet been fully discussed.

The purpose of this paper is to investigate the Kirkwood superposition approximation by means of the general theorem in statistical mechanics. In § 2 using the expansion theorem for the potential of average force, the meaning of the superposition approximation is clarified and its correction term is given in a form of expansion in powers of ρ. In § 3 the lowest order term of this correction is retained and calculated in the case of hard spheres for a special configuration of three particles. The Kirkwood approximation is shown to overestimate the triplet distribution function in this case. Then § 4 is devoted to generalize the method to the case in which the coupling parameter is included and the Kirkwood integral equation with correction is solved by expanding the radial distribution function in powers of ρ. It is confirmed that the exact terms up to ρ^2 are obtained by this procedure. In § 5 a possible representation of triplet distribution function in terms of radial distribution function is discussed and a few terms of this representation are calculated explicitly, of which the first term corresponds to the Kirkwood superposition approximation.

§ 2. Expansion of potential of average force

When a subset of m particles (denoted by $\boldsymbol{M}$) at fixed configuration is chosen out of all the particles in a fluid system, there is a potential of average force act-

ing on this set, averaged over the possible configurations of the remaining particles. If the distribution function normalized to unity of this subset is denoted by $g_m(\boldsymbol{M})$, the potential of average force $W_m(\boldsymbol{M})$ is defined by

$$g_m(\boldsymbol{M}) = \exp[-\beta W_m(\boldsymbol{M})] \tag{1}$$

where $\beta \equiv 1/kT$ with k Boltzmann's constant, T the absolute temperature. In this section we shall briefly explain the expansion theorem proved by Meeron[21] and find the correction term to the superposition approximation in a form of expansion in powers of ρ.

For a one-component system with no long-range order, $W_m(\boldsymbol{M})$ is expanded in powers of ρ as follows[21],

$$W_m(\boldsymbol{M}) = U(\boldsymbol{M}) - \frac{1}{\beta}\sum_{N\geq 1}\frac{\rho^N}{N!}\int Q(\boldsymbol{M}, \boldsymbol{N})\, d(\boldsymbol{N}). \tag{2}$$

Here $U(\boldsymbol{M})$ is the direct interaction potential in the set $\boldsymbol{M}$ and $d(\boldsymbol{N})$ denotes the integration over the coordinates of n particles of the set $\boldsymbol{N}$. The $Q(\boldsymbol{M}, \boldsymbol{N})$ is a sum of products of f functions,

$$f(ij) \equiv \exp[-\beta U(ij)] - 1, \tag{3}$$

where $U(ij)$ is the two-body potential function.* This sum is characterized by**

(a) every particle of the set $\boldsymbol{N}$ is connected to at least two particles of the set $\boldsymbol{M}$, either directly or by two or more independent paths, i.e., paths which involve mutually exclusive sets of intermediate particles,

(b) all particles of the set $\boldsymbol{N}$ are also conneted among themselves independently of the set $\boldsymbol{M}$, i. e., without involving particles of $\boldsymbol{M}$,

(c) particles of $\boldsymbol{M}$ are not directly connected among themselves.

Some of bond diagrams contributing to $W_3(123)$ are given in Fig. 1 and Fig. 2.

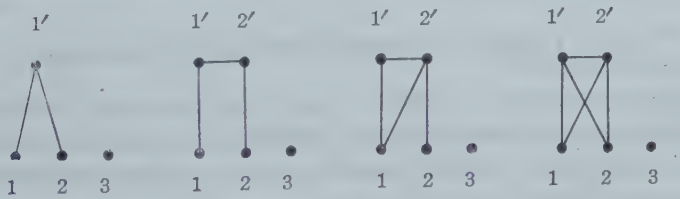

Fig. 1. Diagrams contributing to W_3 (123). The points 1′ and 2′ mean the particles of the set $\boldsymbol{N}$. These diagrams are characterized by the absence of f-bond connected to the particle 3.

It may be easily seen that the summation of all the diagrams of the type shown in Fig. 1 together with $U(12)$ yields $W_2(12)$. The similar diagrams in which the particle 1 or 2 is not connected together with $U(23)$ or $U(31)$ will lead to $W_2(23)$ or $W_2(31)$. Therefore, if we write[22]

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* The potential energy of the system is assumed to be a sum of pair potentials.

** A detailed account may be found in references 11) and 21).

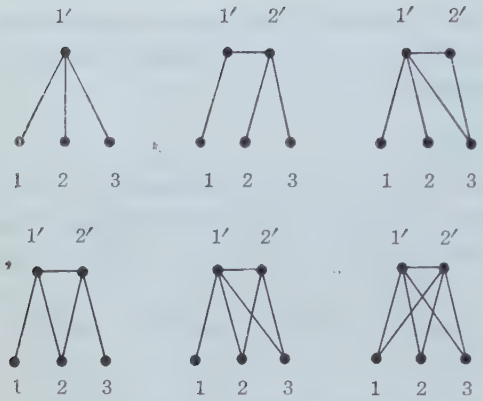

Fig. 2. Diagrams contributing to $W_3(123)$ in which every particle of the set $\boldsymbol{M}$ (=1, 2, 3) is connected by f-bond.

$$W_3(123) = W_2(12) + W_2(23) + W_2(31) + w_3(123), \tag{4}$$

$w_3(123)$ may be characterized by the bond diagrams shown in Fig. 2, i.e.,

$$w_3(123) = -\frac{1}{\beta}\sum_{N\geq 1}\frac{\rho^N}{N!}\int q(3, \boldsymbol{N})\, d(\boldsymbol{N}), \tag{5}$$

where $q(3, \boldsymbol{N})$ is a sum of products of f functions subject to,

(d) every particle of the set (1, 2, 3) is connected to at least one particle of the set $\boldsymbol{N}$,

besides the conditions (a), (b) and (c) mentioned above.

Substituting eq. (4) in eq. (1), we have

$$g_3(123) = g(12)g(23)g(31)\exp[-\beta w_3(123)], \tag{6}$$

where $g(12)$ is the radial distribution function, i.e., $g(12) = \exp[-\beta W_2(12)]$. From this equation it follows that the Kirkwood superposition approximation, $g_3(123) = g(12)g(23)g(31)$, corresponds to assuming that $w_3(123) = 0$, i.e., to neglecting the contributions of the bond diagrams shown in Fig. 2. In other words, $w_3(123)$ represents the correction term to the Kirkwood approximation.

If we retain the lowest order term in eq. (5), eq. (6) becomes

$$g_3(123) = g(12)g(23)g(31)\exp[\rho\int f(11')f(21')f(31')\,d\tau_1'] . \tag{7}$$

§ 3. Validity of superposition approximation

In the previous section, we have derived the correction term to the superposition approximation in a form of power series of ρ. In this section we shall calculate the lowest order term of this correction for a special configuration of three particles in a fluid of hard spheres.

For hard spheres of diameter a, f function is given by

$$f(ij) = -1, \quad \text{for } |\boldsymbol{x}_i - \boldsymbol{x}_j| < a ,$$
$$= 0, \quad \text{for } |\boldsymbol{x}_i - \boldsymbol{x}_j| > a .$$

Therefore, as is seen from eq. (7), the calculation of the correction term reduces to calculating the volume of the region belonging to all three spheres of radius a, of which centers are separated by $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$, $|\boldsymbol{x}_2 - \boldsymbol{x}_3|$ and $|\boldsymbol{x}_3 - \boldsymbol{x}_1|$, respectively. For simplicity, however, we shall consider a special configuration in which these centers form a regular triangle. If we measure the distance in units of a, we obtain from

eq. (7)

$$g_3(123)/g(12)g(23)g(31) \equiv F(r) = \exp[-\rho a^3 \Omega(r)] \tag{8}$$

for this special configuration. Here $\Omega(r)$ implies the volume of the region common to three unit spheres of which centers are separated from each other by r, with $r=|\boldsymbol{x}_1-\boldsymbol{x}_2|/a=|\boldsymbol{x}_2-\boldsymbol{x}_3|/a=|\boldsymbol{x}_3-\boldsymbol{x}_1|/a$.

The calculation of $\Omega(r)$ is rather lengthy, though it is elementary. Here we quote only the result (the derivation is given in the Appendix):

$$\Omega(r)=\pi+\frac{r^2}{6}\sqrt{3-r^2}+\frac{\pi}{8}r(r^2-12)-\frac{1}{4}r(r^2-12)\sin^{-1}\left(\frac{r}{\sqrt{3}\sqrt{4-r^2}}\right)$$
$$+\sin^{-1}\left[\frac{6-2r-r^2}{\sqrt{3}(2-r)\sqrt{4-r^2}}\right]-\sin^{-1}\left[\frac{6+2r-r^2}{\sqrt{3}(2+r)\sqrt{4-r^2}}\right],\quad 1\leq r\leq\sqrt{3},$$
$$=0,\quad r\geq\sqrt{3}. \tag{9}$$

Using this equation, we calculated the function $F(r)$ for $\rho a^3=0.5$ and 1.0, respectively, and the result is shown graphically in Fig. 3. From this figure, it may follow that the superposition approximation is exact for $r\geq\sqrt{3}$ to the extent of the approximation employed here, but the error produced by the superposition approximation increases as r decreases and amounts to about 50% for $r=1$, i. e., for the configuration of closest packing. Furthermore, $F(r)\leq 1$ for $1\leq r\leq\sqrt{3}$, that is, the superposition approximation overestimates the triplet distribution function in this case. However, the numerical value obtained here should not be taken too literally, for we have considered only the lowest order term of correction, neglecting the higher order terms. These terms seem to have the compensating effects, i. e., the tendency to increase $F(r)$, since the contribution of the products of even number of f functions is opposite in sign to that of the lowest order term. Futhermore, if the potential has the attractive part, the f function becomes positive in the region of attraction, thus $F(r)$ may increase even in the lowest order approximation.

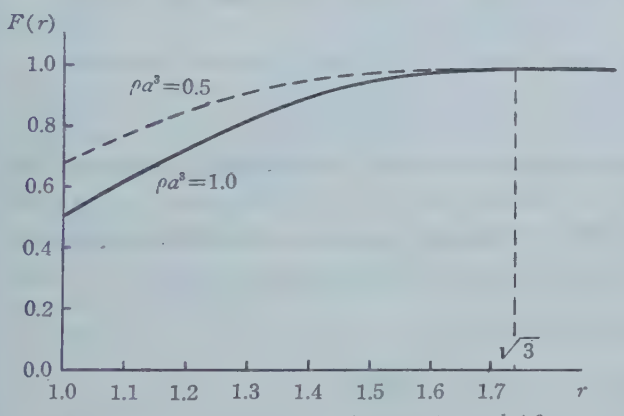

Fig. 3. The function $F(r)$ for $\rho a^3=0.5$ and 1.0.

Although the quantitative feature of the superposition approximation is somewhat obscure as mentioned above, the qualitative feature may be that it overestimates the triplet distribution function for the repulsive potential and underestimates for the attractive potential. To clarify the situation it may be necessary to solve the Yvon-Born-Green integral equation with correction and compare the results with those

obtained without correction. We shall not enter into this problem here, but in the next section we shall solve the Kirkwood integral equation with correction expanding the radial distribution function in powers of ρ.

§ 4. Superposition approximation in Kirkwood's integral equation

In the Kirkwood integral equation for the distribution function, there appears the coupling parameter ξ which is defined so that we have the fully coupled real fluid of N particles for $\xi=1$, but if $\xi=0$, particle 1 does not interact with any of remaining $N-1$ particles. The generalization of the results obtained in § 2 to this case is straightforward if we regard the particle 1 as an impurity and apply the expansion theorem for the two-component system[21]. Then it is readily verified that

$$g_3(123, \xi) = g(12, \xi) g(23) g(31, \xi) \exp[-\beta w_3(123, \xi)], \tag{10}$$

where $w_3(123, \xi)$ is given, as previously done, by eq. (5), but the f function connected to the particle 1 is replaced by

$$f(1i, \xi) = \exp[-\beta \xi U(1i)] - 1. \tag{11}$$

If we retain only the lowest order term as in § 3, we have

$$-\beta w_3(123, \xi) = \rho \int f(11', \xi) f(21') f(31') d\tau_1'. \tag{12}$$

We shall solve the Kirkwood integral equation in the following, using eqs. (10) and (12).

Let us consider the exact Kirkwood integral equation, without superposition,

$$\partial g(12, \xi)/\partial \xi = -U(12) g(12, \xi) + \beta \rho g(12, \xi) \int U(13) g(13, \xi) d\tau_3 - \beta \rho \int U(13) g_3(123, \xi) d\tau_3. \tag{13}$$

In solving this equation, we expand $g(12, \xi)$ as follows,

$$g(12, \xi) = e^{-\beta \xi U(12)} [1 + \rho g_1(12, \xi) + \rho^2 g_2(12, \xi) + \cdots]. \tag{14}$$

Substituting eqs. (10), (12) and (14) in eq. (13) and comparing the coefficient of ρ, we have

$$\partial g_1(12, \xi)/\partial \xi = -\beta \int U(13) e^{-\beta \xi U(13)} f(23) d\tau_3.$$

If we integrate this equation by ξ, we have

$$g_1(12, \xi) = \int f(13, \xi) f(23) d\tau_3. \tag{15}$$

Using eq. (15) and comparing the coefficients of ρ^2, we obtain

$$\frac{\partial}{\partial \xi} g_2(12, \xi) = \frac{\partial}{\partial \xi} \int \{ f(13, \xi) f(14, \xi) f(23) f(24)/2 + f(13, \xi) f(14, \xi) f(23) f(34) + f(13, \xi)[1 + f(23)] f(24) f(34) + f(13, \xi) f(14, \xi) f(23) f(24) f(34)/2 \} d\tau_3 d\tau_4. \tag{16}$$

The integration over ξ from 0 to 1 leads to the exact form for the $g_2(12)$, if one notes that $g_2(12, 0)=f(1i, 0)=0$. Therefore, if one calculates the fourth virial coefficient by using $g(12)$ obtained here, it may be given correctly both via the virial theorem and via the compressibility integral. On the other hand, if the Kirkwood approximation, i.e., $g_3(123, \xi)=g(12, \xi)g(23)g(31, \xi)$ is substituted in eq. (13), the $g_2(12)$ is no longer exact though the $g_1(12)$ is given correctly, and hence the fourth virial coefficient is not given correctly though the third one is correct.

From the above argument, it may follow that the fourth virial coefficient is given exactly if the term of the order of ρ is taken into account in $w_3(123)$. Similarly, if the terms up to ρ^2 are retained in $w_3(123)$, the fifth virial coefficient may be calculated exactly, and so forth. Thus the superposition approximation can be improved up to any virial coefficients as one desires.

§ 5. Possible representation of g_3 in terms of g

Since the integral equations for the distribution functions involve g_3 in addition to g, if it is possible to represent g_3 in terms of g, we shall have an exact equation to determine the radial distribution function and hence all thermodynamic functions will be calculated. In this section we shall discuss this representation, using the results obtained in the previous sections (we consider the case $\xi=1$.)

It is easily seen that g_3 can, in principle, be expressed in terms of g; for since both g_3 and g involve the potential function U, the elimination of U from g_3 and g may give the functional relation of g_3 with g, independently of the potential used. Based on this consideration, we shall calculate a few terms of the expansion of g_3 in terms of g.

Let us first consider the expansion of $W_2(12)$ in powers of ρ:

$$W_2(12)=U(12)-\frac{\rho}{\beta}\int f(11')f(21')\,d\tau_1'-\frac{\rho^2}{\beta}\int[f(11')f(22')$$
$$+f(11')f(21')f(22')+f(11')f(21')f(12')$$
$$+\tfrac{1}{2}f(11')f(12')f(21')f(22')]f(1'2')\,d\tau_1'd\tau_2'+0(\rho^3). \tag{17}$$

We can regard this equation conversely as a defining equation of $U(12)$ in terms of $W_2(12)$ and f functions, and solve $U(12)$ as a function of W_2 and g functions by an iteration procedure:

$$U(12)=W_2(12)+\frac{\rho}{\beta}\int h(11')h(21')\,d\tau_1'-\frac{\rho^2}{\beta}\int[h(11')h(22')$$
$$-\tfrac{1}{2}h(11')h(12')h(21')h(22')]h(1'2')\,d\tau_1'd\tau_2'+0(\rho^3)\,, \tag{18}$$

where

$$h(12)=g(12)-1. \tag{19}$$

From eqs. (3) and (18), we have

$$f(12)=h(12)-\rho g(12)\int h(11')h(21')\,d\tau_1'+0(\rho^2). \tag{20}$$

Then the substitution of eq. (20) in eq. (5) leads to

$$-\beta w_3(123)=\rho\int h(11')h(21')h(31')\,d\tau_1'$$

$$+\frac{\rho^2}{2}\int\sum{}'\left[\prod_{ij}h(i1')h(j2')\right]h(1'2')\,d\tau_1'd\tau_2'+0(\rho^3), \tag{21}$$

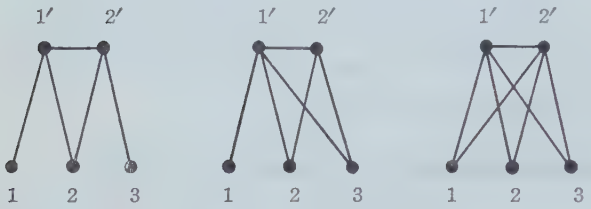

Fig. 4. Diagrammatic representation of h-bond contributing to w_3.

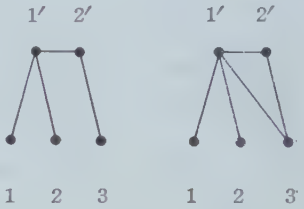

Fig. 5.

where $\sum'$ implies a summation over the bond diagrams of the type shown in Fig. 4. However, the bond diagrams given in Fig. 5, which appear when $w_3(123)$ is expressed in terms of f functions, do not appear in the new series.

It may seem that eq. (21) is a power series of ρ, but, in reality it is not the case, since h function depends on ρ. If eq. (21) is substituted in eq. (6), it is evident that $g_3(123)$ is expressed in terms of y. Obviously, the procedure developed here can be carried out to any desired higher order terms, although the procedure rapidly becomes very tedious. Furthermore, the procedure may be readily extended to the case in which the coupling parameter is not equal to 1.

In this section we have considered the problem in the classical theory of statistical mechanics. However, we believe that the conjecture developed here may be generalized to the case of quantum theory. For instance, the functional relation between the reduced density materices may be obtained by eliminating the potential function from these quantities, as we have done in this section. We hope that such a procedure may prove useful in connection with the quantum mechanical treatments of the many-body problem.

In conclusion the author wishes to express his sincere thanks to Prof. H. Ichimura, Dr. K. Hiroike rnd Dr. T. Morita for their valuable discussions. A part of this work was supported by the Scientific Research Expenditure of the Ministry of Education.

Appendix. Calculation of $\Omega(r)$

If we set the volume of the region cut from the sphere with radius 1 by two plains ① and ② illustrated in Fig. 6 to be $V(p, l, \theta)$, it is easy to see that

$$\Omega(r)=3V(r/2,\ r/2,\ \pi/6), \tag{A1}$$

and therefore the calculation of $\Omega(r)$ reduces to that of $V(p, l, \theta)$.

Let the equation of plain ② in Fig. 6 be

$$y/y_0+z/z_0=1.$$

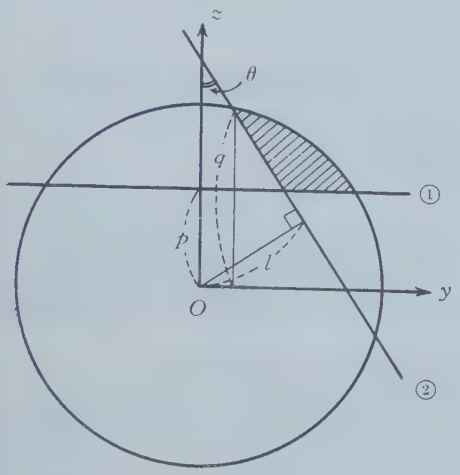

Fig. 6. ① and ② indicate two plains of which perpendicular distances from the origin 0 are p and l, respectively. The volume of shaded region is $V(p, l, \theta)$.

Furthermore, if q denotes the distance illustrated in Fig. 6, $V(p, l, \theta)$ is given by

$$V(p,l,\theta)=\int_p^q\Big[\frac{\pi}{2}(1-z^2)-(1-z^2)\sin^{-1}\Big(\frac{y_0(1-z/z_0)}{\sqrt{1-z^2}}\Big)-y_0(1-z/z_0)\times\sqrt{1-z^2-y_0^2(1-z/z_0)^2}\Big]dz, \tag{A2}$$

for $z_0>1$. The integration of the second term on the right-hand side can be carried out in terms of elementary functions by the use of partial integration, and after a very lengthy calculation, we obtain

$$V(p,l,\theta)=\pi/6\cdot p(p^2-3)\cdot\{1+2/\pi\cdot\sin^{-1}[(p\sin\theta-l)/\sqrt{1-p^2}\cos\theta]\}$$
$$+\pi/6\cdot l(l^2-3)\cdot\{1+2/\pi\cdot\sin^{-1}[(l\sin\theta-p)/\sqrt{1-l^2}\cos\theta]\}$$
$$+2tpl\cos\theta/3-t\sin\theta\cos\theta/3+t^3\sin\theta\cos\theta/3+(\pi-J)/3, \tag{A3}$$

where

$$t^2=[\cos^2\theta+2pl\sin\theta-p^2-l^2]/\cos^2\theta \tag{A4}$$

and

$$J=\sin^{-1}\Big[\frac{(1+l\sin\theta)(1+p)-(l+\sin\theta)^2}{(1+p)\sqrt{1-l^2}\cos\theta}\Big]$$
$$-\sin^{-1}\Big[\frac{(1-l\sin\theta)(1-p)-(l-\sin\theta)^2}{(1-p)\sqrt{1-l^2}\cos\theta}\Big]. \tag{A5}$$

From eqs. (A1), (A3), (A4) and (A5), we immediately have eq. (9) in § 3. It should be noted here that the J given by eq. (A5) can be written in a more symmetrical form:

$$J=\sin^{-1}[2t\cos\theta(pl-\sin\theta)/(1-p^2)(1-l^2)]. \tag{A6}$$

However, this form for J is not convenient, for the value of $\sin^{-1}x$ is not uniquely determined in eq. (A6). On the other hand, in eq. (A5) this value is unique and takes the principal value, i. e., $-\pi/2\leq\sin^{-1}x\leq\pi/2$.

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The Unified Model for Si^{29}

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The weak coupling unified model is applied to the low lying levels of Si^{29}. It is shown that, although the energy levels are predicted quite successfully by the model, it has only qualitative success in explaining other properties such as the magnetic moment or β-decay of P^{29}.

§ 1. Introduction

A considerable amount of experimental information on the properties of the low levels of Si^{29} (excitation energy ≤ 4 Mev) has now become available.[1] The success of the rotational collective model (indicative of large deformation of the nuclear core) in explaining the experimental data for $A=25$ nuclei has led to the application of this model to Si^{29}, with a reasonable amount of success.[2] However, for $A \geq 28$, the occurrence of the closed sub-shells at $N=Z=14$, 16 and the trend of decreasing spheroidicity with increasing A noted in reference 2), suggest that a vibrational collective model may perhaps be more appropriate for discussing the properties of these nuclei. Some time ago Feenberg[3] has considered the case of Si^{29} in terms of the weak coupling collective model. At the time, however, the nuclear level scheme was not so well-known, and a detailed comparison of the theory and the experimental results could not be done. We present here results of a more detailed calculation of some of the properties of the low states of Si^{29}, using assumptions somewhat different from those of Feenberg. Only the states of positive parity are considered in this report.

§ 2. Energy levels and wave functions

We consider Si^{29} nucleus as a spherical core of 14 protons and 14 neutrons filling up the nuclear sub-shells up to $1d_{5/2}$, and the last odd neutron in the $2s_{1/2}$ or $1d_{3/2}$ state. The collective motion of the core is described by the quadrupole surface oscillations, which are quantised, and the states of the nuclear core with 0, 1 and 2 quanta are coupled to the single particle states of the odd neutron. The mathematical formalism for such a case is well known,[4] and will not be described here. The important parameters entering the theory are:

(i) the energy separation of the two states $2s_{1/2}$ and $1d_{3/2}$

$$\Delta = E(1d_{3/2}) - E(2s_{1/2})$$

(ii) $\hbar\omega$, since the excitation energy of the n quanta state of the collective oscillations is given by $(n+5/2)\hbar\omega$, and

(iii) q, the parameter describing the strength of the coupling of the odd neutron to the collective motion of the nuclear core. We have in our notation

$$q^2 = k^2 \cdot \hbar\omega / 8\pi c$$

where c depends upon the rigidity of the nuclear surface, and k is a constant.

In the following calculations, we consider q as a variable parameter, and take $q=0.5$, 1.0 and 2.0. The value of $\hbar\omega$ is chosen from the experimental data on Si^{28}, which shows the first excited state at 1.8 Mev. We interpret this state as the $n=1$ vibrational state of the spherical core, thus giving $\hbar\omega=1.8$ Mev. Since we are only interested in energy levels below 4 Mev, it is clear that states with $n \geq 3$, lying at excitation energies of 5.5 Mev or higher, may be neglected. In absence of a priori knowledge of Δ, we take $\Delta=1.5$ Mev.

The Hamiltonian matrices for $J=1/2$, $3/2$ and $5/2$ are constructed, and the eigenvalues and the eigenfunctions for the lowest two states of each J are evaluated by explicit diagonalisation of the matrices. Table 1 lists the energy levels thus obtained for various values of q, normalised to zero energy for the lowest state $J=1/2$. We find that with our choice of the parameters Δ and $\hbar\omega$, the observed energy levels can be fitted reasonably well by $q \lesssim 1$. The argument against larger values of q is supplied by the excitation energy of the lowest $J=3/2$ state, and the splitting of the two $J=3/2$ states. Our experience with the nuclear shell model suggests that the separation of the $2s_{1/2}$ and $1d_{3/2}$ states could not be much larger than what we have assumed here, viz. 1.5 Mev. It can be seen from the structure of the Hamiltonian matrices that smaller values of Δ would lead to smaller separations between the two $J=5/2$ states, larger separations between $J=3/2$ states, and lower excitation energies for $J=3/2$ states as well as the first excited $J=1/2$ state. Thus we conclude that if the value of $\hbar\omega$ is fixed from the assumption that the first excited state of Si^{28} is a collective vibrational state (so that $\hbar\omega \sim 2$ Mev), and Δ is given by the reasonable value of 1.0–1.5 Mev, the observed energy level scheme

Table 1. Energy (in Mev) of the low-lying levels of Si^{29} for $\Delta=1.5$ Mev, $\hbar\omega=1.8$ Mev. The experimental energies are listed in the last row. Although the spin of the 3.07 Mev state is not exactly known, we have assumed it be 5/2.

J	1/2	1/2*	3/2	3/2*	5/2	5/2*
$q=0$	0	3.30	1.50	1.80	1.80	3.30
$q=0.5$	0	3.10	1.15	2.15	1.75	3.15
$q=1.0$	0	3.23	0.80	2.43	1.68	3.03
$q=2.0$	0	4.32	0.42	2.88	1.74	3.22
Expt.	0	$\geq$4.00	1.28	2.43	2.03	3.07

* unstarred and starred values refer to the lowest and the first excited states of a given J.

Table 2. Eigenfunctions of the low lying states of Si^{29}, for $q=1.0$.

J (nkj)	1/2	3/2	3/2*	5/2	5/2*
(0 0 1/2)	0.920	—	—	—	—
(0 0 3/2)	—	0.709	0.340	—	—
(1 2 1/2)	—	−0.605	0.677	0.919	0.129
(1 2 3/2)	0.377	−0.293	−0.501	−0.085	0.823
(2 0 1/2)	0.081	—	—	—	—
(2 2 1/2)	—	0.103	0.338	−0.021	0.453
(2 4 1/2)	—	—	—	—	—
(2 0 3/2)	—	0.117	−0.023	—	—
(2 2 3/2)	−0.078	0.147	−0.246	0.145	0.272
(2 4 3/2)	—	—	—	0.355	−0.165

* unstarred and starred values refer to the lowest and the first excited states of a given J.

for Si^{29} below 4 Mev, can be fairly well explained with $q\sim 1$. In particular, the spin of the 3.07 Mev state is here predicted to be 5/2.

Table 2 lists the eigenfunctions of some of the states for $q=1.0$. The notation is that (nkj) denotes the amplitude of the state of the nuclear core, with n quanta and spin k, coupled to the single particle state of spin j of the last odd neutron. It may be seen that the weak coupling approximation is fairly good for the states considered here, and the components with $n=2$ have rather small values, indicating convergence, and justifying the neglect of components with $n\geq 3$.

§ 3. Magnetic moment of Si^{29} and β-decay $P^{29}\rightarrow Si^{29}$

With the help of the eigenfunctions as calculated in the previous section we have evaluated the magnetic moment of the ground state of Si^{29}. In this calculation, Schmidt values are taken for the single particle magnetic moments, and the g-factor for the nuclear core is assumed to be 0.5. The values obtained for the magnetic moment are −1.82, −1.61, and −1.20 for $q=0.5$, 1.0 and 2.0 respectively, as compared to the value −1.91 for $q=0$, and the experimental value −0.55. The correction is in the right direction but not large enough. It seems to us that in the weak or intermediate coupling model (and also in the nuclear shell model including mixed configurations) where the correction to the single particle value of the magnetic moment is provided by mixing of the higher state wave functions in the ground state, it will not be possible to reproduce the experimental value of the magnetic moment, because as the experiments show, the next excited $J=1/2$ state lies at least 4 Mev above the ground state, and hence in any such model the mixing of the two states would not be expected to be very large.

The experimental data on the positron decay of the mirror nucleus P^{29} to the low levels of Si^{29} is summarised in reference 2. For the ft-values of the allowed beta-transitions we take the expression,

$$ft = \frac{6500}{|\int 1|^2 + 1.3|\int \sigma|^2}$$

Table 3 lists the values of the matrix element $|\int \sigma|^2$ for the various transitions as obtained from the above expression and the observed ft-values. Also listed are the values of the matrix element as calculated theoretically with the help of the wave functions obtained previously.

Table 3. Experimental and Theoretical values of the matrix element $|\int \sigma|^2$ for the β-decay of P^{29}.

Initial State J_i	Final State J_f	Experiment	Calculated		
			$q=0$	$q=1.0$	$q=2.0$
1/2	1/2	0.180	3	2.34	1.56
1/2	3/2	0.040	0	0.01	0.03
1/2	3/2*	0.15	0	0.02	0.03
1/2	5/2	$\leq$0.03	0	0	0

Again we see that the qualitative effect of introducing the collective vibrations of the core is to improve the shell model predictions, particularly for $q \simeq 1$ we obtain the ratio of the matrix element $|\int \sigma|^2$ for the two states $J=3/2$ as 2 compared to the experimental ratio of ~ 3.5; however, the quantitative discrepancies are very large.

§ 4. Conclusions

We have considered the unified model for Si^{29}, and attempted to improve the predictions of the nuclear shell model by introducing collective vibrations of the nuclear core. It appears that in the weak coupling, for a reasonable realistic choice of the parameters, the observed level scheme for excitation energy < 4 Mev can be well explained. However, when we consider the nuclear properties depending upon the detailed structure of the wave functions of the nuclear states, we find that although a qualitative improvement over the results of the simple nuclear shell model is obtained, the quantitative discrepancies between the theoretical and the experimental results are still large. It seems unlikely that increasing the magnitude of the coupling parameter would remove completely the discrepancies.

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Impurity Effect in Superconductors

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The influence of non-magnetic impurities on superconductivity is investigated on the basis of Bardeen, Cooper and Schrieffer's theory. It is shown that the impurity-scattering of electrons leads to the reduction of the energy difference between the superconducting state and the normal state, and the effective reduction of the electron-electron interaction due to phonon exchange and screened Coulomb force takes place as a result. This effect can account for the reduction of the superconducting transition temperature which is observed for tin by Lynton, Serin and Zucker. The calculated value of the change in the transition temperature is in agreement with the experimental one.

§ 1. Introduction

Lynton, Serin and Zucker[1] have reported that the superconducting transition temperature of tin is lowered when small quantities of other metals are added, by an amount proportional to the residual resistivity but independent of the actual metal added. In connection with the absorption of ultrasonic waves in metals, by the semi-classical treatment, Pippard[2] showed that the impurity-scattering of electrons leads to the reduction of the electron-phonon interaction and suggested that this effect may account for the reduction in the transition temperature of tin caused by alloying.

By the perturbation-theoretic method, we investigate the influence of non-magnetic impurities on superconductivity on the basis of Bardeen, Cooper and Schrieffer's theory.[3] Our investigation is concerned with the nature of the ground state of impure superconducting metals. We can show that the impurity-scattering of electrons leads to the reduction of energy difference between the superconducting state and the normal state, and the effective reduction of the electron-electron interaction due to phonon exchange and screened Coulomb force takes place as a result. Also we can show that, under the assumption of the isotropic impurity-scattering, the BCS theory explains the experimental fact that the change in the transition temperature is proportional to the residual resistivity but independent of the properties of the impurities added. The calculated value of the change in the transition temperature of tin is in semi-quantitative agreement with the experimental one.

§ 2. Calculation of energy difference between the superconducting state and the normal state

The Hamiltonian of the system is taken as follows,

$$H = H_{BCS} + H', \tag{1}$$

$$H' = \sum_{k,k',\sigma} v_{k,k'} C^*_{k,\sigma} C_{k',\sigma}, \tag{2}$$

where H_{BCS} is the Hamiltonian given in BCS, and $v_{k,k'}$ the matrix element of the perturbing potential which originates from the deviation of the periodic potential. Throughout this paper, the notations are same as in BCS.

It is easy to calculate the energy difference between the superconducting state and the normal state in the impure metal. By the use of the tables II and III in BCS, the second order perturbed energy can be written as

$$E_2(\epsilon_0) = -\sum_{k,k'} \frac{|v_{k,k'}|^2}{E+E'} \cdot \frac{1}{2} \cdot \left(1 - \frac{\epsilon\epsilon' - \epsilon_0^2}{EE'}\right). \tag{3}$$

In the limit $\epsilon_0 \to 0$, eq. (3) reduces to the expression for the perturbed energy of the normal state, which is

$$E_2(0) = -\sum_{\epsilon\epsilon'<0} \frac{|v_{k,k'}|^2}{|\epsilon| + |\epsilon'|}. \tag{4}$$

Thus the energy difference between the superconducting state and the normal state can be written as follows,

$$W_0^* = W_0 + E_2(0) - E_2(\epsilon_0), \tag{5}$$

where W_0 is the energy difference of the pure metals. By using eqs. (3) and (4), we get

$$E_2(0) - E_2(\epsilon_0) = \frac{\Omega^2}{(2\pi)^6} \iint |v_{k,k'}|^2 \left\{ \frac{EE' - \epsilon\epsilon' + \epsilon_0^2}{2EE'(E+E')} - \frac{|\epsilon\epsilon'| - \epsilon\epsilon'}{2|\epsilon\epsilon'|(|\epsilon| + |\epsilon'|)} \right\} dk\, dk', \tag{6}$$

where Ω is the volume of the system. In the case of weak coupling, the predominant contribution of the integral of eq. (6) may come from the neighborhood of the Fermi surface. Then, in eq. (6), the matrix element $|v_{k,k'}|^2$ can be replaced by an average value over the Fermi surface, $\langle |v_{k,k'}|^2 \rangle_f$. Hence eq. (6) can be rewritten as follows,

$$E_2(0) - E_2(\epsilon_0) = \frac{\Omega^2}{2(2\pi)^6} \langle |v_{k,k'}|^2 \rangle_f \iint_R \left\{ \frac{EE' - \epsilon\epsilon' + \epsilon_0^2}{EE'(E+E')} - \frac{|\epsilon\epsilon'| - \epsilon\epsilon'}{|\epsilon\epsilon'|(|\epsilon| + |\epsilon'|)} \right\} dk dk'. \tag{7}$$

The integration is over the region R; the region of interaction which is defined by the condition $|\epsilon| < \hbar\omega$, as in BCS. Thus we obtain

$$E_2(0) - E_2(\epsilon_0) = \frac{\Omega^2}{2\pi^4} \left(\frac{mK_0}{\hbar^2}\right)^2 \langle |v_{k,k'}|^2 \rangle_f \int_0^{\hbar\omega} d\epsilon \int_0^{\hbar\omega} d\epsilon' \left\{ \frac{EE' + \epsilon_0^2}{EE'(E+E')} - \frac{1}{|\epsilon| + |\epsilon'|} \right\}, \tag{8}$$

where K_0 is the wave number at the Fermi surface. To the order of $(\epsilon_0/\hbar\omega)^2$, we get

$$E_2(0)-E_2(\epsilon_0)=-\frac{9}{8}N\frac{N\langle|v_{k,k'}|^2\rangle_f}{E_f^2}\hbar\omega\left(\frac{\epsilon_0}{\hbar\omega}\right)^2\log\left(\frac{\hbar\omega}{\epsilon_0}\sqrt{e}\right), \tag{9}$$

where N is the total number of conduction electrons, and E_f the Fermi energy. Then eq. (9) can be rewritten as follows,

$$E_2(0)-E_2(\epsilon_0)=-\frac{9}{32}N\frac{N\langle|v_{k,k'}|^2\rangle_f}{E_f^2}\cdot\frac{\hbar\omega}{N(0)V}\exp\left(-\frac{2}{N(0)V}\right). \tag{10}$$

Substituting eq. (10) in eq. (5), we obtain

$$W_0^*=W_0\left(1-\frac{9}{64N(0)V}\cdot\frac{N}{N(0)\hbar\omega}\cdot\frac{N\langle|v_{k,k'}|^2\rangle_f}{E_f^2}\right). \tag{11}$$

It is seen from eq. (11) that the impurity-scattering of electrons leads to the reduction in the energy difference between the superconducting state and the normal state, that is, the interaction of impurity atoms with the electrons lowers the energy of the normal state more than that of the superconducting state.† The reduction of the electron-electron interaction can be derived from eq. (11). Using eq. (11) and BCS' expression for W_0, we get

$$W_0^*=2N(0)(\hbar\omega)^2\exp(-2/N(0)V^\dagger), \tag{12}$$

where V^* is the electron-electron interaction constant in the case of impure metal and is given by

$$V^*=V\left(1-\frac{3}{32}\cdot\frac{N\langle|v_{k,k'}|^2\rangle_f}{E_f\hbar\omega}\right). \tag{13}$$

Next we shall show that the change in the transition temperature is proportional to the residual resistivity. Using eq. (11) and the empirical relation that $N(0)\cdot(\kappa T_c)^2$ is proportional to W_0, we get

$$\frac{\Delta T_c}{T_c}\sim\frac{9}{128N(0)V}\cdot\frac{N}{N(0)\hbar\omega}\cdot\frac{N\langle|v_{k,k'}|^2\rangle_f}{E_f^2}. \tag{14}$$

Using the free electron value of $N(0)$, we get

$$\frac{\Delta T_c}{T_c}\sim\frac{3}{32}\cdot\frac{1}{N(0)V}\cdot\frac{N\langle|v_{k,k'}|^2\rangle_f}{E_f\hbar\omega}. \tag{15}$$

According to Nordheim's[6] approximation, we put

$$|v_{k,k'}|^2=(N_a/\Omega^2)x(1-x)|\delta v_{k,k'}|^2, \tag{16}$$

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† Recently, in connection with Matthias, Shul and Corenzwit's experiment[4], C. Herring has suggested the same effect in the case of the spin exchange in superconductors. His suggestion is seen in reference (5). The author is indebted to Professor R. Kubo for this point.

where N_a is the total number of ions and x the concentration of impurities. $\delta v_{k,k'}$ is expressed as follows,

$$\delta v_{k,k'}=\int \{v(\boldsymbol{r})-v_i(\boldsymbol{r})\}\, e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\, d\boldsymbol{r}\,, \tag{17}$$

where $v(\boldsymbol{r})$ and $v_i(\boldsymbol{r})$ are the core potentials of the host atoms and the impurity atoms, respectively. From eqs. (14) and (16), it is evident that the change is proportional to the concentration of impurities x when it is small. This result coincides with the experimental fact. It is well known that the residual resistance and the mean free path[7] are given by

$$\rho_r=3\hbar^3/32\pi m e^2 l_r \mathrm{E}_f\,, \tag{18}$$

and

$$\frac{1}{l_r}=\frac{8\pi\Omega_0 m^2}{\hbar^4}\cdot\frac{\mathrm{N}_a^{\,2}}{\Omega^2}x(1-x)\int_0^{\pi}|\delta v_{k,k'}|^2(1-\cos\theta)\,\sin\theta\, d\theta\,, \tag{19}$$

where Ω_0 is the volume of the unit cell, and θ is the angle between the two wave vectors, $\boldsymbol{k}$ and $\boldsymbol{k}'$.

If $|\delta v_{k,k'}|^2$ is replaced by an average value over Fermi surface, i.e., if the impurity-scattering is isotropic over the Fermi surface, eq. (19) can be rewritten as follows,

$$\frac{1}{l_r}=\frac{16\pi^3\Omega_0 m}{\hbar^4}\cdot\frac{\mathrm{N}_a^{\,2}}{\Omega^2}x(1-x)\langle|\delta v_{k,k'}|^2\rangle_f\,. \tag{20}$$

This is consistent with the assumption of BCS that the electron-electron interaction constant $\mathrm{V}_{k,k'}$ is replaced by an average value over the Fermi surface. Substituting eqs. (16) and (20) in eq. (15), we obtain

$$\frac{\Delta\mathrm{T}_C}{\mathrm{T}_C}\sim\frac{9}{128}\cdot\frac{1}{\mathrm{N}(0)\mathrm{V}}\cdot\frac{1}{(r_s a_0)^3}\cdot\frac{\hbar^4}{m^2}\cdot\frac{1}{\kappa\theta}\cdot\frac{1}{l_r\mathrm{E}_f}\,, \tag{21}$$

$$\sim\frac{3}{32\pi^2}\cdot\frac{e^2\hbar}{\mathrm{N}(0)\mathrm{V}}\cdot\frac{1}{(r_s a_0)^3}\cdot\frac{\rho_r}{m(\kappa\theta)}\,, \tag{22}$$

where a_0 is the Bohr radius and r_s is defined by the following relation,

$$(r_s a_0)=(3/4\pi\mathrm{N}_e)^{1/3}\,,$$

where N_e is the number density of electrons. The number of valence electrons per atom and the mass of ions, contained in H_{BCS} of eq. (1), must be replaced by their average values. Therefore, the coefficient of ρ_r in eq. (22) is expressed by the value corresponding to their average ones. When the concentration of impurities is small, however, it will be adequate to calculate the coefficient of ρ_r by using the numerical values of the pure metal. Then the change $\Delta\mathrm{T}_C$ is proportional to the residual resistance but independent of the property of the added impurities. Eq. (22) explains the result observed by Serin et al. For the purpose of numerical

comparison with the experimental results, we apply eq. (21) to the case of tin. Using the values of $N(0)V=0.296$, $r_s=2.21$, $\theta=195^\circ K$ (from Pines[8]), and $E_f=1.131\times10^{-11}$ erg*, we obtain

$$\Delta T_c/T_c \sim 7\times10^{-7}\times(1/l_r). \tag{23}$$

In the case of the alloy whose residual resistance is 1 per cent of the room temperature resistance, the mean free path l_r is of the order of 10^{-4} cm. Then, from (23), we see that transition temperature T_c is decreased by about 0.7 per cent. This value is in agreement with the observed value of 0.7 per cent.

§ 3. Discussion

Pines[8] suggested that the effect of alloying is explained by using the average number of valence electrons per atom and the average mass of ions. This is the coherent part of the impurity effect. Our investigation is concerned with the incoherent part of the impurity effect (impurity-scattering). Lynton, Serin and Zucker's experiment shows that when the concentration of impurities is small, the incoherent part contributes essentially to the change in the transition temperature rather than the coherent part, and that when the concentration is relatively large, the coherent part has the predominant effect of increasing the transition temperature. Our conclusion is that in the case of the small concentration of impurities, the impurity-scattering lowers the energy of the normal state more than that of the superconducting state, and the effective reduction of the electron-electron interaction due to phonon exchange, and screened Coulomb force takes place as a result. The reduction of the transition temperature caused by alloying can be explained by this effect. Also it is seen that under the assumption of the isotropic impurity-scattering over the Fermi surface, the BCS theory can explain the experimental fact that the reduction of the transition temperature is proportional to the residual resistivity but independent of the added impurities. It will be allowable to neglect anisotropic effects and assume the isotropic impurity-scattering over the Fermi surface, in the same approximation as that of the BCS theory that the matrix element $V_{k,k'}$ is replaced by an average value over the Fermi surface. Furthermore, the discussion on the case of the screened impurity-potential, in which $\delta v_{k,k'}$ will not change appreciably over the Fermi surface, leads us to the same assumption.

In order to calculate quantitatively the change of transition temperature caused by alloying, we should investigate the change of free energy difference between the normal state and the superconducting state. Our calculation, however, may be sufficient for explanation of the experimental result. The details of the quantitative calculation will be published in future.

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* This value is evaluated from the specific heat coefficient, γ, which is 4.11×10^{-4} cal/mole/deg^2 for tin (from Serin et al.).

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Global Symmetry and Isotopic Spin Conservation in Non-Leptonic Hyperon Decays

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Possible existence of the universal $V\mp A$ coupling and the global symmetry in the non-leptonic hyperon decays is discussed. The observed deviations from the expected symmetries in actual decays are explained as effects due to the mesonic structure of hyperons. In a close relation with the idea of the global symmetry a new isotopic spin is defined, which is 1/2 for all the baryons and integer for K. The conservation law of this isotopic spin is satisfied in the non-leptonic decays.

§ 1. Introduction

There are many conservation laws such as the conservation of the baryon number, of the charge, of the strangeness, and of the isotopic spin which have been found in elementary particle physics. Some of them are known to be only approximately valid and reasons for their violation are due to either the electromagnetic or the weak decay interactions. It is quite conceivable that there may be more approximate conservation or symmetry laws so far unknown to us, because they are violated due to the strong π or K interaction to an appreciable extent. Among them the global symmetry of baryons has been proposed by Gell-Mann[1] and Schwinger,[2] but the K interaction is supposed to violate it considerably and prevents us from proving the existence of the symmetry.

The isotopic spin change (ΔI) in the non-leptonic decay of hyperons has been studied and the $\Delta I=1/2$ selection rule has been confirmed to be approximately valid.[3] An attempt was made by Prentki et al[4]. and one of the present authors[5] for interpreting this $\Delta I=1/2$ rule as a conservation law $\Delta(I+I')=0$ strictly valid to the non-leptonic decay interaction when the K interation and the electromagnetic interaction are turned off. The small deviation from it may be due to either the electro-magnetic interaction or the K interaction.**

** One way to obtain the deviation is through the electro-magnetic correction on the $\Delta I=1/2$ rule. Another way is to assume presence of decays with $\Delta I\neq 1/2$ even in the bare particle decay interaction. In this case the global symmetry of the π interaction prevents those decays with $\Delta I\neq 3/2$ from occurring as real processes. Then the K interaction violates the symmetry, thus making possible the decays to proceed. See ref. (5), pages 641 and 642.

Our purpose of making further study on the non-leptonic decay of hyperons is to try to disclose yet unknown symmetry or conservation laws, which might be exhibited in the decay interactions of bare baryons. Deviations from the expected laws should be due to either the π or the K cloud around the bare particle (core) and studying of them may give us some insight into properties of the mesonic clouds of hyperons.

§ 2. Experimental data of the non-leptonic hyperon decays

Before proceeding with our program, let us summarize our experimental knowledge about the non-leptonic decays. The observed decay amplitude A and the asymmetry factor α for various hyperon decays are given in Table I.*,**

Table I. Table for values of the decay amplitude and the asymmetry factor

	$\lvert A\rvert^2$	$\lvert\alpha\rvert$	
$\Lambda\to p+\pi^-$	0.78×10^{-14}	~1	$(A_{\Lambda-})$
$\to n+\pi^0$	0.41×10^{-14}	unmeasured	$(A_{\Lambda0})$
$\Sigma^+\to p+\pi^0$	1.04×10^{-14}	$\lvert\alpha_0\rvert\sim1$	(A_0)
$\to n+\pi^+$	1.24×10^{-14}	$\lvert\alpha_+\rvert\sim0$	(A_+)
$\Sigma^-\to n+\pi^-$	1.06×10^{-14}	$\lvert\alpha_-\rvert\sim0$	(A_-)

Since the parity may be violated in those decays, the amplitudes are composed of two parts, A^s and A^p; the former gives a pion in an s-state and the latter gives a pion in a p-state. Gell-Mann and Rosenfeld[6] consider A as an amplitude in a complex plane, namely, $A=A^s+iA^p$. The branching ratio of Λ decay is quite consistent with the $\Delta I=1/2$ rule. The large asymmetry factor of Λ decay ($\alpha\approx1$) means $\lvert A^s\rvert\approx\lvert A^p\rvert$ for $\Lambda\to p+\pi^-$ and indicates that the decay interaction is of the $V\mp A$ type.

The $\Delta I=1/2$ rule applied on the Σ decays imposes a relation on A;

$$-A_+ +\sqrt{2}\,A_0=A_-. \tag{1}$$

The equation (1) and the observed relation

$$\lvert A_+\rvert^2\approx\lvert A_0\rvert^2\approx\lvert A_-\rvert^2 \tag{2}$$

require that A_+, A_-, and $\sqrt{2}\,A_0$ form approximately a $(45^\circ, 45^\circ, 90^\circ)$ triangle in the complex A-plane. This indicates A^s/A^p or the asymmetry factor is different among the three Σ decays. In fact the results of recent measurement on α

* The decay amplitude A is defined by $\lvert A\rvert^2=\hbar\Gamma/2pc$, where Γ is the corresponding decay rate and p is the momentum of the pion emitted in the decay. See ref. (6), page 447.

** The asymmetry factor (α) times polarization of hyperon (P) has been measured, but not α alone. According to the latest experiment done at Berkeley, αP is large for $\Sigma^+\to p+\pi^0$ and very small for $\Sigma^\pm\to n+\pi^\pm$ (private communication from H. Miyazawa). The values for α in Table I are certainly consistent with the measured values of αP, although they may not necessarily be true.

($\alpha_{\pm}\approx 0$, $|\alpha_0|\approx 1$) are consistent with this prediction and require that the two lines of the triangle, A_+ and A_-, nearly coincide with the two axises of the complex A-plane, respectively.

§ 3. Universality and global symmetry of the bare hyperon decays

Our working hypothesis here is to attribute as many symmetry or conservation laws as possible to the bare particle decay interaction, and then to see whether the observed violation of the laws can be explained as due to effects of the mesonic cloud. Since we have only little understanding even of the pion structure of the nucleon, where more experimental information is available, it is fair to say that we have no knowledge on the mesonic structure of hyperons at all. Therefore, what we will emphasize later is to present a possible mesonic structure and a decay scheme attached to it which can qualitatively explain the observed violation of the assumed laws.

The observed large asymmetry ($|\alpha|\approx 1$) in the Λ decay indicates that the decay interaction is approximately of the $V\mp A$ type. If the actual Λ is equal to the bare Λ in most of its time or the effect of mesonic cloud on the decay interaction is negligible, the decay interaction of the bare Λ may also be of the $V\mp A$ type. This $V\mp A$ interaction and its possible universality for the bare Fermion decays have been suggested[8] and may have profound meaning. Therefore, we shall assume that the bare hyperon decays ($\Sigma\to N+\pi$ and $\Lambda\to N+\pi$) are of the $V\mp A$ type.

The observed branching ratio of Λ decay, $(\Lambda\to n+\pi^0)/(\Lambda\to p+\pi^-)\approx 1/2$, is consistent with the $\Delta I=1/2$ selection rule. This will also hold for the bare Λ decay, because the π and K interactions are supposed to be charge independent and do not change ΔI. The $\Delta I=1/2$ rule is not necessarily required for the Σ decays, but the experimental data are consistent with it. Furthermore, if it does not hold, Λ should show, to a certain extent, decays with ΔI different from $1/2$ as Λ can decay indirectly through a virtual Σ decay. In order to preserve symmetry between the Λ and Σ decays as much as possible we shall assume that the $\Delta I=1/2$ rule is universally valid for the non-leptonic hyperon decays.

As discussed in § 2, the $\Delta I=1/2$ rule alone does not determine the Σ decays but gives only eq. (1). According to our working hypothesis we shall further impose a symmetry property for the bare hyperon decays, which is essentially equal to Gell-Mann's global symmetry[1] applied to the decay interaction. The idea is as follows:

Σ^+ and $Y^0=(\Lambda^0-\Sigma^0)/\sqrt{2}$ or $Z^0=(\Lambda^0+\Sigma^0)/\sqrt{2}$ and Σ^- form a pair of baryons, respectively, as p and n or Ξ^0 and Ξ^- do;

$$\Psi_Y=\begin{pmatrix}\Sigma^+\\ Y^0\end{pmatrix},\quad \Psi_Z=\begin{pmatrix}Z^0\\ \Sigma^-\end{pmatrix},\quad \Psi_N=\begin{pmatrix}p\\ n\end{pmatrix},\quad \text{and}\quad \Psi_\Xi=\begin{pmatrix}\Xi^0\\ \Xi^-\end{pmatrix}. \tag{3}$$

The four Ψ's are considered to be an iso-spinor with isotopic spin $1/2$, the iso-space here being different from the ordinary one. Gell-Mann's global symmetry

is to require that the π couplings with each of the four fields are equal among each other:

$$H_\pi = f\{\bar{\Psi}_N \gamma_5 \vec{\tau} \Psi_N \vec{\phi}_\pi + \bar{\Psi}_\Xi \gamma_5 \vec{\tau} \Psi_\Xi \vec{\phi}_\pi + \bar{\Psi}_y \gamma_5 \vec{\tau} \Psi_y \vec{\phi}_\pi + \bar{\Psi}_z \gamma_5 \vec{\tau} \Psi_z \vec{\phi}_\pi\}. \tag{4}$$

The baryons have the identical mass when only the π coupling is present and the actual mass differences among them are due to the K coupling. Here we assume the global symmetry even for the bare hyperon decay and take the following decay interaction:

$$g\{\bar{\Psi}_Y \vec{\tau} \gamma_\mu (1 \pm \gamma_5) \Psi_N \partial_\mu \vec{\phi}_\pi + \bar{\Psi}_Z \vec{\tau} \gamma_\mu (1 \pm \gamma_5) \Psi_\Xi \partial_\mu \vec{\phi}_\pi\} + \text{hermite conjugate.} \tag{5}$$

This globally symmetric interaction contains in it the conservation law of a new isotopic spin quantum number, which is $1/2$ for every baryons and 1 for π. The $\Delta I = 1/2$ rule, where I is the ordinary isotopic spin, and the universal $V \mp A$ coupling are also satisfied.

If there is no correction to the decay interaction (5) due to the π and K couplings, the universal $V \mp A$ law gives the asymmetry factor α equal to ± 1 for all the Σ and Λ decays. Furthermore, the global symmetry gives

$$|A_0|^2 : |A_+|^2 : |A_-|^2 = 1 : 2 : 0. \tag{6}$$

Since the strong π coupling is assumed to be globally symmetric and also does not produce the mass differences among baryons, eq. (6) holds even in the presence of the π coupling. The asymmetry factor, on the other hand, will deviate from ± 1, because the π coupling may enhance or damp A^s more than A^p. Yet α should be the same among all the non-leptonic hyperon decays. Only the K coupling can cause the observed difference of α among them. In the following section we shall discuss modifications of the decay amplitudes A due to the π and K mesonic structure of hyperons and we will show that the observed A could be obtained by taking such effects into account.

§ 4. Effects of the mesonic structure of hyperons on the non-leptonic decay amplitudes

Since the magnitude and phase of A in the complex plane could be quite effected by the presence of the mesonic clouds around the core, let us first try to get an orientation about the structure of the clouds. The actual Λ or Σ is considered to be in a composite state composed of a bare particle state, a bare particle plus one pion state and so on:

$$\begin{aligned} &\Lambda: \tilde{\Lambda},\ \tilde{\Sigma}^{\pm,0}+\pi^{\mp,0},\ldots\ldots \\ &\Sigma^{\pm}: \tilde{\Sigma}^{\pm},\ \tilde{\Lambda}+\pi^{\pm},\ \tilde{\Sigma}+\pi,\ldots\ldots \end{aligned} \tag{7}$$

We consider here only the pion cloud, because to the following discussion it is more important than the K cloud and also it is probably more intense. Then the bare particle $\tilde{\Lambda}$ or $\tilde{\Sigma}$ in eq. (7) should be interpreted as one including the K cloud. Therefore, $\tilde{\Sigma}$ is heavier than $\tilde{\Lambda}$ if the actual mass difference between Σ and Λ is due to the K interaction. Then the energy difference between the Σ and the $\tilde{\Lambda}+\pi$ states is small and the $\tilde{\Lambda}+\pi$ state may have an appreciably large amplitude. On the other hand, the $\tilde{\Sigma}+\pi$ configurations in both Λ and Σ are energetically much less favourable and may be relatively unimportant.

Hereafter we shall take a rather extreme point of view that Λ is nearly equal to $\tilde{\Lambda}$ and Σ is a superposition of $\tilde{\Sigma}$ and $\tilde{\Lambda}+\pi$. Deviations from this assumed structure may be appreciable, but we hope that qualitative aspects of the following discussion still remain. Making one further remark concerning the importance of the K coupling may be relevant here. The K coupling is assumed to be an agent for the mass splitting among baryons and the effect of it is taken into accont through the mass difference between $\tilde{\Sigma}$ and $\tilde{\Lambda}$. Otherwise the $\tilde{\Sigma}+\pi$ configurations in the actual Λ and Σ should be equally important as the $\tilde{\Lambda}+\pi$ configuration in Σ and the following discussion concerning the asymmetry factor α will have no validity.

The bare hyperon decay interaction (5) is written in terms of the bare particle fields,* Ψ_Y, Ψ_Z, Ψ_N, Ψ_Ξ and ϕ_π. Since the bare nucleon (Ψ_N) and the bare pion can form an actual nucleon, the interaction (5) itself can cause, for example, a virtual decay of $\tilde{\Lambda}$ into an actual neutron. This transition does conserve momentum, spin, and the other conserving quantities except energy of the system and may be important as a virtual process. Possible transitions due to the interaction (5) are schematically shown in Fig. 1. The core of hyperon, $\tilde{\Sigma}$ or $\tilde{\Lambda}$, can decay into either a continuum state with one nucleon plus one pion or an actual nucleon state ($\tilde{\Lambda}\to n$ or $\tilde{\Sigma}^{+}\to p$). It is important to notice that only V or A interaction can be effective for $\tilde{\Lambda}\to n$ (or $\tilde{\Sigma}^{+}\to p$), because the relative parity between $\tilde{\Lambda}(\tilde{\Sigma})$ and N is definite and therefore parity is always conserved (or violated)

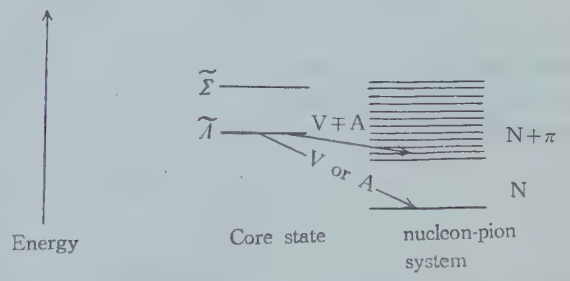


Fig. 1 Schematical representation of possible transitions due to the bare particle decay interaction. (5).

* Since we shall discuss modification of the decay interaction (5) due to the pion cloud only, the decay interaction which we really need is one written not in the bare fields but in the core fields including the K cloud. However, we shall here disregard the difference.

when their relative parity is even (or odd). On the other hand the pion in the continuum state can be either in s or p orbit around the nucleon and Λ^s and Λ^p can be both present for the transitions to the continuum state.

The presence of the transition, the core going into a nucleon alone, is important in reducing the magnitude of asymmetry factor, $|\alpha$, from 1 to 0 for the $\Sigma^+\to n+\pi^+$ and $\Sigma^-\to n+\pi^-$ decays. First order perturbational treatment of eq. (5) does not contain this transition and it appears only when the strong π interaction between the bare nucleon and pion is taken into account.* We shall assume that the transition is very much enhanced by the π interaction.

The decay interaction (5) and our model for the hyperon structure, $\tilde{\Lambda}\approx\Lambda$ and $\Sigma^\pm\approx\tilde{\Sigma}^\pm$ plus $\tilde{\Lambda}+\pi^\pm$, gives the following decays:

	decay scheme	decay amplitude	
Σ decay	$\tilde{\Sigma}^+ \xrightarrow[V\mp A]{} p+\pi^0$	(a_s+ia_p)	
	$\tilde{\Sigma}^+ \xrightarrow[V\mp A]{} n+\pi^+$	$\sqrt{2}\,(a_s+ia_p)$	(8)
	$\tilde{\Lambda}+\pi^\pm \xrightarrow[V \text{ or } A]{} n+\pi^\pm$	ib_p	
Λ decay	$\Lambda \xrightarrow[V\mp A]{} p+\pi^-$	$(a_s'+ia_p')$	
	$\xrightarrow[V\mp A]{} n+\pi^0$	$\sqrt{\tfrac{1}{2}}\,(a_s'+ia_p')$.	

The decay amplitude a_s+ia_p is the amplitude for a $\tilde{\Sigma}$ going into a continuum $N+\pi$ state times the probability amplitude for seeing a bare $\tilde{\Sigma}$ alone in the actual Σ. Because of the $V\mp A$ character, we have $|a_s|\approx|a_p|$. The amplitude ib_p is the amplitude for a $\tilde{\Lambda}$ going into a neutron times the probability amplitude of the $\tilde{\Lambda}+\pi$ configuration in the actual Σ. We have chosen relative parity between Λ and N as even, and therefore only the p-wave amplitude has appeared. This choice is more desirable from our point of view for preserving symmetry among baryons as much as possible, although it is not essential to our discussion. The resultant decay amplitudes for the three Σ decays are

$$A_+=\sqrt{2}\,(a_s+ia_p)+ib_p,$$
$$A_-= \qquad\qquad ib_p, \tag{9}$$

and $$A_0=a_s+ia_p,$$

where $$|a_s|\approx|a_p|. \tag{10}$$

The effect of the mesonic structure of hyperons on their decay amplitudes is expressed solely in terms of b_p/a_p and non-zero value of this parameter modifies

* See the Appendix.

the simple prediction by the universal $V\mp A$ law and the global symmetry applied on the Σ' decay. Now let us examine whether the observed asymmetry factors and the relative magnitudes among A's (in Table I) can be obtained by a suitable choice of the ratio.

Eq. (9) predicts $|\alpha_0|\approx 1$ and $|\alpha_-|\approx 0$ which are consistent with the measurements. The value of b_p, a_p is determined from the requirement, $|\alpha_+|\approx 0$, and it gives

$$\sqrt{2}\,a_p+b_p\approx 0. \tag{11}$$

Therefore, A_- and A_- have only s and p wave amplitudes respectively. As for the decay rates, eq. (11) together with eqs. (9) and (10) give

$$|A_+|^2\approx|A_-|^2\approx|A_0|^2\approx 2a^2, \tag{12}$$

which is just the observed relation among the three amplitudes.

For the Λ decay the observed branching ratio ($\approx 1/2$) is easily explained as a consequence of the decay interaction (5), which has the $\Delta I=1/2$ rule in it. As for the magnitude of decay amplitude, the global symmetry of the decay interaction assures us the nearly equal decay rate for $\Lambda\to p+\pi^-$ and $\Sigma^+\to p+\pi^0$, which is consistent with the experimental result. If we take our model too literally, the ratio $|A_\Lambda-|^2/|A_0|^2(=|a'/a|^2)$ is equal to 1 divided by the probability of the bare $\tilde{\Sigma}$ configuration in an actual Σ'. Theoretically this should be larger than 1 and may conflict with the experimental value, which is a little smaller than 1. Considering our choice of the extremely specific mesonic structure of hyperons, the discrepancy should not be taken too seriously.

§ 5. Discussions

We have succeeded to show how the observed hyperon decays can be reconciled with the assumed laws, namely, the universal $V\mp A$ law and the global symmetry in the bare particle decay. There the $\tilde{\Lambda}+\pi$ configuration in Σ' and the parity conserving decay of $\tilde{\Lambda}$ going into a neutron have played essential role. Although we have practically no quantitative understanding on the mesonic structure of hyperons, our success could perhaps be considered as a certain justification for the validity of the assumed laws and the importance of the $\tilde{\Lambda}+\pi$ configuration in Σ'.

Our difficulty in exploring the symmetry and conservation laws valid for the bare particle decay interaction is that those are, in general, violated to a considerable extent by stronger interactions than the decay one itself. There has been argument by several authors that the stronger interaction would show more symmetry or conservation law than the weaker one. The original idea of the global symmetry was proposed along this line. If nature really follows the arguments then finding these laws would be easier than otherwise. We think that the argument does not necessarily stand and there may be other cases where the weaker interaction shows more symmetry than the stronger one. In fact, our global symmetry is valid for

both the strong π interaction and the weak decay interaction and not valid for the K interaction.

Decays of baryon do not occur by separation of the π or K cloud from the core of baryon. Rather they are due to change of the core itself, accompanied with the change of the isotopic spin, strangeness and so forth. Therefore, we may say that the essential part of decay is in a certain region of the core, although how small the region is is beyond our present knowledge. If the decay interaction possesses some symmetry or conservation laws, these would be of vital importance for future study of the natural law governing the small region of the core.

The decay interaction (5) with the global symmetry has indicated a new possibility for assigning an isotopic spin to each elementary particle, which is different from the ordinary one. In Table II we give a list of values of the new isotopic spin together with the older one.

Table II. Values of the isotopic spin of baryons, π, and K mesons

	N	Λ	Σ	Ξ	K	π
New assignment	1/2	1/2	1/2	1/2	0,1	1
Old assignment	1/2	0	1	1/2	1/2	1

We list only the magnitudes of isotopic spin. The four isotopic spin doublets in the new assignment should be taken as shown in eq. (3). The new isotopic spin is conserved in the π and the weak decay interaction, but not in the K interaction Since the K interaction does not conserve the new isotopic spin, the value of isotopic spin for K itself is not determined through the strong interaction. Rather we require that the conservation of the new isotopic spin is generally valid not only for the bare hyperon decays but also for the bare K decays into π mesons. The $\Delta I=1/2$ rule interpreted as the conservation law of the new isotopic spin gives zero as the isotopic spin of K_1^0 and 1 as those of (K^+, K_2^0, K^-).

Let us ask which assignment of the isotopic spin is more advantageous for our understanding of elementary particles. Since in the old assignment the strong π and K interactions conserve the isotopic spin, the study of production, scattering, and other strong interactions would certainly be much more easily treated in terms of the older one. On the other hand, the new one may have more advantages over the older one in studying the weak decay interaction, which conserves the new isotopic spin when the particles are bare. (See Table III) Furthermore, it may be appropriate to remark that whether the K interaction conserves the usual isotopic spin is not yet decided experimentally. We think that *the two isotopic spins are in complementary relation with each other*; for example, existence of the charge doublets of K meson, (K_+, K_0) and $(\tilde{K}_0, \tilde{K}_-)$, and the K_1^0 and K_2^0 problem would be clearly understood in the usual isotopic spin assignment 1/2 for K. On the other hand the $\Delta I=1/2$ rule in the non-leptonic K decay would be understood more easily in terms of the new isotopic spin assignment, which is zero or 1 for K.

Table III. Tables of isotopic spin conservation and non-conservation for the various types of couplings

	π coupling	K coupling	el-mag. coupling	decay
New isotopic spin	○	×	×	○
Old isotopic spin	○	○(?)	×	×

In § 3 we assumed the decay interation (5) of Yukawa type. We can also take the interaction of Fermi type, which is biquadratic in the baryon fields. Both the universal $V \mp A$ law and the global symmetry can be retained, if we write the interaction as, e.g.,

$$g\{\bar{\Psi}_Y \vec{\tau}\gamma_\mu(1\pm\gamma_5)\Psi_N(\bar{\Psi}_N\vec{\tau}\gamma_\mu\gamma_5\Psi_N+\bar{\Psi}_Y\vec{\tau}\gamma_\mu\gamma_5\Psi_Y)$$
$$+\bar{\Psi}_Z\vec{\tau}\gamma_\mu(1\pm\gamma_5)\Psi_\Xi(\bar{\Psi}_\Xi\vec{\tau}\gamma_\mu\gamma_5\Psi_\Xi+\bar{\Psi}_Z\vec{\tau}\gamma_\mu\gamma_5\Psi_Z)\} \qquad (13)$$
$$+\text{hermitic conjugate.}$$

The conservation of the new isotopic spin is evident in the expression (13) and also the $\Delta I=1/2$ rule is obtained. Whether the Yukawa type interaction or the Fermi one is more fundamental cannot be decided here. If the latter turns out to be a choice, we have to remember that eq. (13) is only an example of those interactions which show the universal $V \mp A$ law, the global symmetry and the conservation of the new isototopic spin.

Finally, we shall make remarks about the observed deviation from the $\Delta I=1/2$ rule. The presence of non-leptonic K^+ decays into two pions show clearly that the rule is only approximately valid. The electro-magnetic interaction certainly is a cause of the violation of the law, although preliminary study of it seems to indicate the effect is not enough to explain the observed K^+ decay rate. We shall discuss the effect in a forthcoming paper which will appear in the Supplement of the Progress of Theoretical Physics. If the electro-magnetic correction turns out to be too small, we have to assume such a bare particle decay interaction that causes transitions not only with $\Delta I=1/2$ but also with ΔI different from $1/2$. It is easy to show that the conservation of the new isotopic spin is not compatible with the presence of such transitions. Therefore, we have to abandon either our global symmetry and conservation of the new isotopic spin in the bare baryon decays or the charge independence of the K coupling which is not yet experimentally proved to good accuracy. What is the cause of the decay of K^+ into two pions remains to be seen.

Appendix

Perturbation calculation of decay amplitudes in eq. (9)

The decay amplitudes (eq. (9)) are obtained through two important assumptions; Σ is in a compound state composed of the $\tilde{\Sigma}$ and the $\Lambda+\pi$ configurations,

and the enhancement of the virtual transition, a Λ going into a neutron, does occur. Considering the great difficulty in making any reliable calculation of the mesonic corrections on the bare particle decay interaction, it is not unreasonable to emphasize the importance of the existence of a model which could reconcile the proposed symmetry laws with the observed decay rates and asymmetry factors of Σ decays, such as ours, where the two assumptions are essential.

However, we shall here present perturbation calculation of the Σ decay amplitudes for comparison. In the perturbation method the decay amplitudes $\boldsymbol{a}$ and $\boldsymbol{b}$ in eq. (8) are obtained from Feynman diagrams, Fig. (A1) and Fig. (A2), respectively. If we make calculation in the static approximation, the amplitudes become

$$\pm \boldsymbol{a}_s \approx \boldsymbol{a}_p \propto g(-i\boldsymbol{\sigma}\boldsymbol{q})\,(2\omega_q)^{-1/2}, \tag{A1}$$

and

$$\begin{aligned}\boldsymbol{b}_p \propto\ & (3g/2)\,(f'/\mu)^2 (2\omega_q)^{-1/2}(2\pi)^{-3}\int d^3k\,(i\boldsymbol{\sigma}\cdot\boldsymbol{k})\,(\Delta_{\Sigma N}-\omega_q-\omega_k)^{-1} \\ & \times(-i\boldsymbol{\sigma}\cdot\boldsymbol{k})\,(\Delta_{\Sigma\Lambda}-\omega_q)^{-1}(-i\boldsymbol{\sigma}\cdot\boldsymbol{q})\,(2\omega_k)^{-1/2} \\ =\ & (3g/2)\,(f'^2/\mu^2)\,(-i\boldsymbol{\sigma}\cdot\boldsymbol{q})\,(2\omega_q)^{-1/2}(\omega_q-\Delta_{\Sigma\Lambda})^{-1} \\ & \times(2\pi)^{-3}\int dk\,(k^2/2\omega_k)\,(\omega_k+\omega_q-\Delta_{\Sigma N})^{-1}.\end{aligned} \tag{A2}$$

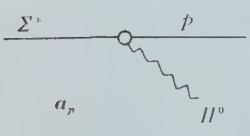
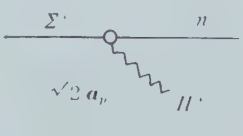


Fig. (A1)



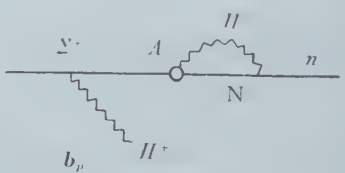
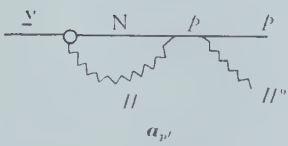


Fig. (A2)



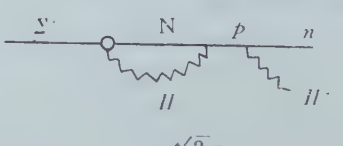


Fig. (A3)

Here, $\Delta_{\Sigma N}$ and $\Delta_{\Sigma\Lambda}$ are the mass differences between Σ and N and Σ and Λ, respectively. $f'^2/4\pi$ ($f'=\mu/2m\cdot f$ is the pseudovector coupling constant between pions and nucleons) is taken as equal to 0.08. ω_q and $\vec{\boldsymbol{q}}$ are the energy and momentum of the emitted pion. ω_k and $\vec{\boldsymbol{k}}$ are the energy and momentum of the virtual pion which appeared in Fig. (A2).

The ratio of the two p-wave amplitudes, $R=\boldsymbol{b}_p/\boldsymbol{a}_p$, becomes

$$\begin{aligned}R=\ & (f'^2/4\pi)\,(3/2\pi)\,\mu^{-2}(\omega_q-\Delta_{\Sigma\Lambda})^{-1}\int dk\,k^4/\omega_k^2 \\ \approx\ & 1.5.\end{aligned} \tag{A3}$$

In obtaining this we have used $\omega_q \approx \Delta_{\Sigma N}$ in eq. (A2) and a cut-off momentum of 5μ for the integral over k.

Thus, our perturbation calculation gets nearly equal magnitude and the same sign for a_p and b_p. In the text we required the relation (11) to hold, which leads to nearly equal magnitude but the different sign for the two amplitudes.

Next we shall examine in more detail the decay amplitude of Σ^* going into a continuum state. The decay of the Feynman diagram, Fig. (A1), gives, of course, nearly equal amount of s and p wave amplitudes. However, there is another decay amplitude of higher order as mesonic correction, e.g., one from the Feynman diagram, Fig. (A3). There Σ^- becomes a single proton, which should be considered as a bare nucleon in a real nucleon plus a pion state. The decays have only p-wave amplitudes and conserve parity. In the text we assumed that those could be neglected.

In the perturbation calculation the decay amplitude a_p' becomes

$$\begin{aligned} a_p' \propto & 3g(f'^2/\mu^2)(2\omega_q)^{-1/2}(2\pi)^{-3}\int d^3k(2\omega_k)^{-1}(-i\sigma\cdot q) \\ & \times \Delta_{\Sigma N}^{-1}(i\sigma\cdot k)(\Delta_{\Sigma N}-\omega_k)^{-1}(-i\sigma\cdot k) \\ = & -3g(f^2/\mu^2)(2\omega_q)^{-1/2}(-i\sigma\cdot q)\Delta_{\Sigma N}^{-1} \\ & \times (2\pi)^{-3}\int d^3k(2\omega_k)^{-1}k^2(\omega_k-\Delta_{\Sigma N})^{-1}. \end{aligned} \tag{A4}$$

Then the ratio, a_p'/a_p, is

$$\begin{aligned} & -(3/\pi)(f'^2/4\pi)\Delta_{\Sigma N}^{-1}\mu^{-2}\int dk\, k^4\omega_k^{-1}(\omega_k-\Delta_{\Sigma N})^{-1} \\ & \approx -5.1. \end{aligned} \tag{A5}$$

Therefore, the parity conserving decays (Fig. (A3)) are not small at all. Besides, it means the existence of the parity conserving decay even for $\Sigma^+\to p+\pi^0$, which was disregarded in the text.

Note added in prooe. The measurement of αP for the Σ decays at Berkley was done by Cool, Cork, Cronin, and Wentzel. Combining their data with the $\Delta I=1/2$ rule, they obtained $\alpha_+=-\alpha_-\leqq \pm(0.03\pm0.11)$ and $\alpha_0=\pm(0.99\pm0.01)$. We are very grateful to Prof. Lofgren for sending their manuscript before publication.

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Determination of the Pion-Nucleon Coupling Constant by means of the Nucleon-Nucleon Dispersion Relation

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Making use of the dispersion relation for nucleon-nucleon scattering, the coupling constant between pion and nucleon is determined from the neutron-proton scattering length in triplet state and the binding energy of the deuteron. The coupling constant $g^2/4\pi$ thus determined coincides with those deduced by other theories.

§ 1. Introduction

Attempts to determine the pion-nucleon coupling constant have been made by various authors. Otsuki and his colleagues[1] were perhaps the first group that has succeeded to bring out this programme in a reliable way, and obtained $g^2/4\pi = 0.065 \sim 0.090$. However, their arguments, based mainly on the electric quadrupole moment of the deuteron, depends on the nature of nuclear forces at short distances, and could not avoid some ambiguities in the course of calculations.* Concerning the pion-nucleon scattering and pion photo-production, calculations were made by Chew and Low[2] by using their own formulation and by Bernardini[3] et al. with the aid of the Kroll-Ruderman theorem.

Recently, the dispersion relations for pion-nucleon scattering of Goldberger and others[4] have enabled us to calculate the quantity in a way independent of the detailed character of meson theory, and many investigations have been made along this line.[5] However, the pion-nucleon scattering data at present contain large experimental errors, and do not permit a very accurate determination, leading to apparently inconsistent conclusions in some cases.[6] At any rate, the striking agreement between these values of the coupling constant determined in various ways gives a strong support to the pion theory of nuclear forces.

The dispersion relations for nucleon-nucleon scattering derived by Miyazawa and the present author[7] provides another approach for this problem. These formulae present a method to calculate the quantity in question *directly* (without solving the Schrödinger equation) from the scattering amplitudes (or phase shifts) of the nucleon-nucleon scattering.

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* Ohmura has shown that the restrictions imposed by them on the nature of nuclear forces at short distances are too stringent and that the lower limit of $g^2/4\pi$ becomes 0.05 when these points are remedied.

Since the experimental data for nucleon-nucleon scattering are more accurate than those for pion-nucleon scattering, especially in the low energy region, and since the binding energy of the deuteron is very well known, it seems likely at first glance that our formula provides a highly accurate means for this purpose. Unfortunately, however, we have no experimental data for this case *separately* for singlet and triplet states except for extremely low energies, and there appears a continuous spectrum in our formula in the unphysical region in contrast with the pion-nucleon case. These two situations give rise to some complications in our treatment, and obscures a part of the merits of our method.

In this connection, the value of $g^2/4\pi$ must be determined in such a way that the low energy scattering amplitudes give the main contribution to our result as will be illustrated in sections 2 and 3, where the effective range approximation plays an essential rôle.

Errors caused by the approximating methods in our calculations are estimated in detail in section 4. Suggestions as to how to increase the accuracy are made in the last section.

§ 2. Formulation

Our fundamental formula is the dispersion relation for nucleon-nucleon scattering in the laboratory system and in the forward direction with one subtraction, i. e.,

$$\frac{D^{(s)}(\omega_1)-D^{(s)}(m)}{\omega_1-m}=8\pi\frac{R^{-1}}{1-R^{-1}r}\frac{1}{m^2(\omega_1-\omega_D)(m-\omega_D)}-\frac{U^{(s)}(\omega_1)-U^{(s)}(m)}{\omega_1-m}$$

$$+\frac{1}{\pi}\mathrm{P}\int_m^\infty\frac{A^{(s)}(\omega)}{(\omega-\omega_1)(\omega-m)}d\omega. \qquad (2\cdot1)$$

All the notations are the same as those in I. The first term on the right-hand side originates in the fact that the two nucleons can form the deuteron, and should therefore be retained only in the case of neutron-proton scattering in triplet state. The second term is the integral of the same function as in the third term with the region of integration $\omega<m$. $U^{(s)}(\omega)$ reduces as was proved in I, to the Fourier transform of the nuclear force potential in the static limit. Therefore the second term can be calculated at least in principle in a field theoretical way and can be represented as a power series of g, the pion-nucleon coupling constant.

If we choose the energy ω_1 sufficiently small, the main contribution comes from the potential of the second order in g, fourth and higher order potential being small corrections to the second order one. We shall take in this paper only, the second and fourth order potentials neglecting all higher order effects.

Thus eq. (2·1) becomes an quadratic equation of $g^2/4\pi$, and we can determine the value of $g^2/4\pi$ from this equation if all the quantities in the other terms, D,

A, R and r, are known. Assuming the charge independence of coupling between pion and nucleon, we can write down the second order nuclear potential as

$$V^{(2)}(\omega) = -\frac{g^2}{\mu^2}\frac{k^2}{k^2+\mu^2}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{e})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{e})(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}), \qquad (2\cdot2)$$

where k is the momentum of the incident nucleon, $m^2+k^2=\omega^2$, and $\boldsymbol{e}$ the unit vector in the direction of $\boldsymbol{k}$. Pseudoscalar coupling and pseudovector coupling give the same result by the so-called equivalence theorem[5] as far as the second order potential is concerned.

As mentioned in section 1, we do not know at present any experiment that deals with the nucleon scattering in singlet and triplet states separately, except for extremely low energies, and therefore we must insert the averaged value of $A^{(s)}(\omega)$ and $D^{(s)}(\omega)$ into (2·1) if the high energy quantities give large contribution to this equation. On the other hand, if the average of the second order potential (2·2) is taken over the singlet and triplet states, it vanishes and the fourth order potential now becomes the main part of $U(\omega)$.

The Fourier transform of the fourth order potential, which arises from a part of the continuous spectrum of $A(\omega)$ in the unphysical region, cannot be calculated in any reliable way from our present stage of knowledge of quantum field theory, and therefore, our formula (2·1) gives no information about the value of the coupling constant if we choose ω_1 so that the high energy data contribute largely to (2·1).

At low energies (" thermal " energies) the neutron-proton scattering cross section is very well known for spin singlet and triplet states separately, and consequently we have a good knowledge about the scattering lengths for these respective states.

Moreover, for low energies, say, below 10 Mev in the centre-of-mass system, the S-matrix (and consequently $D(\omega)$ and $A(\omega)$) can be represented by only one phase shift (the S-wave in the case of singlet state and the so-called α-wave, the S- and D- wave admixture, in the triplet state), contributions of all other partial waves being negligibly small, and for the phase shift of the S-wave (or α-wave) the well-known effective range approximation holds.[9]

Therefore, if we choose ω_1 so small that for the left-hand side of (2·1) the effective range approximation holds and that the contribution of the fourth order potential to $U^{(s)}(\omega_1)-U^{(s)}(m)$ is sufficiently small in comparison with the second order, and if the integral of the last term is sufficiently convergent so that $A(\omega)$ for large value of ω (say, above 10 Mev in c. m. system) contributes to the integral very little, eq. (2·1) can be regarded as a relation of $g^2/4\pi$, the scattering length a and the effective range r (deuteron radius R can be represented by a function of a and r).

This is indeed the case, and we can determine the value of $g^2/4\pi$ from the scattering length in the triplet state and the binding energy B of the deuteron to a certain extent.

§ 3. Method of calculations and result

Denoting the phase shift of the S-wave (or the α-wave) by δ, the dispersive and absorptive part for the singlet state can be represented as follows:

$$D(\omega_c) = \frac{2\pi}{m}\frac{1}{k_c}\sin\delta\cos\delta$$

and

$$A(\omega_c) = \frac{2\pi}{m}\frac{1}{k_c}\sin^2\delta, \tag{3·1}$$

where the subscript c means "in the centre-of-mass system".

For the triplet state, where the effect of the tensor force is not negligible, we can write down the expressions for D and A in the same form as (3·1) provided that δ denotes the phase shift for α-wave, and the D and A the averaged values over the direction of the resultant spin.[10]

As we are mainly concerned with the non-relativistic energies, $k_c = k/2$, and the effective range approximation reads

$$k_c\cot\delta = -\frac{1}{a} + \frac{1}{2}rk_c^2. \tag{3·2}$$

On inserting (3·1) and (3·2) into (2·1), the integration of the last term can be carried out *analytically* without difficulty, and we have no need of a numerical integration of a singular function such as in (2·1) which has often been the source of an enormous error in the pion-nucleon case.

Let us consider the case of the neutron and proton in the singlet state. Inserting the numerical value $a = (23.69 \pm 0.06) \times 10^{-13}$ cm, and $r = 2.1 \sim 2.7 \times 10^{-13}$ cm, and taking $\omega_1 = 1.17$ Mev$+m$ in the laboratory system, we see that the term proportional to the nuclear force is extremely small (less than 1 per cent) compared with other two terms. This is perhaps due to the fact that the scattering length for this state is large and negative and the appearance of the virtual bound state makes the Born approximation (i. e. the Fourier transform of the nuclear potential) very much different from the scattering amplitude $D(\omega_1)$ for such a low energy. Owing to this fact and the large uncertainty of r, we cannot determine the value of the coupling constant from the singlet data, but we can see that our formula (2·1) is consistent with the value of $f^2/4\pi$ determined from other methods.

For the triplet state, the situation is much better. The scattering length is positive and much smaller than that for the singlet state, and the effective range r can be accurately determined by the relation

$$r = 2R^2\left(\frac{1}{R} - \frac{1}{a}\right)^*$$

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* Strictly speaking, the quantity defined by this relation is not r of (3·2), but $\rho(0, -B)$ of Bethe.[11] Similarly, the r in the first term on the right-hand side of (2·1) should be replaced by $\rho(-B, -B)$. Errors caused by the identification of $\rho(0, -B)$ with r and $\rho(-B, -B)$ will be estimated in the next section.

and

$$R^{-1}=(mB)^{1/2}, \tag{3·3}$$

where B is the binding energy of the deuteron.

Inserting the experimental value for a and B known at present,

$$a=(5.377\pm0.023)\times10^{-13}\text{ cm},$$

$$B=2.226\pm0.004\text{ Mev}, \tag{3·4}$$

and taking $\omega_1=m+1.17$ Mev*, we see that the magnitude of the second term on the right-hand side of (2·1) is about 4 per cent of the left-hand side and that the experimental data (3·4) have an enough accuracy to determine the coupling constant.

Concerning the convergence of the integral in the last term of (2·1), we see that the most (98 per cent) of the contribution of the integral comes from the energy region below 10 Mev in the c. m. system, and therefore the use of the effective range approximation proves legitimate. Detailed discussions for this point will be given in the next section.

The second term on the right-hand side of (2·1), the term containing the Fourier transform of the nuclear force, is expanded in a power series of the square of the coupling constant, and the term of the second order in g^2 is immediately given by (2·2). The evaluation of the term proportional to g^4 should be carefully done for, if we calculate it with pseudovector coupling theory the divergence difficulty which cannot be amalgamated into the mass nor the coupling constant appears. On the other hand, if we start with pseudoscalar coupling theory, the main contribution comes from nucleon pair terms which in almost all the cases known at present show a violent disagreement with experiment.

We have therefore decided to calculate it with pv coupling with momentum cut-off at 6μ which has shown in some cases tolerably good agreement with experiment. As the fourth order term itself is a minor correction to the second order term (about 6 per cent) a rough estimation will suffice. Although the Fourier transform of the fourth order potential in pv coupling theory without cut-off itself is a quadratically divergent quantity, it reduces to logarithmic one when the subtraction procedure as in (2·1) is made. Thus this term becomes fairly cut-off-independent, e. g. even if we take the cut-off momentum 3μ instead of 6μ, our result, $g^2/4\pi$, is changed only by 1 per cent. Therefore it is not unreasonable to suppose that our procedure, although crude may it seem, gives a fairly reliable result so far as $g^2/4\pi$ is concerned. All the terms of the orders higher than g^4 are neglected.

The result thus obtained is

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* This energy corresponds to the momentum $k_1=0.05\ mc$. The choice of the value for ω_1 is of course arbitrary so long as the approximation methods described above hold. Indeed, precisely the same result is obtained also when we put e. g. $k_1=0.01\ mc$.

$$g^2/4\pi = 0.080 \pm 0.007. \qquad (3\cdot5)$$

§ 4. Estimation of the errors due to the approximation method

The method of calculations described in the preceding section contains some approximations in certain respects. Since the term in our formula containing the coupling constant is represented by a small difference of large quantities, the accuracy of these approximations must be carefully examined.

In this section, we shall deal with this problem in detail, and find that our result (3·5), although surprisingly accurate may it seem, is subject to some uncertainties.

a) The effect of P-wave

Phase shifts for triplet P state are given by Worthington and others[12] for the case of proton-proton scattering. Charge independence of the nuclear forces prescribes that the same phase shifts apply also in our case. They are so small that when inserted in our formula averaged with respect to resultant spin directions give rise to no appreciable change in our result. This fact suggests that the effects of partial waves of higher angular momentum can be safely neglected.

b) As the integral in the last term of (2·1) extends to high energy region, this term may deviate from the original one if we take a more accurate formula

$$k_c \cot\delta = -\frac{1}{a} + \frac{1}{2} r k_c^2 - P r^3 k_c^4, \qquad (4\cdot1)$$

instead of (3·2).

The numerical value of P is not exactly known at present either experimentally or theoretically, but it is plausible that P does not exceed 0.05.[13] If we assume $P=0.05$, the value of the integration increases by 0.083 per cent and the value of $g^2/4\pi$ decreases by 1.3 per cent.

c) Differences between r and $\rho(-B, -B)$ and $\rho(0, -B)$

When the effective range approximation (3·2) exactly holds, there is no difference between r and $\rho(-B, -B)$ and $\rho(0, -B)$, but if we take into account the last term of (4·1), slight differences between them come forth. They are given by

$$\rho(-B, -B) = \left(1 + 2\frac{\rho^2}{R^2} P\right)\rho(0, -B)$$

and

$$r = \left(1 - 2\frac{\rho^2}{R^2} P\right)\rho(0, -B),$$

and change our result by an appreciable amount. Even if we put the value of P as small as 0.01, the first term on the right-hand side of (2·1) increases by 0.19 per cent, the other terms increasing very slightly, and the $g^2/4\pi$ increases by 9.4 per cent.

§ 5. Conclusion

(3·5) is our main result. This result has been obtained by putting $P=0$, and the probable error indicated here is purely of experimental origin. About 4/5 of it comes from the uncertainty of the triplet scattering length and the rest from that of the binding energy of the deuteron. Inclusion of the last term of (4·1) may change our result to an appreciable amount, and therefore we cannot determine the value of the coupling constant very accurately by our method.

However, it is to be remarked that our result agrees well with those obtained by quite different methods. The good agreement of our result obtained by putting $P=0$ with others may be an indication of the smallness of P. If we can obtain the value of P, in either experimental or theoretical way, the accuracy of our calculation will be much increased and another test of the consistency of the meson theory will be presented.

In the singlet state, where the effect of P is less important, more accurate measurement of the scattering length is desired.

The author thanks Drs. H. Miyazawa and T. Ohmura for valuable discussions. He is also indebted to Yomiuri-Yukawa Fellowship for the financial aid.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in dublicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Lo not forget to couni in enough space for formulas, figures or tables.

A Note on Polarized Cross Section of High Energy $n-d$ Elastic Scattering

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December 6, 1958

In the previous note[1] we pointed out that for high energy $n-d$ elastic scattering the calculated values of differential cross section give similar results for various sets of phase shifts calculated by several authors.[2),3)] That is to say, in spite of the difference between the phase shifts calculated using meson theoretical potential[2] and the ones calculated using entirely phenomenological potential,[3] especially the large difference in the ${}^3P_J$ ($J=0$, 1, 2), ${}^3F_2$, ${}^3S_1$, ${}^3D_1$-state phase shifts and mixing parameters ϵ_J ($J=1$, 2), we can reproduce the experimental differential cross section well. At 90 Mev, which is of our interest, the dominant contributions to the cross sections of the nucleon-nucleon scattering as well as $n-d$ elastic scattering are determined by the ${}^3P_J$, ${}^3F_2$, ${}^3S_1$, ${}^3D_1$-state phase shifts and mixing parameters ϵ_1, ϵ_2. For high energy $n-d$ elastic scattering, the manner of interference and the magnitude of interference terms are quite different for various sets of phase shifts. However, the sticking factor or the fourier transform of the intrinsic deuteron wave function acts to mask the difference between the various sets of phase shifts. So, the calculated differential cross sections become similar.

In this note, we report the results of our calculation of polarized cross sections for the $n-d$ elastic scattering using two sets of phase shifts[2),3)] of the nucleon-nucleon scattering.

The polarized cross section of $n-d$ elastic scattering is given by

$$PI(\theta)=\frac{1}{6}\cdot\frac{16}{9}\left\{\frac{1}{2}(B^*C+C^*B)+\frac{1}{2}(G^*C+C^*G)+\frac{9}{2}(N^*C+C^*N)\right\}S(\Delta k,\theta), \quad (1)$$

where $I(\theta)$ is the $n-d$ elastic differential cross section given by

$$I(\theta)=\frac{1}{6}\cdot\frac{16}{9}(2|N|^2+2|H|^2+\frac{1}{3}|G|^2+\frac{2}{3}|G-N|^2+10|C|^2+\frac{1}{9}|G-2N|^2+\frac{1}{72}|9B+G+N|^2)S(\Delta k,\theta), \quad (2)$$

where $B=B_{nn}+B_{np}$, $C=C_{nn}+C_{np}$, and so forth. B_{nn}, C_{nn}, etc., are the coefficients of $n-n$ scattering spin matrices and B_{np}, C_{np}, etc., the coefficients of $n-d$ scattering spin matrices, respectively. And B, C, etc., were expressed by Wright[4] in terms of the nucleon-

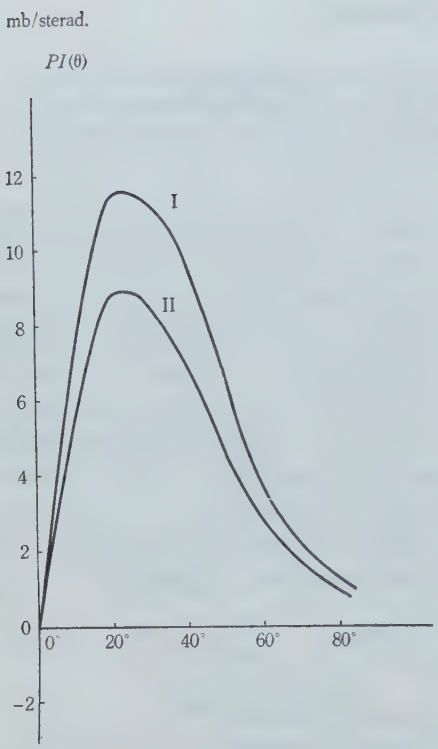

Fig. 1. Polarized cross section of $n-d$ elastic scattering at 90 Mev. The curves are calculated with the equation (1) of the text. Curve I is the result for the set of meson theoretically calculated phase shifts,[2] and curve II is the result for the set of entirely phenomenologically calculated phase shifts.[3] The maximum of polarized cross section comes out at 50° for $n-p$, and 45° for $p-p$ scattering.[2] But, owing to the sticking factor the maximum of $n-d$ elastic polarized cross section comes out at 25°~30° and the calculated polarized cross sections have positive signs for angles which we are interested in.

nucleon scattering phase shifts.

$S(\Delta k, \theta)$ in the right-hand side of eq. (1) and eq. (2) are the sticking factor. When we calculate $S(\Delta k, \theta)$, we disregard the D-state wave function of deuteron. This is consistent with the approximation for eq. (1) and eq. (2) having physical meanings at small angles. To calculate $S(\Delta k, \theta)$, we use the deuteron wave function with hard core[5] in order to be consistent with using sets of phase shifts calculated by using meson theoretical potential or entirely phenomenological potential.

The calculated values of polarized cross section using both sets of phase shifts[2),3)] give similar results (See Fig. 1). However, the difference between these calculated values of the polarized cross section are appreciable in constrast with those calculated values of differential cross section (See Fig. 1 in our previous note[1)]). Now, it is worthy to note that the effects of sticking factor which acts to mask the difference between both sets of phase shifts for differential cross section disappear for the polarization $P \equiv PI(\theta)/I(\theta)$ (not for polarized cross section), so the interference between $n-n$ and $n-p$ scattering appears explicitly.

If we had an accurate experiment for the polarized cross section of $n-d$ scattering in future, eq. (1) would be useful to decide more reasonable set of phase shifts of nucleon-nucleon scattering.

The author would like to express his sincere gratitude to Prof. M. Kobayasi for his kind encouragement, and to the members of his group for helpful discussions. He also would like to acknowledge the correspondences of Dr. J. Sawicki and Dr. L. S. Singh on this problem. The author is indebted to Yomiuri-Yukawa Fellowship for financial aid.

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A Remark concerning the Bogoliubov Theory

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January 5, 1959

In the Bogoliubov theory of superconductivity[1] the statistical matrix does not commute with the operator for the total number of electrons in the system. However, if we introduce the operator P defined by

$$\begin{aligned} P&=+1 \quad \text{if} \quad N_{\text{elec.}}=\text{even} \\ P&=-1 \quad \text{if} \quad N_{\text{elec.}}=\text{odd}, \end{aligned} \tag{1}$$

then the Bogoliubov U-matrix commutes with P. There are Fermi-type and Bose-type excitations in the Bogoliubov theory. Odd numbers of Fermi-type excitations change P to $-P$; even numbers of Fermi-type excitations, and Bose-type excitations in any number, leave P unaltered. All physical operators, for example, the current density operator j, commute with P both before and after the Bogoliubov transformation. As a result, an electromagnetic field can induce transitions only to even numbers of Fermi-type excitations.

If we denote the energy gap in the Fermi-type spectrum by C, then the electromagnetic methods[2] measure the quantity $2C$. On the other hand, the exponential term in the specific heat computed in a straightforward fashion is $\exp(-C/kT)$, not $\exp(-2C/kT)$. There should therefore be a difference of a factor 2 between the "electromagnetic energy gap" and the "thermal energy gap". No such difference is observed.

We suggest that the Bogoliubov theory should be reinterpreted as describing, not a single system, but an ensemble of two systems, one with $P=+1$, the other with $P=-1$. This means that *single Fermi-type excitations in the Bogoliubov theory, as well as all odd numbers of such excitations, are to be regarded as unphysical and should be omitted.* Since the operator P commutes with all other operators in the theory, this procedure is consistent within the theory.

We can visualize the situation by means of the following simple analogy to the ordinary band theory of solids. We start with a single-particle energy spectrum with a gap of size $2C$ at the Fermi energy μ, as shown in Fig. 1a. The least possible excitation energy is then $2C$, corresponding to the excitation of an electron just below the gap into a state just above the gap. The Fermi energy μ (at zero temperature) lies halfway in the gap, as shown by the dotted line.

When we transform to hole theory in the usual fashion, we plot $|\varepsilon_k-\mu|$ rather than ε_k itself. The resulting energy spectrum is shown in Fig. 1b. The gap appears to be of size C rather than $2C$. However, now the least possible excitation is a combined excitation of an electron, with $k>k_F$, and a hole, with $k<k_F$, and the excitation energy is again $2C$, as it ought to be. "Single excitations" are impossible, and the same

applies to all odd numbers of Fermi-type excitations.

The analogy to the Bogoliubov theory can be made even closer by including the Bose-type excitations into the picture. If we draw the Bose-type spectrum as energy per particle, i. e., we draw the quantity $\frac{1}{2}\varepsilon_c(p)$ rather than the energy $\varepsilon_c(p)$ of the pair of particles making up the boson, then the spectrum starts at the Fermi energy μ, goes up linearly with $|\boldsymbol{p}|$, and ceases when $\frac{1}{2}\varepsilon_c(p)$ exceeds C. This is shown in Fig. 2a in the ordinary way of looking in Fig. 2b with the use at it, of hole theory.

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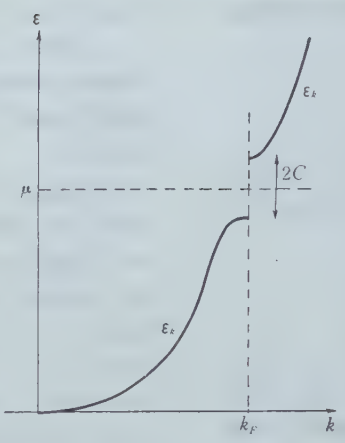

Fig. 1a

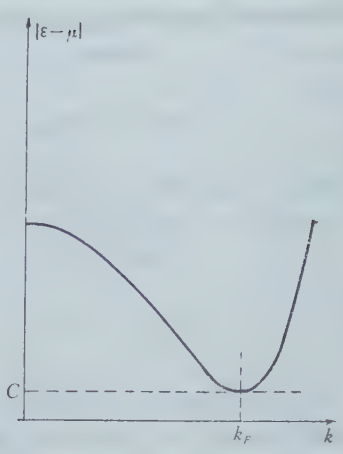

Fig. 1b

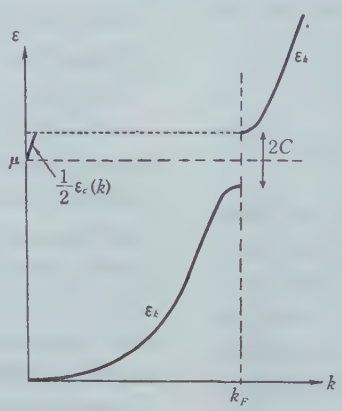

Fig. 2a

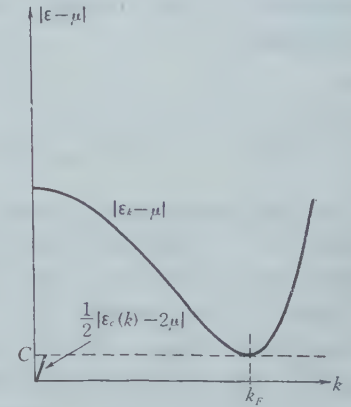

Fig. 2b

Comment on (p, n) Cross Sections

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December 13, 1958

Recently, Howe[1] measured the excitation cross sections of (p, n) reaction from ^{63}Cu, ^{65}Cu, ^{64}Zn, ^{66}Zn, and ^{68}Zn and suggested that the large variations from isotope to isotope in the magnitude of the (p, n) cross section seem to be an indication of the occurrence of interactions other than compound nucleus formation process. In that paper, the (p, n) cross sections were measured at energies up to 12 Mev, and the cross

sections were compared with the total reaction cross sections based on Shapiro's argument[2] that the (p, n) cross section is the same as the total reaction cross section in the low energy region since the neutron emission mode is considered as the most predominant one so far as the energy is not very high. Although this argument is certainly justified for many cases, careful note on binding energy of the last proton must be taken in discussing the reaction mechanism.

The purpose of the present note is concerned with the fact that the large variations from isotope in the magnitude of the (p, n) cross section, as pointed out by Howe, may be or may not be explained by the use of compound nucleus picture. The calculations are made for the isotopes of Ni, Cu and Zn from among the many available data[3] on (p, n) cross section since the variations from isotope to isotope of these nuclei considerably differ from each other in spite of these elements are in neighboring atomic number.

The method of the calculation, based on the usual compound nucleus model, is same as before. In the evaluation of the excitation cross sections, the nuclear radius is assumed to be $R=1.5\times A^{1/3}$ *fermi* and the constant a which appears in level density formula, is chosen as $a=2\ \mathrm{Mev}^{-1}$ for all elements undertaken. The cross sections for compound nucleus formation by proton are estimated from Shapiro's paper,[2] and the separation energies are determined by the use of Wapstra's table.[4] In the numerical integration to obtain the excitation cross sections, the evaluations at low excitation energies become somewhat uncertain because of the technique of the numerical calculation and then one must devote efforts to the determination of the threshold energies in (p, n) reactions. The threshold energy for (p, n) reaction is given by $E_{th}=S_n-S_p$, where S_n and S_p are the separation

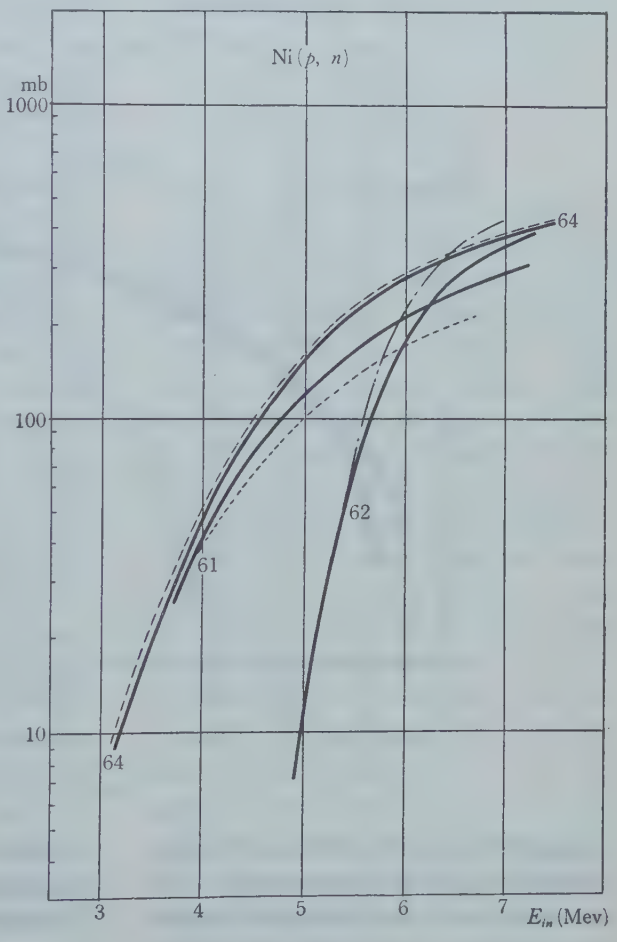

Fig. 1

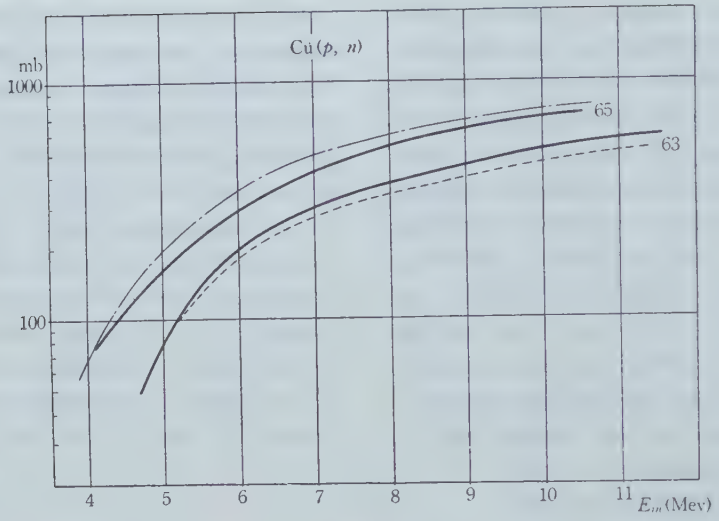

Fig. 2

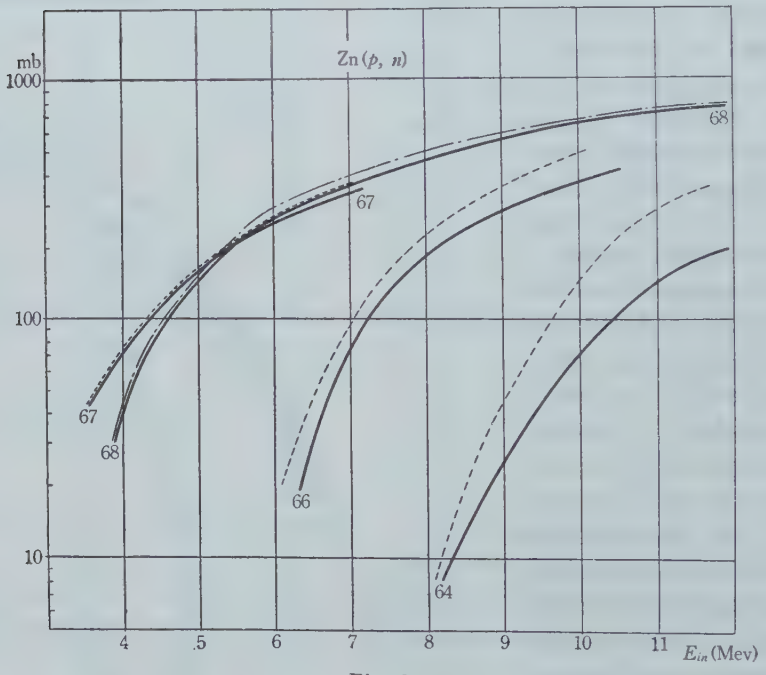

Fig. 3

energies of neutron and proton from the compound nucleus, respectively. For the most cases, the difference between the threshold energies determine the variation from isotope to isotope in the magnitude of the cross section at low excitation energy region, and with increasing the excitation energy the varia-

tion is governed by the competitive processes of other kinds of particle or of secondary emitted particles, in other words, the threshold energies of other decay modes play an important role for the behavior of the excitation function.

The comparisons between the experimental cross sections and the calculated ones are made in Figs. 1, 2, and 3, where the calculated cross sections are represented by the solid curves and the numbers added to each curves refer to the mass numbers of target nuclei. From these figures, it is apparent that the calculations based on the compound nucleus model may explain the variations from isotope to isotope in the magnitude of the (p, n) cross section fairly well. For the cases of Cu and Zn (Figs. 2 and 3), some considerable disagreements in absolute values of the cross sections are observed. As the reason for these disagreements, one may consider the two facts; one is the discrepancy between the experimentally determined threshold energy and the estimated one, and the other is the fact that in the present calculations for all cases the decay probabilities are calculated for compound nucleus of ^{64}Cu except for the relations of the separation energies and of the even-odd rule. If the calculations are made more rigorously for each cases, one may expect better agreement between experiment and calculation. Thus we may conclude that the large variations from isotope to isotope in the magnitude of (p, n) reaction, as measured by Howe, are not the evidence for the non-compound nuclear process but offer an example as to the usefulness of the compound nucleus picture in such an energy region.

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Momentum Dependence of the Healing Distance

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November 13, 1958

One of the merits[1] of Brueckner's method is to furnish the theoretical basis to the validity of the independent particle model for nucleus. Indeed, Bethe and Goldstone[2] have shown that the application of the exclusion principle in the intermediate state of the constituent particles should play the most important role for the validity of I. P. M.; the inter-nucleon force "wounds" the nucleon wave function violently when the nucleon distance is much less than the range of nuclear force, but as the nucleon distance increases, the wound would be, in virtue of the exclusion principle, "healed" up and consequently the wave function would approach to the free particle wave function, not to that asymptotic form with a phase shift. The distance in which the wounded wave function may nearly come in accordance with the free particle wave

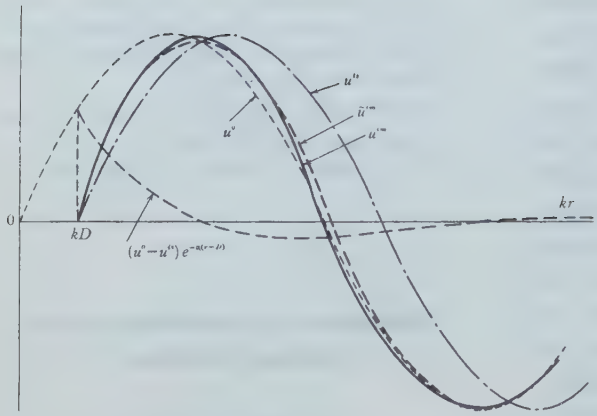

Fig. 1. The schematical behaviors of s-state wave function of the two particles interacting through hard core potential.

- - - - - - - free particle pair : u^0
— — — — isolated particle pair : u^{is}
— — — — imbedded particle pair : $\tilde{u}^{im}$
——— imbedded particle pair : u^{im}

function has been called "healing distance" by Weisskopf et al.[3] and found to be equal at most to the mean nucleon distance in the nuclear matter.

In this paper, a plausible analytic expression is given to this healing distance and its momentum dependence is discussed quantitatively based on Bethe-Goldstone's equation which is, for an imbedded nucleon pair whose center-of-mass is assumed to be at rest, given by

$$\frac{1}{M^*}(V^2+k^2)\psi_k(\boldsymbol{r})=V(\boldsymbol{r})\psi_k(\boldsymbol{r})$$
$$-\frac{1}{(2\pi)^3}\int_0^{k_F}d\boldsymbol{k}'\int_0^{\infty}d\boldsymbol{r}'\,V(\boldsymbol{r}')\psi_k(\boldsymbol{r}')\,e^{i\boldsymbol{k}'(\boldsymbol{r}-\boldsymbol{r}')}, \tag{1}$$

where M^* denotes the reduced mass, $\boldsymbol{k}$'s correspond to the relative momentum and the Pauli principle is taken into account by the last term. As has been noted by Weisskopf et al.,[3] the restriction of the center-of-mass momentum $\boldsymbol{K}=0$ would not change the main features of relative wave functions for general case. Following the procedure of Bethe and Goldstone, we can solve this equation and get the s-state wave function in the closed form, assuming the two-nucleon potential $V(r)$ consisting of only hard core with core radius D. Such potential is of course different from the real nuclear forces, but for the purpose of discussing the effect of the Pauli principle the following results would be reliable for the actual situation, too.

$u_k(r)=kr\psi_k(r)$ is given in the approximation $(Dk_F)^4\ll 1$ by

$$u_k(r)=A[u_0-(1/2\pi)(u_1+u_2-u_3-u_4)], \tag{2}$$

where

$$u_0=\sin k(r-D),$$
$$u_1=\sin k(r-D)[Sia(r-D)+Sib(r-D)+SiaD+SibD],$$
$$u_2=\cos k(r-D)[Cia(r-D)-Cib(r-D)-CiaD+CibD],$$
$$u_3=\sin k(r-D)[Sia(r+D)+Sib(r+D)-SiaD-SibD],$$
$$u_4=\cos k(r-D)[Cia(r-D)-Cib(r+D)-CiaD+CibD],$$

with

$$A=k_F[1-(1/\pi)\{SiaD+SibD)\cos kD-(CiaD-CibD)\sin kD\}]^{-1},$$

$$a=k_F+k, \quad b=k_F-k,$$

and Ci, Si are the integralcosine and integralsine functions as are tabulated in the book of Jahnke-Emde.

The behavior of $u_k(r)$ is shown in Fig. 1.

Now, we wish to rewrite $u_k(r)$ approximately in the following form:

$$\tilde{u}_k(r)=\begin{cases}\sin kr-\Big(\sin kr-\sin k(r-D)\Big) & \\ \qquad\times e^{-\alpha(r-D)}, & r\geqq D \\ 0, & r<D,\end{cases} \tag{3}$$

where the second term in the right-hand side is the difference between the unperturbed wave function $u_k^0(r)=\sin kr$ and the isolated pair particles wave function $u_k^{is}(r)$, which should correspond to the magnitude of the "wound" due to the interaction. The schematical behavior of this approximate function $\tilde{u}_k(r)$ is also illustrated in Fig. 1, from which we can see that α corresponds just to the inverse of the healing distance λ_k.

Expanding the solution $u_k(r)$ and its approximate expression $\tilde{u}_k(r)$ both about the core radius D and rearranging them in the power series of kD, then we get

$$u_k(r)=A\Big[k(r-D)-\Big(k(r-D)\Big)^3/6 +(kD)\Big(k(r-D)\Big)^3/12+\cdots\cdots\Big], \tag{4}$$

$$\tilde{u}_k(r)=(1+\alpha D)\Big[k(r-D) -\Big(k(r-D)\Big)^3/6+O\Big((kD)^5\Big)+\cdots\Big], \tag{5}$$

with

$$A=k_F\Big[1-(Dk_F/\pi)\Big(2-\kappa\ln\frac{1+\kappa}{1-\kappa}\Big) +(Dk_F/\pi)^3\Big(\frac{15\kappa^2-1}{9}-\frac{\kappa^3}{6}\ln\frac{1+\kappa}{1-\kappa}\Big) +O\Big((k_FD)^5\Big)+\cdots\cdots\Big]^{-1}$$

and

$$\kappa=k/k_F.$$

Restricting ourselves in the region of $\kappa^2\lesssim Dk_F$, in accordance with the assumption $(Dk_F)^4\ll 1$ involved in the solution (2) of the Bethe-Goldstone equation, we can get the next relation from the comparison of the above equations (4) and (5),

$$1+\alpha D=A.$$

Thus, the healing distance is given as follows:

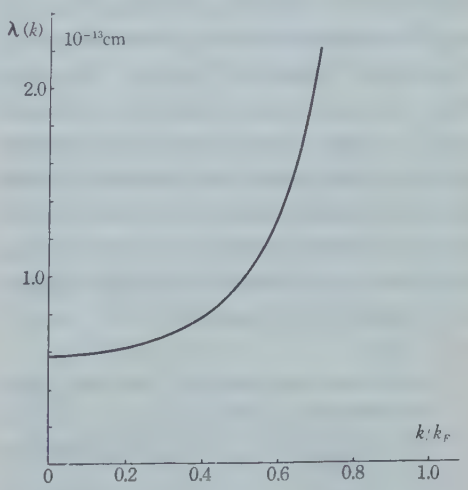

Fig. 2. The momentum dependence of the healing distance. The hard core radius is assumed to be 0.4×10^{-13} cm and consequently this figure should be regarded with confidence in the region of $k/k_F\lesssim 0.7$.

$$\lambda(k)=D\left[\frac{\pi}{Dk_F}\left(2-\kappa\ln\frac{1+\kappa}{1-\kappa}\right)^{-1}-1\right],$$

$$\text{for } \kappa^2\lesssim Dk_F. \qquad (6)$$

In Fig. 2, the momentum dependence of the healing distance is shown for the assumed hard core radius 0.4×10^{-13} cm.

For the imbedded pair nucleons, it is found that the wave function of a nucleon pair moving with the relative momentum less than $0.6k_F$ is healed up within the mean nucleon distance in nuclei and it seems, therefore, that such nucleons are moving nearly independently. On the other hand, for those nucleons which have the relative momentum nearly equal to Fermi momentum k_F, the healing distance is so large compared with mean nucleon distance and consequently the nucleon pairs seem to behave against the neighbouring nucleons just like as isolated nucleon pairs in free scattering states. Thus, it may be concluded from the local Thomas-Fermi gas aspects of nuclear structure that the nucleons inside the nucleus move almost independently. However in its surface region, the situation would be considerably different and there would be some strong tendency to the nucleon clustering about which we shall discuss in a forthcoming paper.

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On the Intermediate Resonance Levels in the Nuclear Reactions

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Nuclear reactions at intermediate energy region have been interpreted by means either of direct reaction[1] or of the statistical model.[2] The direct reactions are such processes in which the incident particle interacts only with a special nucleon in the target nucleus, while in the case of the statistical model nuclear reactions proceed mainly through compound nucleus formation. However, one may reasonably think the mechanism of nuclear processes in this energy region lies somewhere between the two mechanisms. That is to say, it may be supposed that in actual reactions the incident particle interacts strongly with several but not all nucleons in the target nucleus. Indeed, there are experimental facts[3] supporting this mechanism. In nuclear reactions at intermediate energies level spacings are shown to be 100～300 kev and level widths 50～100 kev. For example, Fig. 1 shows the experimental data of Ni(p, p')Ni* obtained at the Institute of Nuclear Study, Tokyo University.[3] In this note, we should like to interpret these data by a nuclear model based on the above mentioned mechanism.

In our model, we assume that the nucleons in the target nucleus, which

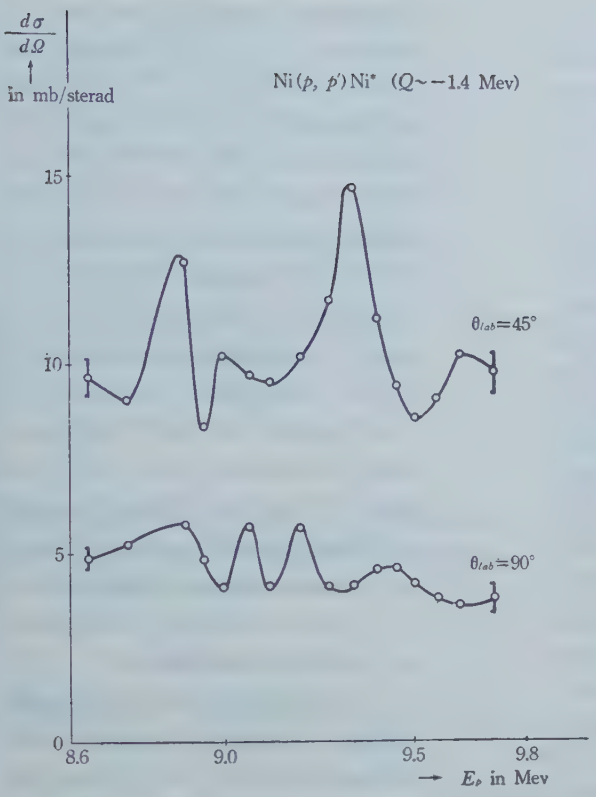

Fig. 1. The differential cross sections of inelastically scattered protons as a function of the incident proton energy E_p. Residual nucleus are left at first excited state. There appear four intermediate resonance levels.

consists of A-nucleons, can be separated into two groups, an inert core and N-outer nucleons. We further assume that the inert core always remains in its ground state and its effect upon the incident particle as well as the outer nucleons is approximately given by a certain average potential acting on them. We further assume that all the remaining interactions are two body forces. With these assumptions, our model Hamiltonian is written as follows,

$$H_M = T_p + H_r + V_{pr} + H_c + U_p + U_r, \qquad (1)$$

where T_p is the kinetic energy of the incident particle, H_r and H_c are the outer-nucleon and the inert-core Hamiltonians respectively. V_{pr} is the sum of interactions between incident particle and the outer nucleons, U_p and U_r are the potentials acting on the incident particle and the outer nucleons from the inert core, respectively.

Now, we consider a fictitious light nucleus containing $(N+1)$ nucleons. This number of nucleons corresponds to the sum of the outer nucleons and the incident particle. Using the same notation as that of the outer nucleons and incident particle, the Hamiltonian of this light nucleus is written as follows,

$$H_g = T_p + H_r + V_{pr}. \qquad (2)$$

Then, we consider the relation between the light nucleus and the model system. If U_p and U_r in the right-hand side of (1) are constant inside the internal region of the model system, the eigenvalues of the model Hamiltonian H_M correspond one to one with the eigenvalues of H_g, except an additional constant. As is well known, the radius of the model system is proportional to $(A+1)^{1/3}$, while the radius of the light nucleus to $(N+1)^{1/3}$. As the level spacings are proportional to inverse square of the nuclear radius, the level spacings D_μ of the model system are narrower than that of the light nucleus, $D_{\mu 0}$. Similar considerations show that the excitation energy E of the model system is larger by the

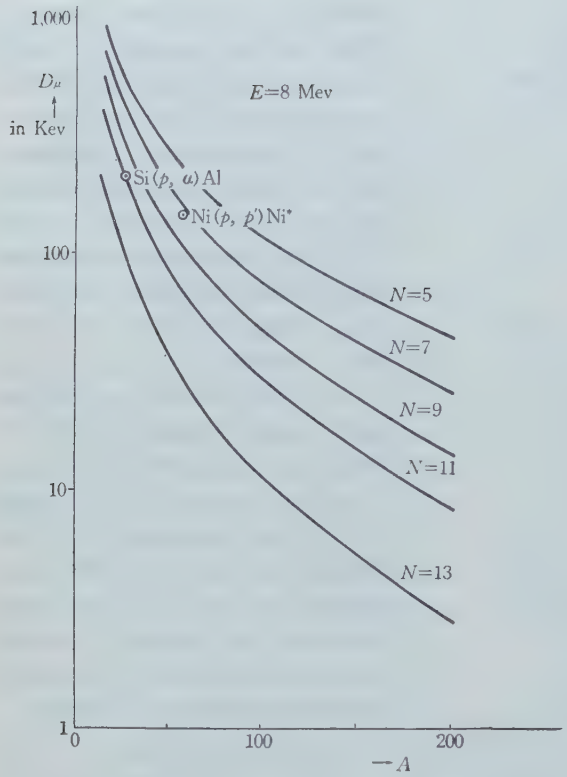

Fig. 2. Level distances of intermediate resonances *vs* mass numbers of compound nucleus. Experimental data are shown by circles, and five curves are calculated one for incident energy E=8 Mev. These two data, Ni(p, p')Ni* and Si(p, α), are explained by N=7~11 outer nucleon numbers.

same ratio than the corresponding energy E' of the light nucleus. These relations are roughly estimated as

$$D_\mu(E) \sim D_{\mu 0}(E') \cdot \left(\frac{N+1}{A+1}\right)^{2/3} \quad (3)$$

and

$$E' \sim E \cdot \left(\frac{A+1}{N+1}\right)^{2/3}, \quad (4)$$

respectively.

We are interested in determining N which depends on the incident energy E and the mass number A in order to reproduce the experimental level spacings. Weisskopf formula[4] was used to estimate $D_{\mu 0}$. The results are shown in Fig. 2. The calculated curves are fit with the experimental values if we take N=7~11. In other words, the levels with intermediate widths which are shown in Fig. 2, suggest strongly the existence of some resonating groups with 7~11 nucleons in the nucleus.

The experiments available for the evidence of such resonating groups, however, are still very few. Thus it is too early to say anything about the actual energy- and mass number dependence of N. Furthercoming experiments will clarify these points.

This model seems to suggest various interesting theoretical investigations, such as a generalization of the intermediate coupling theory proposed by Lane, Thomas and Wigner.[5] These problems will be discussed later on.

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New Equation of Spin 1/2 Particles and Isomorphism between Various Transformations

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In consideration of the Fermi-interaction, Pauli[1] proposed two transformations,

$$\text{(P-I)} \quad \psi' = a\psi + b\gamma_5\psi^p \quad (|a|^2+|b|^2=1),$$

$$\text{(P-II)} \quad \psi' = \exp(i\lambda\gamma_5)\psi,$$

under which the neutrino equation is left invariant. ψ^p is the charge conjugate function of ψ. Meanwhile, Gürsey[2] showed that these transformations are isomorphic to the space-rotation and the gauge transformation of the first kind. He obtained these results by using his "wave matrix formalism".[3] However, not only are the greater part of physicists not accustomed to this formalism, but also it is doubtful that these results are always correct independent of the representations of the Dirac matrices, because the formalism was constructed by referring to the special representations of γ- and σ-matrices.

The purpose of this letter is twofold. First, let the wave function ϕ be a suitable linear combination of ψ and ψ_p. When ψ satisfies the Dirac equation, ϕ satisfies a new field equation which includes ϕ^p in the mass term. Serpe and Case[4] have shown that the Weyl-Jehle equation is equivalent to the Majorana equation for hermitian field. Similarly, the new equation is equivalent to the Dirac equation for non-hermitian field. Secondly, if we investigate the transformations of ϕ corresponding to the various transformations of ψ, Gürsey's results are easily obtained. Consequently, the derivation essentially depends on the suitable combination ϕ of ψ and ψ^p, and not on the wave matrix. The general formulation of the wave matrix theory can be constructed by re-interpretation of the direct-product representations of γ-matrices, and indeed, if we apply this formalism to ϕ, we can immediately derive Gürsey's wave matrix.

The wave function ϕ and its charge conjugate ϕ^p are defined by

$$\phi = \psi_+ - \gamma_4\psi_-^p, \quad \phi^p = \psi_-^p + \gamma_4\psi_-,$$

where

$$\psi_\pm = \gamma_\pm\psi, \quad \psi_\pm^p = (\psi_\pm)^p = \gamma_\mp\psi^p,$$

and

$$\gamma_\pm = (1\pm\gamma_5)/2.$$

If ψ obeys the Dirac equation $(\gamma_\mu\partial_\mu + \kappa)\psi = 0$, ϕ satisfies the generalized Jehle

equation $(\partial_4+i\Sigma_j\partial_j)\phi+\kappa\gamma_5\phi^p=0$ in which $\Sigma_j=-i\epsilon_{jkl}\gamma_k\gamma_l/2$ is a set of 4×4-Pauli matrices. In the case of the Majorana abbreviation $\psi=\psi^p$, if we choose the special representation

$$\gamma_j=\begin{pmatrix}0 & -i\sigma_j\\ i\sigma_j & 0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

it is readily verified that the equation reduces to the equation (16) of Case's paper in the sub-space corresponding to the eigenvalue 1 of γ_5. Using our method, we can easily deduce the rather peculiar transformation laws, given by Case, corresponding to space-reflection and time-reversal.

Corresponding to the proper Lorentz transformation of ψ,

$$\psi'=\exp(ia_k\Sigma_k+b_kL_k)\psi, \qquad (1)$$

ϕ transforms according to

$$\phi'=\exp(iA_k\Sigma_k)\phi, \qquad (1')$$

in which a_k and b_k are real and A_k is a complex 3-vector, and $L_j=i\gamma_j\gamma_4$. Phase- and γ_5-transformation (P–II) of ψ correspond respectively to γ_5- and phase-transformation of ϕ. Hitherto, the Pauli- (P–I) and TKT-transformation[5]* have been proposed to generalize a phase transformation (the gauge transformation of the first kind), but we propose the following as a general gauge transformation,

$$\psi'=(A+B\gamma_5)\psi+(C+D\gamma_5)\psi^p, \qquad (2)$$

in which A, B, C and D are the complex numbers satisfying the relation

$$|A|^2-|B|^2-|C|^2+|D|^2=1$$

and $\quad AB^*=CD^*.$

Corresponding to this 6-parameter transformation, ϕ is tranformed by

$$\phi'=\exp(iA_k\wp_k)\phi, \qquad (2')$$

where A_k is a complex 3-vector. $\wp_k$ is a set of 4×4-Pauli metrices defined by

$$\wp_1=-\gamma_5,\quad \wp_2=-i\gamma_1\gamma_2\gamma_3,\quad \wp_3=\gamma_4.$$

Indeed, Σ_j and $\wp_j$ satisfy Pauli's commutation relation, $\Sigma_j\Sigma_k=i\epsilon_{jkl}\Sigma_l+\delta_{jk}$ and $\wp_j\wp_k=i\epsilon_{jkl}\wp_l+\delta_{jk}$, but L_j is not a set of Pauli matrices because it satisfies the relation $L_jL_k=i\epsilon_{jkl}\Sigma_l+\delta_{jk}$. Consequently, (1′) and (2′) have the same form which shows the isomorphism of the proper Lorents transformation and the general gauge transformation. This is an extension of Gürsey's correspondence. The general γ_5-transformation of ψ is given by

$$\psi'=\exp(iA_k'\wp_k)\psi,$$

and corresponds to the transformation of ϕ

$$\phi'=(A'+B'\gamma_5)\phi+(C'+D'\gamma_5)\phi^p.$$

Therefore, the general gauge transformation and the general γ_5-transformation are reciprocal to each other with respect to ψ and ϕ.

Here, we call attention to the fact that the Dirac equation is not left invariant under the general gauge transformation and general γ_5-transformation. There are two view-points. One is that under some restrictions the invariancy

* The TKT-transformation is introduced by Takahashi, Kamefuchi and Tanaka in order to interpret the electromagnetic interaction of the Jauch field. It is a generalization of the usual phase transformation, and is given by
$\psi'=a\psi+b\psi^p \quad (|a|^2-|b|^2=1)$.

of equation is regained, e. g. for $\kappa=0$ the equation is left invariant with respect to the Pauli transformation and γ_5-transformation though it is not so for general $\kappa(\neq 0)$. The other view-point is that the Dirac equation is not satisfactory and must be generalized so as to be left invariant under general transformation. Baryons may obey such equations.

The detailed discussions will appear in this journal in the near future.

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Origin of Cosmic Rays Revealed in the Rigidity Spectrum of Primary Particles

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It has now become very probable that the rigidity spectra of various nuclei in the primary cosmic radiation are similar to each other; especially the existence of their maxima all at a rigidity of around 1.6 GV as well as their gradual decreases towards lower rigidities are noticeable.[1] When we regard these features as being caused outside the solar influences (for the discussions of the other possibility see reference 1)), we can impose certain important restrictions on the possible mechanisms effective in producing the primary cosmic radiation.

In order to obtain a spectrum such as observed from one of thermal type which, in the energy region under consideration, increases monotonously towards lower energies, a mechanism is needed to cut-off the lower energy part of the latter. This process, called B hereafter, has to act together with some acceleration process A. From the approximate coincidence of the maximum positions in the rigidity spectra of the primary p, α and C, N, O components, it is quite natural to assume that the effects of both A and B on the primary cosmic ray particles depend mainly on the magnetic rigidity of the particle. Since the B process is more responsible than the A process for the occurrence of the maximum, we discuss some possibilities of the B process.

First of all, it can be conjectured that the processes responsible for the observed spectrum shape cannot be operative uniformly over the entire Galaxy, since the overall average matter density is very low and only the ionization loss acts as the predominant B process. We, then, get the results in conflict with the observation as has been argued in reference 1).

We thus have to consider certain

spatially limited regions of our Galaxy, called active regions hereafter, as the places where A and B processes operate, the latter taking part in modifying the spectrum from that inside the active regions. The next question is whether the observed spectrum is in a steady state or in a transient one. When we consider the fact that we are observing a cummulative effect over a very long period, we arrive at the conclusion that a certain B process is operative steadily in some active regions of our Galaxy.

One of the B processes of the required type may be the absorption in the active region combined with the temporary confinement of the particles by, for instance, magnetic clouds, by which the spectrum of emitted particles is modified from that of confined ones. This type of B process has been shown[2] to modify the spectrum to some extent in the case of the Forbush decrease. Since the association of some celestial body with the active region is quite conceivable, we exploit this possibility a little further. This is, in fact, the case in the solar production of cosmic rays, and a similar situation may be considered as responsible also for the substantial part of the galactic cosmic radiation, if we refer to the speculations that the extended atmosphere of red giant and super giant stars[3] or the shocks taking place between pairs of binary stars[4] could be conceivable local sources.

We now describe qualitatively how effective the B process is in the accelerating region containing absorbers, leaving a quantitative study to a forthcoming paper. The particles trapped in this region undergo the scattering by turbulent magnetic fields in the course of acceleration and are lost out of this region by escape or the absorption. Taking the linear dimension of this region as L and the scattering mean free path as l, we estimate the mean time for escape of the particles of velocity v as $t_e \simeq L^2/lv$. Since the particles of lower rigidities, hence of shorter l's, stay longer in the region, they have more chances to hit the absorbers. Therefore, we can expect the B process that preferentially cuts out the particles of low rigidities. In order to see if the absorption is effective enough, we estimate the mean lifetime for the absorption. If the particles are trapped in a volume of L^3 in which there are absorbers with an effective cross section of S, the mean absorption time is obtained as $t_a \simeq L^3/Sv$. The ratio of t_e to t_a, $t_e/t_a \simeq S/Ll$, shows how the B process becomes effective as the rigidity decreases.

In the extended atmosphere of the giant stars S may not be much smaller than L^2, so that t_e/t_a seems to be large enough. In the binary star system, S may be as small as $10^{-4}\ L^2$, so that only too small a value of l can give a large enough value of t_e/t_a. It is interesting to note that the particles thus absorbed may be responsible for the generation of light elements, as has been shown in § 4 of reference 3). In this respect our model seems to get an independent support. It should also be remarked that the acceleration process taking place together with the above B process does not have to be an unrealistic one. For example, the rate of energy gain may be such that it stays low in the lowest energy region, then increases

with energy linearly or so and levels off at high energies. Such an energy dependence is quite reasonable and may be accounted for in terms of the betatron mechanism and its realistic modifications.

In addition to the simplest model above, there are some other possibilities, if we assume special A processes. One of them is a combination of A and B processes in such a way that the particles accelerated from a low rigidity drops out of the acceleration rather rapidly and only a small part undergoes an efficient acceleration, so that they reach high rigidities without appreciable loss. Then the acceleration stops at a certain rigidity; consequently the particles accumulate just below this limiting rigidity. Such a mechanism may be operative in the so-called magnetic mirrors.

A more special and probably unrealistic mechanism is the resonance acceleration such as the one proposed in Appendix V, (iv) of reference 3). This certainly produces a valley in the rigidity spectrum.

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Equation of State of Classical Electron Gas

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In a recent paper by Ichikawa,[1] the long range correlation effects of Coulomb interaction are shown to increase the free energy of the Debye-Hückel limiting law by 22%. If this is true, a similar conclusion may be drawn by the usual cluster expansion method, without recourse to the collective coordinate representation. Although Mayer[2] has carried out the order estimation of the contributions of complicated clusters other than ring clusters and shown that they are of higher orders, his conclusion is not fully justified in the case of electron gas, since some of terms considered by him diverge for purely Coulomb interactions without hard cores. Thus it will be worth while to investigate the higher order terms in a systematic way, free from the divergence difficulties. We wish here to present an exact evaluation of the free energy term which is expected to be of the next higher orders than the lowest one, i. e., the Debye-Hückel law.

If we define the function $S(\rho)$ by

$$S(\rho)=\sum_{n=1}^{\infty}\frac{\beta_n}{n+1}\rho^{n+1} \qquad (1)$$

with β_n the irreducible cluster integral and ρ the particle number density, the equation of state, the Helmholtz free

energy, etc., are easily derived from this function. According to Mayer,[2] $S(\rho)$ is expanded as follows,

$$S(\rho)=S_0(\rho)+\sum_{m=2}^{\infty}\frac{\rho^m}{m!}\frac{1}{V}\times\int\sum_{m\geqq i>j\geqq 1}{}'\prod_{ij}\frac{[-\beta q_{ij}]^{k_{ij}}}{(k_{ij})!}d(m). \quad (2)$$

Here the $S_0(\rho)$ is the contribution of the ring clusters and $\sum'$ implies the summation over prototype graphs with m branch points,[3] $d(m)$ the integration over the branch points and k_{ij} the number of bonds between the branch points i and j. The potential function q_{ij} is given by

$$q(r)=\varepsilon^2 e^{-\kappa r}/r \quad (3)$$

in the case of Coulomb interaction. $S_0(\rho)$ is known to lead to the Debye-Hückel law, i. e.,

$$S_0(\rho)/\rho=\lambda/3 \quad (4)$$

with $\lambda=2\sqrt{\pi}\beta^{3/2}\rho^{1/2}\varepsilon^3$ which is the non-dimensional parameter characteristic of the classical electron gas. If we further consider the graphs $m=2$ in eq. (2), each term is shown to be divergent. However, if *all* the terms $m=2$, i. e., *all* the graphs of watermelon type,[4),5)] are summed up, the result is no longer divergent and is given by

$$\frac{S_2(\rho)}{\rho}=2\pi\rho\int_0^{\infty}\left[e^{-\beta q(r)}-1+\beta q(r)-\frac{\beta^2 q^2(r)}{2}\right]r^2dr.$$

The similar expression was already derived by Yukhnovsky[6] from another point of view. An analytical evaluation of the integral leads to

$$S_2(\rho)/\rho=1/2-\lambda/8-\lambda^2/6\times\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r/r!(n-r)!\times\lambda^{n/2}(r+1)^{n/2}\times[2/\lambda(r+1)\cdot K_{|n-2|}(2\sqrt{\lambda(r+1)})+2(r+1)\sqrt{\lambda(r+1)}\times K_{|n-3|}(2\sqrt{\lambda(r+1)})],$$

where $K_n(x)$ is the Neumann function of purely imaginary argument. An explicit calculation yields

$$S_2(\rho)/\rho=\lambda^2(1/12\cdot\log 3+\gamma/6-11/72)+\lambda^2\log\lambda/12+O(\lambda^3)$$

where γ is Euler's constant. Thus the contribution of graphs with 2 branch points never leads to the term of the order λ and turns out to be of higher orders than that of $S_0(\rho)$. It seems that the more complicated graphs do not lead to the term of the order λ and the Debye-Hückel law is exact with respect to this order for the purely Coulomb interactions as well as for the one with hard cores. The detail of this article will soon be published.

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Field Theoretical Interpretation of Multiple Meson Production

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The historical development of the multiple production of particles has recently been surveyed by Koba and Takagi.[1] In the present note we shall give the formula for the cross-section of multiple production of mesons under the following three assumptions:

a) The interaction between a nucleon and a pion is present only for a finite time interval 0 to t. The transition matrix is assumed to be

$$U(\infty,\ t)=U(0,\ -\infty)=1, \qquad (1)$$

namely no interaction exsists for $-\infty$ to 0, and t to ∞.

b) The free Hamiltonian H_0 is regarded as a C-number for the above mentioned time-interval $(0,\ t)$. In this approximation we have

$$U(t,\ 0)=e^{-iH_0 t}e^{iHt}=e^{iH_I t}, \qquad (2)$$

where H_I is the interaction Hamiltonian.

c) The concept of renormalization is assumed to get only the relative cross-section of n pion production.

By virtue of a), b) and of the symmetrical pseudoscalar interaction, we can exactly obtain the following matrix element, for example, a positive pion collision with a nucleon.

$$\begin{aligned} M&=\langle\chi_f|U(\infty,\ -\infty)|\chi_i\rangle \\ &=\{\sqrt{2}\,k_0/(2\pi)^3\} \\ &\times\int\langle\chi_f|\phi^*(\boldsymbol{x}')\,(\bar{\psi}(\boldsymbol{x})\,e^{ig\gamma_4\gamma_5\tau\phi(\boldsymbol{x})})_p|\chi_0\rangle \\ &\times\beta_{p\sigma}a_\sigma^{\nu}(\boldsymbol{K})\,e^{i\boldsymbol{K}\boldsymbol{x}+i\boldsymbol{k}\boldsymbol{x}'}d\boldsymbol{x}\,d\boldsymbol{x}', \end{aligned} \qquad (3)$$

where χ_0 is the free vacuum, a is a Dirac spinor, and $\boldsymbol{K}$, $\boldsymbol{k}$ are the momentum of the incident nucleon and the pion respectively. $\phi^*(\boldsymbol{x}')$ and $\bar{\psi}(\boldsymbol{x})$ are the operators of the meson and nucleon fields for $t=0$. From the above matrix element we have the relative cross-section of $(n-1)$ pion production (the number of final mesons, $n=n_+ +n_- +n_0$, n_+, n_- and n_0 are the numbers of positive, negative and neutral pions respectively), such as,

$$\begin{aligned} \frac{d\sigma}{d\boldsymbol{K}'}&\sim\frac{(gt)^{2n+2}t^{-2}}{2^n(2\pi)^{3n}} \\ &\times\frac{\{K_0K_0'+(-)^{n-1}(\boldsymbol{K}\cdot\boldsymbol{K}'+m^2)\}}{K_0k_0K_0'} \\ &\times\{F(n_+,\ n_-+1,\ n_0)\,(n_-+1)\}^2 \\ &\times G(n+1,\ n_0) \\ &\times\int\prod_{i=1}^{n}\delta^4(K+k-K'-\sum_{i=1}^{n}k_i)\,dk_i/k_{i0}. \end{aligned} \qquad (5)$$

Here the following relation has been used,

$$\begin{aligned} e^{ig\gamma_4\gamma_5\tau\phi(\boldsymbol{x})}&=Z\sum_{n=1}(ig\gamma_4\gamma_5)^{n+1} \\ &:\overbrace{\phi_0(\boldsymbol{x})\cdots}^{n_0},\ \overbrace{\phi(\boldsymbol{x})\cdots}^{n_-+1},\ \overbrace{\phi^*(\boldsymbol{x})\cdots}^{n_+}: \\ &\times F(n_+,\ n_-+1,\ n_0)\,G(n+1,\ n_0), \end{aligned} \qquad (6)$$

† On leave from Osaka University, Osaka.

where

$$F(n_+,\ n_-+1,\ n_0)=\Gamma\{(n_0+\delta_{n_0}+1)/2\}$$
$$\times\{(n-n_0+1+\delta_{n+1}-\delta_{n_0})/2\}!$$
$$\times\sqrt{2}^{\,n_0-n}/n_+!(n_-+1)!\,n_0!$$
$$\times\Gamma\{(n+2+\delta_{n+1})/2\},$$
$$G(n+1,\ n_0)=1-\delta_{n_0}(1-\delta_{n+1})$$

with $\delta_n=\begin{cases}0\\1\end{cases}$ for $\begin{matrix}n=\text{even}\\n=\text{odd.}\end{matrix}$

Here Z is a renormalization constant and the symbol, : : represents Wick's S-product.

The covariant integration with respect to the phase space in the formula (5) is evaluated by the saddle point method.[2] Namely it runs to

$$\int\prod_{i=1}^{n}\delta^3(\boldsymbol{K}'+\sum_{i=1}^{n}\boldsymbol{k}_i)\,\delta(W-\sum_{i=1}^{n}k_{i_0})\,d\boldsymbol{k}_i/k_{i_0}$$
$$=(2\pi^2)^n\mu^{2n-4}n^{-2}(a/\gamma)^{3/2}/(2\pi)^2J(a)^{1/2}$$
$$\times\exp[n\log\{-H_1^{(2)}(-ia)/a\}$$
$$+n\gamma-\boldsymbol{K}'^2a/2n\mu^2\gamma], \qquad (7)$$

where a is the solution of the following equation,

$$f(x)=2/x+\{iH_0^{(2)}(-ix)/H_1^{(2)}(-ix)\}$$
$$=W/n\mu=\gamma$$

and

$$J(x)=1+2/x^2$$
$$-(3/x)\{iH_0^{(2)}(-ix)/H_1^{(2)}(-ix)\}$$
$$-\{iH_0^{(2)}(-ix)/H_1^{(2)}(-ix)\}^2.$$

From the equations (5) and (7) we can obtain quantitative results. The average value of n becomes 3.7, for example, in π^-+p collisions at 5 Bev when t is taken to be the collision time and $g^2/4\pi=15$. This value is in fairly good agreement with experimental one,[3] but unfortunately the angular distribution of nucleon in finite prongs is nearly isotropic in c. m. system.

In a similar way we can derive the cross-sections in nucleon-nucleon collisions and in nucleon pair annihilations. One would find the differences of the present treatment from those of LOW[4] or Fukuda and Takeda's theory,[5] namely in the present work the interaction between nucleons is not replaced by a potential, the nucleon recoils are taken into account and its self energy part is also considered. Furthermore, it will be seen that, contrary to Fermi's statistical theory,[6] we have evaluated the matrix element from a point of view of meson theory and the integration over the phase space is covariantly performed. Details of this note will be reported in near future in three parts: pion-nucleon collision, nucleon-nucleon collision and antinucleon-nucleon annihilation.

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Molecular Processes induced by μ^- Mesons in Hydrogen Bubble Chamber, II

——Mechanisms of Formation of μ^- Mesonic Hydrogen Molecular Ion†——

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December 23, 1958

In this note we report the result of our qualitative study on the mechanism of formation of a μ^- mesonic molecular ion by the low energy collision of a mesonic proton or deuteron atom with a proton or deuteron bound in an ordinary hydrogen molecule. This was made as a preliminary study for detailed calculations based on approximations similar to those used in the previous paper.[1]

First we mention on the molecular ion formation by collision of a μ^- mesonic deuteron atom with a proton. The mesonic $(HD)^+$ has two bound states, one being the ground state Φ_S (whose dissociation energy >187 ev) with the total orbital angular momentum $L=0$ and the other being the excited state Φ_P (whose dissociation energy $>(44$ ev) with $L=1$. Then $(HD)^+$ in the ground state can be formed by two ways. One is by the direct transition from the incident state Φ_I to the ground state Φ_S. The other is by the indirect transition through the excited state Φ_P. In these transitions some other quanta than p, d, and μ^- under consideration must be ejected because both the total energy and the total momentum must be conserved. We here consider only two cases of ejection of a photon (radiative process) and of ejection of a $1s$ electron bound to the proton (Auger's process), because these two are expected to be most important. First, we consider the direct transition. Since the bombarding energy is very low, Φ_I is essentially of s character for small values of the $p-d$ distance under consideration. Then the direct transition is essentially the so-called 0–0 transition, so that the radiative process is almost impossible. Consequently the transition occurs exclusively by Auger's process. Next, we consider the indirect transition. In this case, the radiative and Auger's processes are both possible, and each step of transitions turns out to be an electric dipole transition from angular momentum arguments. In the present preliminary calculation, we assumed for Φ's simple forms similar to those used by Jackson.[2] The result of this calculation is given in Table I.

Table I. Reaction rate of $(HD)^+$ formation in pure liquid hydrogen (in sec^{-1}).

	$\Phi_I \to \Phi_S$	$\Phi_I \to \Phi_P$	$\Phi_P \to \Phi_S$
Auger's process	10^{-1}	10^{4}	10^{10}
Radiative process	0	10^{-1}	10^{6}

From Table I we see that $(HD)^+$ will be formed mainly by making two steps of transitions $\Phi_I \to \Phi_P \to \Phi_S$ ejecting two

† The main part of this note was published in *Progress Report of Research Group for the Study of Molecular Structure in Japan*, No. 7 (December, 1957).

electrons bound in an ordinary hydrogen molecule. This conclusion is consistent with Skyrme's one[3] that some other mechanisms than the direct transition will be important; but it is inconsistent with Jackson's one.[2] Jackson estimated the reaction rate of the direct transition and found a very large value ($\sim 10^7$ sec^{-1}). Then he concluded that a μ^- mesonic molecular ion will be formed by the direct transition. However, Jackson's method of treating the direct transition is very difficult to be justified, as will be explained in our full paper. Our present calculation will also be too rough to be accepted quantitatively, but our conclusion seems to be correct qualitatively.

Next, we will touch simply on the mechanism of formation of the μ^--mesonic $(HH)^+$ and $(DD)^+$. In these cases the indirect transition through the Φ_P state ($E1$ transition) is forbidden at the limit of zero bombarding energy, as is seen by considering the symmetry character of Φ_I, Φ_P, and Φ_S with respect to the exchange of the two nuclei. Then we expect from the above argument on $(HD)^+$ that the reaction rate will be very small compared with that of $(HD)^+$. However, this expectation seems to be open to discussion, for our qualitative study on bound states of $(HH)^+$ based on a Morse oscillator model tells that $(HH)^+$ may have an extremely loosely bound state with $L=0$ or a resonance level in the neighbourhood of zero bombarding energy. Then we can expect a possibility that the low energy scattering of a μ^- mesonic proton atom by a proton exhibits a resonance and the amplitude of the correct incident wave function Φ_I for values of the $p-p$ distance of the order of the μ^- mesonic Bohr radius is far larger than that of the Born approximation used in our present estimation. For $(DD)^+$, on the other hand, it has four bound states, two with $L=0$, one with $L=1$, and one with $L=2$, and then not only the direct transition $\Phi_I \to \Phi_S$ but also several indirect transitions such as $\Phi_I \to \Phi_P \to \Phi_S$, $\Phi_I \to \Phi_D \to \Phi_S$ (excited) $\to \Phi_S$ are possible.

In these circumstances, we wish finally to stress that the mechanism of molecular ion formation is rather different from isotope to isotope. The result of our detailed calculation based on an approximation similar to (I)[4] will be published in the near future.

1) M. Shimizu, Y. Mizuno and T. Izuyama, Prog. Theor. Phys. **20** (1958), 777; which will be hereafter referred to as (I).
2) J. D. Jackson, Phys. Rev. **106** (1957), 330.
3) T. H. R. Skyrme, Phil. Mag. **2** (1957), 910.
4) S. Cohen, D. L. Judd, and R. J. Riddel, Phys. Rev. **110** (1958), 1471;
They have also calculated the reaction rate of molecular ion formation. Their calculation seems very similar to our detailed one, though the details of the mechanism and the calculation are not reported so far.

Errata

Dispersion Relations and Vertex Properties in Perturbation Theory

K. Symanzik

Prog. Theor. Phys. **20** (1958), 690

a) In the formula following (10) the last term should read $7\mu^2/64$ instead of $F\mu^2/64$.

b) The statement in footnote *11) that the general method gives $\Delta^2 \leq 2M^2+4M\varepsilon+O(\varepsilon^2)$ is due to a calculational error. The correct value is $\Delta^2 \leq 2M^2+O(\varepsilon^2)$ and, therefore, the conclusion of the footnote is invalid. Cp. Y. Nambu, "Limitations on Momentum Transfer in Dispersion Relations of Scattering Amplitudes", to appear in Nuovo Cimento.

Errata

The Clobsch-Gordan Coefficients

S. Datta MAJUMDAR

Prog. Theor. Phys. **20** (1958), 798

The following corrections should be made.

p. 800, 11 th line...The equation should read as follows

$$\Psi_{jm} = \cdots = e^{im\zeta}\chi_{jm}$$

p. 801, eq. (10)...Delete the comma in the middle of the equation and close up.

p. 902...In the line immediately after eq. (13) $(m_1m_2|j_1j_2)$ should be replaced by $(m_1m_2|jj)$.

p. 802...4th line from the bottom should read as follows

$$\times F(-j_1-j_2+j,\ -j_1+j_2+j+1\,;\ -2j_1\,;\ 1)$$

p. 803...At the end of the last line "$\chi=1$"

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Dynamical Properties of *s-d* Interaction

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The dynamical properties of two spin systems composed of conduction electron spins and localized *d*-spins interacting by exchange are investigated in connection with the electron-spin resonance of *d*-spins with the free electron *g*-value. It seems to us that some basic considerations are necessary for the problem along the line of the widely accepted idea that the *s-d* interaction itself does not lead to any relaxation time and shift. The usual adiabatic approximation is applied to this idea. A modified damping equation of the Bloch type as well as Bloch's relaxation theory are also useful, but these are not enough for a complete solution of the problem. A certain controversy involved in recent experimental and theoretical work at Berkeley concerning the electron spin resonance in Cu-Mn dilute alloy is pointed out and examined. We believe that the possibility could not be eliminated that the spin-lattice relaxation time of conduction electrons in the alloy due to mechanisms other than the *s-d* interaction is longer than previous authors have assumed.

§ 1. Introduction

The importance of the exchange interaction between conduction electrons and localized incomplete *d*-shells, so-called *s-d* interaction, has recently received increasing attention both from experimental and theoretical points of view. In particular, Yosida[1] has investigated theoretically the origin of anomalous magnetic and electric properties of Cu-Mn or Ag-Mn alloys taking into account this interaction, and has given an explanation for the absence of a large paramagnetic resonance shift which was anticipated by Owen, Browne, Knight, and Kittel[2] but was not found in their experiments of the electronspin resonance on manganese in copper. Yosida has also presented an explanation for the absence of a large excess Knight shift of the nuclear magnetic resonance on copper in Cu-Mn alloys, saying that the cross perturbational effects of the *s-d* interaction and the hyperfine interaction should be responsible for the broadening of the resonance line rather than for the shift. A detailed calculation of the line width has been made by Behringer[3] on this basis. Yosida[4] has further given a theoretical interpretation of the anomalous electrical resistivity and magnetoresistance found in these alloys at low temperatures in terms of the effect due to the *s-d* interaction. The success of his theory in the resistance problem may be regarded as a direct evidence of the interaction. According to these two authors the main features of Cu-Mn alloys can thus be understood fairly well if one assumes a value of the *s-d* exchange integral in the alloys nearly equal to the one

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in free Mn^{+} ions. The experimental investigation of impurity states of various transition elements in copper has been extended by Sugawara[5] using nuclear magnetic resonance techniques.

Owen, Brown, Arp, and Kip[6] have reported the results of their experiment on the magnetic susceptibility and the electron spin resonance in those alloys. In order to explain the mechanism of the resonance anomaly found at low temperatures they argued that the polarization of conduction electron spins is highly concentrated at each manganese ion and produces the local anisotropy field for the d-core spins of manganese. In this argument they referred to Kittel and Mitchell's theory[7] which states that, if the coupling of conduction electron spins with the lattice is stronger than with the d-core spins, the above mentioned local field may be effective as a shift observable in the d-core spin resonance experiment.

This idea has a close relation to Yosida's theory, because the localized character of the polarization is one of its main assertions. But Yosida's theory implies the fact, though he did not mention it explicitly, that this type of polarization can never contribute to a shift in the d-core spin resonance experiment, since in such dynamical processes the magnetization of conduction electrons does not stay at its thermal equilibrium position, but precesses following the d-core magnetization so that its direction is always kept parallel to the latter. This picture is essentially based upon the so-called adiabatic approximation.[8]

However, it is not clear how Kittel-Mitchell's theory and Yosida's theory are mutually related, because the latter author did not state the significance of the spin-lattice relaxation of conduction electrons in the treatment. In fact, Yosida's treatment of minimizing the total diagonal energy involves an isothermal picture for the spin system of conduction electrons, and is therefore different from the adiabatic approximation in its original sense; it assumes implicitly the shortness of the spin-lattice relaxation of conduction electrons.

Generally speaking, the dynamical problem of the s-d interaction is much more complicated than that of the nuclear relaxation through the hyperfine interaction in metals, since the time scale for the motion of the conduction electron spins may be comparable to that of the localized d-spins, so that the observable shift and broadening may not necessarily be considered in a simple analogy to the nuclear magnetic resonance in metals. In fact, we may expect that the effective shift and line width are considerably reduced from the corresponding value in the case of isothermal limit as functions of the ratio,

$$x=\frac{\text{Relaxation time through } s\text{-}d \text{ interaction}}{\text{Relaxation time through other mechanism}}.$$

This is shown in section 4.

The calculation used in that section is essentially phenomenological and has a certain ambiguity in treating the spin-lattice relaxation. Nevertheless, we regard it as a necessary step to obtain a complete understanding of the problem, because the

experimental data of the line width at paramagnetic region seem to indicate that the spin-lattice relaxation time in copper is not short enough as previous theories are applicable.

§ 2. Experimental facts and existing theories

2.1 First, we refer to the work of Owen, Browne, Knight, and Kittel,[2] who made the following argument.

The polarization of conduction electrons in the substance is given by

$$M_s^\circ = \chi_p (H + \lambda M_d^\circ) , \tag{2·1}$$

where χ_p is the Pauli paramagnetic susceptibility, λ the molecular field coefficient of the interaction between conduction electron spins and d-core spins, and M_d° the magnetization of total d-core spins along the direction of the applied constant field H. As is well known, χ_p is given by

$$\chi_p = 2\mu_B^2 n(E_f) , \tag{2·2}$$

where $n(E_f)$ is the state density per unit volume at the Fermi surface of the s-band for one spin direction, and is equal to $3n/4E_f$ for the free electron model, n being the number of conduction electrons per unit volume. The molecular field coefficient λ is expressed in terms of the zero-Fourier component $J(0)$ of the s-d exchange integral as

$$\lambda = J(0)/g\mu_B^2 N , \tag{2·3}$$

N being the number of Cu atoms per unit volume. Then the effective fields acting on each d-core spin of manganese and on each nuclear spin of copper are written as

$$H_d = \lambda M_s^\circ = \lambda\chi_p H + \lambda^2 \chi_p M_d^\circ \tag{2·4}$$

$$H_n = \alpha M_s^\circ = \alpha\chi_p H + \alpha\lambda\chi_p M_d^\circ \tag{2·5}$$

respectively, where

$$\alpha = A_0/2Ng_N\mu_N\mu_B \tag{2·6}$$

is related to the hyperfine constant of copper in the metal. Therefore, one might expect the additional resonance shifts

$$\Delta H_d^{(2)}/H = \lambda^2 \chi_p M_d^\circ/H = \lambda^2\chi_p\chi_d \tag{2·7}$$

for the electron spin resonance, and

$$\Delta H_n^{(2)}/H = \alpha\lambda\chi_p M_d^\circ/H = \alpha\lambda\chi_p\chi_d \tag{2·8}$$

for the nuclear resonance, besides the ordinary first order shift and the Knight shift i.e.

$$\Delta H_d^{(1)}/H = \lambda\chi_p = (2/g)\,(n(E_f)/N)\cdot J(0) \tag{2·9}$$

$$\Delta H_n^{(1)}/H=\alpha\chi_p=(1/g_N)(\mu_B/\mu_N)(n(E_f)/N)A_0 \qquad (2\cdot10)$$

respectively. The additional shifts, (2·7) and (2·8), are directly proportional to the equilibrium value of the magnetization M_d° which depends on the temperature and the concentration of d-cores. The ratio of these shifts to the first order shifts is given by

$$\lambda\chi_d\simeq(J(0)/3kT)(N_0/N)gS(S+1) \qquad (2\cdot11)$$

which is related to the paramagnetic susceptibility of the contributions from d-core spins, where N_0 is the number of Mn ions per unit volume. Even at low concentration of manganese in copper, say one per cent, this ratio is expected to be of the order of one at 100°K. However, these additional shifts were not observed experimentally, and therefore the s-d exchange interaction in the alloy looks to be much weaker than that in a free ion of manganese.

In the d-electron spin resonance, the second order shift of the form (2·7) should be observable, if the spin-lattice relaxation of conduction electrons due to the proper mechanism is shorter than the one due to the s-d interaction, so that conduction electron spins may be kept at the equilibrium polarization as given by eq. (2·1).

This last assertion is due to Kittel and Mitchell's argument[7] made in connection with the ferromagnetic resonance shift in ferromagnetic metals. Yosida[1] did not make any argument concerning this assertion, but showed rather in a different way that the excess field (2·7) cannot contribute to a line shift in the d-electron spin resonance. The assumption made implicitly in Yosida's argument is that the second part of the polarization proportional to the molecular field coefficient λ in eq. (2·1) follows instantaneously the temporal change in the total magnetization of d-electron spins, so that eq. (2·1) should be replaced by the following vector equation

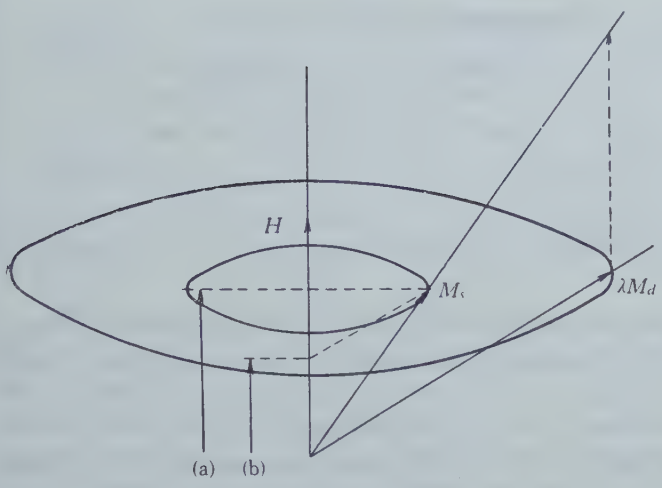
$$M_s(t)=\chi_p(H+\lambda M_d(t)) . \qquad (2\cdot13)$$

This is illustrated in Fig. 1. One sees at once that the effective field $\lambda^2\chi_p M_d(t)$ cannot produce a torque on $M_d(t)$. The difference of Kittel-Mitchell's shift and Yosida's shift is also shown there.

This interpretation comes from the following treatment in Yosida's theory. First, we take into account the diagonal part of the s-d interaction with respect to the wave number k of the Bloch states, and, combining it with the kinetic energy and the Zeeman energy of conduction electrons, we minimize the total first order energy. This leads to the uniform polarization indicated by (2·1). At the same time, we obtain a gain of the polarization energy in the following form

$$\Delta H_p^{(1)}=-\tfrac{1}{2}\chi_p(H+\lambda M_d^z)^2 , \qquad (2\cdot13)$$

which is supposed to take part in determining the motion of the d-electron magnetization. But, we must also take into account the non-diagonal part of the s-d interaction so as to complete the perturbational treatment. This gives the additional



(a) Kittel-Mitchell's shift
(b) Yosida's shift

Fig. 1.

terms of the polarization energy, $-\frac{1}{2}\chi_p(\lambda M_d^x)^2$ and $-\frac{1}{2}\chi_p(\lambda M_d^y)^2$ of the parts of the interaction which are non-diagonal with respect to the Zeeman states but are diagonal to the Bloch states. These terms add up, together with $\Delta H_p^{(1)}$, to the following isotropic energy

$$\Delta H^{(1)} = -\tfrac{1}{2}\chi_p(H+\lambda M_d)^2, \qquad (2\cdot14)$$

which may be looked as the effective Hamiltonian for the motion of M_d. Eq. (2. 12) can be regarded as a phenomenological expression of this procedure.

2. 2 Before making further examination we briefly review the experimental results and some related discussions.

(1) *The line shift and width of Mn-electron spin resonance at paramagnetic region.*

As was reported by Owen, Browne, Arp, and Kip[6], the fine structure and the hyperfine structure are absent and the observed g-shift is quite small, less than 1 %. From the theoretical side, however, Yosida predicted that the first order shift (2·9) should be observable though the second order shift (2·7) should not. Yosida admitted that the theoretical value of this shift is several times larger than the observed shift, when one uses the free ion value of the exchange integral J_0 in eq. (2·9).

The observed line width above 77°K is approximately proportional to the temperature, which is a strong indication that the spin-lattice relaxation through the s-d interaction is responsible for this broadening, but the observed width is only about one several tenth of the value estimated by the Korringa formula[9].

(2) *Localization of the polarization of conduction electron spins.*

The absence of the second part Knight shift (2·8) was explained by Hart[10] and Yosida[1] in terms of the localization of the polarization. From their argument, the second part of the uniform polarization in (2·1) is completely cancelled by the perturbation of the wave function almost everywhere just except the vicinity of the position where each d-core spin is located.

(3) *Static susceptibility.*

If the simple molecular field approximation is applied, we obtain the Curie-Weiss law[2]

$$\chi=\chi_p+\frac{C(1+\lambda\chi_p)^2}{T-\theta_p} \tag{2·15}$$

where

$$C=\frac{N_0}{3k}g^2\mu_B^2S(S+1) \tag{2·16}$$

$$\theta_p=\frac{1}{3k}\left(\frac{N_0}{N}-\frac{2n(E_f)}{N}\right)J^2(0)S(S+1). \tag{2·17}$$

The paramagnetic Curie temperature θ_p is positive in this approximation. But, if Yosida's theory is strictly valid in explaining the absence of the second order Knight shift, we must expect that this positive θ_p is almost cancelled out by the second order perturbation of the non-diagonal part of the *s-d* interaction. This is because the complete expression of θ_p obtained by the moment-expansion method, when applied to the *s-d* interaction, is proportional to the first moment of the Cu-nuclear resonance line predicted by Yosida's formula, provided that the mean distribution of Mn atoms around an Mn atom is the same as that around a Cu atom. Therefore, the existence of positive θ_p experimentally confirmed for the alloys of Mn concentration more than 1% must be accounted for by certain other origins.

(4) *Mn-electron spin resonance at low temperatures.*

There is a serious discrepancy between two pictures illustrated in Fig. 1 in connection with the mechanism of the anomalous shift.

(5) *Anomalous decrease of resistivity at low temperatures.*

The mechanism of the phenomenon has been explained by Yosida[4] under the simple molecular field assumption. Above the Néel temperature a conduction electron is scattered by each *d*-core isotropically with respect to that spin states, so that the cross section is proportional to $S(S+1)$, whereas at absolute zero it is proportional to S^2, because *d*-core spins are completely ordered. He derived the corresponding resistivity difference assuming the free electron model and the free ion value of $J(0)$. His result is 1/3 of the observed value in Cu-Mn alloys. We note here, however, that the state density at the Fermi surface of the *s*-band in copper may be 1.4 or 1.9 times as large as the one expected from the free electron model[11]. If we take the effective mass of the conduction electrons in copper 1.4 times as large as the true mass, the estimated value of the resistivity difference becomes more consistent, since the resistivity is proportional to the square of effective mass.

Thus the experimental facts seem to present us several problems to be re-examined more carefully. An important question is concerning the length of the spin-lattice relaxation time of conduction electrons, which determines the applicability of various pictures.

§ 3. Consideration from thermodynamical point of view

The total system under consideration is composed of three components, namely

$\boldsymbol{M}_d$ = (spin-system of d-cores)

$\boldsymbol{M}_s$ = (spin-system of conduction electrons)

$\boldsymbol{L}$ = (translational motion of conduction electrons).

The translational motion of conduction electrons is generally very fast. Roughly speaking, it is characterized by the frequency $E_f/\hbar$. Therefore, the dynamical behavior of the slow spin-system may be described by the usual adiabatic approximation.

Suppose that the total Hamiltonian for the total system is given by

$$H=H_L(q)+H_M(Q)+H_I(q\,Q), \tag{2·1}$$

where q and Q represent the coordinates of the fast system L and the slow spin system $\boldsymbol{M}$ respectively. $H_L(q)$ represents the dynamical motion of the fast system, i.e. the kinetic energy of conduction electrons, $H_M(Q)$ the Zeeman energy for the both spin-system, and $H_I(q\,Q)$ the interaction between them. Then, the principle of adiabatic approximation states that the Schrödinger equation for the two systems can be separated into two parts:

$$\{H_L(q)+H_I(q\,Q)\}\psi_{nQ}=W_n(Q)\psi_{nQ} \tag{2·2}$$

$$\{W_n(Q)+H_M(Q)\}\phi_{n\Gamma}=E_{n\Gamma}\phi_{n\Gamma}\,. \tag{2·3}$$

This couple of equations states that the system L polarizes according to the instantaneous configuration Q, giving an adiabatic potential $W_n(Q)$ which takes part in determining the motion of M. As a consequence, the heat transfer processes between L and M are neglected.

In the present problem of the s-d interaction, $H_I(q\,Q)$ represents the exchange interaction between conduction electron spins and d-core spins, for which the adiabatic approximation is ideally satisfied, so far as there is no other magnetic interaction and the g-factors of conduction electrons and d-core spins are just equal to each other, so that the relation

$$[H_M,\ H_{s\text{-}d}]=0 \tag{2·4}$$

is fulfilled.

In this case the frequency spectrum of $H_{s\text{-}d}$ is a δ-function having a sharp peak at zero-frequency on the frequency scale of H_M, so that the precessional motion in the spin-system due to the Zeeman energy does not induce any transition between the levels of adiabatic potential, or, in other words, the adiabatic approximation in this case is not an approximation. Accordingly, the s-d interaction neither affects the Zeeman levels nor produces any heat transfer process from the spin-system to its surrounding, so that we do not expect any shift and any relaxation through the s-d interaction in the resonance problem.

From this consideration we see that the energy minimization treated by Yosida

cannot be understood by the adiabatic approximation. In that treatment, the spin-system of conduction electrons $\boldsymbol{M}_s$ is regarded as completely in thermal equilibrium at a definite temperature T which characterizes the thermal equilibrium of the system $\boldsymbol{L}$. In this sense the fast system $\boldsymbol{L}$ plays a role of heat reservoir for the dynamical system, so that we may call $\boldsymbol{L}$ a lattice system. At the same time, Yosida's picture may be called an isothermal picture for conduction electron spins. As a result of this picture, the coordinates of the conduction electron spins are completely eliminated from the dynamical equation, which necessarily leads to the first order shift (2·9) for d-spin resonance. This situation is shown in Fig. 2. The difference between the adiabatic and the isothermal picture rises primarily in the strength of thermal contact between the systems. In the isothermal picture the transfer of heat from the system $\boldsymbol{M}_s$ to the lattice $\boldsymbol{L}$ takes place in a moment, so that the s-d interaction is fully effective on the relaxation for the dynamical system $\boldsymbol{M}_d$. For this picture, therefore, it is necessary that the spin-lattice relaxation time of conduction electrons is sufficiently short; this must be an implicit assumption in Yosida's theory.

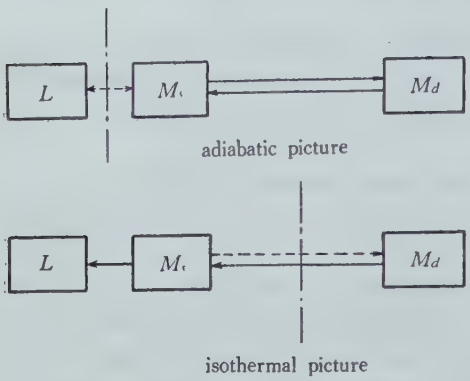


Fig. 2.

In actual cases we may expect, more or less, an intermediate situation between the two idealized pictures, where the s-d interaction may be partially effective on the shift and broadening observable in d-spin resonance.

§ 4. Phenomenological equation for two spin systems

4. 1 In order to make an interpolation between the two idealized pictures in the dynamical problems we consider a couple of phenomenological equations of the Bloch type.[12] This is written as

$$\dot{M}_s=\gamma M_s\times(H+\lambda M_d)-\frac{1}{T_{sL}}(M_s-\bar{M}_s)-\frac{1}{T_{sd}}M_s+\frac{1}{T_{ds}}M_d \qquad (4\cdot1)$$

$$\dot{M}_d=\gamma' M_d\times(H+\lambda M_s)+\frac{1}{T_{sd}}M_s-\frac{1}{T_{ds}}M_d\,. \qquad (4\cdot2)$$

Here M_s and M_d are the total magnetization vectors of conduction electron spins and d-electron spins respectively, and γ and γ' are corresponding magneto-mechanical ratios which are assumed to be just equal to each other through the present section.

First of all we note the several assumptions required for these equations. These are as follows:

(1) The strength of thermal contact may be characterized by three time constants, namely

T_{sL}: spin-lattice relaxation time of conduction electron due to the mechanism other than the *s-d* exchange.

T_{sd}: spin-lattice relaxation time of conduction electrons due to the *s-d* exchange.

T_{ds}: spin-lattice relaxation time of *d*-cores due to the *s-d* exchange.

(2) It should be satisfied that

$$\dot{M}=0, \quad M=M_s+M_d, \tag{4·3}$$

when $T_{sL}=\infty$, i.e. zero-strength of the contact between the system of conduction electron spins and the lattice. This is because these equations should be consistent with the original equation of motion.

(3) In the equilibrium state each magnetization M_s, M_d has a definite value M_s° and M_d° respectively, and the law of detailed balance

$$M_s^\circ/T_{sd}=M_d^\circ/T_{ds} \tag{4·4}$$

is satisfied.

(4) In eq. (4·1) $\overline{M}_s$ is the final value of M_s, towards which the relaxation of M_s proceeds. For the equilibrium state eqs. (4·1) and (4·2) provide solutions

$$M_s=M_s^\circ, \quad M_d=M_d^\circ . \tag{4·5}$$

From this requirement we see that the equilibrium value of $\overline{M}_s$ is equal to M_s°. But in the non-equilibrium state it is not necessarily true to expect that

$$\overline{M}_s=M_s^\circ . \tag{4·6}$$

This is the most serious ambiguity inherent in the phenomenological treatment. In fact, there is another possibility to put

$$M_s=M_s^\circ+\chi_p\lambda(M_d-M_d^\circ) \tag{4·7}$$

which implies that the magnetization M_s is always relaxing to its instantaneous equilibrium value.* In any case, it is important to note that the phenomenological treatment itself does not concern what are the actual values of M_s° and M_d°, so far as the requirements (4·4) and (4·5) are satisfied.

(5) In the non-equilibrium state the deviations of M_s and M_d from their equilibrium values are assumed to be small, so that the terms of quadratic or higher order in the deviations

$$\delta M_s=M_s-M_s^\circ, \qquad \delta M_d=M_d-M_d^\circ$$

may be neglected.

Then eqs. (4·1) and (4·2) are separated into transverse and longitudinal com-

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* This standpoint is expressed by Kasuya[13]. About this point we had a controversy which remains unsolved. We hope that a detail of his theory will be published in future.

ponents and are linearized for the deviations as follows:

$$\dot{M}_s^+ = -i\gamma(H_z+\lambda M_d^\circ)M_s^+ + i\gamma(H_+ +\lambda M_d^+)M_s^\circ - \frac{1}{T_{sL}}(M_s^+ - \overline{M}_s^+)$$
$$-\frac{1}{T_{sd}}M_s^+ + \frac{1}{T_{ds}}M_d^+ \qquad (4\cdot8)$$

$$\dot{M}_d^+ = -i\gamma(H_z+\lambda M_s^\circ)M_d^+ + i\gamma(H_+ +\lambda M_s^+)M_d^\circ$$
$$+\frac{1}{T_{sd}}M_s^+ - \frac{1}{T_{ds}}M_d^+ \qquad (4\cdot9)$$

for transverse components, where

$$M_s^\pm = M_s^x \pm iM_s^y, \qquad M_d^\pm = M_d^x \pm iM_d^y, \qquad (4\cdot10)$$

and

$$H_\pm = H_1 e^{\pm i\omega t} \qquad (4\cdot11)$$

are the external oscillating field.

$$\delta\dot{M}_s^z = -w_s(M_s^\circ + \delta M_s) - \frac{1}{T_{sL}}(\delta M_s^z - \delta\overline{M}_s^z)$$
$$-\frac{1}{T_{sd}}\delta M_s^z + \frac{1}{T_{ds}}\delta M_d^z \qquad (4\cdot12)$$

$$\delta\dot{M}_d^z = -w_d(M_d^\circ + \delta M_d) + \frac{1}{T_{sd}}\delta M_s^z - \frac{1}{T_{ds}}\delta M_d^z \qquad (4\cdot13)$$

for longitudinal components, where w_s and w_d have the order of magnitude H_1^2, which are taken into account in the problem of saturation.

These equations are direct extension of the nuclear relaxation in metals through hyperfine interaction, which was treated by Heitler, Teller,[14] and Overhauser.[15] According to their results T_{sd} and T_{ds} are given by

$$\frac{1}{T_{sd}} = \frac{2\pi}{\hbar}\frac{2n(E_f)}{N}\langle J^2\rangle_f \frac{N_0}{N}\frac{2}{3}S(S+1) \qquad (4\cdot14)$$

$$\frac{1}{T_{ds}} = \frac{2\pi}{\hbar}\frac{2n(E_f)}{N}\langle J^2\rangle_f \frac{n(E_f)}{N}kT \qquad (4\cdot15)$$

where $\langle J^2\rangle_f$ means the appropriate average of the square of the s-d exchange integral over the Fermi surface. These are based on the second order perturbational theory, and satisfy the relation

$$\chi_p/T_{sd} = \chi_d/T_{ds} \qquad (4\cdot16)$$

which is to be compared to eq. (4·4). We shall come back to this point later in connection with the choice of the final value $\overline{M}_s$ in the phenomenological equations.

4.2 Eqs. (4·8) and (4·9) are rewritten in a following form:

$$\begin{pmatrix} M_s^+ \\ M_d^+ \end{pmatrix} = -D_+ \begin{pmatrix} M_s^+ \\ M_d^+ \end{pmatrix} + i\gamma H_+ \begin{pmatrix} M_s^\circ \\ M_d^\circ \end{pmatrix}, \tag{4·17}$$

where

$$-D_+ = i \begin{pmatrix} \omega_\gamma & \omega_\beta \\ \omega_\alpha & \omega_\delta \end{pmatrix}. \tag{4·18}$$

When the oscillating field $H_+(t)$ is inserted adiabatically at $t=-\infty$, the solution of eq. (4·17) is given by

$$\begin{pmatrix} M_s^+ \\ M_d^+ \end{pmatrix}_t = \int_{-\infty}^{t} e^{-(t-t')D_+} i\gamma H_+(t') \begin{pmatrix} M_s^\circ \\ M_d^\circ \end{pmatrix} dt', \tag{4·19}$$

which describes the resonance under the condition (4·11). The resonance frequency and the effective damping constant are determined by solving the determinantal equation

$$\det|i\omega + D_+| = 0. \tag{4·20}$$

If the two characteristic frequencies $\omega_\pm$ of the solution satisfy the relation

$$Im(\omega_-) \gg Im(\omega_+) \tag{4·21}$$

we may neglect the heavily damped component $e^{i\omega_- t}$ in the expression (4·19), and obtain its asymptotic form.

This becomes

$$\begin{pmatrix} M_s^+ \\ M_d^+ \end{pmatrix}_t = i\gamma H_1 \begin{pmatrix} 1 - \dfrac{\omega_\delta - \omega_\gamma}{\omega_+ - \omega_-} & \dfrac{2\omega_\beta}{\omega_+ - \omega_-} \\ \dfrac{2\omega_\alpha}{\omega_+ - \omega_-} & 1 + \dfrac{\omega_\delta - \omega_\gamma}{\omega_+ - \omega_-} \end{pmatrix} \int_{-\infty}^{t} e^{-i\omega_+(t-t')+i\omega t} \begin{pmatrix} M_s^\circ \\ M_d^\circ \end{pmatrix} dt'. \tag{4·22}$$

When the observed spin-resonance in the metal is regarded as the one due to localized d-core spins, only the fourth component in the matrix (4·22) is effective on the absorption, and so we obtain

$$\chi''(\omega) = Im\left[\left(1 + \frac{\omega_\delta - \omega_\gamma}{\omega_+ - \omega_-}\right) \frac{i\gamma}{i(\omega + Re\omega_+) + Im\omega_+}\right] M_d^\circ. \tag{4·23}$$

If there is a necessity to take into account the microwave skin-effect in analysing the experimental data, one may apply Dyson's theory[16] of the electron-spin resonance in metals in the case of infinite diffusion time, where the fundamental resonance frequency and the relaxation time are considered to be given by

$$\omega_{res} = -Re(\omega_+) \tag{4·24}$$

$$1/T_{2eff}=Im(\omega_+)\,. \tag{4·25}$$

The solution of the determinantal equation (4·20) is written as

$$\omega_\pm=\tfrac{1}{2}(\omega_\tau+\omega_\delta)\pm\tfrac{1}{2}[(\omega_\tau-\omega_\delta)^2+4\omega_\alpha\omega_\beta]^{1/2} \tag{4·26}$$

where ω_+ is chosen so as to satisfy the inequality (4·21).

Now we calculate the explicit form of this frequency according to the given model.

case A. $\overline{M}=M_s^\circ$

$$-D_+=i\begin{pmatrix}\omega_\tau=-\gamma H_z+(i/T_{sL})-(\gamma\lambda-iw)M_d^\circ & \omega_\beta=(\gamma\lambda-iw)M_s^\circ\\ \omega_\alpha=(\gamma\lambda-iw)M_d^\circ & \omega_\delta=-\gamma H_z-(\gamma\lambda-iw)M_s^\circ\end{pmatrix}. \tag{4·27}$$

Here we write

$$1/T_{sd}=wM_d^\circ,\qquad 1/T_{ds}=wM_s^\circ, \tag{4·28}$$

for the sake of simplicity.

Inserting each parameter given by (4·27) into eq. (4·26), $\omega_\pm$ become

$$\omega_\pm=-\gamma H_z-\frac{1}{2}(\gamma\lambda-iw)(M_s^\circ+M_d^\circ)-\frac{i}{2}\frac{1}{T_{sL}}$$
$$\pm\frac{1}{2}\left[\left\{(\gamma\lambda-iw)(M_s^\circ+M_d^\circ)-\frac{i}{T_{sL}}\right\}^2+4(\gamma\lambda-iw)M_s^\circ\frac{i}{T_{sL}}\right]^{1/2}. \tag{4·29}$$

These may be simplified under the assumption

$$M_s^\circ\ll M_d^\circ\,,$$

which is actually valid so far as the observed resonance is regarded as the one due to the d-core spins. These become

$$\omega_+\cong-\gamma H_z+\frac{(\gamma\lambda-iw)M_s^\circ i/T_{sL}}{(\gamma\lambda-iw)M_d^\circ-i/T_{sL}} \tag{4·30}$$

$$\omega_-\cong-\gamma H_z-(\gamma\lambda-iw)M_d^\circ+\frac{i}{T_{sL}} \tag{4·31}$$

where the inequality (4·21) is automatically fulfilled. From eqs. (4·24) and (4·25) the shift and the broadening are given by

$$\Delta\omega=-Re\left[\frac{(\gamma\lambda-iw)M_s^\circ i/T_{sL}}{(\gamma\lambda-iw)M_d^\circ-i/T_{sL}}\right]=\frac{(T_{sd}/T_{sL})^2}{(1+T_{sd}/T_{ds})^2+(\gamma\lambda M_d^\circ T_{sd})^2}\gamma\lambda M_s^\circ \tag{4·32}$$

$$\frac{1}{T_{2eff}}=Im\left[\frac{(\gamma\lambda-iw)M_s^\circ i/T_{sL}}{(\gamma\lambda-iw)M_d^\circ-i/T_{sL}}\right]=\frac{(1+\gamma\lambda M_d^\circ T_{sd})^2T_{sd}/T_{sL}+(T_{sd}/T_{sL})^2}{(1+T_{sd}/T_{sL})^2+(\gamma\lambda M_d^\circ T_{sd})^2}\frac{1}{T_{ds}}. \tag{4·33}$$

In a similar way we can estimate the effective longitudinal relaxation time T_{1eff} by

assuming saturation. Under the stationary condition the left-hand sides of eqs. (4·12) and (4·13) are equated to be zero, and the deviation δM_d^z is solved as a term of H_1^2.

This becomes

$$\delta M_d^z = -\frac{(w_s M_s^\circ + w_d M_d^\circ)/T_{sd} + w_d M_d^\circ / T_{sL}}{w_d(1/T_{sL}+1/T_{sd})+1/T_{sL}T_{ds}} . \tag{4·34}$$

Here we assume that the effective power absorption of conduction electron spins from microwave is small compared with that of d-spins, i.e.

$$w_s \ll w_d ,$$

that the solution (4·34) is approximated to

$$M_d^z = -\frac{w_d T_{sL} T_{ds}(1/T_{sL}+1/T_{sd})}{1+w_d T_{sL} T_{ds}(1/T_{sL}+1/T_{sd})} M_d^\circ . \tag{4·35}$$

This is compared with the usual saturation formula for one spin system, which is written as

$$M_d^z = -\frac{wT_1}{1+wT_1} M_d^\circ .$$

Therefore

$$\frac{1}{T_{1eff}} = \frac{T_{sd}}{T_{sL}+T_{sd}} \frac{1}{T_{ds}} . \tag{4·36}$$

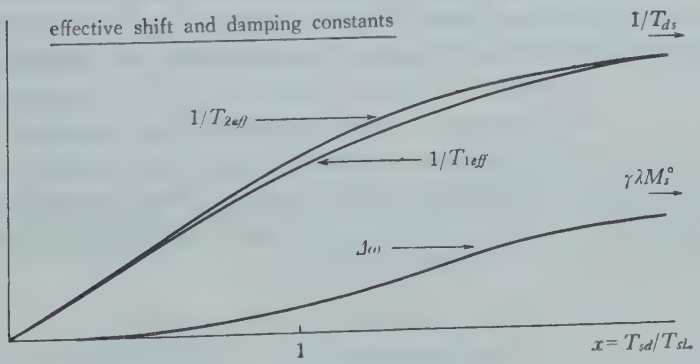

Fig. 3.

These results are graphically shown in Fig. 3 as a function of the parameter $x=T_{sd}/T_{sL}$.

§ 5. Discussion and summary

The results obtained in the preceding section contain two independent parameters $x=T_{sd}/T_{sL}$ and M_s°, which are unknown so long as we confine ourselves to the

phenomenological treatment. The most important question there is concerning the value of M_s° which determines the shift and the relaxation time in the shortest limit of T_{sL}. When M_s° assumes the value

$$\chi_p(H+\lambda M_d^\circ)\,, \tag{5·1}$$

then eq. (4·32) gives Kittel-Mitchell's shift (2·7) in the case $x\to\infty$, which contradicts Yosida's theory. In this case, further, the calculation of T_{ds} in the second order perturbation of the *s-d* exchange is no more valid because of the restriction of the detailed balance (4·4). For the consistency with that relation, we must expect

$$\frac{1}{T_{ds}}=\frac{2\pi}{\hbar}\frac{2n(E_f)}{N}\langle J^2\rangle_f\frac{n(E_f)}{N}kT(1+\lambda\chi_d)\,, \tag{5·2}$$

instead of the Korringa formula (4·15). Or, in the reversed sense, the second order perturbational theory in calculating the relaxation time requires the assumption

$$M_s^\circ=\chi_pH\,. \tag{5·3}$$

This leads to Yosida's shift in the case $x\to\infty$.

This ambiguity rises in the uncertainty of the question to what extent the *s-d* exchange field may be regarded as a fluctuating quantity and to what extent it actually contains the uniform field acting on conduction electron spins. The second order perturbational theory (4·14) and (4·15) assumes that the *s-d* exchange produces nothing but a fluctuation, while the equilibrium value (5·1) implies that it actually acts on conduction electron spins as a uniform field like the external field.

In the present paper we cannot answer to this question, because it requires more detailed calculations from the given Hamiltonians of the spin-lattice interaction as well as of the *s-d* exchange. Instead of this we here introduce an important idea to modify the phenomenological treatment which was proposed by Kasuya.[13] This asserts that, if the *s-d* exchange produces a uniform field steadily acting on conduction electron spins, then the spin-lattice relaxation of conduction electrons should occur as if the polarization M_s keeps its direction parallel the instantaneous field $\chi_p(H+\lambda M_d)$ (see Fig. 1). This corresponds to

case B.
$$M_s=M_s^\circ+\chi_p\lambda(M_d-M_d^\circ) \tag{5·4}$$

$$M_s^\circ=\chi_p(H+\lambda M_d^\circ)\,. \tag{5·5}$$

$$-D_+=i\begin{cases}\omega_\tau=-\gamma H_z+\dfrac{i}{T_{sL}}-(\gamma\lambda-i\omega)M_d^\circ & \omega_\beta=(\gamma-i\omega)M_s^\circ-i\left(\dfrac{\lambda\chi_p}{T_{sL}}\right)\\ \omega_\alpha=(\gamma\lambda-i\omega)M_d^\circ & \omega_\delta=-\gamma H_z-(\gamma\lambda-i\omega)M_s^\circ\end{cases}. \tag{5·6}$$

This leads to the results just same as (4·32), (4·33), and (4·35), where the parameter M_s° is replaced by χ_pH, i.e.

$$\Delta\omega = -\frac{(T_{sd}/T_{sL})^2}{(1+T_{sd}/T_{sL})^2+(\gamma\lambda M_d{}^\circ T_{sd})^2}\gamma\lambda\chi_p H \tag{4·32′}$$

$$\frac{1}{T_{2eff}} = \frac{(1+\gamma\lambda M_d{}^\circ T_{sd})^2 T_{sd}/T_{sL}+(T_{sd}/T_{sL})^2}{(1+T_{sd}/T_{sL})^2+(\gamma\lambda M_d{}^\circ T_{sd})^2}\frac{1}{T_{ds}} \tag{4·33′}$$

$$\frac{1}{T_{1eff}} = \frac{T_{sd}}{T_{sL}+T_{sd}}\frac{1}{T_{ds}} \tag{4·35′}$$

with the results of the second order perturbation theory for the relaxation time

$$\frac{1}{T_{sd}} = \frac{2\pi}{\hbar}\frac{2n(E_f)}{N}\langle J^2\rangle_f \frac{N_0}{N}\frac{2}{3}S(S+1) \tag{4·14}$$

$$\frac{1}{T_{ds}} = \frac{2\pi}{\hbar}\frac{2n(E_f)}{N}\langle J^2\rangle_f \frac{n(E_f)}{N}kT\,. \tag{4·15}$$

Here we summarize the results:

(1) For the dynamical problem such as resonance there are two entirely different pictures in treating the spin-system of conduction electrons; i.e. adiabatic and isothermal pictures. In the adiabatic limit we do not expect any shift and any broadening of the *d*-spin resonance line through the *s-d* interaction. On the other hand, in the isothermal limit, we have two different predictions, i.e. Kittel-Mitchell's shift and Yosida's shift. As far as we know, there is no reliable answer to the question which shift is actually correct to be expected.

(2) From the phenomenological point of view based on the thermodynamical consideration it is possible to make an interpolation between two idealized pictures, i.e. between the adiabatic picture and the isothermal picture of either K-M theory and Yosida's theory. In any case the criterion of the isothermal picture is given by

$$T_{sL} < T_{sd}{}^*\,. \tag{5·7}$$

This means that the *s-d* interaction is almost ineffective on the shift as well as on the line broadening, unless the above condition is fulfilled. For instance, the time scale of Cu-Mn dilute alloys (1% Mn in Cu) can be written as follows:

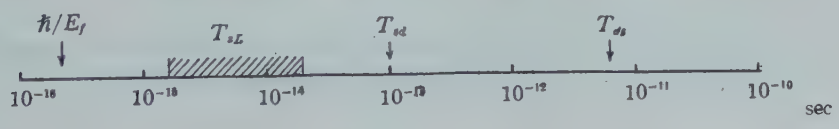

Fig. 4.

One sees that the length of T_{sL} which well satisfies the isothermal condition is fallen into a quite limited range.

(3) If one assumes that the equilibrium value of the total magnetization $M_s{}^\circ$ is given by

* Numerically this criterion does not depend on the choice of $M_s{}^\circ$.

$$M_s^\circ = \chi_p(H+\lambda M_d^\circ) \tag{5·8}$$

then the difference between K-M theory and Yosida's theory consists in the direction towards which M_s is relaxing in every moment. In both cases the calculation of the spin-lattice relaxation time based on the usual second order perturbation theory is wrong, and therefore neither of them is more convincing. To obtain a more definite conclusion it is necessary to estimate the higher order effect of the *s-d* interaction.

In this connection it is noted that Yosida's theory implies that the magnetization of conduction electrons is spatially not uniform. This is far from the simple expression (5·8). From his picture M_s° is rather only the Pauli paramagnetism $\chi_p H$ almost everywhere just except the vicinity of each *d*-core, and the dynamical behavior of the localized part of the magnetization might be understood from the fact that conduction electrons adiabatically follow the *d*-core spins. The greatest trouble is, therefore, how to reconcile the relaxation theory with that effect of localization.

(4) We make a discussion from experimental points of view. The magnitude of the *s-d* exchange integral previously anticipated by Berkeley group for Cu-Mn dilute alloy was about 1/10 or so of the one for free M_n^+ ion because of the screening effect. This has been modified by the precise analysis of the resistivity anomaly and of the line width of the nuclear magnetic resonance. As was shown by Yosida and Behringer, the *s-d* exchange integral in the alloy is consistent with the free ion value which is equal to

$$J_0 = 3.5\times10^{-13}\ \text{erg}. \tag{5·9}$$

This is also confirmed by Sugawara[5].

The corresponding relaxation time T_{sd} is generally very short, which may be expected as

$$T_{sd} = 2\times10^{-13} f\ \text{sec}, \tag{5·10}$$

where f is the atomic per cent of manganese in copper. Accordingly, the isothermal condition

$$T_{sL} < T_{sd}$$

is considerably strict even for the specimen of the lowest concentration of manganese used in the experiment. If this limiting case is valid, the shift and T_{2eff} are estimated as

$$\Delta H/H = \lambda\chi_p = 3.3\times10^{-2} \quad \text{by Yosida's theory} \tag{5·11}$$

$$= \lambda\chi_p(1+\lambda\chi_d) \cong 3.4\times f/T \quad \text{by K-M theory,} \tag{5·12}$$

where T is the absolute temperature.

$$T_{2eff} = T_{ds} = 5\times10^{-10}\,T\ \text{sec.} \quad \text{by Korringa formula}$$

$$(=7\times10^{-12}\ \text{sec, at } T=77^\circ\text{K}) \tag{5·13}$$

$$=5\times10^{-10}\,T^{-1}(1+\lambda\chi_d)^{-1}=5\times10^{-12}f^{-1} \qquad (5\cdot14)$$

by assuming the K-M theory.

The shift (5·11) is several times as large as the observed shift for Cu-Mn alloy at paramagnetic region, even though about 1% experimental error is assumed. The effective relaxation time (5·13) is based on the formula (4·15). This is independent of the concentration of manganese. The observed value of T_{2eff} is about $1\sim5\times10^{-10}$ sec. at 77°K, and is several ten times as long as the estimated value (5·13). It is even longer than the value (5·14) expected from the formula (5·2) which is consistent with the K-M theory. On the other hand, if T_{sL} is longer than T_{sd}, we may expect that the *s-d* interaction is not fully effective on these observable quantities. For instance, when T_{sL} is assumed to be 10 times as long as T_{sd}, eqs. (4·32) and (4·33) tell us that

$$\frac{\Delta H}{H}\simeq\frac{1}{100}\,\frac{\lambda M_s^{\circ}}{H} \qquad (5\cdot15)$$

$$T_{2eff}\simeq10T_{ds}\,. \qquad (5\cdot16)$$

The experimental results are not quite precise so that we cannot determine whether the K-M theory or Yosida's theory suits better. But, in any way, there are considerable discrepancies between the observed data and the predicted values from these theories. In this connection it is interesting to note that the observed line width at 77°K is slightly concentration dependent, which has a tendency that the specimen of the higher concentration shows the narrower line. This may be expected from our formula (4·33) or Fig. 3, provided that T_{sL} is longer than T_{sd} and does not depend so much on the concentration of manganese.

(5) The conduction electron spin resonance in metallic copper has not yet been observed, and so the spin-lattice relaxation time of the conduction electrons in copper is regarded as considerably short compared with the one in alkali metals. Elliott[17] has discussed on the basis of the spin-orbit coupling theory that T_{sL} in noble metal is about 10^{-10} sec at room temperature, which seems to be still longer than the actual length of T_{sL} in copper. Further, there is a possibility that the actual length of T_{sL} in the alloy is more or less shortened from that in pure copper.

It seems to us, however, that the length of T_{sL} in the alloy less than 10^{-13} sec at 77°K is too short, and is rather unrealistic. If this might be the case, much larger effect must be expected on the electrical resistivity. The plausible length of T_{sL} will be

$$T_{sL}\sim10^{-12}\text{ sec at }77^{\circ}\text{K}\,.$$

Acknowledgement

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Expansion Formulae of the Free Energy and Distribution Functions in Powers of the One Particle Distribution Function

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A new method is proposed which enables one to convert the formulae for a uniform multi-component system to those for a non-uniform one. As an application of the method, the expansion formulae of the free energy, potentials of average force and distribution functions for a non-uniform multi-component system are derived in a simple and systematic way. The formulae are obtained both for a continuous and for a lattice system. The formula obtained for the free energy for a continuous system coincides with that given by Arinstein, and the first few terms of the formulae for the free energy and the pair distribution function for a lattice system are identical to those given by Yvon.

A procedure to convert the formulae for a continuous system to those for a lattice system and *vice versa* is given.

The phase change between solid, liquid and gas will be discussed on the basis of the expansion formulae obtained here, in a separate paper.

§ 1. Introduction

The expansion formulae of the free energy and distribution functions in powers of the one particle distribution function are long and well-known for a uniform one-component continuous system.*[1,2] The corresponding formulae for a uniform one-component lattice system* will be obtained by the same method, as has been noticed by Fuchs[3]. For a uniform multi-component continuous system, the expansion formula of the free energy has long been known[5] and that of the distribution functions was derived lately by Meeron[6]. It seems, however, to be improper to apply the ordinary virial expansion formulae to non-uniform systems such as the solid state, the system of two phases in equilibrium, the ordered states of solid solutions or antiferromagnetic Ising system, etc. In such cases, use of the expansion formulae in powers of the one particle distribution function will be more suitable. In fact, for a non-uniform lattice system, the first few terms of the expansion of the free energy and the pair distribution function in powers of the one

* By a continuous system we mean a system which is composed of particles whose coordinates are continuous variables: the examples are ordinary fluids and solids. In contrast, by a lattice system we mean a system which is composed of particles whose coordinates are discrete variables, only taking the coordinates of lattice sites of a lattice: the examples are a classical lattice gas, Ising system and models for solid solutions[3,4].

particle distribution function have been obtained by Yvon[4]. For a non-uniform continuous system, the expansion formula of the free energy was lately given by Arinstein[7] in full terms, though without detailed derivation. The derivation of the expansion formulae will be much simpler and more direct for uniform systems than for non-uniform systems, and the expansion formulae for uniform cases have already been derived, as mentiond above. Then it will be interesting if we can obtain the formulae for the non-uniform systems by such a method as starting with the formulae obtained for the uniform systems, without any effort to repeat or generalize the derivation for the uniform cases. In this paper, it will be shown that such a method exists in fact: *The formulae for a non-uniform system* will be shown to be obtained by applying *the formulae for a uniform system* to *a uniform system properly constructed*. The method will be illustrated for the cases of the expansion formulae of the free energy, potentials of average force and distribution functions in powers of the one particle distribution function.

In § 2 will be given a procedure to convert the formulae for a lattice system to those for a continuous system and *vice versa*. By the aid of the procedure introduced in § 2, the expansion formulae of the free energy, potentials of average force and distribution functions for multi-component continuous and lattice systems will be obtained in § 3 and § 4. The procedure used in § 3 and § 4 is merely the application of the known virial expansion formulae for a uniform multi-component lattice system to a lattice system properly constructed.

§ 2. Relations between quantities for a continuous system and those for a lattice system

In order to deal with both continuous and lattice systems at the same time, it is convenient to introduce the transformation procedure which enables one to transform the quantities for a continuous system to those for a lattice system and *vice versa*. In this section, we shall discuss this transformation procedure, which is to be used in the following sections.

For this purpose, we consider a continuous system of volume V, which consists of L cells of equal volume $\Delta = V/L$ and whose potential energy is dependent only on *which cells are occupied*, independently of the detailed positions in the cells. So if we denote the coordinate and species of the ith particle by $\boldsymbol{r}_i$ and ν_i and the set $(\boldsymbol{r}_i, \nu_i)$ by $\boldsymbol{x}_i$, and the coordinate of the center of the cell including $\boldsymbol{r}_i$ by $\boldsymbol{R}_i$ and the set $(\boldsymbol{R}_i, \nu_i)$ by $\boldsymbol{X}_i$, then the potential energy of our system above stated satisfies

$$\Phi(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N) = \Phi(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N). \tag{2·1}$$

The configurational part of the partition function of our system is

$$Z_C = \iint\cdots\int d\boldsymbol{r}_1 \cdots d\boldsymbol{r}_N \exp[-\Phi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N)/kT]$$

$$= \Delta^N \cdot \sum_{R_1=1}^{L} \cdots \sum_{R_N=1}^{L} \exp[-\Phi(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N)/kT] = \Delta^N \cdot Z_L,$$

where $\sum_{R=1}^{L}$ means that the sum is to be taken over all coordinates of the center of the cells. Now we assume that the centers of the cells constitute a lattice, then Z_L is the partition function of the lattice system, of L lattice sites, of which the potential energy is given by $\Phi(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N)$.

The n particle distribution function of our system is

$$\begin{aligned}\rho_C^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) &= \iint \cdots \int d\boldsymbol{r}_{n+1} \cdots d\boldsymbol{r}_N \exp[-\Phi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N)/kT]/Z_C \\ &= \Delta^{N-n} \cdot \sum_{R_{n+1}=1}^{L} \cdots \sum_{R_N=1}^{L} \exp[-\Phi(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N)/kT]/\Delta^N Z_L \\ &= \rho_L^{(n)}(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n)/\Delta^n ;\end{aligned}$$

$\rho_L^{(n)}(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n)$ is the n particle distribution function of a lattice system.

As the result, we can transform the formulae for the continuous system to those for the lattice system by the procedure of replacements:

$$\boldsymbol{r}_i \longrightarrow \boldsymbol{R}_i, \qquad \int d\boldsymbol{r}_i \longrightarrow \Delta \cdot \sum_{R_i=1}^{L}, \tag{2·2}$$

then

$$Z_C \longrightarrow \Delta^N \cdot Z_L, \quad \rho_C^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) \longrightarrow \rho_L^{(n)}(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n)/\Delta^n. \tag{2·3}$$

Reversely, if we consider the continuous system as a lattice system with infinitesimal Δ, then we can transform the formulae for a lattice system to those for a continuous system by the procedure reverse to that just introduced.

§ 3. Free energy

The systems considered in the following are those of which the potential energy is given by the sum of potential energies between pair of particles:

$$\begin{aligned}&\Phi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N) = \sum_{N \geq i > j \geq 1} \phi(\boldsymbol{x}_i, \boldsymbol{x}_j), \\ &\phi(\boldsymbol{x}_i, \boldsymbol{x}_j) = \phi_{\nu_i \nu_j}(\boldsymbol{r}_i, \boldsymbol{r}_j)\end{aligned} \tag{3·1}$$

for continuous systems and

$$\begin{aligned}&\Phi(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N) = \sum_{N \geq i > j \geq 1} \phi(\boldsymbol{X}_i, \boldsymbol{X}_j), \\ &\phi(\boldsymbol{X}_i, \boldsymbol{X}_j) = \phi_{\nu_i \nu_j}(\boldsymbol{R}_i, \boldsymbol{R}_j)\end{aligned} \tag{3·2}$$

for lattice systems.

Let us start with the virial expansion formula of the free energy A_C for a uniform multi-component continuous system of volume V and temperature T,

composed of N particles of which N_α, N_β,$\cdots$ and N_τ particles are of the species α, β,$\cdots$ and τ, respectively:[5]

$$\frac{A_C}{kT}=V\sum_{\nu=\alpha}^{\tau}\rho_\nu\ln\left(\frac{h^2}{2\pi m_\nu kT}\right)^{3/2}\frac{\rho_\nu}{e}-\sum_{n_\alpha=0}^{\infty}\cdots\sum_{n_\tau=0}^{\infty}{}'\cdot\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu^{n_\nu}}{n_\nu!}\int\int\cdots\int d\boldsymbol{r}_1\cdots d\boldsymbol{r}_n\sum_{n\geq i>j\geq 1}^{(A)}\prod f_{ij}$$
$$(n\equiv n_\alpha+\cdots+n_\tau) \tag{3·3}$$

with

$$\begin{aligned}f_{ij}&=f(\boldsymbol{x}_i,\boldsymbol{x}_j)=f_{\nu_i\nu_j}(\boldsymbol{r}_i,\boldsymbol{r}_j)\\&=\exp[-\phi(\boldsymbol{x}_i,\boldsymbol{x}_j)/kT]-1\\&=\exp[-\phi_{\nu_i\nu_j}(\boldsymbol{r}_i,\boldsymbol{r}_j)/kT]-1\end{aligned} \tag{3·4}$$

where $\rho_\nu=N_\nu/V$. Here, A on $\sum^{(A)}$ implies the specification by Mayer[1,8] that the sum is to be taken over all products which are more than singly connected; strictly speaking, the term of $n=2$ is the exception where f_{ij} is taken.

The corresponding formula of the free energy A_L for a uniform multi-component lattice system, of L lattice sites, is obtained from (3·3) by an application of the procedure (2·2) and (2·3) as

$$\begin{aligned}\frac{A_L}{kT}&=-\ln Z_L=N\ln\Delta-\ln Z_C=N\ln\Delta+\frac{A_C}{kT}-\sum_{\nu=\alpha}^{\tau}N_\nu\ln\left(\frac{h^2}{2\pi m_\nu kT}\right)^{3/2}\\&=L\sum_{\nu=\alpha}^{\tau}\rho_\nu\ln\frac{\rho_\nu}{e}-\sum_{n_\alpha=0}^{\infty}\cdots\sum_{n_\tau=0}^{\infty}{}'\cdot\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu^{n_\nu}}{n_\nu!}\cdot\sum_{R_1=1}^{L}\cdots\sum_{R_n=1}^{L}\sum_{n\geq i>j\geq 1}^{(A)}\prod F_{ij}\end{aligned} \tag{3·5}$$
$$(n\equiv n_\alpha+\cdots+n_\tau)$$

with

$$\begin{aligned}F_{ij}&=F(\boldsymbol{X}_i,\boldsymbol{X}_j)=F_{\nu_i\nu_j}(\boldsymbol{R}_i,\boldsymbol{R}_j)\\&=\exp[-\phi(\boldsymbol{X}_i,\boldsymbol{X}_j)/kT]-1\\&=\exp[-\phi_{\nu_i\nu_j}(\boldsymbol{R}_i,\boldsymbol{R}_j)/kT]-1\end{aligned} \tag{3·6}$$

where $\rho_\nu=N_\nu/L$.

Now, we turn to the derivation of the expansion formulae of the free energy for a lattice system of L lattice sites, in which the one particle distribution function $\rho(\boldsymbol{X})=\rho_\nu(\boldsymbol{R})$ ($\nu=\alpha,\cdots,\tau$; $\boldsymbol{R}=1,\cdots,L$) is prescribed. In order to deal with this system in the following, we shall regard the particles of the same species on the different lattice sites as the particles of different species, and take our system as if it were a lattice site composed of particles of the species 1α, 1β,$\cdots$, 1τ, 2α,$\cdots$, $L\alpha$,$\cdots$, $L\tau$ (cf. Fig. 1), and then construct a uniform multi-component lattice system of M lattice sites, composed of $MN=M\sum_{R=1}^{L}\sum_{\nu=\alpha}^{\tau}\rho_\nu(\boldsymbol{R})$ particles, of which $M\rho_\nu(\boldsymbol{R})$ are of the species $\boldsymbol{R}\nu$ ($\boldsymbol{R}=1,\cdots,L$; $\nu=\alpha,\cdots,\tau$) (cf. Fig. 2). Then, for the lattice ensemble constructed in this way, the expansion formula for the uniform

system is applicable and we shall be able to obtain the corresponding formula for the non-uniform lattice system.

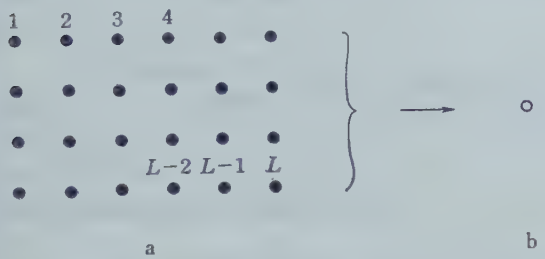

Fig. 1. The system of L lattice sites under consideration. ●'s are occupied by particles of the species, $\alpha, \beta, \cdots, \tau$. The one particle distribution function is $\rho_\nu(\boldsymbol{R})$ at lattice site $\boldsymbol{R}$ ($\boldsymbol{R}=1, \cdots, L$).

This *system* can be taken as a *lattice site* ○, occupied by particles of the species, $1\alpha, 1\beta, \cdots, 1\tau, \cdots, L\alpha, \cdots, L\tau$. The density is $\rho_{R\nu}=\rho_\nu(\boldsymbol{R})$.

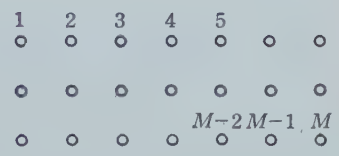

Fig. 2. The ensemble of M systems of Fig. 1. ○ denotes a system of Fig. 1.

This *ensemble* can be taken as a *uniform lattice system*, of M lattice sites ○, composed of the particles of the species, $1\alpha, \cdots, 1\tau, \cdots, L\alpha, \cdots, L\tau$. The density is $\rho_{\mathcal{R}\nu}=\rho_\nu(\boldsymbol{R})$ for all lattice sites $\mathcal{R}$ ($\mathcal{R}=1, \cdots, M$).

Since the free energy is an extensive quantity, the free energy of the ensemble (system) of Fig. 2 is M times the free energy A_L of the system of Fig. 1a. Therefore, the application of (3·5) to the system given in Fig. 2 leads to

$$\frac{MA_L}{kT}=M\sum_{R=1}^{L}\sum_{\nu=\alpha}^{\tau}\rho_{R\nu}\ln\frac{\rho_{R\nu}}{e}$$
$$-\sum_{n_{1\alpha}=0}^{\infty}\cdots\sum_{n_{1\tau}=0}^{\infty}\cdots\sum_{n_{L\alpha}=0}^{\infty}\cdots\sum_{n_{L\tau}=0}^{\infty}\cdot\prod_{R=1}^{L}\prod_{\nu=\alpha}^{\tau}\frac{\rho_{R\nu}^{n_{R\nu}}}{n_{R\nu}!}\cdot\sum_{\mathcal{R}_1=1}^{M}\cdots\sum_{\mathcal{R}_n=1}^{M}\sum_{n\geq i>j\geq 1}^{(A)}\prod\mathcal{F}_{ij},$$
$$(n\equiv\textstyle\sum_{R=1}^{L}\sum_{\nu=\alpha}^{\tau}n_{R\nu}) \qquad (3\cdot 7)$$

with

$$\mathcal{F}_{ij}=\mathcal{F}(\mathcal{K}_i, \mathcal{K}_j)=\exp[-\phi(\mathcal{K}_i, \mathcal{K}_j)/kT]-1,$$
$$\phi(\mathcal{K}_i, \mathcal{K}_j)=\phi(\boldsymbol{X}_i, \boldsymbol{X}_j)\cdot\delta_{\mathcal{R}_i,\mathcal{R}_j}, \qquad (3\cdot 8)$$

where $\mathcal{R}_i$ denotes a lattice site (system) in Fig. 2 occupied by the ith particle and $\mathcal{K}_i$ the set $(\mathcal{R}_i, \boldsymbol{X}_i)=(\mathcal{R}_i, \boldsymbol{R}_i, \nu_i)$. The fact that $\mathcal{F}_{ij}$ is zero if $\mathcal{R}_i\neq\mathcal{R}_j$ reduces (3·7) to

$$\frac{A_L}{kT}=\sum_{R=1}^{L}\sum_{\nu=\alpha}^{\tau}\rho_\nu(\boldsymbol{R})\ln\frac{\rho_\nu(\boldsymbol{R})}{e}$$
$$-\sum_{n_{1\alpha}=0}^{\infty}\cdots\sum_{n_{1\tau}=0}^{\infty}\cdots\sum_{n_{L\alpha}=0}^{\infty}\cdots\sum_{n_{L\tau}=0}^{\infty}\cdot\prod_{R=1}^{L}\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu(\boldsymbol{R})^{n_{R\nu}}}{n_{R\nu}!}\sum_{n\geq i>j\geq 1}^{(A)}\prod F_{ij}, \qquad (3\cdot 9)$$
$$(n\equiv\textstyle\sum_{R=1}^{L}\sum_{\nu=\alpha}^{\tau}n_{R\nu})$$

where $\rho_{R\nu}=\rho_\nu(\boldsymbol{R})$ is used. Here, noticing that the interchange between $i\nu$ and $j\nu$

does not change the sum $\sum^{(A)} \Pi F_{ij}$, we have

$$\frac{A_L}{kT} = \sum_{\nu=\alpha}^{\tau} \sum_{R=1}^{L} \rho_\nu(\boldsymbol{R}) \ln \frac{\rho_\nu(\boldsymbol{R})}{e}$$

$$- \sum_{n_\alpha=0}^{\infty} \cdots \sum_{n_\tau=0}^{\infty} \cdot \prod_{\nu=\alpha}^{\tau} \frac{1}{n_\nu!} \cdot \sum_{\boldsymbol{R}_1=1}^{L} \cdots \sum_{\boldsymbol{R}_n=1}^{L} \rho_{\nu_1}(\boldsymbol{R}_1) \cdots \rho_{\nu_n}(\boldsymbol{R}_n) \cdot \sum_{n \geq i > j \geq 1}^{(A)} \Pi F_{ij}$$

$$(n \equiv n_\alpha + \cdots + n_\tau) \qquad (3 \cdot 10)$$

where $\nu_1 = \nu_2 = \cdots = \nu_{n_\alpha} = \alpha$, $\nu_{n_\alpha+1} = \cdots = \nu_{n_\alpha+n_\beta} = \beta, \cdots$; which is the resultant expansion formula for a lattice system.

The analogous formula for a continuous system can be obtained by applying the procedure (2·2) and (2·3) in the reverse direction to (3·10), and we have

$$\frac{A_C}{kT} = \sum_{\nu=\alpha}^{\tau} \int d\boldsymbol{r} \rho_\nu(\boldsymbol{r}) \ln \left(\frac{h^2}{2\pi m_\nu kT} \right)^{3/2} \frac{\rho_\nu(\boldsymbol{r})}{e}$$

$$- \sum_{n_\alpha=0}^{\infty} \cdots \sum_{n_\tau=0}^{\infty} \cdot \prod_{\nu=\alpha}^{\tau} \frac{1}{n_\nu!} \cdot \iint \cdots \int d\boldsymbol{r}_1 \cdots d\boldsymbol{r}_n \rho_{\nu_1}(\boldsymbol{r}_1) \cdots \rho_{\nu_n}(\boldsymbol{r}_n) \sum_{n \geq i > j \geq 1}^{(A)} \Pi f_{ij} .$$

$$(n \equiv n_\alpha + \cdots + n_\tau) \qquad (3 \cdot 11)$$

In the above, we have taken two particles of the same species at different positions as if they were of different species.* For systems in thermal equilibrium, the condition of the identity of the particles is described by the equality of the chemical potentials, which means that they can be connected to the same particle bath without change in the thermodynamic state of the system. This condition for a lattice system is

$$\frac{\delta}{\delta \rho_\nu^{(1)}(\boldsymbol{R})} \frac{A_L}{kT} = \frac{\mu_\nu}{kT} = \text{a value for all } \boldsymbol{R} ; \qquad (3 \cdot 12)$$

μ_ν is the chemical potential of the particles of the species ν and is to be determined by

$$N_\nu = \sum_{R=1}^{L} \rho_\nu(\boldsymbol{R}), \qquad (\nu = \alpha, \cdots, \tau). \qquad (3 \cdot 13)$$

For a continuous system, this condition is

$$\frac{\delta}{\delta \rho_\nu^{(1)}(\boldsymbol{r})} \frac{A_C}{kT} = \frac{\mu_\nu}{kT} = \text{a value for all } \boldsymbol{r} \text{ in } V, \qquad (3 \cdot 14)$$

and μ_ν is to be determined by

$$N_\nu = \int d\boldsymbol{r} \rho_\nu(\boldsymbol{r}), \quad (\nu = \alpha, \cdots, \tau). \qquad (3 \cdot 15)$$

When we have $\rho_\nu^{(1)}(\boldsymbol{R})$ or $\rho_\nu^{(1)}(\boldsymbol{r})$ prescribed, it will satisfy (3·12) or (3·14).

* What used in eq. (3·10) is only the indifference of the potential energy.

In turn, if we wish to treat a non-uniform system of given V or L, T and N_ν's, by starting with (3·10) or (3·11), then (3·12) with (3·13) or (3·14) with (3·15) is the condition of the thermal equilibrium and determines the values of $\rho_\nu^{(1)}(\boldsymbol{R})$ or $\rho_\nu^{(1)}(\boldsymbol{r})$.*

It should be noted, here, that equations (3·12) and (3·14), describing the condition of the thermal equilibrium, are nothing but the extremum condition of the free energy (3·10), and (3·11) for the variations by $\rho_\nu(\boldsymbol{R})$ and $\rho_\nu(\boldsymbol{r})$, respectively, under the restriction, (3·15) and (3·17), for the number of particles in the system.

§ 4. Distribution functions and potentials of average force

For the case of the distribution functions and potentials of average force, we will use the procedure used in the previous section for the case of the free energy. That is, we apply the virial expansion formula of the distribution functions or potentials of average force to the system as constructed in Fig. 2 which is the ensemble of the systems of Fig. 1, the system under consideration. Here, we will treat the expansion of the potentials of average force. As to the expansion formula of the distribution functions, we will only give the results.

The potential of average force of n particles for a continuous system is defined by

$$W^{(n)}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)=-kT\ln[\rho^{(n)}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)/\rho^{(1)}(\boldsymbol{x}_1)\cdots\rho^{(1)}(\boldsymbol{x}_n)] \tag{4·1}$$

in terms of the distribution functions of n and 1 particles $\rho^{(n)}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)$ and $\rho^{(1)}(\boldsymbol{x})=\rho(\boldsymbol{x})=\rho_\nu(\boldsymbol{r})$. For a lattice system, it is

$$W^{(n)}(\boldsymbol{X}_1,\cdots,\boldsymbol{X}_n)=-kT\ln[\rho^{(n)}(\boldsymbol{X}_1,\cdots,\boldsymbol{X}_n)/\rho^{(1)}(\boldsymbol{X}_1)\cdots\rho^{(1)}(\boldsymbol{X}_n)] \tag{4·2}$$

where $\rho^{(n)}(\boldsymbol{X}_1,\cdots,\boldsymbol{X}_n)$ and $\rho^{(1)}(\boldsymbol{X})=\rho(\boldsymbol{X})=\rho_\nu(\boldsymbol{R})$ are the n and 1 particle distribution functions, respectively.

The virial expansion formula of the potential of the average force of n particles is given by[6,9]

$$-\frac{W^{(n)}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)}{kT}=-\sum_{n\geq i>j\geq 1}\frac{\phi(\boldsymbol{x}_i,\boldsymbol{x}_j)}{kT}+\sum_{s_\alpha=0}^{\infty}\cdots\sum_{s_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu^{s_\nu}}{s_\nu!}$$

$$\cdot\iint\cdots\int d\boldsymbol{r}_{n+1}\cdots d\boldsymbol{r}_{n+s}\sum_{\substack{n+s\geq i>j\geq n+1\\ n+s\geq k\geq n+1\\ n\geq\kappa\geq 1}}^{(W)}\prod f_{ij}f_{k\kappa}\,. \tag{4·3}$$

$$(s=s_\alpha+\cdots+s_\tau)$$

Here, W on $\sum^{(W)}$ implies that the sum is to be taken over all products for which each particle of the set $\{\boldsymbol{x}_{n+1},\cdots\boldsymbol{x}_{n+s}\}$ is connected to at least two of $\{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n\}$ by

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* Compare with the situation for a uniform systems where we could start directly with (3·5) or (3·3): (3·5) or (3·3) is written in terms of the thermodynamical quantities, V or L, T and N_ν's, alone, while it is not the case with (3·10) and (3·11).

an independent path and also for which the particles $\{\boldsymbol{x}_{n+1},\cdots,\boldsymbol{x}_{n+s}\}$ are connected independently of the particles $\{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n\}$.

The virial expansion formula for a lattice system is obtained by an application of the procedure (2·2) and (2·3), as

$$-\frac{W^{(n)}(\boldsymbol{X}_1,\cdots,\boldsymbol{X}_n)}{kT}=-\sum_{n\geq i>j\geq 1}\frac{\phi(\boldsymbol{X}_i,\boldsymbol{X}_j)}{kT}+\sum_{s_\alpha=0}^{\infty}\cdots\sum_{s_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{\rho_\nu^{s_\nu}}{s_\nu!}$$

$$\cdot\sum_{\boldsymbol{R}_{n+1}=1}^{\infty}\cdots\sum_{\boldsymbol{R}_{n+s}=1}^{\infty}\sum_{\substack{n+s\geq i>j\geq n+1\\ n+s\geq k\geq n+1\\ n\geq\kappa\geq 1}}^{(W)}\prod\quad F_{ij}\,F_{k\kappa}\,. \tag{4·4}$$

$$(s\equiv s_\alpha+\cdots+s_\tau)$$

We apply this (4·4) to the system of Fig. 2——the ensemble of the systems of Fig. 1——, then we have

$$-\frac{W^{(n)}(\mathcal{K}_1,\cdots,\mathcal{K}_n)}{kT}=-\sum_{n\geq i>j\geq 1}\frac{\phi(\mathcal{K}_i,\mathcal{K}_j)}{kT}+\sum_{s_{1\alpha}=0}^{\infty}\cdots\sum_{s_{1\tau}=0}^{\infty}\cdots\sum_{s_{L\alpha}=0}^{\infty}\cdots\sum_{s_{L\tau}=0}^{\infty}\cdot\prod_{R=1}^{L}\prod_{\nu=\alpha}^{\tau}\frac{\rho_{R\nu}{}^{n_{R\nu}}}{n_{R\nu}!}$$

$$\cdot\sum_{\mathcal{R}_{n+1}=1}^{M}\cdots\sum_{\mathcal{R}_{n+s}=1}^{M}\sum_{\substack{n+s\geq i>j\geq n+1\\ n+s\geq k\geq n+1\\ n\geq\kappa\geq 1}}^{(W)}\prod\quad\mathcal{F}_{ij}\,\mathcal{F}_{k\kappa}\,. \tag{4·5}$$

$$(s\equiv\textstyle\sum_{R=1}^{L}\sum_{\nu=\alpha}^{\tau}s_{R\nu})$$

Since $\phi(\mathcal{K}_i,\mathcal{K}_j)$ and $\mathcal{F}_{ij}$ are zero when $\mathcal{R}_i\neq\mathcal{R}_j$, (4·5) has the form of a sum of contributions from lattice sites (systems) $\mathcal{R}(\mathcal{R}=1,\cdots,M)$; if the set $(\mathcal{K}_1,\cdots,\mathcal{K}_m)$ is the set of the total of coordinates for which $\mathcal{R}_i$ is equal to $\mathcal{R}$ among $(\mathcal{K}_1,\cdots,\mathcal{K}_n)$, then the contribution from the lattice site (system) $\mathcal{R}$ is

$$-\frac{W^{(m)}(\boldsymbol{X}_1,\cdots,\boldsymbol{X}_m)}{kT}=-\sum_{m\geq i>j\geq 1}\frac{\phi(\boldsymbol{X}_i,\boldsymbol{X}_j)}{kT}+\sum_{s_{1\alpha}=0}^{\infty}\cdots\sum_{s_{1\tau}=0}^{\infty}\cdots\sum_{s_{L\alpha}=0}^{\infty}\cdots\sum_{s_{L\tau}=0}^{\infty}{}'\cdot\prod_{R=1}^{L}\prod_{\nu=\alpha}^{\tau}\frac{\rho_{R\nu}{}^{n_{R\nu}}}{n_{R\nu}!}$$

$$\cdot\sum_{\substack{m+s\geq i>j\geq m+1\\ m+s\geq k\geq m+1\\ m\geq\kappa\geq 1}}^{(W)}\prod\quad F_{ij}\,F_{k\kappa}\,. \tag{4·6}$$

$$(s\equiv\textstyle\sum_{R=1}^{L}\sum_{\nu=\alpha}^{\tau}s_{R\nu})$$

Noticing that the interchange between $i\nu$ and $j\nu$, $(\nu=\alpha,\cdots,\tau)$, does not change the sum $\sum^{(W)}\Pi F_{ij}F_{k\kappa}$, we have

$$-\frac{W^{(m)}(\boldsymbol{X}_1,\cdots,\boldsymbol{X}_m)}{kT}=-\sum_{m\geq i>j\geq 1}\frac{\phi(\boldsymbol{X}_i,\boldsymbol{X}_j)}{kT}+\sum_{s_\alpha=0}^{\infty}\cdots\sum_{s_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{1}{n_\nu!}$$

$$\cdot\sum_{\boldsymbol{R}_{m+1}=1}^{L}\cdots\sum_{\boldsymbol{R}_{m+s}=1}^{L}\rho_{\nu_{m+1}}(\boldsymbol{R}_{m+1})\cdots\rho_{\nu_{m+s}}(\boldsymbol{R}_{m+s})\sum_{\substack{m+s\geq i>j\geq m+1\\ m+s\geq k\geq m+1\\ m\geq\kappa\geq 1}}^{(W)}\prod F_{ij}\,F_{k\kappa}, \tag{4·7}$$

$$(s\equiv s_\alpha+\cdots+s_\tau)$$

where $\nu_{m+1}=\cdots=\nu_{m+s_\alpha}=\alpha,\ \nu_{m+s_\alpha+1}=\cdots=\nu_{m+s_\alpha+s_\beta}=\beta,\cdots$.

An application of the procedure (2·2) and (2·3) in the reverse direction gives the formula for a continuous system

$$-\frac{W^{(m)}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_m)}{kT}=-\sum_{m\geq i>j\geq 1}\frac{\phi(\boldsymbol{x}_i,\boldsymbol{x}_j)}{kT}+\sum_{s_\alpha=0}^{\infty}\cdots\sum_{s_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{1}{n_\nu!}$$
$$\cdot\iint\cdots\int d\boldsymbol{r}_{m+1}\cdots d\boldsymbol{r}_{m+s}\,\rho_{\nu_{m+1}}(\boldsymbol{r}_{m+1})\cdots\rho_{\nu_{m+s}}(\boldsymbol{r}_{m+s})\sum_{\substack{m+s\geq i>j\geq m+1\\ m+s\geq k\geq m+1\\ m\geq\kappa\geq 1}}^{(W)}\prod f_{ij}f_{k\kappa}$$
$$(s\equiv s_\alpha+\cdots+s_\tau) \tag{4·8}$$

(4·7) and (4·8) are the expansion formulae we have been trying to get.

The expansion formulae for the distribution functions can be obtained by the same procedure by starting with the virial expansion formulae given by Meeron[6]. The results obtained are

$$\frac{\rho^{(m)}(\boldsymbol{X}_1,\cdots,\boldsymbol{X}_m)}{\rho^{(1)}(\boldsymbol{X}_1)\cdots\rho^{(1)}(\boldsymbol{X}_m)}=\exp\left[-\sum_{m\geq i>j\geq 1}\frac{\phi(\boldsymbol{X}_i,\boldsymbol{X}_j)}{kT}\right]\cdot\left\{1+\sum_{s_\alpha=0}^{\infty}\cdots\sum_{s_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{1}{n_\nu!}\cdot\right.$$
$$\left.\cdot\sum_{\boldsymbol{R}_{m+1}=1}^{L}\cdots\sum_{\boldsymbol{R}_{m+s}=1}^{L}\rho_{\nu_{m+1}}(\boldsymbol{R}_{m+1})\cdots\rho_{\nu_{m+s}}(\boldsymbol{R}_{m+s})\sum_{\substack{m+s\geq i>j\geq m+1\\ m+s\geq k\geq m+1\\ m\geq\kappa\geq 1}}^{(G)}\prod F_{ij}F_{k\kappa}\right\}$$
$$(s\equiv s_\alpha+\cdots+s_\tau) \tag{4·9}$$

for a lattice system, and

$$\frac{\rho^{(m)}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_m)}{\rho^{(1)}(\boldsymbol{x}_1)\cdots\rho^{(1)}(\boldsymbol{x}_m)}=\exp\left[-\sum_{m\geq i>j\geq 1}\frac{\phi(\boldsymbol{x}_i,\boldsymbol{x}_j)}{kT}\right]\cdot\left\{1+\sum_{s_\alpha=0}^{\infty}\cdots\sum_{s_\tau=0}^{\infty}\cdot\prod_{\nu=\alpha}^{\tau}\frac{1}{n_\nu!}\cdot\right.$$
$$\left.\cdot\iint\cdots\int d\boldsymbol{r}_{m+1}\cdots d\boldsymbol{r}_{m+s}\rho_{\nu_{m+1}}(\boldsymbol{r}_{m+1})\cdots\rho_{\nu_{m+s}}(\boldsymbol{r}_{m+s})\sum_{\substack{m+s\geq i>j\geq m+1\\ m+s\geq k\geq m+1\\ m\geq\kappa\geq 1}}^{(G)}\prod f_{ij}f_{k\kappa}\right\}$$
$$(s\equiv s_\alpha+\cdots+s_\tau) \tag{4·10}$$

for a continuous system. Here, G on $\sum^{(G)}$ implies that the sum should be taken over all products for which each particles of the set $\{\boldsymbol{x}_{m+1},\cdots,\boldsymbol{x}_{m+s}\}$ or $\{\boldsymbol{X}_{m+1},\cdots,\boldsymbol{X}_{m+s}\}$ are connected at least to two of $\{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_m\}$ or $\{\boldsymbol{X}_1,\cdots,\boldsymbol{X}_m\}$ by an independent path.

§ 5. Conclusion

We have obtained the expansion formulae of the free energy, potentials of average force and distribution functions in powers of the one particle distribution function both for a lattice and for a continuous system, starting from the known virial expansion formulae of the respective quantities for uniform multi-component systems. The main results of this paper are eqs. (3·10), (4·7) and (4·9) with (3·6) for a lattice system and eqs. (3·11), (4·8) and (4·10) with (3·4) for a continuous system. Eq. (3·11), the expansion formula of the free energy for a continuous system, is identical to that given by Arinstein.[7] First few terms of eqs. (3·11) and (4·10) for $m=2$, the expansion formulae of the free energy and the two particle distribution function for a lattice system, are identical to those given by Yvon.[4]

If there are quantities of which virial expansion formulae are known for uniform multi-component system, their expansion formulae in powers of the one particle distribution function will be obtained by the procedure developed in this paper.

In conclusion, we will give some comments on the method used in the text. We have arranged the replicas of our system in a lattice form, but without introducing the interaction between the replicas. One might think that, without the interaction between the replicas, the arrangement on the lattice could not be essential in the derivation. On such a standpoint, we would have adopted an alternative method of derivation, which is different in the order of reasonings but essentially equivalent to that in the text. First, we consider such a uniform lattice system, of M equivalent lattice sites, in which the interaction between the lattice sites is absent, and apply the formulae for a uniform lattice system to this system. As the lattice system just considered may be taken simply as an ensemble of M relpicas of one of lattice sites, the resultant formulae may also be taken as those for this ensemble. Then, if we have had this one of lattice sites corresponded to a lattice system, may it be uniform or non-uniform, we would have obtained the formulae for the ensemble of M replicas of the lattice system and then those for the lattice system, which have just been aiming at.

The constructionof the theory of phase transition between solid, liquid and gas, both for a lattice and for a continuous system, on the basis of the expansion formulae obtained in this paper will be discussed in a separate paper.[10]

The author wishes to express his cordial thanks to Mr. R. Abe for valuable discussions on this work.

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Nuclear Photo-Reaction and Excitation Modes of Nuclei*

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Analyzing the excitation functions given by the experimental researches on the nuclear photo-reaction, it is found that there are two types of excitation mode of nuclei; the extra particle excitation and the core excitation. The careful comparison of the predictions by the single particle model with experimental data shows that these two types of excitation mode are distinguished in the photo-reaction of the light nuclei. It is indicated that the resonances observed in the (γ, n) reactions of C^{13}, N^{14} and F^{19} and $N^{14}(\gamma, p)C^{13}$ in the energy region of 10~15 Mev are essentially due to the extra particle excitation.

In the case of medium weight and heavy nuclei the resonances due to the extra particle excitation should appear, in many cases, at the energies lower than the thresholds for the (γ, n) or (γ, p) reactions.

On the other hand, the giant resonance is characterized by the core excitation mode. For the description of the core the possibility that the collective model may be rather reasonable than the simple independent particle model is indicated. And further, the distinction between these two models are discussed in detail.

§ 1. Introduction

In the photo-reactions of the light nuclei, there exist some gross resonances besides the "giant resonance". Namely, those are observed in the (γ, n) reactions of Be^9, C^{13}, N^{14} and F^{19}, and $N^{14}(\gamma, p)C^{13}$ in the energy region of 10~15 Mev.[1,2,3,4,5] In the photo-reactions of the heavy nuclei, even though such gross resonances may exist there, they have not been observed until now. In this paper we pay our attention to the gross resonances and it is our very aim to investigate the excitation mode characterizing them.

First, let's consider the success of "shell model" and "optical model" in the other fields of nuclear study. As is well known, nuclear shell model has achieved great success in explaining the ground states and the low-lying excited states of many nuclei. The success means that at least in these states the interaction between nucleons in a nucleus does not play so great part as having been thought before. We should also like to take note of "optical model" which was successful in explaining the elastic scattering of neutron from the nucleus. Till then the idea of "compound nucleus" proposed by N. Bohr had been predominant in nuclear reaction

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theories and so it had been believed that the interaction between nucleons in a nucleus was extremely strong. Against this compound theory, Feshbach, Porter and Weisskopf[6] succeeded in representing systematically the gross structure in the elastic scattering of neutron by various nuclei using the optical potential. The depth of the potential used then was $42(1+0.05\,i)$ Mev in the nucleus. Its imaginary part is unexpectedly small from the view-point of "strong interaction". There we can know the fact that the nucleons in a nucleus move unexpectedly independently each other even in the excited states with the excitation energies of $8\sim 10$ Mev. Accordingly, first we shall put this fact to practical use in the process of researches for the excitation mode in the phenomena of nuclear photo-reaction.

In § 2 we shall try to apply the single particle model to the extra nucleon of C^{13} and obtain the cross section for $C^{13}(\gamma, n)C^{12}$. The result will show a resonance-like peak in the E1-excitation curve. It will be compared with the gross resonance observed in $C^{13}(\gamma, n)C^{12}$ at the γ-ray energy 13.5 ± 1 Mev.[2] Moreover, in § 3 we apply this model to various nuclei to investigate the systematic which is found in the results for the photo-disintegrations of them. Actually we find the same results as seen in the case of $C^{13}(\gamma, n)C^{12}$ for the E1-excitation functions for $N^{14}(\gamma, n)N^{13}$, $N^{14}(\gamma, p)C^{13}$ and $F^{19}(\gamma, n)F^{18}$. We point out that the resonance-like peaks correspond to the gross resonances mentioned at the beginning of this paragraph. Guth and Mullin[1] had already made the similar attempt for $Be^{9}(\gamma, n)Be^{8}$. Then they showed that their result agreed with the experimental result if they assumed the potential depending on the state. Accordingly it should be right to say that the resonance of $Be^{9}(\gamma, n)Be^{8}$ essentially consists of the same mechanism as the photo-reactions of other light nuclei.

Thus we indicate that in the nuclear excitation caused by γ-ray there exists the mode to excite the extra particle, hardly disturbing the motion of the other nucleons. In the same manner the single particle model, moreover, could be applied to the photo-reactions of medium weight and heavy nuclei. Then, in the E1-excitation functions of these nuclei there should also appear the resonance-like peaks similar to what did in the excitation functions of the light nuclei. The peaks, however, should lie often in the energy region lower than the thresholds for (γ, n) and (γ, p) of these nuclei.

On the other hand, from the integrated cross section of the giant resonance we can certainly infer that the giant resonance is due to the excitation of many nucleons in the nucleus. There the electric dipole absorption mechanism in the giant resonance region cannot be explained by assuming that all nucleons interact with each other only through the nuclear potential. Actually, basing on the same assumption Burkhardt[7] had found a strong resonance at the energy lower by about 9 Mev than the giant resonance in the E1-excitation function for $Cu^{64}(\gamma, n)Cu^{63}$. For various nuclei the giant resonance energy is much higher in comparison with the prediction by the simple independent particle model so far as the nuclear potential is assumed to be velocity-independent.[9]

From this discrepancy it may be inferred that the nuclear potential is velocity-dependent as indicated by Wilkinson,[9] Weisskopf[15] and Rand.[16] But it is uncertain whether the introduction of the velocity-dependence alone can settle the above discrepancy essentially.

In § 4 we indicate the possibility that the dipole vibration model may be rather reasonable than the simple independent particle model. The dipole vibration model was proposed by Goldhaber and Teller[8] and developed by Gallone and Businaro,[19] Takagi and the author[20] and Fujita.[21] They assumed that γ-ray excites a motion in which the bulk of protons moves in one direction while neutrons in the opposite direction. And further, we discuss the distinction between the collective model and the simple independent particle model. It may be interesting that the Majorana exchange force existing in two-body force plays an important role in this discussion.

The conclusion we draw in this paper will be stated in § 5.

§ 2. The photo-disintegration in the single particle model

Nuclear state excited by γ-ray may be supposed to be generally very complicated, because all the nucleons in the target nucleus have the possibility of absorbing γ-ray. Nevertheless, in the nuclear excitation by γ-ray there is the mode to excite single nucleon, hardly disturbing the motion of the other nucleons. In this section and the next section we shall show the fact. We have already known one of the examples endorsing the above fact in the phenomena of $Be^9(\gamma, n)Be^8$. Guth and Mullin[1] had introduced a model to explain the phenomena in the energy region near the threshold for $Be^9(\gamma, n)Be^8$. They, namely, assumed that the ground state of Be^9 consists both of two α-particles and one extra neutron. And they supposed that the incident γ-ray should excite mainly the extra neutron in the energy region near the threshold and the γ-absorption by α-particles should chiefly contribute to the giant resonance. They, in practice, calculated the cross section for the γ-absorption by the extra neutron, and the result successfully agreed with experimental data. While the giant resonance in $Be^9(\gamma, n)Be^8$ had not been analyzed at all, it is well corresponding with that in $He^4(\gamma, n)He^3$. For the answer to the question whether such a phenomenon is due to speciality of Be^9 or not, we can't help depending on the analysis of the other photo-reactions. Now we shall draw the conclusion to this problem through examining the excitation functions for $C^{13}(\gamma, n)C^{12}$, $N^{14}(\gamma, n)N^{13}$, $N^{14}(\gamma, p)C^{13}$ and $F^{19}(\gamma, n)F^{18}$.

$C^{13}(\gamma, n)C^{12}$

The excitation function for $C^{13}(\gamma, n)C^{12}$ measured by Cook[2] shows two resonance energies $E_r = 13.5 \pm 1$ Mev and ~ 22 Mev, with the half widths $\Gamma = 5 \pm 1$ Mev and ~ 8 Mev respectively. Unfortunately the angular distribution and the energy distribution of the neutrons haven't yet been measured by now.

Comparing this excitation function with that for $C^{12}(\gamma, n)C^{11}$,[2,10] it is found that the excitation function for $C^{12}(\gamma, n)C^{11}$ is well corresponding with the giant

resonance in $C^{13}(\gamma, n)C^{12}$. So we may expect as follows: two resonances in $C^{13}(\gamma, n)C^{12}$ are due respectively to the excitation of the C^{12}-core and of the extra neutron moving around it. The above expectation is partly grounded on the fact that the separation energy of neutron for C^{13} is by far smaller than that for C^{12}. And the spin and parity of C^{13} in the ground state can be explained with this consideration.

Now, we apply the following model to the extra neutron to compute the cross section for the (γ, n) reaction. Namely we assume that the extra neutron interacts with the other nucleons only through nuclear potential. When a photon energy $\hbar\omega$ is incident in the z-direction, the differential cross section for the photo-disintegration caused by electric 2^L-pole radiation is as follows,

$$\frac{d\sigma_L}{d\Omega}=\frac{8\pi^3}{2j+1}\left(\frac{c}{\omega}\right)\rho(k)\sum_{l_1'}\sum_{l_2'}\sum_{j_1'}\sum_{j_2'}\sum_{L'} i^{L'}(-1)^{j+j_1'-j_2'+1/2}F_L^*(l_2'\,j_2';\,lj)$$
$$\times F_L(l_1'\,j_1';\,lj)\sum_m(LLm-m|LLL'0)\,W(Lj_2'Lj_1';\,jL')$$
$$\times Z(l_2'\,j_2'\,l_1'\,j_1';\,\tfrac{1}{2}L')\,P_{L'}(\cos\theta), \tag{1}$$

where

$$F_L(l'j';\,lj)=(-)^{1/2+L-l-j'}f_L\left(\frac{i\omega}{c}\right)^L\sqrt{\frac{(2l+1)\,(2L+1)\,(2j'+1)\,(2j+1)}{4\pi}}$$
$$\times(Ll00|Ll'l'0)\,W(l'j'lj;\,\tfrac{1}{2}L)\int_0^\infty \bar{R}_{fl'j'}R_{ilj}\,r^{L+2}\,dr \tag{2}$$

and

$$\rho(k)=Mk/8\pi^3\hbar^2. \tag{3}$$

There M and k mean the mass and the wave number of the extra neutron respectively. "j" and "l" are the total and orbital angular momentum of the neutron in the ground state. R_{ilj} is the radial wave function of the particle in the bound state and $R_{fl'j'}$ is that in the final state where the particle is emitted from the nucleus with the orbital angular momentum "l'" and the total angular momentum "j'". And f_L is

$$f_L=\sqrt{\frac{2\pi(2L+1)\,(L+1)}{L}}\cdot\frac{1}{(2L+1)!!}\,e_L, \tag{4}$$

where e_L is the effective charge.

The total cross section is

$$\sigma_L=2\frac{Mk}{\hbar^2}\left(\frac{\omega}{c}\right)^{2L-1}f_L^{\,2}(2l+1)\sum_{l'}\sum_{j'}(2j'+1)[(Ll00|Ll'l'0)]^2$$
$$\times[W(l'j'lj;\,\tfrac{1}{2}L)]^2\times\left|\int_0^\infty \bar{R}_{fl'j'}R_{ilj}\,r^{L+2}\,dr\right|^2. \tag{5}$$

When nuclear potential does not contain the term of spin-orbit coupling, eq. (1) can be written as follows,

$$\frac{d\sigma_L}{d\Omega}=4\pi^2(-)^{l+1}f_L^2\left(\frac{\omega}{c}\right)^{2L-1}\rho(k)\sum_{l_1'}\sum_{l_2'}\sum_{L'}i^{L'}(LL1-1|LLL'0)\,(Ll00|Ll_2'0)$$
$$\times(Ll00|Ll_1'0)G_L^*(l_2';\,l)G_L(l_1';\,l)Z(l_2'Ll_1'L;\,lL')\,P_{L'}(\cos\theta), \tag{6}$$

where

$$G_L(l';\,l)=\int_0^\infty \overline{R}_{fl'}R_{il}\,r^{L+2}\,dr. \tag{7}$$

The total cross section is given as follows[11],

$$\sigma_L=\frac{2Mk}{\hbar^2}\left(\frac{\omega}{c}\right)^{2L-1}f_L^2\sum_{l'}[\,(lL00|lLl'0)\,]^2\times|G_L(l';\,l)\,|^2. \tag{8}$$

When nuclear potential is of square well, $G_1(l';\,l)$ can be integrated analytically, as it is shown in Appendix (1). The result is

$$G_1(l';\,l)=\frac{\hbar^2}{2M}\left[\frac{1}{E_f+W'-E_0-W}-\frac{1}{E_f-E_0}\right]R^2F(R)$$
$$+\frac{\hbar^4}{2M^2}\left[\frac{1}{(E_f+W'-E_0-W)^2}-\frac{1}{(E_f-E_0)^2}\right]G(R)$$
$$+\frac{\hbar^4}{4M^2}\left[\frac{(a^2+K^2)R^2}{(E_f+W'-E_0-W)^2}-\frac{(k^2-b^2)R^2}{(E_f-E_0)^2}\right]I(R)$$
$$(l'=l\pm1), \tag{9}*$$

where

$$F(R)=R\left[R_{fl'}\frac{dR_{il}}{dr}-\frac{dR_{fl'}}{dr}R_{il}\right]_{r=R}, \tag{10}$$

$$G(R)=\left[rR_{fl'}\frac{dR_{il}}{dr}+r\frac{dR_{fl'}}{dr}R_{il}+r^2\frac{dR_{fl'}}{dr}\frac{dR_{il}}{dr}+\frac{1}{2}\left\{l'(l'+1)-l(l+1)\right\}\right.$$
$$\left.\times\left\{rR_{fl'}\frac{dR_{il}}{dr}-r\frac{dR_{fl'}}{dr}R_{il}\right\}+\left[1-\tfrac{1}{2}\left\{l'(l'+1)-l(l+1)\right\}\right]R_{fl'}R_{il}\right]_{r=R} \tag{11}$$

and

$$I(R)=R_{fl'}(R)\,R_{il}(R). \tag{12}$$

There a, b and K are given from the following equations.

$$a^2=(2M/\hbar^2)\,(W-|E_0|),$$
$$b^2=2M|E_0|/\hbar^2$$

and

$$K^2=k^2+2MW'/\hbar^2.$$

* R is the range of the potential.

$W(W')$ is the depth of the potential in the initial (final) state and $|E_0|$ is the binding energy of the neutron.

We can also calculate $G_2(l'; l)$ by the way shown in Appendix (2), provided that nuclear potential is velocity-independent. The result is

$$G_2(l'; l) = 2\frac{\hbar^2}{M} R^3 \frac{W}{(\hbar\omega)^2}[R_{fl'} R_{il}]_{r=R} + \frac{2\hbar^4}{M^2} R^2 \frac{W}{(\hbar\omega)^3}\left\{\frac{dR_{fl'}}{dr} R_{il} - R_{fl'}\frac{dR_{il}}{dr}\right\}_{r=R}$$
$$+ [l'(l'+1) - l(l+1)]\frac{\hbar^4}{M^2} R \frac{W}{(\hbar\omega)^3}[R_{fl'} R_{il}]_{r=R}$$
$$\text{(for } l'=l\pm 2 \text{ and } l'=l). \qquad (13)$$

Let us use eq. (5) to compute the cross section for the absorption by the extra neutron of electric dipole radiation. The parameters of nuclear potential chosen then are as follows,

$$R = 3.29\times 10^{-13} \text{ cm}$$

and

$$V(\text{Mev}) = -W_c - W_{so}(\boldsymbol{l}\cdot\boldsymbol{s}) \qquad (r<R)$$
$$= 0 \qquad (r>R),$$

where we take $W_c=31.1$ and $W_{so}=2.4$. $W_{so}(\boldsymbol{l}\cdot\boldsymbol{s})$ is the spin-orbit coupling term. By the way we shall adopt 1.4×10^{-13} cm for r_0 and $13.2\times A^{-2/3}$ for W_{so} in this section and the next. V is chosen so as to give the binding energy properly. The configuration of the extra neutron in the ground state is $1p_{1/2}$. Our assumption that the nuclear potential is of square well may simplify the actual one, because from the elastic scattering of neutron from nuclei the nuclear potential has been known not to plunge to zero abruptly at $r=R$ and the spin-orbit coupling term to be confined to the surface of the nucleus. For the computation of $\int_0^\infty R_{fl'j'} R_{ilj} r^3 dr$ we can use the eq. (9) with the eqs. (10), (11) and (12).

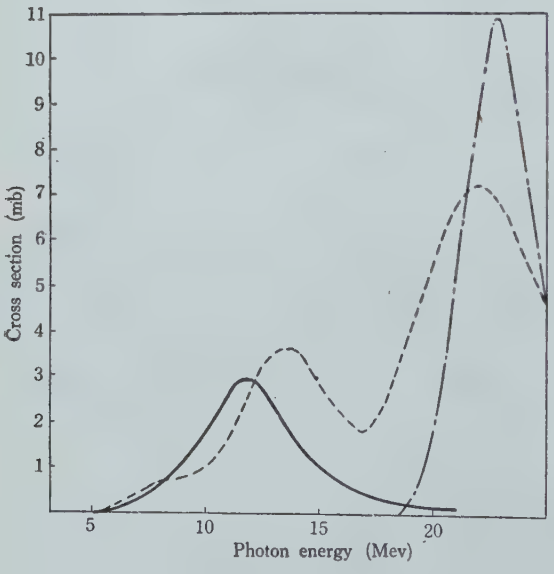

Fig. 1. The excitation functions for $C^{13}(\gamma, n)C^{12}$ and $C^{12}(\gamma, n)C^{11}$. "----" means the experiment and the solid line does the computation of the cross section for the neutron emitted in $d_{3/2}$-wave after absorbing electric dipole radiation in $C^{13}(\gamma, n)C^{12}$, "—·—·—" means the experiment for $C^{12}(\gamma, n)C^{11}$.

Fig. 1 is the cross section for the photo-disintegration of this system, where the extraneutron is emitted in the $d_{3/2}$-wave. It is to be noted that there appears a resonance-like peak in the excitation curve. The

resonance energy E_r, the peak cross section σ_m and the half width Γ are given as follows,

$$E_r = 11.8 \text{ Mev},$$

$$\sigma_m = 3 \text{ mb}$$

and

$$\Gamma = 4.6 \text{ Mev}.$$

There are some differences between this result and the experiment that indicates a gross resonance at 13.5 Mev. They may suggest that even in the energy region in question the excitation of the C^{12}-core contributes somewhat to the excitation function in the background. In other words, under the influence of the extra neutron the C^{12}-core is not so stable for the photo-disintegration as when there exists it alone. Our result however is dependent on the value chosen for nuclear radius. Namely, choosing the smaller value for nuclear radius, the higher resonance energy is obtained. Therefore it seems that the differences are not serious enough to deny our model. It may be rather proper to say that the 13.5 Mev-resonance in $C^{13}(\gamma, n)C^{12}$ is due to the jump of the extra neutron, hardly disturbing the inner motion of the C^{12}-core.

At the same time, our model predicts the angular distribution of $(1+\frac{3}{2}\sin^2\theta)$ though it is not yet measured. The strength with which the neutron is emitted in s-wave is far weaker than in d-wave.

Fig. 2 shows the cross section for the emission of the neutron caused by E2.* It is remarkable that at 26 Mev there appears such a peak as seen in the E1-excitation curve. It is because the amplitude of the f-wave in the nucleus is especially large in the energy region near 26 Mev. The value of the cross section, however, is small. Basing on these results, we can indicate as follows; as if "optical potential" represented systematically the gross resonances appearing in the elastic scattering of neutron from nucleus, the single particle model predicts the systematic appearance of the gross resonances in nuclear photo-reaction.

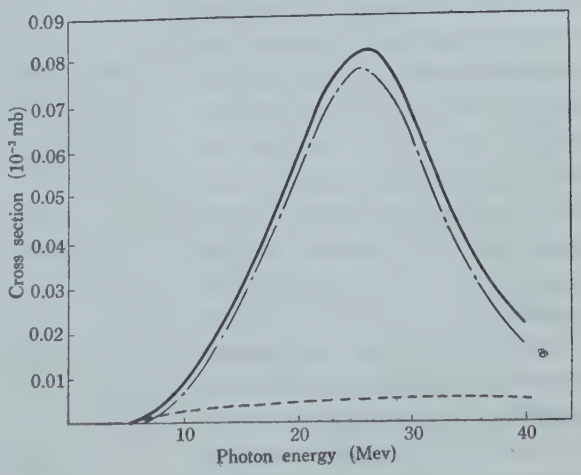

Fig. 2. The cross section for absorption by the single neutron of electric quadrupole radiation in $C^{13}(\gamma, n)C^{12}$; (the computation).
"– – – – –" is for the neutron emitted in f-wave, "– – – –" for the neutron emitted in p-wave and the solid line is the total of the two cross sections.

* In this case we neglect the spin-orbit coupling and so compute the cross section using eqs. (8) and (13).

§ 3. The investigation of the single particle model

$N^{14}(\gamma, n)N^{13}$ and $N^{14}(\gamma, p)C^{13}$

In this case, let us adopt the following model. Namely it is assumed that N^{14} consists of the C^{12}-core and two nucleons moving independently each other around it. The configurations of two extra nucleons are both $1p_{1/2}$ in the ground state. For the range of nuclear potential we take $R=3.37\times10^{-13}$ cm. On the other hand, $W_c=37.3$ Mev and 33.6 Mev are chosen for the (γ, n) and (γ, p) reactions respectively. The difference between the two W originates from the difference between the separation energy of neutron and of proton. In the final state we neglect the configuration interaction between the emitted nucleon and the residual nucleus. Moreover, for the (γ, p) reaction we do not take into account the effect of the Coulomb force. So in the energy region lower than the Coulomb barrier our result may have to be modified somewhat.

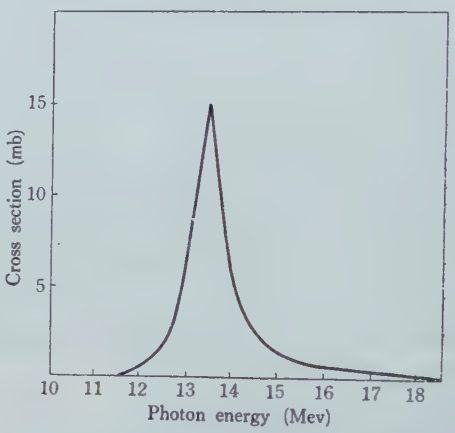

Fig. 3. The excitation function for $N^{14}(\gamma, n)N^{13}$; (the computation).
The cross section is for the single neutron emitted in $d_{3/2}$-wave after absorbing electric dipole radiation.

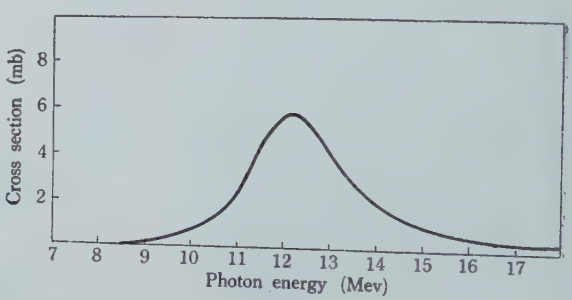

Fig. 4. The excitation function for $N^{14}(\gamma, p)C^{13}$; (the computation).
The cross section is for the single proton emitted in $d_{3/2}$-wave after absorbing electric dipole radiation.

Our results for the electric dipole absorption cross section are shown in Fig. 3 and Fig. 4, where we assume that nuclear potential contains the spin-orbit term as well as in the case of $C^{13}(\gamma, n)C^{12}$. Expectedly, such a resonance as seen in $C^{13}(\gamma, n)C^{12}$ appears also in each of $N^{14}(\gamma, n)N^{13}$ and $N^{14}(\gamma, p)C^{13}$. It originates from the circumstances that the amplitude of the $d_{3/2}$-wave in the nucleus is the largest at 12~14 Mev in the case of N^{14}. The parameters characterizing the resonances are as follows. In $N^{14}(\gamma, n)N^{13}$

$$E_r=13.5 \text{ Mev}, \quad \sigma_m=15 \text{ mb} \quad \text{and} \quad \Gamma=0.9 \text{ Mev},$$

while in $N^{14}(\gamma, p)C^{13}$

$$E_r = 12.2 \text{ Mev}, \quad \sigma_m = 5.8 \text{ mb} \quad \text{and} \quad \Gamma = 2.3 \text{ Mev}.$$

On the other hand, we shall attend to the experimental data. The excitation function for $N^{14}(\gamma, p)C^{13}$ shows sharp and strong resonances at $E_r \simeq 8{,}9$ and 10 Mev respectively.[5] The resonances at 8 Mev and 9 Mev may correspond with the levels at 8.06 Mev and 9.18 Mev measured from the inverse reaction $C^{13}(p, \gamma)N^{14}$ respectively. Those resonances should be understood to form the fine structure of the excitation function, so they must be averaged when it is compared with our result. In the higher energy region the excitation function once decreases and then forms the giant resonance at $E_r \simeq 24$ Mev. For the excitation function for $N^{14}(\gamma, n)N^{13}$, the experimental data measured by Johns et al.[12] show a peak at about 12.5 Mev. It is consistent with the prediction by the single particle model. Their result for the $\Gamma\sigma_m$ of the peak is about 1/12 as much as the theoretical value. But the absolute value of the cross section is suspected to be underestimated in their experiment. Actually the cross section of the giant resonance measured then is 1/5 as much as that of other light nuclei. From this we should infer that the actual discrepancy between the theory and the experiment is not so large as is above-mentioned. It is also reported recently that there exist some resonances in the region from 10.5 Mev to 16 Mev.[3] Namely, in this region, there appear a peak at $\sim$12.7 Mev of half-width $\sim$1 Mev and a peak at $\sim$11.5 Mev of half-width $\sim$0.3 Mev. There is some evidence of a very small peak at $\sim$10.8 Mev. Therefore, our result that in each of the (γ, n) and the (γ, p) reactions there should exist a resonance in the region of 12$\sim$14 Mev seems to be satisfactory on the whole.

Our result also indicates that there is a difference between the resonances of the (γ, n) and the (γ, p) reactions in comparison of their shapes while they take almost the same resonance energy. The shape of the resonance seen in (γ, p) is sharper than that in (γ, n). It is due to our choosing different value for the W in each case. The shape is changeable to the value chosen for the depth and range of nuclear potential. Nevertheless $\sigma_m\Gamma$ indicates the same value in both cases of (γ, n) and (γ, p). That is $\sigma_m\Gamma = 13.5$ mb-Mev in the former and $\sigma_m\Gamma = 13.3$ mb-Mev in the latter.

Moreover, we can predict that in this case the angular distribution of the emitted nucleon is the same as in $C^{13}(\gamma, n)C^{12}$;

$$\sim 1 + \tfrac{3}{2}\sin^2\theta.$$

$F^{19}(\gamma, n)F^{18}$

In the excitation curve of $F^{19}(\gamma, n)F^{18}$ it had been measured that there is a shoulder-like thing in the energy region lower than the giant resonance. The measurement, however, had not been precise enough to make clear whether it should really be a shoulder or a gross resonance giving the appearance of a shoulder. It was the experiment by Taylor et al.[4] that resolved the above question. Namely they found a gross resonance at $\sim$12.2 Mev of the half width $\sim$2 Mev. The peak cross section σ_m is 2.3 mb.

Now we shall apply the single particle model to this reaction in order to make clear whether this resonance should be due to the excitation of the extra particle or not. There the range of nuclear potential and W_c are chosen to be 3.74×10^{-13} cm and 43.7 Mev respectively. We choose $1d_{5/2}$ for the configuration of the neutron in the ground state.

Fig. 5 shows the excitation function for $F^{19}(\gamma, n)F^{18}$ obtained from eqs. (5) and (9). It indicates a very sharp peak at 14 Mev with the $\sigma_m\Gamma=15$ mb-Mev.* It seems to be unaccountable at a glance that this peak is by far sharper than the peak seen in the γ-reaction of C^{13} or N^{14}. In $F^{19}(\gamma, n)F^{18}$, however, the peak consists of the nuclear excitation from the $d_{5/2}$ state to the $f_{7/2}$ state. So the angular distribution also is different from that in $C^{13}(\gamma, n)C^{12}$ or $N^{14}(\gamma, n)N^{13}$ and is $\sim(6+5\sin^2\theta)$.

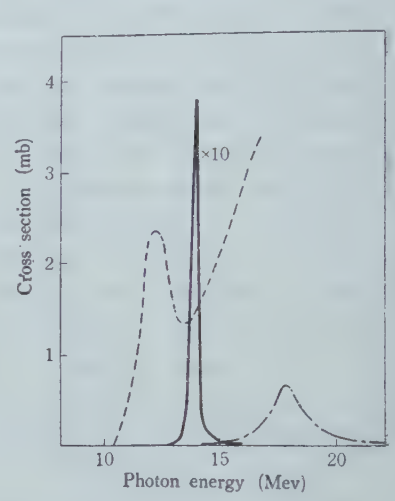

Fig. 5. The excitation function for $F^{19}(\gamma, n)F^{18}$. "- - - - -" is the experimental data taken from Taylor, Robinson and Haslam[4]. The solid line and "— — — — —" means the computation of the cross section for the single neutron emitted in $f_{7/2}$-wave and $f_{5/2}$-wave after absorbing electric dipole radiation.

Our result for Γ is too small in comparison with experiment. But the value of $\sigma_m\Gamma$ should have been measured more precisely than that of σ_m or Γ, taking account of the errors in the experiment. Our result for $\sigma_m\Gamma$ is only thrice as large as the experiment. The resonance energy calculated is higher by 2 Mev than the experiment. This discrepancy, however, does not seem to essentially deny the single particle model. For our result for the resonance energy was lower by 3 Mev than the experiment in $C^{13}(\gamma, n)C^{12}$ and the predictions by this model are not systematically different from the experiment in the resonance energy, E_r, for these nuclei. In short, though the gross resonance at 12 Mev in $F^{19}(\gamma, n)F^{18}$ is not represented with the single particle model so properly as in $C^{13}(\gamma, n)C^{12}$, the existence of the mode to excite the extra particle hardly disturbing the motion of the other nucleons is concluded to be an immovable fact.

Thus in the photo-reactions of the light nuclei we can find the existence of the mode to excite the extra nucleon hardly disturbing the motion of the others. In the same manner the single particle model, moreover, could be applied to the photo-reactions of medium weight and heavy nuclei. Then in the excitation curves for these reactions caused by electric dipole radiation there should also appear resonance-like peaks similar to what did in the excitation curves for the light nuclei. The peak energy is inversely proportional to nuclear radius on the whole. In other

* The cross section for the neutron emitted in $f_{5/2}$-wave has a peak at $\sim$18 Mev.

words it is in proportion to $A^{-1/3}$. So the peaks should lie in the energy region lower than the thresholds for (γ, n) and (γ, p) of those nuclei, so they could not be observed in the experiments for the (γ, n) and (γ, p) reactions of medium weight and heavy nuclei. If we want to recognize their existence we are rather to examine the (γ, γ) and (γ, γ') reactions.

And it is to be noted that for the resonance at 12.2 Mev in $F^{19}(\gamma, n)F^{18}$ the half width computed is too small and our result for $\sigma_m \Gamma$ is thrice as large as the experiment. This discrepancy may be attributed to our nelecting the interaction of the specific nucleon with the others. We consider that the interaction connects closely with the correlation of the nucleons in the core, which we indicate in § 4.

§ 4. The giant resonance and the core excitation mode

As we referred in § 1, the giant resonance is essentially due to the core excitation and its resonance energy for the various nuclei is much higher than the prediction by the independent particle model so far as the nuclear potential is assumed to be velocity-independent.

The above matter can be understood easily in the special case that the nuclear potential is of harmonic oscillator:

$$H=\sum_n\left(\frac{1}{2M}\boldsymbol{p}_n^2+\frac{1}{2}k\boldsymbol{r}_n^2\right). \tag{14}$$

Then, Brink[13] indicated that the Hamiltonian can be transformed as follows,

$$H=H_{\text{int}}+\frac{1}{2AM}\boldsymbol{P}^2+\frac{1}{2}AM\omega_c^2\boldsymbol{R}^2+\frac{1}{2(NZ/A)M}\boldsymbol{\pi}^2+\frac{1}{2}\frac{NZ}{A}M\omega_a^2\boldsymbol{\xi}^2, \tag{15}$$

$$H_{\text{int}}=T_{\text{int}}+U_{\text{int}},$$

$$T_{\text{int}}=\sum_n\frac{1}{2M}\boldsymbol{p}_n^2-\frac{1}{2AM}\boldsymbol{P}^2-\frac{1}{2(NZ/A)M}\boldsymbol{\pi}^2$$

and

$$U_{\text{int}}=\sum_n\frac{1}{2}M\omega_i^2\boldsymbol{r}_n^2-\frac{1}{2}AM\omega_i^2\boldsymbol{R}^2-\frac{1}{2}\frac{NZ}{A}M\omega_i^2\boldsymbol{\xi}^2$$

where

$$\boldsymbol{R}=\frac{1}{A}\sum_n\boldsymbol{r}_n,\quad \boldsymbol{P}=\sum_n\boldsymbol{p}_n,$$

$$\boldsymbol{\xi}=\frac{1}{Z}\sum_i\boldsymbol{r}_i-\frac{1}{N}\sum_j\boldsymbol{r}_j,\quad \boldsymbol{\pi}=\frac{N}{A}\sum_i\boldsymbol{p}_i-\frac{Z}{A}\sum_j\boldsymbol{p}_j$$

and

$$\omega_c=\omega_a=\omega_i=\omega\left(=\sqrt{\frac{k}{M}}\right). \tag{16}$$

i and j mean the i-th proton and the j-th neutron respectively, and H_{int} is quite independent of the centre of mass and the dipole vibration coordinates. From the above equations Brink pointed out that the Goldhaber-Teller collective mode is contained already within the shell model description. The classical angular frequency of the dipole vibration is just the same as that of one particle motion, ω. ω may be determined in terms of nuclear radius $R=r_0A^{1/3}$ from the following equation.[14]

$$\int \Psi^*(\boldsymbol{r}_n-\boldsymbol{R})^2\Psi d\tau=3/5\cdot R^2. \qquad (17)$$

Now, let us pay our attention to the closed-shell nuclei ($N=Z=2, 8, 20,\cdots\cdots$). In these nuclei the wave function for the ground state can also be written as follows,

$$\Psi=\Phi(\boldsymbol{\zeta})\cdot\psi_d(\boldsymbol{\xi})\cdot\psi_c(\boldsymbol{R}), \qquad (18)$$

where Φ contains only proton-proton and neutron-neutron relative coordinates while ψ_c and ψ_d are the lowest S state wave functions for the centre of mass motion and the dipole vibration. Then the relation between ω and R is obtained from eq. (17).

$$\hbar\omega=\frac{5\hbar^2}{3Mr_0^2}A^{-2/3}\{\sum_n C_n(n+\tfrac{3}{2})/A-(3/2A)\}. \qquad (19)$$

C_n is the number of nucleons with quantum number n. Following Levinger's estimation[14] that $\sum_n C_n(n+\frac{3}{2})/A\simeq 0.87\ A^{1/3}$, we obtain

$$\hbar\omega=42\ A^{-1/3}\{1-1.7\ A^{-4/3}\}\ \text{Mev}. \qquad (19')$$

Here the numerical result is based on $r_0=1.2\times10^{-13}$ cm.

If this model is not incomplete, $\hbar\omega$ should correspond to the giant resonance energy. The theoretical value, however, is lower than the experiment as is seen in Table 1.

This discrepancy may suggest the velocity-dependence of the nuclear potential, which was indicated by Wilkinson,[9] Weisskopf[15] and Rand.[16] The velocity-dependence is certainly made clear, not in detail enough, also from the experiments for the elastic scattering of nucleon from nuclei.[17] But it is not sure whether the introduction of the velocity-dependence alone can settle the above discrepancy essentially. Then, it will also be significant to study another possibility that the classical angular frequency of the dipole vibration, ω_d, may be not always the same as that of one particle motion in the actual nucleus.

The nuclear radius is assured to be the same, provided that there is the following relation between ω_d and ω_i even if $\omega_d\neq\omega$, $\omega_i\neq\omega$ and $\omega_c=\omega$.

$$(1/\hbar\omega_i)=[\{\sum_n C_n(n+3/2)/A-(3/2A)\}(1/\hbar\omega) - (3/2A)(1/\hbar\omega_d)]/[\sum_n C_n(n+3/2)/A-(3/A)]. \qquad (20)$$

On the other hand, it is to be noted that in this special case the potential part can also be written, as Bloch[18] has indicated, in the following form.

$$U_{\text{int}}+\frac{1}{2}\frac{NZ}{A}M\omega_d^2\xi^2+\frac{1}{2}AM\omega^2\boldsymbol{R}^2$$
$$=\sum_{i<i'}A_{pp}(\boldsymbol{r}_i-\boldsymbol{r}_{i'})^2+\sum_{j<j'}A_{nn}(\boldsymbol{r}_j-\boldsymbol{r}_{j'})^2+\sum_{ij}A_{np}(\boldsymbol{r}_i-\boldsymbol{r}_j)^2+\tfrac{1}{2}AM\omega^2\boldsymbol{R}^2$$

where

$$A_{np}=\frac{M\omega_d^2}{2A}\quad\text{and}\quad A_{pp}=A_{nn}=\frac{M}{2A}(2\omega_i^2-\omega_d^2).$$

A_{nn}, A_{pp} and A_{np} are dependent on mass number A and so $A_{nn}(\boldsymbol{r}_j-\boldsymbol{r}_{j'})^2$, for instance, does not correspond to the neutron-neutron force, as it is. But we may consider A_{nn}/A_{np} to be the ratio of the effective strength of the neutron-neutron interaction to the neutron-proton in the nucleus. When two-body force is given,

$$(V_W+V_MP_M)J(r),$$

we may estimate for A_{nn}/A_{np} and A_{pp}/A_{np} as follows,

$$\frac{A_{nn}}{A_{np}}=\frac{A_{pp}}{A_{np}}\simeq\left(V_W+\frac{n_+-n_-}{n_++n_-}V_M\right)\Big/\left(V_W+\frac{n_+'-n_-'}{n_+'+n_-'}V_M\right),\qquad(21)$$

where n_+ (n_-) is the number of symmetric (anti-symmetric) neutron-neutron pairs and n'_+ (n'_-) is the number of symmetric (anti-symmetric) neutron-proton pairs in the nucleus. P_M is the Majorana exchange operator.

From eqs. (20) and (21) we can obtain the following expression for $\hbar\omega_d$ and $\hbar\omega_i$.

$$\hbar\omega_d=\hbar\omega\left[\sqrt{\frac{1}{1-x}}\{\sum_n C_n(n+\tfrac{3}{2})/A-(3/A)\}+(3/2A)\right]/\{\sum_n C_n(n+\tfrac{3}{2})/A-(3/2A)\}\qquad(22)$$

and

$$\hbar\omega_i=\sqrt{1-x}\,\hbar\omega_d$$

where

$$x=\frac{A^2-16}{4A(A-2)}\cdot\frac{V_M}{V_W+(4/A)V_M}.$$

The values of $\hbar\omega_d$ and $\hbar\omega_i$ when $V_W=V_M$ and $r_0=1.2\times10^{-13}$ cm is given in Table 1.

Here we can indicate that the tendency of $\omega_d>\omega>\omega_i$ is rather reasonable when there exists the Majorana exchange force in two-body force. When $\omega_d>\omega>\omega_i$, the model is no longer the independent particle model, but may correspond rather to the collective model, though $\hbar\omega_d$ is still smaller than the experiment for the giant resonance energy. In order to make clear the relation between those models we examine the wave function Ψ' in the case that $\omega_d>\omega>\omega_i$ and $\omega_c=2\omega_i-\omega_d$. The wave function Ψ' is different from that in the above case of $\omega_d>\omega>\omega_i$ and $\omega_c=\omega$, but it may be understood that they are equivalent to each other so far as we mention the matter of the internal motion of the nucleus. Namely, they give

Table 1. The theoretical value for $\hbar\omega$, $\hbar\omega_d$ and $\hbar\omega_i$. $\hbar\omega$ is obtained from the eq. (19′), and $\hbar\omega_d$ and $\hbar\omega_i$ from the eq. (22).

A	$\hbar\omega$ (Mev)	$\hbar\omega_d$ (Mev)	$\hbar\omega_i$ (Mev)
4	19.3	19.3	19.3
8	18.7	20.4	18.5
16	16.0	18.2	16.0
40	12.2	14.0	12.2
80	9.7	11.1	9.7
125	8.4	9.7	8.4
216	7.0	8.1	7.0

the same expectation for any operator which does not contain the coordinate or the momentum of the centre of gravity.

Then, it is worth notice that Ψ' can be written as follows,

$$\Psi' = \prod_{ij} F(\boldsymbol{r}_i, \boldsymbol{r}_j) \cdot \Psi_0 .$$

where

$$F(\boldsymbol{r}_i, \boldsymbol{r}_j) = \left\{ \frac{\omega_d(2\omega_i - \omega_d)}{\omega_i^2} \right\}^{3/A^2} \exp\left[\frac{8\beta^2}{A} \cdot \left(\frac{\boldsymbol{r}_i + \boldsymbol{r}_j}{2} \right)^2 \right] \exp\left[-\frac{2\beta^2}{A} (\boldsymbol{r}_i - \boldsymbol{r}_j)^2 \right]$$

and

$$\beta^2 = \frac{AM}{4\hbar}(\omega_d - \omega_i) .$$

Ψ_0 is the wave function of the oscillator shell model where the nuclear potential is $\frac{1}{2} M\omega_i^2 \boldsymbol{r}_n^2$. Thus we know that the collective model deviates from the simple independent particle model and suggests the correlation of the nucleons in the nucleus.

In the above discussion we neglected the Coulomb force and the repulsive force of short range in two-body force. Moreover, the arguments cannot be settled so simply as the above, in the case that the nuclear potential is of square well and contains the term of spin-orbit coupling. The dipole vibration mode, then, can be no longer independent of the other modes. The coupling should correspond to the half width of the giant resonance.

In the E2 absorption mechanism we can also find the distinction between the simple independent particle model and the collective model. In the collective model the interaction of the nucleus with electric quadrupole radiation is described as follows,

$$I \simeq -\sqrt{\frac{2\pi\hbar c^2}{\omega}} \cdot \frac{e}{2} \cdot \left(\frac{i\omega}{c} \right)^2 \left\{ \sum_i \left(\boldsymbol{\varepsilon} \cdot \boldsymbol{r}_i - \frac{1}{Z'} \sum_{i'}' \boldsymbol{r}_{i'} \right) \left(\boldsymbol{n} \cdot \boldsymbol{r}_i - \frac{1}{Z'} \sum_{i''}' \boldsymbol{r}_{i''} \right) \right.$$

$$\left. + Z' \left(\frac{N'}{Z' + N'} \right)^2 (\boldsymbol{\varepsilon} \cdot \boldsymbol{\xi})(\boldsymbol{n} \cdot \boldsymbol{\xi}) \right\} .$$

$\boldsymbol{n}$ is the unit vector in the incident direction and $\boldsymbol{\varepsilon}$ the poralization vector of the radiation.* So, for E2 we can indicate two types of excitation mode—one of them is due to the dipole vibration and another due to the individual particles.

The second excited level of the dipole vibration is situated in the energy region twice as high as the giant resonance. So there should be observed the resonance absorption due to the collective motion of E2 in that energy region.

For the absorption by the individual particles we can obtain the estimate using the model stated in § 2. For O^{16} we compute the cross section for the E2 absorption by the proton in $p_{1/2}$ level. The result indicates a resonance at $E_\gamma \sim 26$ Mev. Thus the collective model predicts two characteristic resonances for the electric quadrupole absorption.

§ 5. Conclusion

In the photo-reactions of the light nuclei we succeeded in finding an excitation mode, namely the mode to excite the extra nucleon without very much disturbing the motion of the others. In the phenomena of $C^{13}(\gamma, n)C^{12}$ we could see the fact that the excitation mode exists almost in a pure form. The 13.5 Mev-resonance is characterized by the excitation mode. The resonance energy, the half width and the peak cross section can be explained with the single particle model. Though there certainly exists a little difference, it is not the essential one but is depending on the parameter chosen then.

The excitation functions near the thresholds for $N^{14}(\gamma, n)N^{13}$, $N^{14}(\gamma, p)C^{13}$ and $F^{19}(\gamma, n)F^{18}$ are not inconsistent with the predictions by the single particle model. But the extra particle excitation mode does not seem to be kept so purely as $C^{13}(\gamma, n)C^{12}$. In particular the resonance in $F^{19}(\gamma, n)F^{18}$ is by far broader than our result. This discrepancy may be attributed to our neglecting the interaction between the specific nucleon and the other nucleons. For $N^{14}(\gamma, n)N^{13}$ and $N^{14}(\gamma, p)C^{13}$ we cannot draw the definite conclusion unless the experimental analysis is done more in detail, but our assertion that at $11 \sim 13$ Mev there should exist a gross resonance characterized by the extra particle excitation mode does not seem to be inconsistent with experiment.

The above conclusion also ought to be experimentally ascertained from the study of the angular distributions.

In medium weight and heavy nuclei the resonance due to the extra particle excitation mode should, in many cases, be lower than the thresholds for the (γ, n) and (γ, p). So, whether there exists such a resonance should be ascertained by examining the (γ, γ) and (γ, γ') reactions. The extra particle excitation mode, however, may not keep a pure form in those nuclei as well as F^{19}.

On the other hand in the giant resonance region it is rather the core particle than the extra particle that mainly contributes to the excitation by γ-ray. And as

* N' and Z' mean the number of the neutrons and protons in the core.

Wilkonson indicated, the independent particle model basing on the supposition of the velocity-dependence of the nuclear potential may be profitable to interprete various aspects for the giant resonance. We indicated that in some case, however, the dipole vibration model proposed by Goldhaber and Teller and developed by Takagi et al. is a reasonable description for the core particles rather than the simple independent particle model. And further we could make clear the features of the collective model through examining the ground state wave function and the prediction for the E2 absorption.

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Appendix (1)

In order to calculate $G_L(l', l)$, let us use $\boldsymbol{u}_{nl}(r)=r\boldsymbol{R}_{nl}(r)$ instead of $\boldsymbol{R}_{nl}(r)$. $\boldsymbol{u}_{nl}$ satisfies the following equation.

$$\boldsymbol{H}_{nl}\boldsymbol{u}_{nl}\equiv\left(\frac{1}{2M}p_r^2+V_n(r)+\frac{l(l+1)\hbar^2}{2Mr^2}\right)\boldsymbol{u}_{nl}=E_{nl}\boldsymbol{u}_{nl}, \tag{A·1}$$

where

$$p_r=-i\hbar\frac{d}{dr}$$

and

$$\begin{aligned} V_n(r)&=-W_n \quad (r<R)\\ &=0 \quad (r>R). \end{aligned} \tag{A·2}$$

For the purpose of simplifying the calculation, we define two operators H_l and $F_l^{l'}$. Namely,

$$H_l\equiv\frac{1}{2M}p_r^2+\frac{l(l+1)\hbar^2}{2Mr^2} \tag{A·3}$$

and

$$F_l^{l'}\equiv-\frac{i\hbar}{M}p_r+\frac{\hbar^2}{2Mr}[l'(l'+1)-l(l+1)]. \tag{A·4}$$

Then

$$H_{l'}r-rH_l=F_l^{l'} \tag{A·5}$$

and

$$H_{l'}F_l^{l'}-F_l^{l'}H_l=-\frac{\hbar^4}{2M^2r^3}\left[\{l'(l'+1)-l(l+1)\}^2-2\{l'(l'+1)+l(l+1)\}\right]. \tag{A·6}$$

When $l'=l\pm1$, the right side of eq. (A·6) vanishes.

On the other hand, we can obtain the following equations.

$$\int_0^R \boldsymbol{u}_{n'l'}[H_{l'}\, r - rH_l]\boldsymbol{u}_{nl}\, dr$$

$$= -\frac{\hbar^2}{2M}\left[\boldsymbol{u}_{n'l'}\frac{d(r\boldsymbol{u}_{nl})}{dr} - \frac{d\boldsymbol{u}_{n'l'}}{dr}(r\boldsymbol{u}_{nl})\right]_0^R + (E_{n'l'} + W_{n'} - E_{nl} - W_n)\int_0^R \boldsymbol{u}_{n'l'}\, r\boldsymbol{u}_{nl}\, dr \tag{A·7}$$

and

$$\int_0^R \boldsymbol{u}_{n'l'}[H_{l'}F_l^{l'} - F_l^{l'}H_l]\boldsymbol{u}_{nl}\, dr$$

$$= -\frac{\hbar^2}{2M}\left[\boldsymbol{u}_{n'l'}\frac{d(F_l^{l'}\boldsymbol{u}_{nl})}{dr} - \frac{d\boldsymbol{u}_{n'l'}}{dr}(F_l^{l'}\boldsymbol{u}_{nl})\right]_0^R + (E_{n'l'} + W_{n'} - E_{nl} - W_n) \times \int_0^R \boldsymbol{u}_{n'l'}F_l^{l'}\boldsymbol{u}_{nl}\, dr. \tag{A·8}$$

Eq. (A·7) is transformed into the following equation by using eq. (A·5).

$$(E_{n'l'} + W_{n'} - E_{nl} - W_n)\int_0^R \boldsymbol{u}_{n'l'}\, r\boldsymbol{u}_{nl}\, dr$$

$$= \frac{\hbar^2}{2M}\left[\boldsymbol{u}_{n'l'}\frac{d(r\boldsymbol{u}_{nl})}{dr} - \frac{d\boldsymbol{u}_{n'l'}}{dr}(r\boldsymbol{u}_{nl})\right]_0^R + \int_0^R \boldsymbol{u}_{n'l'}F_l^{l'}\boldsymbol{u}_{nl}\, dr. \tag{A·9}$$

Therefore, from equations (A·6) and (A·8)

$$(E_{n'l'} + W_{n'} - E_{nl} - W_n)\int_0^R \boldsymbol{u}_{n'l'}F_l^{l'}\boldsymbol{u}_{nl}\, dr$$

$$= \frac{\hbar^2}{2M}\left[\boldsymbol{u}_{n'l'}\frac{d(F_l^{l'}\boldsymbol{u}_{nl})}{dr} - \frac{d\boldsymbol{u}_{n'l'}}{dr}(F_l^{l'}\boldsymbol{u}_{nl})\right]_0^R \quad (l' = l \pm 1). \tag{A·10}$$

As a result, we can get the following equation from (A·9) and (A·10).

$$\int_0^R \boldsymbol{u}_{n'l'}\boldsymbol{u}_{nl}\, r dr = \frac{\hbar^2}{2M}\cdot\frac{1}{E_{n'l'} + W_{n'} - E_{nl} - W_n} \times \left\{\left[\boldsymbol{u}_{n'l'}\frac{d(r\boldsymbol{u}_{nl})}{dr} - \frac{d\boldsymbol{u}_{n'l'}}{dr}(r\boldsymbol{u}_{nl})\right]_0^R + \frac{1}{E_{n'l'} + W_{n'} - E_{nl} - W_n}\left[\boldsymbol{u}_{n'l'}\frac{d(F_l^{l'}\boldsymbol{u}_{nl})}{dr} - \frac{d\boldsymbol{u}_{n'l'}}{dr}(F_l^{l'}\boldsymbol{u}_{nl})\right]_0^R\right\} \quad (l' = l \pm 1). \tag{A·11}$$

The above equation can be transformed into more general form as follows,

$$\int_0^R \boldsymbol{u}_{n'l'}\boldsymbol{u}_{nl}\,r\,dr$$

$$=\frac{\hbar^2}{4M}\frac{1}{E_{n'l'}+W_{n'}-E_{nl}-W_n}\left\{\left[\boldsymbol{u}_{n'l'}\frac{d(r\boldsymbol{u}_{nl})}{dr}-\frac{d\boldsymbol{u}_{n'l'}}{dr}(r\boldsymbol{u}_{nl})-\boldsymbol{u}_{nl}\frac{d(r\boldsymbol{u}_{n'l'})}{dr}\right.\right.$$

$$\left.+\frac{d\boldsymbol{u}_{nl}}{dr}(r\boldsymbol{u}_{n'l'})\right]_0^R+\frac{1}{E_{n'l'}+W_{n'}-E_{nl}-W_n}\left[\boldsymbol{u}_{n'l'}\frac{d(F_{l'}^{l'}\boldsymbol{u}_{nl})}{dr}\right.$$

$$\left.\left.-\frac{d\boldsymbol{u}_{n'l'}}{dr}(F_l^{l'}\boldsymbol{u}_{nl})+\boldsymbol{u}_{nl}\frac{d(F_{l'}^{l}\boldsymbol{u}_{n'l'})}{dr}-\frac{d\boldsymbol{u}_{nl}}{dr}(F_{l'}^{l}\boldsymbol{u}_{n'l'})\right]_0^R\right\}$$

$$(l'=l\pm1). \qquad (A\cdot12)$$

Eq. (A·12) may be written by using $\boldsymbol{R}_{nl}$ instead of $\boldsymbol{u}_{nl}$. That is,

$$\int_0^R \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^3\,dr$$

$$=\frac{\hbar^2}{2M}\frac{1}{E_{n'l'}+W_{n'}-E_{nl}-W_n}\left\{\left[r^3\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}-r^3\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}\right]_0^R\right.$$

$$+\left(\frac{\hbar^2}{M}\right)\frac{1}{E_{n'l'}+W_{n'}-E_{nl}-W_n}\left[r\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}+r\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}+r^2\frac{d\boldsymbol{R}_{n'l'}}{dr}\frac{d\boldsymbol{R}_{nl}}{dr}\right.$$

$$+\frac{1}{2}\left\{l'(l'+1)-l(l+1)\right\}\left\{r\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}-r\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}\right\}$$

$$\left.\left.+\left[1+\frac{1}{2}(a^2+K^2)\,r^2-\frac{1}{2}\{l'(l'+1)-l(l+1)\}\right]\boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\right]_0^R\right\}$$

$$(l'=l\pm1). \qquad (A\cdot13)$$

There a and K are obtained from the following equations.

$$\left.\begin{aligned} a^2&=\frac{2M}{\hbar^2}(E_{nl}+W_n)\\ \text{and}\qquad\\ K^2&=\frac{2M}{\hbar^2}(E_{n'l'}+W_{n'}).\end{aligned}\right\} \qquad (A\cdot14)$$

The integral in the region of $r>R$ can be done in the same manner. The result is,

$$\int_R^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^3\,dr=\frac{\hbar^2}{2M}\frac{1}{E_{n'l'}-E_{nl}}\left\{\left[r^3\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}-r^3\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}\right]_R^\infty+\frac{\hbar^2}{M}\frac{1}{E_{n'l'}-E_{nl}}\right.$$

$$\left.\times\left[r\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}+r\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}+r^2\frac{d\boldsymbol{R}_{n'l'}}{dr}\frac{d\boldsymbol{R}_{nl}}{dr}+\frac{1}{2}\{l'(l'+1)-l(l+1)\}\right.\right.$$

$$\times\left\{r\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}-r\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}\right\}+\left[1+\frac{1}{2}(k^2-b^2)\,r^2-\frac{1}{2}\{l'(l'+1)-l(l+1)\}\right]$$

$$\times\boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\Big]_R^\infty\Big\}$$

$$(l'=l\pm1), \tag{A·15}$$

where

$$\left.\begin{aligned} b^2&=\frac{2M}{\hbar^2}|E_{nl}|\\ k^2&=\frac{2M}{\hbar^2}E_{n'l'}. \end{aligned}\right\} \tag{A·16}$$

and

As a result, $\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^3\,dr$ is given from (A·13) and (A·15) as follows,

$$\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^3\,dr=\frac{\hbar^2}{2M}\left[\frac{1}{E_{n'l'}+W_{n'}-E_{nl}-W_n}-\frac{1}{E_{n'l'}-E_{nl}}\right]R^2F(R)$$

$$+\frac{\hbar^4}{2M^2}\left[\frac{1}{(E_{n'l'}+W_{n'}-E_{nl}-W_n)^2}-\frac{1}{(E_{n'l'}-E_{nl})^2}\right]G(R)$$

$$+\frac{\hbar^4}{4M^2}\left[\frac{(a^2+K^2)R^2}{(E_{n'l'}+W_{n'}-E_{nl}-W_n)^2}-\frac{(k^2-b^2)R^2}{(E_{n'l'}-E_{nl})^2}\right]I(R)$$

$$(l'=l\pm1), \tag{A·17}$$

where

$$F(R)=R\left[\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}-\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}\right]_{r=R}, \tag{A·18}$$

$$G(R)=\left[r\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}+r\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}+r^2\frac{d\boldsymbol{R}_{n'l'}}{dr}\frac{d\boldsymbol{R}_{nl}}{dr}\right.$$

$$+\frac{1}{2}\left\{l'(l'+1)-l(l+1)\right\}\left\{r\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}-r\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}\right\}$$

$$\left.+\left[1-\frac{1}{2}\{l'(l'+1)-l(l+1)\}\right]\boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\right]_{r=R} \tag{A·19}$$

and

$$I(R)=[\boldsymbol{R}_{n'l'}\cdot\boldsymbol{R}_{nl}]_{r=R}. \tag{A·20}$$

Appendix (2)

In the special case that $W_{n'}=W_n$, let us calculate $\int_0^\infty \boldsymbol{R}_{n'l'}(r)\boldsymbol{R}_{nl}(r)\,r^3\,dr$ $(l'=$

$l\pm1$) and $\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^4\,dr$ ($l'=l\pm2$ and $l'=l$). At first we pay our attention to the following relations existing between $\boldsymbol{H}_l$ and r. That is to say, since $\boldsymbol{H}_l=\frac{1}{2M}p_r^2+V+\frac{l(l+1)\hbar^2}{2Mr^2}$,

$$\boldsymbol{H}_{l'}\,r-r\boldsymbol{H}_l=-\frac{i\hbar}{M}p_r+\frac{\hbar^2}{2Mr}[l'(l'+1)-l(l+1)] \qquad \text{(A·21)}$$

and

$$\boldsymbol{H}_{l'}(\boldsymbol{H}_{l'}\,r-r\boldsymbol{H}_l)-(\boldsymbol{H}_{l'}\,r-r\boldsymbol{H}_l)\boldsymbol{H}_l$$
$$=\frac{\hbar^2}{M}\frac{dV}{dr}+\frac{\hbar^4}{4M^2r^3}\Big[\{l'(l'+1)-l(l+1)\}^2-2\{l'(l'+1)+l(l+1)\}\Big]. \qquad \text{(A·22)}$$

The second term of the right side of (A·22) vanishes in the case of $l'=l\pm1$. Since nuclear potential $V(r)$ is of square well.

$$\frac{dV}{dr}=W\delta(r-R). \qquad \text{(A·23)}$$

On the other hand,

$$\int_0^\infty \boldsymbol{u}_{n'l'}\{\boldsymbol{H}_{l'}(\boldsymbol{H}_{l'}\,r-r\boldsymbol{H}_l)-(\boldsymbol{H}_{l'}\,r-r\boldsymbol{H}_l)\boldsymbol{H}_l\}\boldsymbol{u}_{nl}\,dr$$
$$=(E_{n'l'}-E_{nl})^2\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^3\,dr. \qquad \text{(A·24)}$$

Therefore, from equations (A·22), (A·23) and (A·24) $\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^3\,dr$ is given as follows,

$$\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^3\,dr=\frac{\hbar^2}{M}R^2\frac{W}{(E_{n'l'}-E_{nl})^2}[\boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}]_{r=R}$$
$$(l'=l\pm1). \qquad \text{(A·25)}$$

Eq. (A·25) corresponds with the special case[11] of eq. (A·17), where we take $W_n'=W_n$.

In a similar manner, we can obtain the following equation. That is,

$$(E_{n'l'}-E_{nl})^2\int_0^\infty \boldsymbol{u}_{n'l'}\boldsymbol{u}_{nl}\,r^2\,dr=\frac{2\hbar^2}{M}\int_0^\infty \boldsymbol{u}_{n'l'}\boldsymbol{u}_{nl}\Big(\frac{dV}{dr}\Big)r\,dr+I, \qquad \text{(A·26)}$$

where

$$I=-\frac{2\hbar^2}{M^2}\int_0^\infty \boldsymbol{u}_{n'l'}p_r^2\boldsymbol{u}_{nl}\,dr-\frac{i\hbar^3}{M^2}\{l'(l'+1)-l(l+1)\}\int_0^\infty \boldsymbol{u}_{n'l'}\frac{1}{r}p_r\boldsymbol{u}_{nl}\,dr$$

$$+\frac{\hbar^4}{4M^2}\Big[\{l'(l'+1)-l(l+1)\}^2-2\{l'(l'+1)-l(l+1)\}-8l(l+1)\Big]$$

$$\times\int_0^\infty \boldsymbol{u}_{n'l'}\,\boldsymbol{u}_{nl}\frac{1}{r^2}\,dr. \qquad (A\cdot 27)$$

Especially in the case of $l'=l\pm 2$ or $l'=l$

$$(E_{n'l'}-E_{nl})\cdot I=-\frac{2\hbar^4}{M^2}\int_0^\infty \boldsymbol{u}_{n'l'}\Big(\frac{dV}{dr}\Big)\frac{d\boldsymbol{u}_{nl}}{dr}\,dr+\frac{2\hbar^4}{M^2}\int_0^\infty \frac{d\boldsymbol{u}_{n'l'}}{dr}\Big(\frac{dV}{dr}\Big)\boldsymbol{u}_{nl}\,dr$$

$$+\frac{\hbar^4}{M^2}\{l'(l'+1)-l(l+1)\}\int_0^\infty \boldsymbol{u}_{n'l'}\Big(\frac{dV}{dr}\Big)\boldsymbol{u}_{nl}\frac{1}{r}\,dr$$

$$=\frac{2\hbar^4}{M^2}\int_0^\infty\Big(\frac{dV}{dr}\Big)\Big\{\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}-\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}\Big\}r^2\,dr$$

$$+\frac{\hbar^4}{M^2}\{l'(l'+1)-l(l+1)\}\int_0^\infty \boldsymbol{R}_{n'l'}\Big(\frac{dV}{dr}\Big)\boldsymbol{R}_{nl}\,r\,dr. \qquad (A\cdot 28)$$

Therefore, the following equation is obtained from equations (A·26) and (A·28). Namely,

$$(E_{n'l'}-E_{nl})^2\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^4\,dr$$

$$=\frac{2\hbar^2}{M}\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\Big(\frac{dV}{dr}\Big)r^3\,dr+\frac{2\hbar^4}{M^2(E_{n'l'}-E_{nl})}\Big[\int_0^\infty\Big(\frac{dV}{dr}\Big)\Big\{\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}$$

$$-\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}\Big\}r^2\,dr+\frac{1}{2}\{l'(l'+1)-l(l+1)\}\int_0^\infty \boldsymbol{R}_{n'l'}\Big(\frac{dV}{dr}\Big)\boldsymbol{R}_{nl}\,r\,dr\Big]$$

$$(l'=l\pm 2 \text{ and } l'=l). \qquad (A\cdot 29)$$

Then $\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^4\,dr$ can be given by using eq. (A·23) as follows,

$$\int_0^\infty \boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}\,r^4\,dr=\frac{2\hbar^2 WR^3}{M(E_{n'l'}-E_{nl})^2}[\boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}]_{r=R}$$

$$+\frac{2\hbar^4}{M^2(E_{n'l'}-E_{nl})^3}WR^2\Big\{\frac{d\boldsymbol{R}_{n'l'}}{dr}\boldsymbol{R}_{nl}-\boldsymbol{R}_{n'l'}\frac{d\boldsymbol{R}_{nl}}{dr}\Big\}_{r=R}$$

$$+\{l'(l'+1)-l(l+1)\}\frac{\hbar^4 WR}{M^2(E_{n'l'}-E_{nl})^3}[\boldsymbol{R}_{n'l'}\boldsymbol{R}_{nl}]_{r=R}$$

$$(l'=l\pm 2 \text{ and } l'=l). \qquad (A\cdot 30)$$

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Quasi-Deuteron Model of Atomic Nuclei. I

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Quasi-deuteron model is investigated, taking account of pair correlation functions in the ground state wave function of atomic nuclei. The formalism of this type makes it possible to use the cluster development method conveniently for the calculations of the expectation values or matrix elements by the model wave functions.

Pair nucleon correlation functions are determined phenomenologically but self-consistently, by making use of partly the experimental information about two nucleon systems and partly the theoretical considerations for multiple scattering problem developed by Brueckner, Bethe and their collaborators.

In this paper the properties of nuclear matter are investigated as the preliminary discussion. Two nucleon correlation functions are investigated in detail for every possible spin and isospin states of the nucleon pairs. The effect of the exclusion principle on these correlation functions is given in explicit form and it is shown that on the one hand the exclusion principle plays the most important role for the appearance of the shell structure, and on the other hand it may have some relationships with alpha particle formation on the nuclear surface.

With the use of our ground state wave function, the saturation of nuclear volume energy and that of nuclear density are discussed quantitatively and the velocity dependent one particle potential is constructed self-consistently.

High energy nuclear reactions, which the independent particle model without correlation fails to explain, are discussed and it is shown that the nucleon momentum distribution in nuclei is given correctly by our model.

§ 1. Introduction

The two kinds of difficulties have hitherto prevented us from the complete solution of the nuclear problem; one is the insufficiency of our knowledge about the interactions between nucleons, and the other is the difficulties encountered in the solution of the many body problem. Recent refinements and improvements of experimental knowledges have revealed many shortcomings of the simple phenomenological "models" of the nuclei, and the "theory" of the nuclei is now compelled to face the difficult problem of the solution of a many body problem of nucleons interacting strongly in the atomic nuclei.

The application of the formal theory of multiple scattering on the one hand, and the adaption of the variational principle on the other hand have been proposed. The former method has been energetically developed by Brueckner[1], Bethe[2] and their

collaborators, while many elaborate works have been performed also with the latter method by Jastrow,[3] Yamada and Iwamoto[4] et al.

Before going into discussion of the relation between our method and those treatments, it might be worthwhile to give a very brief outline of the above-mentioned two methods comparing with each other. The method based on the formal theory of multiple scattering has a merit in the treatment of problems in an orthodox way, starting from the fundamental two-body forces and standing on a well defined basis. However, in practice, it would be hardly possible to accomplish the program completely. Various simplifying assumptions or approximations would be indispensable in order to get to some goal anyway, and such situation gives rise to controversies such as: to what kind of physical concept do such assumptions correspond? Or, to what extent do those approximations spoil the merit of the method? etc. In spite of the careful approach taken by Bethe,[2] there remains something still in question. Moreover, so complicated mathematical manipulations necessary in this method might restrict its application to the study of the actual finite size nuclei or of the nuclear reactions without further modifications or assumptions.

On the contrary, the treatment by the variational principle encounters no substantial obstacles in carrying out some programs with the use of the cluster development method. This method is therefore very powerful in investigating various problems, such as, the effect of the hard core, pair correlations and three-body clusters, and so on. Needless to say, however, any variational treatment is marred by the ambiguity in choosing a trial function.

Now let us go back to our own subject of this paper. Our aim is to invent a new practical approach to the study of atomic nucleus which might possess their merits and at the same time be free from some shortcomings of the two methods mentioned above. In other word, we attempt to propose a model wave function which not only has a sufficiently simple form to be applicable to actual calculations, but also is plausible and has a sufficient theoretical background.

The starting point of our procedure is to define the ground state wave function of a nucleus as follows:

$$\Psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots \boldsymbol{x}_A) = \mathcal{A} \prod_{\alpha} \varphi_\alpha(\boldsymbol{x}_i) \prod_{\alpha<\beta} f_{\alpha\beta}(\boldsymbol{x}_i, \boldsymbol{x}_j) \prod_{\alpha<\beta<\gamma} f_{\alpha\beta\gamma}(\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k) \cdots\cdots, \quad (1)$$

in which $\boldsymbol{x}_i$ is the ensemble of coordinates of the i-th particle, α specify nucleon quantum state, and $\mathcal{A}$ symbolizes the antisymmetrization. $\varphi_\alpha(\boldsymbol{x}_i)$ denotes the independent particle orbital wave function of the i-th particle which moves in the self-consistent one-body potential and is in the α state. $f_{\alpha\beta}(\boldsymbol{x}_i, \boldsymbol{x}_j)$ denotes the "two-particle function" which is defined between the i-th and j-th nucleon being in the α and β state, respectively, etc.

Yamada and Iwamoto[4] have recently shown that the wave function similar to (1) makes it possible to describe the expectation value of the Hamiltonian in the cluster development method. $\varphi_\alpha(\boldsymbol{x}_i)$ describes the independent nature and is a shell wave function or a plane wave function, corresponding to a finite size nucleus or

in an infinite nuclear matter, respectively. For the reasonable interpretation of high energy phenomena, it has been shown by Brueckner, Eden and Francis[5] that the inter-nucleon correlations as are represented by $f_{\alpha^3}(\boldsymbol{x}_i, \boldsymbol{x}_j)$ seem to be indispensable. At the nuclear surface, the effect of Pauli's exclusion principle might differ considerably from the situation inside the nuclear matter and consequently there would be some strong tendency to the nucleon clustering, the fact of which may be of importance to alpha particle decay, etc. $f_{\alpha^3\tau}$, $f_{\alpha^3\tau\delta}$,…… would play essential roles in such phenomena.

Our main subjects are to determine $f_{\alpha^3}(\boldsymbol{x}_i, \boldsymbol{x}_j)$, $f_{\alpha^3\tau}(\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k)$,……by simultaneously making use of the experimental knowledges about two nucleon systems and the light nuclei on the one hand, and the theoretical methods already developed in the Brueckner's theory on the other hand.

In this paper, as the first attempt in this direction, we shall discuss the properties of nuclear matter. Accordingly, nuclear wave function (1) is approximated by the Slater determinant describing the degenerated Fermi gas, multiplied by the product of the two-particle functions $f_{\alpha^3}(\boldsymbol{x}_i, \boldsymbol{x}_j)$. It must of course be duly antisymmetrized as a whole. Considering the insufficiency of our knowledge about nuclear forces and also for the sake of the convenience of computations, we shall throughout use the following two-body potentials which has been adjusted so as to reproduce the experimental two-nucleon data correctly at low energies:

$$V(r_{ij}) = 1/2(1+P^M)[{}^3V(r_{ij}){}^3P + {}^1V(r_{ij}){}^1P] \text{ : Serber force,} \qquad (2)$$

in which P^M is a Majorana operator, ${}^3P$ and ${}^1P$ denote the projection operators referring to spin triplet and spin singlet state respectively, and

$${}^\lambda V(r) = \begin{cases} {}^\lambda V_0\, e^{-\mu(r-D)} & r \geq D, \\ \infty & r < D, \quad \lambda = 3 \text{ or } 1, \end{cases}$$

where D is the hard core radius assumed to be 0.4×10^{-13} cm, while

$${}^3V_0 = -475.0 \text{ MeV}, \quad {}^3\mu = 2.521\times10^{13}\ \text{cm}^{-1}. \text{ for spin triplet state,}$$

$${}^1V_0 = -330.8 \text{ MeV}, \quad {}^1\mu = 2.402\times10^{13}\ \text{cm}^{-1}. \text{ for spin singlet state,}$$

have been given by Kikuta, Morita and Yamada.[6]

§ 2. Determination of the two-particle functions

We shall determine the phenomenological two-particle functions $f_{\alpha^3}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ through the following three steps: first, we consider a two body problem in free space taking no account of the effects from the surrounding nuclear matter, second, we take account of the Pauli principle, and finally we take account the interactions with other nucleons as far as they can be described by the effective mass, whose quantity is to be calculated self-consistently (§ 4).

(*a*) Distortion of partial waves

The wave function of two nucleons in free space which are respectively in the state α and β is, if the interaction is switched off, given by

$$\phi_\alpha(\boldsymbol{r}_1)\ \phi_\beta(\boldsymbol{r}_2), \tag{3}$$

where

$$\phi_\alpha(\boldsymbol{r}_1)=(1/\sqrt{\Omega})\exp(i\boldsymbol{k}_\alpha\,\boldsymbol{r}_1)\,v_\pm(\sigma_z)\,u_\pm(\tau_3)$$

is normarized in the volume Ω.

Separating the center-of-mass coordinate and the relative coordinate, this wave function can be written as

$$\begin{aligned}\phi_\alpha(\boldsymbol{r}_1)\phi_\beta(\boldsymbol{r}_2)&=(1/\sqrt{\Omega})\ \exp\ (i\boldsymbol{K}\boldsymbol{R})\ (1/\sqrt{\Omega})\exp(i\boldsymbol{k}\boldsymbol{r})\,S_{\alpha\beta}(1,\ 2)\\&\equiv(1/\sqrt{\Omega})\exp(i\boldsymbol{K}\boldsymbol{R})\phi_k(\boldsymbol{r}),\end{aligned} \tag{4}$$

where

$$\boldsymbol{K}=\boldsymbol{k}_\alpha+\boldsymbol{k}_\beta,\ \ \boldsymbol{k}=(\boldsymbol{k}_\alpha-\boldsymbol{k}_\beta)/2,\ \ \boldsymbol{R}=(\boldsymbol{r}_1+\boldsymbol{r}_2)/2,\ \ \boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2, \tag{5}$$

and $S_{\alpha\beta}(1,\ 2)$ denotes the spin and isospin wave function of the system of two nucleons. When the interaction is switched in, the wave function (4) suffers a marked modification in the relative motion. The solution of the Schrödinger equation with the central potential given by (2) can be written

$$\psi_k(\boldsymbol{r})=(1/\sqrt{\Omega})\sum_{l=0}^{\infty}R_l(r)\,P_l(\cos\theta)\,S_{\alpha\beta}(1,\ 2).^{*} \tag{6}$$

The radial part may have the following asymptotic form:

$$R_l(r)\xrightarrow{r\to\infty}(1/kr)\,(2l+1)\,i^l e^{i\delta_l}\sin(kr-l\pi/2+\delta_l), \tag{7}$$

where δ_l is the phase shift of the l-th partial wave.

Though the rigorous analytic solution for $R_l(r)$ is impossible, we may construct its approximate expression as follows:

$$R_l(r)=(2l+1)\,i^l\,e^{i\delta_l}\,L_l(kr), \tag{8}$$

$$L_l(kr)\equiv(1/kr)[\sin(kr-l\pi/2+\delta_l)-\eta_l(k,r)], \tag{8'}$$

introducing a modification term $\eta_l(k,\ r)$, which will later be determined phenomenologically. $\eta_l(k,r)$ should work so that $R_l(r)$ might reproduce the behavior of actual

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* In general, there exists spin-dependent force, then the angular momentum expansion is written as follows:

$$\psi_{k,s,m}(\boldsymbol{r})=\sum_J\sum_l\sqrt{4\pi(2l+1)}\cdot i^l\mathrm{C}(ls0m|lsJm)\sum_{l'}R_{ll'Js}(r)\cdot F_{Jl's}{}^m(\hat{r}),$$

Where $F_{Jl's}{}^m(\hat{r})$ is an eigenfunction of the total angular momentum, C is a Clebsch-Gordan coefficient and $R_{ll'Js}$ is the radial function for total angular momentum J, spin s, angular momentum l'. In this case the formalism becomes complicated, but our treatments do not encounter any serious trouble.

wave function in the region of the small r and behaves as (7) with increasing r. Thus,

$$\psi_k(\boldsymbol{r}) = (1/\sqrt{\Omega})\sum_{l=0}^{\infty}(2l+1)\,i^l \cdot e^{i\delta_l} L_l(kr)\,P_l(\cos\theta)\,S_{\alpha\beta}(1,\,2). \tag{9}$$

Comparing (9) with

$$\phi_k(\boldsymbol{r}) = (1/\sqrt{\Omega}) \cdot \sum_{l=0}^{\infty}(2l+1)\,i^l \cdot j_l(kr)\,P_l(\cos\theta)\,S_{\alpha\beta}(1,\,2), \tag{10}$$

we now rewrite (9) as follows:

$$\psi_k(\boldsymbol{r}) = (1/\sqrt{\Omega})[e^{i\boldsymbol{kr}} + u_k^{(s)}(\boldsymbol{r}) + u_k^{(p)}(\boldsymbol{r}) + \cdots\cdots]S_{\alpha\beta}(1,\,2), \tag{11}$$

where

$$u_k^{(l)}(\boldsymbol{r}) = (2l+1)\,i^l \cdot [e^{i\delta_l} L_l(kr) - j_l(kr)] \cdot P_l(\cos\theta). \tag{12}$$

$u_k^{(l)}(\boldsymbol{r})$ represents the distortion of the l-th partial wave from the plane wave caused by the two-body force, and contributes to the real scattering of the l-th partial wave.

(b) Behavior near the hard core

Let us here determine the modification term $\gamma_l(k,r)$. The significant information about the behavior of the wave function near the hard core come out from the deuteron wave functions. $^3S_1$-wave functions hitherto computed by various methods are given in Fig. 1.

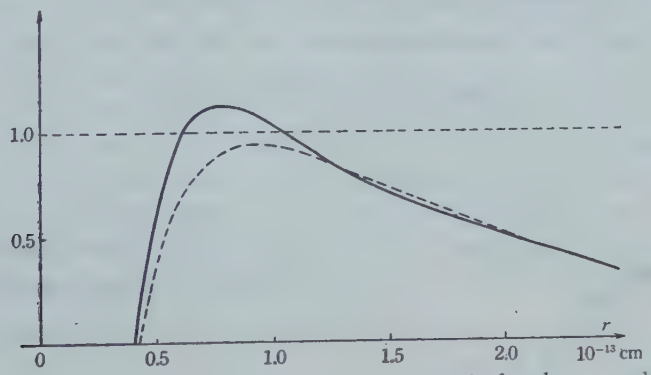

Fig. 1. The radial wave functions ($^3S_1$-state) for the ground state of the deuteron.
——: Hulthen type wave function obtained by the variational principle.
....: The solution as given from the Blatt-Biedenharn's phenomenological potential.[7]

Hulthen type wave function of Fig. 1 can be expressed as

$$\psi_d = 1/r(e^{-a(r-D)} - e^{-b(r-D)}), \tag{13}$$

in which parameters a and b are determined by the variational principle as follows,

$$a = R_d^{-1} = 0.2317 \times 10^{13}\text{cm}^{-1},$$
$$b = (3/\rho)[1 + 4/(9\alpha\rho)]^{-1} = 3.036 \times 10^{13}\text{cm}^{-1}, \tag{13'}$$

where

$R_d = 4.316 \times 10^{-13}$ cm ; deuteron radius, $\rho = r_0 - 2D$,

$r_0 = 1.704 \times 10^{-13}$cm ; triplet effective range, $D = 0.4 \times 10^{-13}$cm ; core radius,

${}^3\alpha = (5.378 \times 10^{-13}\ \text{cm})^{-1}$; inverse triplet scattering length.

Rearranging (13) as

$$\psi_d(r) = 1/r(1 - e^{-(b-a)(r-D)})e^{-a(r-D)}, \tag{14}$$

with

$$b - a = 2.804 \times 10^{13}\text{cm}^{-1}, \text{ for spin triplet state,} \tag{14'}$$

we see that the second factor of (14) determines dominantly the behavior of the S-wave function near the hard core and the third factor determines the asymptotic form giving rise to a deuteron effective radius R_d.

For the singlet state, we conveniently utilize the relation (13'); there is no bound state, then $a = 0$ and from the values of $r_0 = 2.4 \times 10^{-13}$cm, ${}^1\alpha = (-23.69 \times 10^{-13}\text{cm})^{-1}$, we get

$$b - a = 1.64 \times 10^{13}\text{cm}^{-1}, \text{ for spin singlet state.} \tag{14''}$$

From this consideration, for S-wave we assume

$$\eta_0(k, r) = \begin{cases} \sin(kr + \delta_0), & r \leq D, \\ \sin(kD + \delta_0)e^{-\beta(r-D)}, & r > D. \end{cases} \tag{15}$$

Then, from (8)

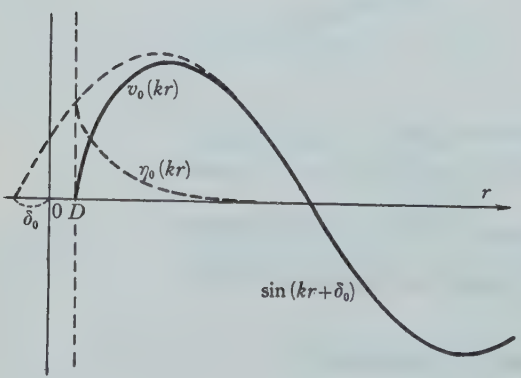

Fig. 2. S-wave part of the isolated two nucleon wave function. (schematical). $v_0(kr)$ is defined by $L_0(kr) = v_0(kr)/kr$.

$$L_0(kr) = \begin{cases} 0, & r \leq D, \\ 1/kr[\sin(kr + \delta_0) - \\ \sin(kD + \delta_0)e^{-\beta(r-D)}, & r > D, \\ 1/kr \sin(kr + \delta_0), & r \to \infty. \end{cases} \tag{16}$$

Comparing (14) with (11), (12) and (16), we can see that the distortion parameter β corresponds to the value of $b - a$. Thus, from eqs. (14'), (14'') and Fig. 1, it may be reasonable to choose

$$\beta = \begin{cases} 2.80 \times 10^{13}\text{cm}^{-1}, & \text{for } {}^3S_1 \text{ state,} \\ 1.64 \times 10^{13}\text{cm}^{-1}, & \text{for } {}^1S_0 \text{ state.} \end{cases} \tag{17}$$

Behavior of $L_0(kr)$ is schematically shown in Fig. 2.

We can introduce the analogous modifications also for the higher partial wave functions. Thus the wave functions of the relative motion are obtained as follows:

$$\psi_k(\boldsymbol{r})=\begin{cases} 0, & r\leq D, \\ e^{ikr}+\sum_{l=0}^{\infty}(2l+1)i^l\{e^{i\delta_l}\cdot L_l(kr)-j_l(kr)\}P_l(\cos\theta), & r>D. \\ e^{ikr}+\sum_{l=0}^{\infty}(2l+1)i^l\cdot(1/kr)\{e^{i\delta_l}\sin(kr-l\pi/2+\delta_l)-\sin(kr-l\pi/2)\} & \\ \qquad\times P_l(\cos\theta), & r\to\infty. \end{cases} \quad (18)$$

A simple rearrangement shows that the asymptotic form of the above equation is nothing but the familiar form,

$$\psi_k(\boldsymbol{r})\to e^{ikr}+(e^{ikr}/r)f(\theta), \text{ as } r\to\infty,$$

with

$$f(\theta)=1/2ik\cdot\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta).$$

(*c*) Effects from the surrounding nuclear matter: Pauli's exclusion principle

So far, we have discussed the situation of the isolated two nucleon system. Let us now consider the same two nucleon system imbedded and moving in the nuclear matter, and introduce the necessary corrections into the wave functions obtained above. We shall carry out this procedure by two steps. The first is the consideration of the effect of Pauli's principle which is to replace ordinary matrix elements by the difference of the direct and exchange matrix elements, and to exclude "filled" intermediate states. The former procedure involves only a simple change of notation, and the latter we shall take into account here. The second step is the multiple scattering effect, which will be discussed later.

The Hamiltonian for the two particles is

$$H(1,2)=H_0(1;2)+V_{12},$$

where

$$H_0(1;2)=H(1)+H(2), \quad H(1)\phi_n(1)=\varepsilon_n\phi_n(1), \text{ etc.}$$

We can define a set of product wave function from two orthonormal sets $\phi_n(1)$ and $\phi_n(2)$ as

$$\Phi_n(1,2)=\phi_n(1)\cdot\phi_n(2), \quad (n=0,1,2,\cdots\cdots)$$

which satisfies

$$H_0\Phi_n=E_n\Phi_n.$$

The exact wave function of the two nucleon sytem satisfies

$$H\Psi=E\Psi.$$

Here we introduce the wave matrix

$$\Omega=1+\frac{P}{e}V\Omega,$$

where

$$e=E-H_0, \qquad \frac{P}{e}=\sum_{n\neq 0}\frac{|\Phi_n\rangle\langle\Phi_n|}{E-E_n}.$$

Then, the following relation is held as well-known,

$$\Psi=\Omega\Phi_0=\Phi_0+\frac{P}{e}V\Psi,$$

where Φ_0 is the wave function of the chosen states α and β to which the system would go if the interaction V_{12} were adiabatically turned off.

Let us assume that these states are separable to the relative motion and the center-of-mass motion, and define as follows:

$$\Phi_n=\Phi_n(R)\cdot\varphi_n(r), \quad \Psi=\Psi(R)\psi(r),$$

which satisfy

$$H_0(r)\varphi_n(r)=\epsilon_n\cdot\varphi_n(r), \quad H(r)\cdot\psi(r)=\epsilon\cdot\psi(r),$$

where

$$H_0(1,2)=H_0(r)+H_0(R), \quad H(r)\simeq H_0(r)+V_{12}(r).$$

Further, we neglect the center-of-mass motion*) and assume the Hermiticity of H_0 and the consequent orthonormality of Φ_n: $\langle\Phi_m|\Phi_n\rangle=\delta_{mn}$, which follows

$$\psi(\boldsymbol{r})=\varphi_0(\boldsymbol{r})+\sum_{n\neq 0}\frac{\varphi_n(\boldsymbol{r})\langle\varphi_n(\boldsymbol{r}')|V(\boldsymbol{r}')|\psi(\boldsymbol{r}')\rangle}{\epsilon-\epsilon_n}. \tag{19}$$

For the isolated nucleon pairs, $H_0=T_1+T_2$ and all the intermediate state can be taken except the chosen states α and β. Therefore (19) is written explicitly as:

$$\psi_k(\boldsymbol{r})=e^{i\boldsymbol{k}\boldsymbol{r}}+\frac{P}{(2\pi)^3}\int_0^\infty d\boldsymbol{k}'\int_0^\infty d\boldsymbol{r}'\,\frac{V(\boldsymbol{r}')\psi_k(\boldsymbol{r}')e^{i\boldsymbol{k}'(\boldsymbol{r}-\boldsymbol{r}')}}{(k^2-k'^2)/M}, \tag{20}$$

where

$$\boldsymbol{k}=(\boldsymbol{k}_\alpha-\boldsymbol{k}_\beta)/2.$$

---

*) Weisskopf et al.[9] have remarked such approximation and noted that the more detailed calculation of cases with the center-of-mass momentum $\boldsymbol{K}\neq 0$ has shown that the main features of the solution are the same as in the case $\boldsymbol{K}=0$. Our present aim of approaching the Bethe-Goldstone equation is not to make use of the results themselves, but to get qualitative information about the effect of Pauli's exclusion principle. So far, the following discussions based on such information would not be so affected by this approximation.

We can transform (20) into the differential equation

$$1/M(\nabla^2+k^2)\psi_k(\boldsymbol{r}) = 1/(2\pi)^3\int_0^\infty d\boldsymbol{k}'\int_0^\infty d\boldsymbol{r}' V(\boldsymbol{r}')\psi_k(\boldsymbol{r}')e^{i\boldsymbol{k}'(\boldsymbol{r}-\boldsymbol{r}')}$$

$$= \int_0^\infty d\boldsymbol{r}'\delta(\boldsymbol{r}-\boldsymbol{r}')V(\boldsymbol{r}')\psi_k(\boldsymbol{r}')$$

$$= V(\boldsymbol{r})\psi_k(\boldsymbol{r}). \tag{20'}$$

This is just an ordinary Schrödinger equation of two-body scattering problem. Consequently its solution $\psi_k(\boldsymbol{r})$ is nothing but the one given by (18).

For the imbedded nucleon pairs, Pauli's exclusion principle excludes the occupied intermediate states of the pair nucleons and the summation in (19) should be carried out only outside the Fermi sphere, which modifies (20) to the expression

$$\psi_k(\boldsymbol{r}) = e^{i\boldsymbol{k}\boldsymbol{r}} + \frac{1}{(2\pi)^3}\int_{k_F}^\infty d\boldsymbol{k}'\int_0^\infty d\boldsymbol{r}'\frac{V(\boldsymbol{r}')\psi_k(\boldsymbol{r}')e^{i\boldsymbol{k}'(\boldsymbol{r}-\boldsymbol{r}')}}{(k^2-k'^2)/M}, \tag{21}$$

where now the principal value operator P can be dropped. The corresponding differential equation comes out to be

$$\frac{1}{M}(\nabla^2+k^2)\psi_k(\boldsymbol{r}) = 1/(2\pi)^3\int_{k_F}^\infty d\boldsymbol{k}'\int_0^\infty d\boldsymbol{r}'\ V(\boldsymbol{r}')\psi_k(\boldsymbol{r}')e^{i\boldsymbol{k}'(\boldsymbol{r}-\boldsymbol{r}')}$$

$$= \int_0^\infty d\boldsymbol{k}'\int_0^\infty d\boldsymbol{r}' - \int_0^{k_F} d\boldsymbol{k}'\int_0^\infty d\boldsymbol{r}' \tag{21'}$$

$$= V(\boldsymbol{r})\psi_k(\boldsymbol{r}) - 1/(2\pi)^3\int_0^{k_F} d\boldsymbol{k}'\int_0^\infty d\boldsymbol{r}' V(\boldsymbol{r}')\psi_k(\boldsymbol{r}')e^{i\boldsymbol{k}'(\boldsymbol{r}-\boldsymbol{r}')},$$

the consequences of which has been explored by Bethe-Goldstone,[8] Weisskopf et al.[9] and others. The main consequence from the modified Schrödinger equation (21') is as follows: near the hard core, the wave function suffers strong distortion following the rapid variation, which means the high momentum components are effective in the k'-integration of (21'). Therefore,

$$\int_{k_F}^\infty d\boldsymbol{k}' \approx \int_0^\infty d\boldsymbol{k}', \text{ or } \quad \int_0^{k_F} d\boldsymbol{k}^2 \approx 0, \text{ for } r \gtrsim D, \tag{22}$$

which implies that the wave function (21) is almost the same as (20) near the hard core. Thus the situation discussed in (***b***) does not suffer serious changes from the exclusion principle. On the other hand, the asymptotic form of the wave function must be substantially influenced by the exclusion principle. In fact, in consequence of the short range nature of the nuclear force $V(\boldsymbol{r}')$, the contribution from the region of large $\boldsymbol{r}'$ is almost negligible, hence in the asymptotic region $(r\to\infty)$, the argument $\boldsymbol{k}'(\boldsymbol{r}-\boldsymbol{r}')$, of exponential is so large and the contributions to the integral would come only from the region of very small $\boldsymbol{k}'$, were it not cut off

by the exclusion principle. Thus, for the asymptotic form,

$$\int_0^{k_F} d\boldsymbol{k}' \approx \int_0^{\infty} d\boldsymbol{k}', \text{ or } \int_{k_F}^{\infty} d\boldsymbol{k}' \approx 0, \text{ for } r \to \infty. \tag{23}$$

Hence, (21) becomes

$$\psi_k(\boldsymbol{r}) \to e^{ikr}, \text{ for } r \to \infty. \tag{24}$$

This means that the wave function contains no real scattered wave at all but only a distortion near the scattering center, that is to say, the exclusion principle forbids the elastic scattering of any nucleons in the Fermi sea.

Now, let us turn back to our problem. In conformity with above conclusion, we must modify the distorted S-wave (16) in the following form:

$$L_0(kr) = 1/kr[\{\sin kr + (\sin(kr+\delta_0) - \sin kr)e^{-\alpha r}\} - \{\sin kD + (\sin(kD+\delta_0) - \sin kD)e^{-\alpha D}\}e^{-\beta(r-D)}]. \tag{25}$$

Instead of (18), then, the correction terms defined by (11), (12) now come out to be

$$u_k^{(s)}(r) = \begin{cases} 0, & r \leq D, \\ e^{i\bar{\delta}_0} L_0(kr) - j_0(kr), & \lambda \gtrsim r > D, \\ 0, & r \gg \lambda, \end{cases} \tag{26}$$

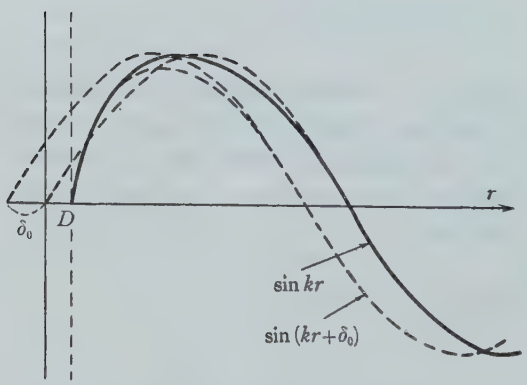

Fig. 3 The modified S-wave function.
—— the imbedded nucleon pair
.... the isolated nucleon pair

where

$$\bar{\delta}_0 = \delta_0 \ e^{-\alpha r}, \qquad \lambda = \alpha^{-1},$$

and λ corresponds to the so-called "healing distance" introduced by Weisskopf et al[9]. The magnitude of which, from the analysis of wave function (20), we have given as[10]

$$\lambda = D\left[\frac{\pi}{Dk_F}\left(2 - \kappa \ln\frac{1+\kappa}{1-\kappa}\right)^{-1} - 1\right], \quad \text{for } \kappa^2 \lesssim Dk_F, \tag{27}$$

where $\kappa = k/k_F$. Conveniently, we shall take the mean value

$$\bar{\lambda} = 1.24 \times 10\text{cm}^{-13}. \tag{27'}$$

The feature of $L_0(kr)$ is shown schematically in Fig. 3.

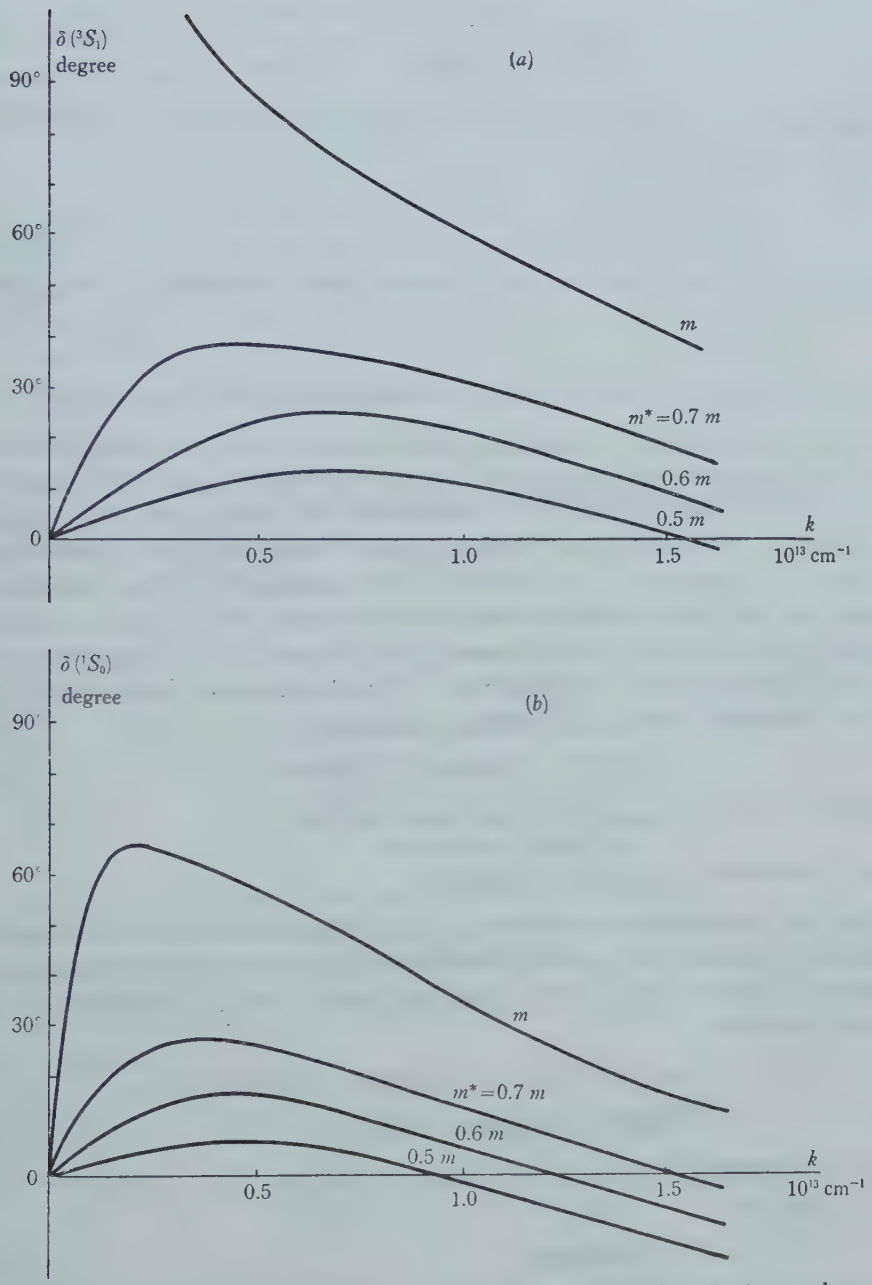

Fig. 4 The phase shifts δ_0^* which have been computed from the effective nuclear force (2). (a) For triplet S state. (b) For singlet S state.

It will not be too difficult to carry out the similar modification for higher partial waves to get

$$L_l(kr)=\begin{cases} 0, & r\leq D, \\ j_l(kr), & r\gg\lambda, \end{cases} \tag{28}$$

and then the wave function of the relative motion (18) should be replaced by

$$\psi_k(\boldsymbol{r})=e^{ikr}+\sum_l u_k^{(l)}(\boldsymbol{r})$$

$$=\begin{cases} 0, & r\leq D, \\ e^{ikr}+\sum_l(2l+1)i^l\{e^{i\delta_l}L_l(kr)-j_l(kr)\}\cdot P_l(\cos\theta), & D<r\lesssim\lambda, \\ e^{ikr}, & r\gg\lambda. \end{cases} \tag{29}$$

(*d*) Effects from the surrounding nuclear matter: effective mass

As has been shown by Brueckner and Levinson[1], the considerable parts of the interactions between the chosen system and the other particles of the nuclear matter can be included through the introduction of a one particle common potential V_c, which has to be determined self-consistently from the wave functions $\psi_k(\boldsymbol{r})$ themselves. In general, V_c comes out to be quadratically velocity dependent and allows us the use of the effective mass approximation. In conformity with these arguments, we replace the nucleon mass M in (20) and (21) by an effective mass M^*, and further, replace the phase shift δ_l by the effective phase shift δ_l^*, which should be computed from the Schrödinger equation

$$1/M^*(p^2+k^2)\psi_k(\boldsymbol{r})=V(\boldsymbol{r})\psi_k(\boldsymbol{r}). \tag{30}$$

Some results of such computations are shown in Fig. 4.

(*e*) Definition of two-body functions

From our fundamental assumption (1) for the ground state wave function of the nucleus, we define the two-body function as follows: the wave function of the nucleon pair 1 and 2 which are in quantum states α and β, respectively, is given in nuclear matter by

$$\Psi_{\alpha\beta}(1,2)=\Phi_{\alpha\beta}(1,2)f_{\alpha\beta}(1,2)S_{\alpha\beta}=\phi_\alpha(1)\phi_\beta(2)f_{\alpha\beta}(1,2)S_{\alpha\beta}$$
$$=\Phi_K(\boldsymbol{R})\varphi_k(\boldsymbol{r})f_{\alpha\beta}(1,2)S_{\alpha\beta}, \tag{31}$$

where each definition is as previously given in (1) and (5). This wave function is also written as

$$\Psi_{\alpha\beta}(1,2)=\Psi_K(\boldsymbol{R})\psi_k(\boldsymbol{r}) \tag{31'}$$

with $\psi_k(\boldsymbol{r})$ given in (29) and referred also to (30). In accordance with the arguments in § 2, (c), we obtain, by comparing (31) and (31'), the following relation:

$$f_{\alpha\beta}(1,2)=f_k(\boldsymbol{r})=1+[u_k^{(s)}(\boldsymbol{r})+u_k^{(p)}(\boldsymbol{r})+u_k^{(d)}(\boldsymbol{r})+\cdots]e^{-ikr} \tag{32}$$

$$\simeq 1+u_k^{(s)}(\boldsymbol{r})e^{-ikr}. \tag{32'}$$

This function depends on the relative distance, relative momentum and spin-isospin values of the nucleon pair. It involves the effective mass implicitly through the effective phase shift which is the only one parameter to be determined self-consistently. Throughout the following calculations, we shall use the approximation (32′). But it should be remarked that (32′) *not only includes a distorted S-wave, but also the unperturbed higher angular momentum partial waves* ($l \geq 1$). It is worth noting that contributions of the higher partial waves come mainly from the region of r much larger than the core radius where the distortions of wave functions are already healed up by virtue of the effect of the exclusion principle. Consequently, it is expected that (32′) comes out to be a good approximation. Some features of $f_{\alpha\beta}(r)$ are shown in Fig. 5.

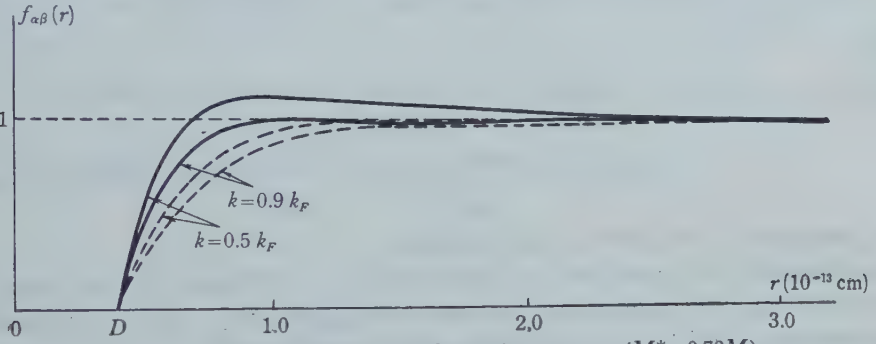

Fig. 5 Two-body functions for the various states. ($M^*=0.73M$)
—— spin triplet state
····· spin singlet state

§ 3. Correlation functions

The expectation value of nuclear potential of the form

$$V=\sum_{\alpha<\beta} V_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)\mathcal{P}, \tag{33}$$

with

$$\mathcal{P}=a_0+a_\sigma\,\sigma_\alpha\sigma_\beta+a_\tau\tau_\alpha\tau_\beta+a_{\sigma\tau}\sigma_\alpha\sigma_\beta\tau_\alpha\tau_\beta$$

can be written as

$$\langle V\rangle=\frac{A(A-1)}{2}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2 \sum_{\substack{\alpha<\beta \\ \alpha'<\beta'}}\Big(S_{\alpha\beta}(1,2)\,|g(\boldsymbol{r}_1,\boldsymbol{r}_2)|\,S_{\alpha'\beta'}(1,2)\Big) \times\Big(S_{\alpha'\beta'}(1,2)\,V(\boldsymbol{r}_1,\boldsymbol{r}_2)\,S_{\alpha\beta}(1,2)\Big), \tag{34}$$

where

$$\Big(S_{\alpha\beta}(1,2)\,|g(\boldsymbol{r}_1,\boldsymbol{r}_2)|\,S_{\alpha\beta}(1,2)\Big)=\int d\boldsymbol{x}_3 d\boldsymbol{x}_4\cdots d\boldsymbol{x}_A \Psi^*(\boldsymbol{r}_1,\boldsymbol{r}_2,S(1,2),\boldsymbol{x}_3,\cdots\boldsymbol{x}_A)\,\Psi(\ ,,\ ) \tag{35}$$

is the so-called "correlation matrix" as denoted by L. Rosenfeld.[11]
In the perturbation theory, the nuclear wave function Ψ comes out to be

$$\Psi=1/\sqrt{1+\sum_{\lambda}|C_{\lambda}|^2}(\Psi_0+\sum_{\lambda}C_{\lambda}\Psi_{\lambda}), \tag{36}$$

where Ψ_0 and Ψ_λ are the Slater determinants corresponding to the ground state and excited states configurations of the Fermi gas. The two first approximations of the correlation matrix are thus found to be

$$\begin{aligned}(S_{\beta\alpha}(1,2)|g|S_{\alpha\beta}(1,2))^{(0)}&=1/A(A-1)\sum_{k_\alpha,k_\beta}^{(occ)}\phi_{k_\alpha}{}^*(\boldsymbol{r}_1)\phi_{k_\beta}{}^*(\boldsymbol{r}_2)\\&\times[(S_{\alpha\beta}(1,2))|1|S_{\alpha\beta}(1,2))\phi_{k_\alpha}(\boldsymbol{r}_1)\phi_{k_\beta}(\boldsymbol{r}_2)\\&-(S_{\alpha\beta}(1,2)|P^\sigma P^\tau|S_{\alpha\beta})\phi_{k_\beta}(\boldsymbol{r}_1)\phi_{k_\alpha}(\boldsymbol{r}_2)],\end{aligned} \tag{37a}$$

$$\begin{aligned}(S_{\alpha\beta}(1,2)|g|S_{\alpha\beta}(1,2))^{(1)}&=1/A(A-1)2\\&\times\mathcal{R}\sum_{k_\alpha k_\beta k_{\alpha'}k_{\beta'}}^{(occ)(unocc)}\phi_{k_\alpha}{}^*(\boldsymbol{r}_1)\phi_{k_\beta}{}^*(\boldsymbol{r}_2)\phi_{k_{\alpha'}}(\boldsymbol{r}_1)\phi_{k_{\beta'}}(\boldsymbol{r}_2)(S_{\alpha\beta}(1,2)|C_\lambda|S_{\alpha'\beta'}(1,2)),\end{aligned} \tag{37b}$$

where

$$\begin{aligned}(S_{\alpha\beta}(1,2)|C_\lambda|S_{\alpha'\beta'}(1,2))=&\frac{(S_{\alpha\beta}(1,2)|\not{\mathcal{D}}|S_{\alpha'\beta'}(1,2))\ (k_\alpha k_\beta|V(r_1,r_2)|k_{\alpha'}k_{\beta'})}{E_\alpha+E_\beta-E_{\alpha'}-E_{\beta'}}\\&-\frac{(S_{\alpha\beta}(1,2)|\not{\mathcal{D}}P^\sigma P^\tau|S_{\alpha'\beta'}(1,2))\ (k_\alpha k_\beta|V(r_1r_2)|k_{\beta'}k_{\alpha'})}{E_\alpha+E_\beta-E_{\alpha'}-E_{\beta'}},\end{aligned} \tag{38}$$

is the coefficient of the perturbation treatment.

In these formula, the summation is to be extended, according to the indication (*occ*) or (*unocc*), over all occupied or all unoccupied individual states of momentum $\boldsymbol{k}$ in the configuration Ψ_0, each state being counted only once.

As well known, Volz calculated the diagonal matrix elements of g by (37) and obtained the results as schematically indicated in Fig. 6.[12]

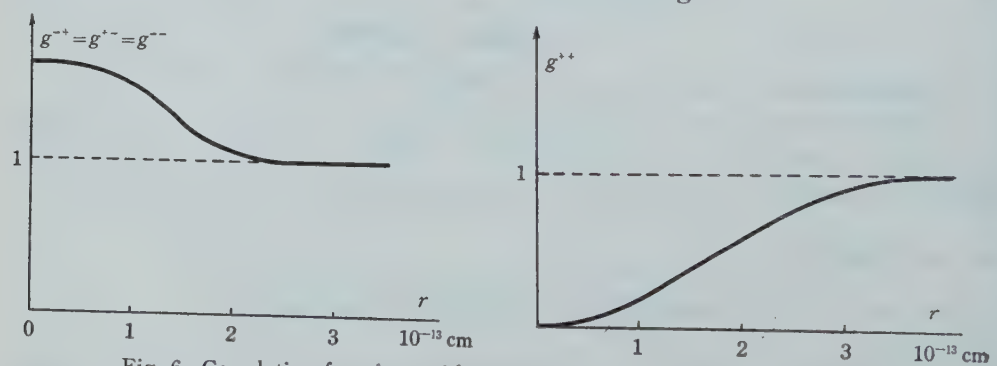

Fig. 6 Correlation functions with respect to I. P. M. aspects of nucleus.

Obviously Volz's calculations fail when the hard core is present in the nuclear potential.

In our model, two-dody wave function $\phi_{k_\alpha}(\boldsymbol{r}_1)\ \phi_{k_\beta}(\boldsymbol{r}_2)$ in (37) should be replaced by $\Psi_{\alpha\beta}(1,2)\ f_{\alpha\beta}(1,2)$, thus we obtain

$$\Big(S_{\alpha\beta}(1,2)|g(r)|S_{\alpha\beta}(1,2)\Big)=\frac{1}{A(A-1)}\ \frac{1}{\Omega^2}\sum_{k_\alpha k_\beta}^{(occ)}g^{(\alpha,\beta)}(\boldsymbol{r})=\frac{1}{\Omega^2}\ \frac{3}{2k_F{}^3}\int_0^{k_F}P(k)g_k^{(\alpha,\beta)}(r)dk, \tag{39}$$

where

$$P(k)=\frac{k^2}{2}\left\{(k/k_F)^3-3(k/k_F)+2,\right\}, \tag{40}$$

$$g^{(\alpha,\beta)}(\boldsymbol{r})=f_{\alpha\beta}{}^*(\boldsymbol{r})[(S_{\alpha\beta}|1|S_{\alpha\beta})f_{\alpha\beta}(\boldsymbol{r})-(S_{\alpha\beta}|P^\sigma P^\tau|S_{\alpha\beta})f_{\beta\alpha}(\boldsymbol{r})]. \tag{41}$$

Substituting $f_{\alpha\beta}(\boldsymbol{r})$ from (32), we get the correlation functions for two nucleons of specified spin, isospin and relative momentum:

$$\begin{cases} g_k^{(3,3)}(r)=1-j_0(2kr)-2|{}^3u_k^{(p)}(r)|^2+\cdots\cdots, & (42\cdot\text{a}) \\ g_k^{(3,1)}(r)=1+j_0(kr)+2|{}^3u_k^{(s)}(r)|^2+4{}^3u_k^{(s)}(r)\,j_0(kr)+2[{}^3u_k{}^{*(p)}(r)\,j_0(kr) \\ \qquad +{}^3u_k^{(s)}(r)\,{}^3u_k{}^{*(p)}(r)]+{}^3u_k{}^{*(p)}(r)[{}^3u_k^{(p)}(r)+{}^3u_k{}^{*(p)}(r)]+\cdots, & (42\cdot\text{b}) \\ g_k^{(1,3)}(r)=\textit{same as } g_k^{(3,1)}(r) \textit{ except the exchange of } 3 \textit{ and } 1, & (42\cdot\text{c}) \\ g_k^{(1,1)}(r)=\textit{same as } g_k^{(3,3)}(r) \textit{ except the exchange of } 3 \textit{ and } 1, & (42\cdot\text{d}) \end{cases}$$

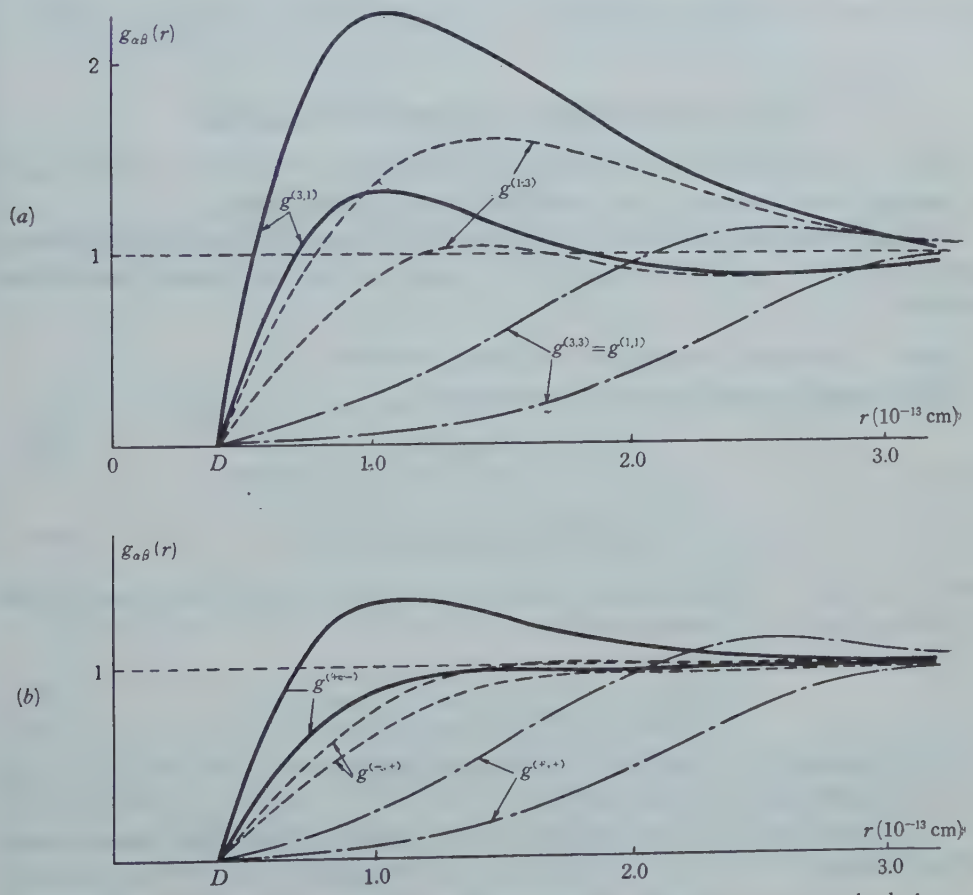

Fig. 7 The correlation functions for two nucleons of specified spin, isospin and relative momentum. Notations should be referred to eqs. (42) and (43). $M^*=0.73M$.

where $g_k^{(3,1)}(r)$ denotes the correlation function for the nucleon pair of spin triplet and isospin singlet state, etc.

It is easy to translate (42.a) to (42.d) in the following form,

$$\begin{cases} g_k^{(+,+)}(r) = 1 - j_0(2kr) - 2|{}^3u_k^{(p)}(r)|^2 + \cdots\cdots, & (43\cdot a) \\ g_k^{(+,-)}(r) = 1 + 2\,{}^3u_k^{(s)}(r)\,j_0(kr) + |{}^3u_k^{(s)}(r)|^2 - |{}^3u_k^{(p)}(r)|^2 + \cdots\cdots, & (43\cdot b) \\ g_k^{(-,+)}(r) = 1 + 2\,{}^1u_k^{(s)}(r)\,j_0(kr) + |{}^1u_k^{(s)}(r)|^2 - |{}^1u_k^{(p)}(r)|^2 + \cdots\cdots, & (43\cdot c) \\ g_k^{(-,-)}(r) = 1 + [{}^3u_k^{(s)}(r) + {}^1u_k^{(s)}(r)]\,j_0(kr) + 1/2[|{}^3u_k^{(s)}(r)|^2 + |{}^1u_k^{(s)}(r)|^2 + \cdots\cdots \\ \qquad\qquad = 1/2\,(g^{(+,-)} + g^{(-,+)}), & (43\cdot d) \end{cases}$$

where $g^{(+,-)}(r)$ denotes the correlation function for the nucleon pair of spin even and isospin odd state, etc. In fact, the following relations hold,

$$\begin{aligned} g(r) &= 9g^{(3,3)} + 3g^{(3,1)} + 3g^{(1,3)} + g^{(1,1)} \\ &= 4g^{(+,+)} + 4g^{(+,-)} + 4g^{(-,+)} + 4g^{(-,-)}, \text{ etc} \end{aligned} \qquad (44)$$

Results of the computations obtained by the approximations (32′) are shown in Fig. 7.

§ 4. Volume energy and nuclear saturation

In § 2, the effective mass was left undetermined. We shall carry it out in this section in the course of the calculation of the volume energy.

In our procedure, the *course* of treatment is based on Brueckner-Levinson's approach, and the *actual calculation* was carried out by the cluster development method.

(*a*) The cluster development; Effective potentials

The nuclear volume energy is represented as

$$E = (\Psi,\ H\Psi)/(\Psi, \Psi), \qquad (45)$$

where

$$H = -(\hbar^2/2M)\cdot\sum_i \nabla_i^2 + \sum_{\alpha<\beta} V_{\alpha\beta}(r_{ij}), \qquad (46)$$

where $V_{\alpha\beta}(r_{ij})$ is two-body nuclear potential which has already been given by (2). Ψ is, according to (1), assumed to be

$$\Psi = \frac{1}{\sqrt{A!}} \sum_{\left(\cdots\alpha,\beta,\cdots\right)}^{\left(1,2,\cdots i,j\cdots A\right)} \varepsilon_p \prod_\alpha^A \phi_{p\alpha}(\boldsymbol{x}_i) \prod_{\alpha<\beta}^A f_{\alpha\beta}(\boldsymbol{x}_i,\ \boldsymbol{x}_j), \qquad (47)$$

where the summation is taken over all the permutations $P = \left(\cdots\alpha,\beta,\cdots\right)^{\left(1,2,\cdots i,j\cdots A\right)}$. ε_p is $+1$ or -1 according as P is even or odd.

The volume energy E can be expanded in conformity with the cluster expansion as follows:

$$E = E^{(1)} + E^{(2)} + E^{(3)} + \cdots\cdots. \qquad (48)$$

$E^{(1)}$ corresponds to a Fermi gas energy, $E^{(2)}$ arises from the two-body clusters and $E^{(3)}$ consists of the correction terms due to the three-body clusters, etc. $E^{(1)}$, $E^{(2)}$, $E^{(3)}, \cdots$ are found to be as follows:

$$E^{(1)}=\sum_{\alpha}\int\phi_\alpha^*(\boldsymbol{r}_1)\,(-\hbar^2/2M\nabla_1^2)\phi_\alpha(\boldsymbol{r}_1)\,d\boldsymbol{r}_1, \tag{49}$$

$$E^{(2)}=1/2\sum_{\alpha,\beta}\Big[\sum_{\binom{i,j}{\alpha,\beta}}\varepsilon_p\int\phi_\alpha^*(\boldsymbol{r}_1)\phi_\beta^*(\boldsymbol{r}_2)f_{\alpha\beta}^*(\boldsymbol{r}_1,\boldsymbol{r}_2)H_{\alpha\beta}(1,2)\phi_i(\boldsymbol{r}_1)\phi_j(\boldsymbol{r}_2)f_{ij}(\boldsymbol{r}_1,\boldsymbol{r}_2)d\boldsymbol{r}_1d\boldsymbol{r}_2, \tag{50}$$

$$E^{(3)}=E_1^{(3)}+E_2^{(3)}+E_3^{(3)}+E_4^{(3)}, \tag{51}$$

where

$$E_1^{(3)}=-\sum_{\alpha,\beta,\gamma}\Big[\sum_{\binom{i,j}{\alpha,\beta}}\varepsilon_p\int\phi_\alpha^*(\boldsymbol{r}_1)\phi_\beta^*(\boldsymbol{r}_2)f_{\alpha\beta}^*(\boldsymbol{r}_1,\boldsymbol{r}_2)H_{\alpha\beta}(1,2)\phi_i(\boldsymbol{r}_1)\phi_j(\boldsymbol{r}_2)f_{ij}(\boldsymbol{r}_1,\boldsymbol{r}_2)$$
$$\times d\boldsymbol{r}_1\,d\boldsymbol{r}_2\sum_{\binom{j,k}{\beta,\gamma}}\phi_\beta^*(\boldsymbol{r}_2)\phi_\gamma^*(\boldsymbol{r}_3)h_{\beta\gamma}(2,3)\phi_j(\boldsymbol{r}_2)\phi_k(\boldsymbol{r}_3)d\boldsymbol{r}_2\cdot d\boldsymbol{r}_3\Big], \tag{51·a}$$

$$E_2^{(3)}=1/2\sum_{\alpha,\beta,\gamma}\Big[\sum_{\binom{i,j,k}{\alpha,\beta,\gamma}}\varepsilon_p\int\phi_\alpha^*(\boldsymbol{r}_1)\phi_\beta^*(\boldsymbol{r}_2)\phi_\gamma^*(\boldsymbol{r}_3)f_{\alpha\beta}^*(\boldsymbol{r}_1,\boldsymbol{r}_2)h_{\beta\gamma}^*(2,3)$$
$$\times[2+h_{\gamma\alpha}^*(3,1)]H_{\alpha\beta}(1,2)\phi_i(\boldsymbol{r}_1)\phi_j(\boldsymbol{r}_2)\phi_k(\boldsymbol{r}_3)f_{ij}(\boldsymbol{r}_1\boldsymbol{r}_2)h_{jk}(2,3)$$
$$\times[2+h_{ki}(3,1)]d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,d\boldsymbol{r}_3\Big], \tag{51·b}$$

$$E_3^{(3)}=1/4\sum_{\alpha,\beta,\gamma,\delta}\Big[\sum_{\binom{i,j,k,l}{\alpha,\beta,\gamma,\delta}}\varepsilon_p\int\phi_\alpha^*(\boldsymbol{r}_1)\phi_\beta^*(\boldsymbol{r}_2)\phi_\gamma^*(\boldsymbol{r}_3)\phi_\delta^*(\boldsymbol{r}_4)f_{\alpha\beta}^*(\boldsymbol{r}_1\boldsymbol{r}_2)h_{\gamma\delta}(3,4)$$
$$\times H_{\alpha\beta}(1,2)\phi_i(\boldsymbol{r}_1)\phi_j(\boldsymbol{r}_2)\phi_k(\boldsymbol{r}_3)\phi_l(\boldsymbol{r}_4)f_{ij}(\boldsymbol{r}_1,\boldsymbol{r}_2)h_{kl}(3,4)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,d\boldsymbol{r}_3\,d\boldsymbol{r}_4\Big], \tag{51·c}$$

$$E_4^{(3)}=1/6\sum_{\alpha,\beta,\gamma}\Big[\sum_{\binom{i,j,k}{\alpha,\beta,\gamma}}\varepsilon_p\int\phi_\alpha^*(\boldsymbol{r}_1)\phi_\beta^*(\boldsymbol{r}_2)\phi_\gamma^*(\boldsymbol{r}_3)f_{\alpha\beta}^*(\boldsymbol{r}_1,\boldsymbol{r}_2)f_{\beta\gamma}^*(\boldsymbol{r}_2,\boldsymbol{r}_3)f_{\gamma\alpha}^*(\boldsymbol{r}_3,\boldsymbol{r}_1)$$
$$\times[-\hbar^2/2M(\nabla_1^2+\nabla_2^2+\nabla_3^2)]\phi_i(\boldsymbol{r}_1)\phi_j(\boldsymbol{r}_2)\phi_k(\boldsymbol{r}_3)$$
$$\times f_{ij}(\boldsymbol{r}_1,\boldsymbol{r}_2)f_{jk}(\boldsymbol{r}_2,\boldsymbol{r}_3)f_{ki}(\boldsymbol{r}_3,\boldsymbol{r}_1)d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,d\boldsymbol{r}_3\Big], \tag{51·d}$$

and

$$H_{\alpha\beta}(1,2)=-(\hbar^2/2M)(\nabla_1^2+\nabla_2^2)+V_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_2), \tag{51·e}$$

$$h_{\alpha\beta}(1,2)=f_{\alpha\beta}^*(\boldsymbol{r}_1,\boldsymbol{r}_2)f_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_2)-1. \tag{51·f}$$

$\sum'$ means the sum over the permutations $\binom{i,j,k,l}{\alpha,\beta,\gamma,\delta}$ excluding 1, (α,β), (γ,δ) and $(\alpha,\beta)(\gamma,\delta)$. Here, we are confining our subjects to the nuclear matter with an equal number of protons and neutrons, and the Coulomb forces are omitted. By virtue of our fundamental assumption, the calculations are, though elaborate more definite and foresighted compared with the formal procedure of the multiple scattering aspects.

The results of the volume energy per nucleon is given by

$$\varepsilon = E/A = \varepsilon^{(1)} + \varepsilon^{(2)} + \varepsilon^{(3)} + \cdots\cdots, \tag{52}$$

where

$$\varepsilon^{(1)} = E^{(1)}/A = (3/5)\,(\hbar^2 k_F^{\,2}/2M), \tag{53}$$

$$\varepsilon^{(2)} = E^{(2)}/A = (1/2A\Omega)\cdot\sum_{\alpha,\beta}^{(occ)}\int U^{(\alpha,\beta)}(\boldsymbol{r})\,d\boldsymbol{r}$$

$$= (1/2A\Omega)\sum_{k_\alpha k_\beta}^{<k_F}\int[9U_k^{(3,3)}(\boldsymbol{r}) + 3U_k^{(3,1)}(\boldsymbol{r}) + 3U_k^{(1,3)}(\boldsymbol{r}) + U_k^{(1,1)}(\boldsymbol{r})]d\boldsymbol{r}, \tag{54}$$

in which $U_k(\boldsymbol{r})$ may be regarded as the "effective potential" to the single particle motion in the nuclear matter and given as:

$$U_k^{(\alpha,\beta)}(\boldsymbol{r}) = \psi_{k_\alpha\beta}{}^*(\boldsymbol{r})\,S_{\alpha\beta}{}^*[(-\hbar^2/M)\,(\nabla_r{}^2 + 2i\boldsymbol{k}\nabla_r) + V_{\alpha\beta}(\boldsymbol{r})]\,(\psi_{k_\alpha\beta}(\boldsymbol{r})\,S_{\alpha\beta} - \psi_{k_\beta\alpha}(\boldsymbol{r})\,S_{\beta\alpha})$$
$$= f_{\alpha\beta}{}^*(\boldsymbol{r})\,S_{\alpha\beta}{}^*[(-\hbar^2/M)\,(\nabla_r{}^2 + 2i\boldsymbol{k}\nabla_r) + V_{\alpha\beta}(\boldsymbol{r})]\,(f_{\alpha\beta}(\boldsymbol{r})\,S_{\alpha\beta} - e^{2i\boldsymbol{k}\boldsymbol{r}}\cdot f_{\beta\alpha}(\boldsymbol{r})\,S_{\beta\alpha}), \tag{55}$$

with

$$\boldsymbol{k} = (\boldsymbol{k}_\alpha - \boldsymbol{k}_\beta)/2, \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2.$$

Substituting (32′) in (55), we get

$$U_k^{(3,3)}(\boldsymbol{r}) = 0, \tag{56·a}$$

$$U_k^{(1,1)}(\boldsymbol{r}) = 0, \tag{56·b}$$

$$U_k^{(3,1)}(\boldsymbol{r}) = 2(-\hbar^2/M)\,({}^3u_k^{(s)}(\boldsymbol{r}) + e^{-i\boldsymbol{k}\boldsymbol{r}})\Big(\nabla_r{}^2\,{}^3u_k^{(s)}(\boldsymbol{r}) + k^2\,{}^3u_k^{(s)}(\boldsymbol{r})\Big) + {}^3V(\boldsymbol{r})\,g_k^{(3,1)}(\boldsymbol{r}), \tag{56·c}$$

$$U_k^{(1,3)}(\boldsymbol{r}) = 2(-\hbar^2/M)\,({}^1u_k^{(s)}(\boldsymbol{r}) + e^{-i\boldsymbol{k}\boldsymbol{r}})\Big(\nabla_r{}^2\,{}^1u_k^{(s)}(\boldsymbol{r}) + k^2\,{}^1u_k^{(s)}(\boldsymbol{r})\Big) + {}^1V(\boldsymbol{r})\,g_k^{(1,3)}(\boldsymbol{r}). \tag{56·d}$$

Replacing the double summations with respect to $\boldsymbol{k}_\alpha$, $\boldsymbol{k}_\beta$ by integrations, we get

$$\varepsilon^{(2)} = (1/2\pi^2)\int_0^{k_F} P(k)\,dk\int_0^\infty r^2\,dr[9U^{(3,3)}(k,r) + 3U^{(3,1)}(k,r) + 3U^{(1,3)}(k,r) + U^{(1,1)}(k,r)], \tag{57}$$

where

$$U^{(3,3)}(k,r) = 0 \tag{58·a}$$

$$U^{(1,1)}(k,r) = 0 \tag{58·b}$$

$$U^{(3,1)}(k,r) = 2(-\hbar^2/M)\Big({}^3u^{(s)}(k,r) + j_0(kr)\Big)\Big(\nabla_r{}^3\,{}^2u^{(s)}(k,r) + k^2\,{}^3u^{(s)}(k,r)\Big)$$
$$+ {}^3V(r)\,g^{(3,1)}(k,r), \tag{58·c}$$

$$U^{(1,3)}(k,r) = 2(-\hbar^2/M)\Big({}^1u^{(s)}(k,r) + j_0(kr)\Big)\Big(\nabla_r{}^2\,{}^1u^{(s)}(k,r) + k^2\,{}^1u^{(s)}(k,r)\Big)$$
$$+ {}^1V(r)\,g^{(1,3)}(k,r), \tag{58·d}$$

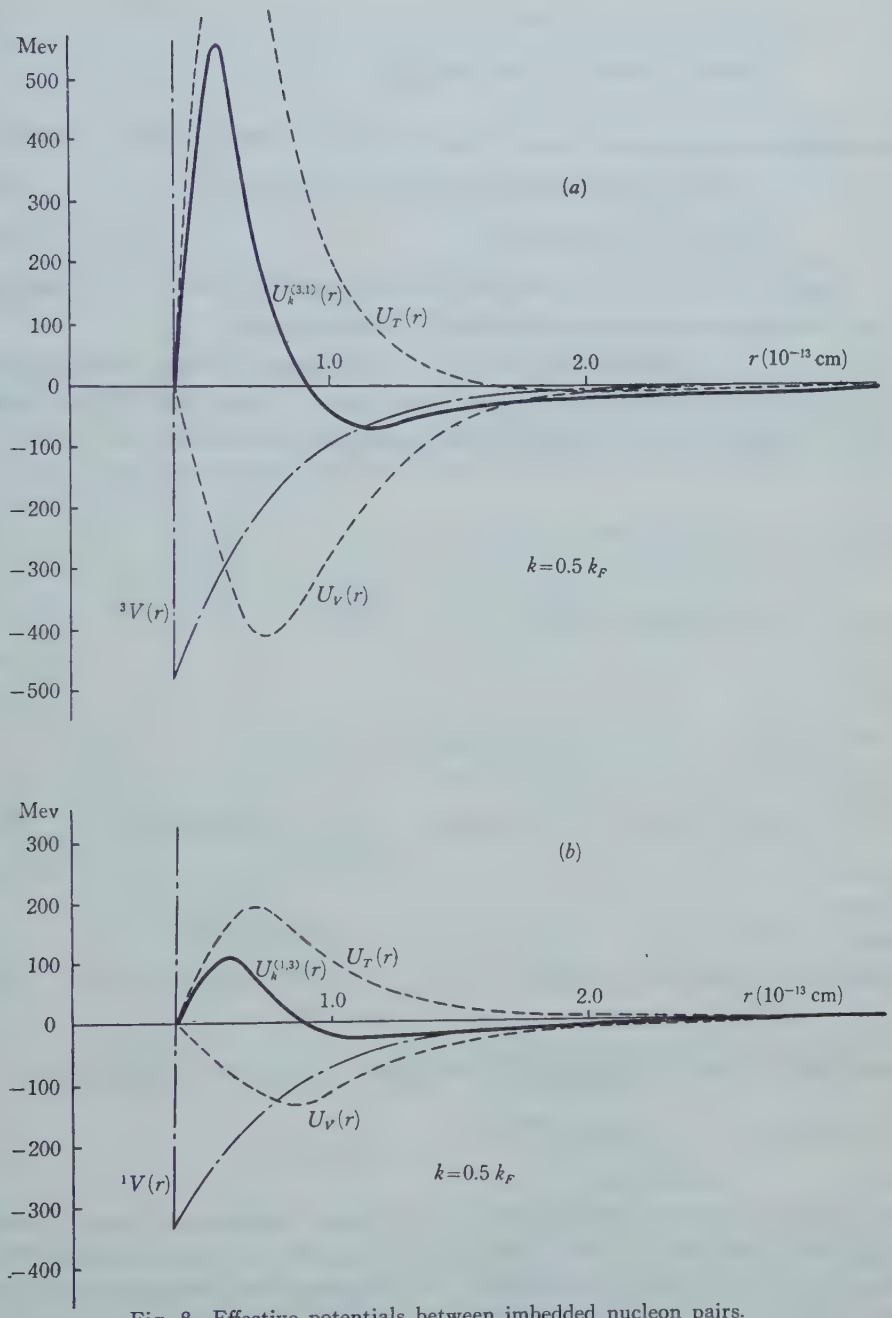

Fig. 8 Effective potentials between imbedded nucleon pairs.
(a) spin triplet, isospin singlet
(b) spin singlet, isospin triplet
($M^*=0.73M$, $k=0.5k_F$)

with

$$U^{(\alpha,\beta)}(k, r)=\int U_k^{(\alpha,\beta)}(\boldsymbol{r})\, d\omega \Big/ \int d\omega.$$

Results of our computations of effective potentials are shown in Fig. 8, where

$$U_T(k, r)=2(-\hbar^2/M)\,(u+j_0(kr))\,(\nabla_r^2 u+k^2 u), \tag{59}$$

$$U_V(k, r)=V(r)g(k, r), \tag{60}$$

and $U(k, r)=U_T(k, r)+U_V(k, r)$ are plotted separately for some spin, isospin states.

(***b***) Self-consistent common potential: Effective mass

In § 2, (d), we have introduced a common potential V_C which should represent the influence of the other particles to the chosen pair. Using the effective potential (56), let us calculate this average common potential self-consistently. We define as

$$\begin{aligned} V_C(M^*, k_\alpha) &= \sum_\beta \int U_k^{(\alpha\,\beta)}(M^*, \boldsymbol{r})\, d\boldsymbol{r} \\ &= 1/2\pi \int_0^\infty Q(k_\alpha, k)\, dk \int_0^\infty r^2 dr \left\{ 3U^{(3,1)}(M^*, k, r)+3U^{(1,3)}(M^*, k, r) \right\}, \end{aligned} \tag{61}$$

where

$$U^{(\alpha,\beta)}(M^*, k, r)=2(-\hbar^2/M^*)\left\{u^{(s)}(\delta_0^*, k, r)+j_0(kr)\right\}$$

$$\times\left\{\nabla_r^2\, u^{(s)}(\delta_0^*, k, r)+k^2\, u^{(s)}(\delta_0^*, k, r)+V'^{(\alpha,\beta)}(r)\, g^{(\alpha,\beta)}(\delta_0^*, k, r)\right\}, \tag{62}$$

and

$$Q(k_\alpha, k)=\begin{cases} 2k, & 0<k<(k_F-k_\alpha)/2, \\ (1/k_\alpha)\,\{k_F^2/4-(k_\alpha/2-k)^2\}, & |k_F-k_\alpha|/2<k<(k_F+k_\alpha)/2, \\ 0, & (k_F+k_\alpha)/2<k. \end{cases} \tag{63}$$

In these effective mass approximation, the effective mass M^* is defined by

$$\frac{1}{k_\alpha^*}\left[\frac{\partial V_C(M^*, k_\alpha)}{\partial k_\alpha}\right]_{k_\alpha=k_\alpha^*}=\frac{1}{M^*}-\frac{1}{M}. \tag{64}$$

As the higher momentum components give the main contribution to the distortion near the hard core as has been discussed in § 2, (***c***), we may choose k_α^* being nearly equal to k_F (Fermi momentum). In this way, the velocity dependent one-particle potential is computed self-consistently after the fifth iteration. The result is shown in Fig. 9.

It is found out from the figure that the effective mass M^* takes the value

$$M^*=0.73M, \tag{65}$$

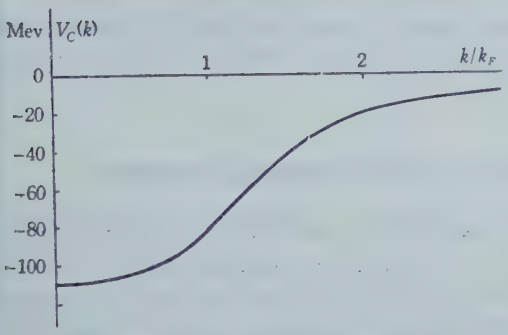

Fig. 9 The velocity dependent one-particle potential. ($M^*=0.73M$)

according to k_α^* nearly equal to k_F.

(*c*) Nuclear saturation

From eqs. (53), (57) and (62), we can derive the volume energy as the function of nuclear density $\rho=(4\pi/3r_0^3)^{-1}$, where r_0 is a mean radius of a nucleon and related to nuclear radius $R=A^{1/3}r_0$ and also to Fermi momentum $k_F=1.523/r_0$. Using the effective mass $M^*=0.73M$, the results of the calculation are shown in Fig. 10, where $\varepsilon^{(1)}$ is the mean kinetic energy of a particle in the ground state of a Fermi gas given by (53), $\varepsilon_T^{(2)}$ is the effective kinetic energy due to two body correlations given by

$$\varepsilon_T^{(2)}=(2\pi^2)^{-1}\int_0^{k_F}P(k)\ dk\int_0^{\infty}r^2\ dr\ 3\left\{U_T^{(3,1)}(k,r)+U_T^{(1,3)}(k,r),\right\} \qquad (66)$$

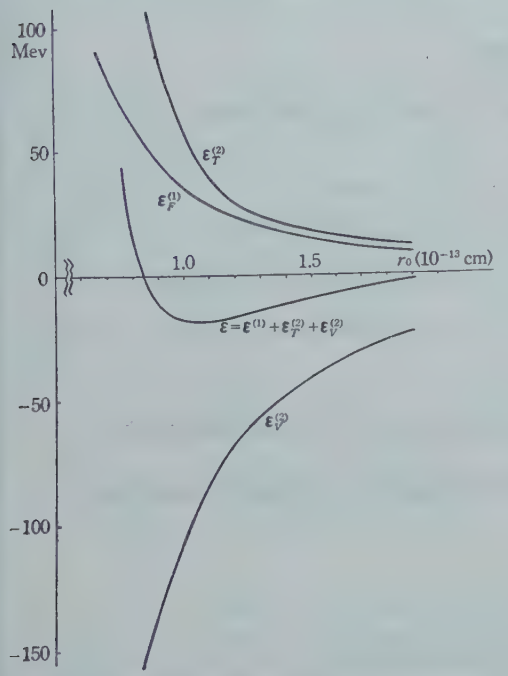

Fig. 10 Nuclar saturation according to the groumd state wave function (1). $\varepsilon^{(1)}$ is the mean kinetic energy of a Fermi gas particle, $\varepsilon_T{}^{(2)}$ is the kinetic energy due to two-body correlation, $\varepsilon_V{}^{(2)}$ is the pure effective potential energy and $\varepsilon=\varepsilon^{(1)}+\varepsilon_T{}^{(2)}+\varepsilon_V{}^{(2)}$. The effective mass is taken as $0.73M$.

and $\varepsilon_V{}^{(2)}$ is the pure effective potential energy given by

$$\varepsilon_V^{(2)}=(2\pi^2)^{-1}\int P(k)\ dk\int r^2\,dr\,3$$
$$\times\left\{U_V^{(3,1)}(k,r)+U_V^{(1,3)}(k,r)\right\}. \qquad (67)$$

It can be seen that the minimum of the volume energy per particle is found to be -18.3 MeV at a density value which is equivalent to a nuclear radius of $0.94\,A^{1/3}\times10^{-13}$cm. The separate contributions to the volume energy per particle are

$$\varepsilon^{(1)}=32.5\ \mathrm{MeV}.\quad \varepsilon_T{}^{(2)}=48.6\ \mathrm{MeV},$$
$$\varepsilon_V{}^{(2)}=-99.4\ \mathrm{MeV}, \qquad (68)$$

and $\varepsilon=\varepsilon^{(1)}+\varepsilon_T{}^{(2)}+\varepsilon_V{}^{(2)}=-18.3$ MeV. The comparison of our results with experimental data ($R=1.12\,A^{1/3}\times10^{-13}$ cm, $\varepsilon=-15.1$ MeV) and the results obtained by Breuckner and Gammel[1] suggests that we have obtained a fairly good agreement. No doubt, it is desirable to

replace our effective nuclear force (2) by a more realistic potential consisting of the central, tensor and, if necessary, spin-orbit force. Also we must still reconsider about our simplifying assumption for the effective mass in which we have chosen $k_\alpha^* \simeq k_F$.

§ 5. Many-body cluster terms and Pauli's exclusion principle

The main corrections to the volume energy (68) come from $E^{(3)}$ (51) which are cluster integrals over the configuration space of three or four particles. In order to see whether the corrections are likely to be small and how the Pauli principle works in this respect, we shall evaluate the main correction terms (three-body cluster $\mathcal{E}_1^{(3)}+\mathcal{E}_2^{(3)}$) using the function $f_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ which makes $E^{(1)}+E^{(2)}$ minimum. From (51, a) and (51, b) we obtain

$$\mathcal{E}_1^{(3)}+\mathcal{E}_2^{(3)}=(1/A)\,(E_1^{(3)}+E_2^{(3)})=\mathcal{E}_\Lambda+\mathcal{E}_\Delta, \tag{69}$$

where $\mathcal{E}_\Lambda$, which consists of $\mathcal{E}_1^{(3)}$ and a part of $\mathcal{E}_2^{(3)}$, is written as:

$$\mathcal{E}_\Lambda=-1/A\Omega^3\sum_{k_\alpha k_\beta k_\tau}\int d\boldsymbol{r}_1\, d\boldsymbol{r}_2\, d\boldsymbol{r}_3\, h^{(\beta\tau)}(2,3)\,U^{(\alpha,\beta;\tau)}(1,2\,;\,3), \tag{70}$$

and $\mathcal{E}_\Delta$, which is the rest part of $\mathcal{E}_2^{(3)}$, is given by

$$\mathcal{E}_\Delta=1/2A\Omega^3\sum_{k_\alpha k_\beta k_\tau}\int d\boldsymbol{r}_1\, d\boldsymbol{r}_2\, d\boldsymbol{r}_3\, h^{(\beta\tau)}(2,3)h^{(\tau,\alpha)}(3,1)\,U^{(\alpha,\beta;\tau)}(1,2;3), \tag{71}$$

in which the effective potential $U^{(\alpha,\beta,\tau)}(1,2\,;\,3)$ is given by

$$U_\Delta^{(\alpha,\beta;\tau)}(1,2;3)=-\{U_D(1,2)e^{ik_{\tau\alpha}r_{31}}-U_X(1,2)e^{i(k_\alpha r_{12}+k_\beta r_{23}+k_\tau r_{31})}\}, \tag{70'}$$

$$U_\Delta^{(\alpha,\beta;\tau)}(1,2;3)=4U(1,2)-\{U_D(1,2)\,(e^{ik_{\beta\tau}r_{23}}+e^{ik_{\tau\alpha}r_{31}})-U_X(1,2)$$
$$\times(e^{i(k_\alpha r_{13}+k_\beta r_{21}+k_\tau r_{32})}+e^{i(k_\alpha r_{12}+k_\beta r_{23}+k_\tau r_{31})})\} \tag{71'}$$

with

$$U_D(1,2)=\sum_{\alpha,\beta} f_{\alpha\beta}^*(\boldsymbol{r}_{12})S_{\alpha\beta}^*\{-\hbar^2/M^*(\nabla_r^2+i\boldsymbol{k}_{\alpha\beta}\nabla_r)+V_{\alpha\beta}(\boldsymbol{r}_{12})\}f_{\alpha\beta}(\boldsymbol{r}_{12})S_{\alpha\beta},$$

$$U_X(1,2)=\sum_{\alpha,\beta} f_{\alpha\beta}^*(\boldsymbol{r}_{12})S_{\alpha\beta}^*\{-\hbar^2/M^*(\nabla_r^2+i\boldsymbol{k}_{\alpha\beta}\nabla_r)+V_{\alpha\beta}(\boldsymbol{r}_{12})\}f_{\beta\alpha}(\boldsymbol{r}_{12})S_{\beta\alpha},$$

$$U(1,2)=U_D(1,2)-U_X(1,2)e^{ik_{\alpha\beta}r_{12}},$$

where

$$\boldsymbol{k}_{\alpha\beta}=\boldsymbol{k}_\alpha-\boldsymbol{k}_\beta, \qquad \boldsymbol{r}_{12}=\boldsymbol{r}_1-\boldsymbol{r}_2, \text{ etc},$$

and $\sum_{\alpha,\beta}$ denotes the summation over spin and isospin states. $h^{(\alpha,\beta)}(1,2)$ has already been given by (51, f). The calculations were carried out along the line indicated in the papers of Yamada-Iwamoto.[4] They are tedious but straightforward. The results are shown in Table 1.

Though we are not too confident about the values in Table 1, it might be con-

Table 1. Correction energies from the three-body clusters.

type of term	$\varepsilon_T^{(3)}$ in MeV	$\varepsilon_V^{(3)}$ in MeV	$\varepsilon^{(3)}=\varepsilon_T^{(3)}+\varepsilon_V^{(3)}$ in MeV
A	12.5	−9.4	3.1
Δ	− 6.9	4.4	−2.5

cluded that the correction energies due to higher clusters are fairly small in the nuclear matter.

Further, in order to investigate the effect of Pauli's exclusion principle, we have repeated the above calculations making use of the two-body functions $f_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ derived from (18) which do not contain the effects due to Pauli's exclusion principle and are, of course, equivalent to the one derived from (29) when α tends to zero, corresponding to the infinite healing distance. The result of such calculations without using the Pauli principle shows that $\mathcal{E}_A^{(3)}+\mathcal{E}_\Delta^{(3)}$ gives rise to an additional energy of about 15 MeV per particle as the difference between the correction energies $\mathcal{E}_T^{(3)}$ and $\mathcal{E}_V^{(3)}$, each being of the order of 300 MeV.

We can clearly recognize this feature from (70) and (71). The magnitudes of these terms are largely determined by the integrals[13]

$$\int r h_{\alpha\beta}(r)\,dr, \quad \text{or} \quad \int r\ h_{\alpha\beta}(r)\,s\,h_{\beta\tau}(s)\ ds\,dr, \tag{72}$$

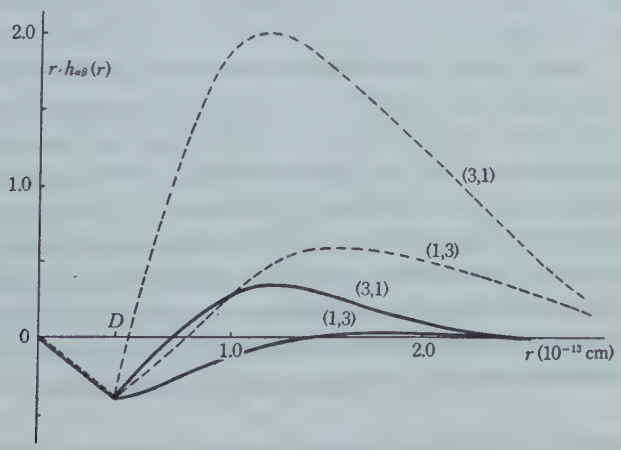

Fig. 11 The behaviors of $rh(r)$ which largely determines the magnitude of correction energies.
—— with the effect of the Pauli principle
····· without the effect of the Pauli principle

and these in turn depend upon the function $rh(r)$. In Fig. 1, we present a comparison of the two functions $rh(r)$, one arising from the $f_{\alpha\beta}(r)$ in which the Pauli principle is taken into account and derived from (29), and the other from $f_{\alpha\beta}(r)$ in which the Pauli principle is not taken into account and derived from (18).

It is clear that the large correction energy $\varepsilon^{(3)}$ arising from the two-body functions derived from (18) are due to the large broad peak of $rh(r)$, which becomes considerably suppressed by the effect of the Pauli principle.

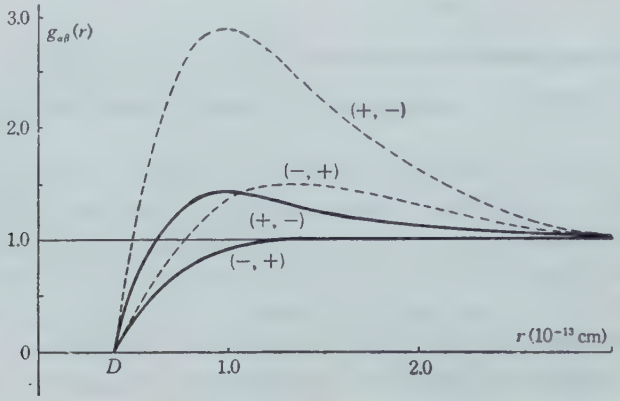

Fig. 12 Correlation functions with or without the effect of the Pauli principle.
—— with the effect of the Pauli principle
····· without the effect of the Pauli principle ($k=0.5k_F$)

It may also be instructive to see how the correlation function $g_{\alpha\beta}(r)$ changes its shape through the effect of the Pauli principle, which are shown in Fig. 12.

We can also conclude from Fig. 12 that in the situation where the healing distance λ takes a large value, the two-body correlation functions have a higher and broader peak. The value $\lambda=1.24\times10^{-13}$cm, which has been made use of throughout in this paper, is the mean value of the one appropriate for dealing with a nuclear matter of infinite extension. For an actual nucleus, however, especially in its surface region, the situation would be considerably different. If we suppose that in the nuclear surface region the healing distance of nucleon wave function happens to have some large value than in the inside, it may be concluded from the above discussions that nucleons in the surface region would have a stronger tendency to clustering, and this clustering in the surface region may be of importance in such phenomena as alpha decay, deuteron pick-up, stripping and (p, α) reaction, etc.

§ 6. Application to high energy nuclear reaction phenomena

The information from the high energy nuclear reactions is indispensable for investigation of the nuclear structure more into details ; the wave length of a nucleon moving with the kinetic energy of about 20 MeV is nearly equal to the mean nucleon distance in nuclei. The tentative analysis carried out by Brueckner, Eden and Francis[5] has shown that the cross sections of high-energy nuclear reactions depend on the nucleon momentum distribution which was found to be much different from that for the shell model of the nucleus and thus provides with strong evidence

or the existence of marked correlation in the nuclear ground state wave function. Assuming the deuteron aspects of the correlation functions between neutron and proton, Levinger[14] and Dedrick[15] have analyzed the high energy photoeffects fairly well. Coincidentally, in the recent experiments which have been carried out at the MIT synchrotron laboratory[16], ejected neutron-proton pairs from nuclei irradiated with a 340 MeV bremsstrahlung beam were observed and their angular correlation was measured. This has been intensively approached by Gottfried[17].

(*a*) Nucleon momentum distribution in the nucleus.

Here we try to show that our ground state wave function (1) provides with a relevant nucleon momentum distribution which can be represented at high energies by a single function giving reasonable agreement with all such experiments on, for example, nuclear photoeffect, deuteron pickup and proton-proton scattering in a nucleus, as has been discussed by BEF[5].

The assumptions necessary for the approach of these high energy reactions are: (1) the impulse approximaion, (2) the partial closure approximation with respect to the nuclear ground state wave function and (3) omission of the antisymmetrization between emitted particles and residual nucleus. Then the cross sections of these reactions depend on the ground state wave function through a factor

$$f(\boldsymbol{k}_1, \boldsymbol{k}_2) = 2/A^2 \sum_{\alpha,\beta} \left| \int (1/2\pi)^3 e^{-i\boldsymbol{k}_1\boldsymbol{r}_1} e^{-i\boldsymbol{k}_2\boldsymbol{r}_2} \psi_0^{(\alpha,\beta)}(\boldsymbol{r}_1, \boldsymbol{r}_2)\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \right|^2, \qquad (73)$$

where $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ are the final momenta of the two ejected nucleons, while $\psi_0(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is the initial wave function of the two nucleons which in our model is given by

$$\psi_0^{(\alpha\beta)}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \phi_{k_\alpha}(\boldsymbol{r}_1)\phi_{k_\beta}(\boldsymbol{r}_2) f_k^{(\alpha,\beta)}(\boldsymbol{r}) S_{\alpha,\beta}(1, 2).$$

Assuming that the center-of-mass moves freely through the Fermi gas,

$$\psi_0^{(\alpha\beta)}(\boldsymbol{r}_1, \boldsymbol{r}_2) \simeq e^{i\boldsymbol{K}\boldsymbol{R}} \psi_k^{(\alpha,\beta)}(\boldsymbol{r}),$$

we find

$$f(\boldsymbol{k}_1, \boldsymbol{k}_2) = 2/A^2 \sum_{\alpha,\beta} \left| \int (1/2\pi)^3 \exp[i(\boldsymbol{k}_2 - \boldsymbol{k}_1)\boldsymbol{r}/2] \psi_k^{(\alpha,\beta)}(\boldsymbol{r})\, d\boldsymbol{r} \right|^2 \cdot \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_\alpha - \boldsymbol{k}_\beta).$$

A further simplification is possible when $\boldsymbol{k}_1, \boldsymbol{k}_2 \gg \boldsymbol{k}_\alpha, \boldsymbol{k}_\beta$ and consequently when the momenta are nearly conserved, $\boldsymbol{k}_1 \fallingdotseq -\boldsymbol{k}_2$. We then get (dropping the delta function) the final result

$$f(\boldsymbol{k}, -\boldsymbol{k}) = g(\boldsymbol{k}) = 2/A^2 \sum_{\alpha,\beta} \left| \int (2\pi)^{-3/2} \exp(-i\boldsymbol{k}\boldsymbol{r}) \psi_k^{(\alpha\beta)}(\boldsymbol{r})\, d\boldsymbol{r} \right|^2. \qquad (74)$$

Substituting the wave functions (29) in $\psi_k^{(\alpha\beta)}(\boldsymbol{r})$, we have calculated $g(\boldsymbol{k})$ in our model. The radial integration is straightforward, but the state integration has to be carried out numerically. The result is shown in Fig. 13.

The experimental momentum distribution can be reproduced with fair accuracy by a Gaussian function

$$g_{exp}(k)=(1/\alpha^2\pi^{3/2})\exp(-k^2/\alpha^2), \tag{75}$$

where the value of the parameter α is given as $\alpha^2/2M\approx 14$ MeV. This is also shown in Fig. 13. We find that for greater values of the momentum k (corresponding to the energies per particle of from 75 to 150 MeV), our nuclear wave function predicts correctly the experimental momentum distribution.

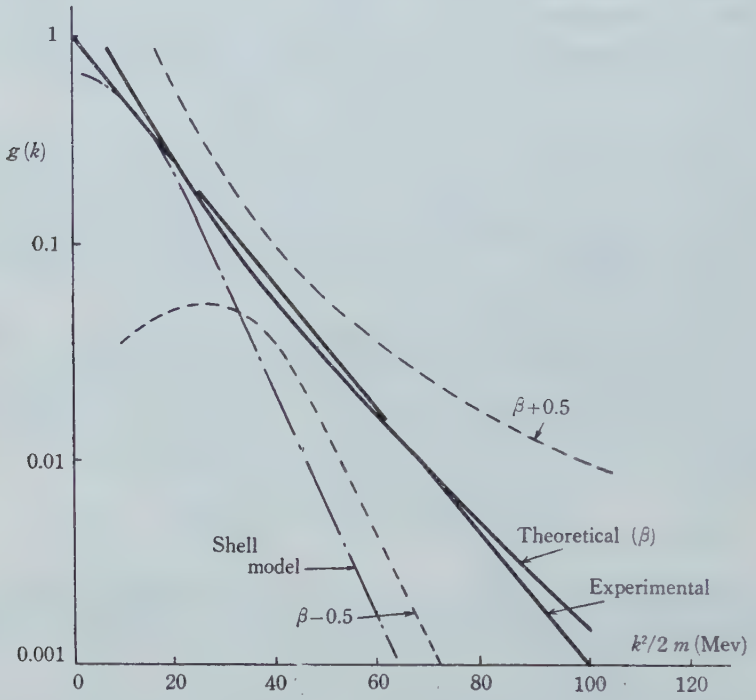

Fig. 13 The nucleon momentum distribution in a nucleus. The distribution come out from the shell model wave function is also illustrated, which has been calculated by BEF.

(***b***) Choice of the distortion parameter β

In § 2, (***b***), we have determined the value of the distortion parameter β from the analysis of deuteron wave function. The high momentum components of the nucleon distribution function derived above is now found to be so sensitive to this parameter that the comparison of $g(k)$ with experiment may furnish a powerful mean for testing the reliability of the value of β.

For this purpose, we shall make use of a simplified two-body wave function, where all partial waves other than the S-wave are neglected,

$$\psi_k(\boldsymbol{r})=1/\sqrt{\Omega}\,\{j_0(kr)+u_k^{(s)}(\boldsymbol{r})\}=1/\sqrt{\Omega}\cdot L_0(k,\,r).$$

As we are interested in the high energy portion of the momentum distribution function, only the behavior of the wave function near the hard core is of importance, where we may put $\exp(-\alpha r)\simeq\exp[-\alpha(r-D)]\approx 1$, then

$$L_0(k, r) \simeq 1/kr\{\sin(kr+\delta_0) - \sin(kD+\delta_0)e^{-\beta(r-D)}\}$$
$$= \cos\delta_0/kr\{\sin kr + \tan\delta_0 \cos kr - (\sin kD + \tan\delta_0 \cos kD)e^{-\beta(r-D)}\}.$$

From the effective range theory, $\tan\delta_0 \simeq -k/a$, where a is the inverse of scattering length. Using this relation, we get

$$L_0(k, r) = -(1/r\sqrt{a^2+k^2})[-\{\cos kr - ar\, j_0(kr)\} + \{\cos kD - aD\, j_0(kD)\}e^{-\beta(r-D)}].$$

As a further approximation, we assume near the hard core

$$\cos kr - ar\, j_0(kr) \simeq e^{-ar}.$$

Thus, the wave function is simplified into

$$\psi_k(\boldsymbol{r}) \simeq (1/\Omega)(1/r)(e^{-aD}/\sqrt{a^2+k^2})\{e^{-a(r-D)} - e^{-\beta(r-D)}\}. \qquad (76)$$

Correspondingly, the nucleon momentum distribution function can be expressed in a simple form as

$$g(p) = 2/A^2 \sum_{\alpha,\beta} \frac{2}{\pi}\frac{1}{\Omega}\frac{1}{a^2+k^2}\frac{1}{p^2}e^{-2aD}\left|\frac{(a\beta-p^2)(\beta-a)\sin Dp + p(\beta^2-a^2)\cos Dp}{(a^2+p^2)(\beta^2+p^2)}\right|^2. \qquad (77)$$

An even further simplification is attained by the approximation $1/\langle a^2+k^2\rangle_{AV} = 4/k_F^2$ and then the summation $\sum_{\alpha,\beta}$ becomes only over spin and isospin state. The change of $g(p)$ caused by the variation of β is illustrated in Fig. 13.

It may be safely concluded from Fig. 13 that the value of the distortion parameter β determined in § 2, (***b***) seems to be fairly reliable.

§ 7. Concluding remarks

It has been our subject to construct such a ground state wave function of nucleus that may not only maintain the characters of the independent particle model wave function, but also include the correlations between the constituents. The starting point of our procedure was to define the ground state wave function as given by (1). In this paper, we have discussed the two-body correlation functions in the case of nuclear matter, where three parameters have played the important role. They were

(1) the distortion parameter "β" related to nuclear forces,
(2) the healing parameter "α" related to Pauli's exclusion principle,
(3) the effective mass "M^*" related to the velocity-dependent common potential.

Taking account of the empirical information and making use of the *methods*—not *results*—of the theoretical approaches hitherto published, we have carried out the determination of these parameters quantitatively.

It has been shown that our model wave function seems to predict some of the properties of nuclear matter correctly such as saturation character of binding energy

and of nuclear density, and the nucleon momentum distribution in nuclei. Moreover, the physical background for the independent particle aspects of the nuclear structure has been brought to light in a concrete and explicit form just as has been emphasized by Yamada and Iwamoto[4]. That is, as has been illustrated in Fig. 8, the kinetic part $U_T(k, r)$ of the effective two-body potential, which comes out from (59) due to the correlation between nucleons and the coupling effect between correlation and the single particle motion, plays a decisive role in making the effective potential much weaker than the real potential. This should be one of the reasons why the independent particle model explains some of the properties of nuclear matter fairly well in spite of the much singular and short range real inter-nucleon forces. This kinetic part of the effective potential gives rise to the effective kinetic energy $\varepsilon_T^{(2)}$, which plays, as can be seen from Fig. 10, an important role in the saturation phenomena, and without which no reasonable saturation properties would come out from the Serber force.

It is remarkable that the above results are the consequences of the application of the cluster development method with the wave function (1). From the method of the formal theory approach of Brueckner et al., however, it would be difficult to give such a concrete form to $U_T(k, r)$ as in our model. It has also been shown in § 5 that, by virtue of the cluster development method, the evaluation of the higher order correlation terms could be carried out, though laborious and tedious, with no obscurity as in formal theory. In fact, we have shown explicitly in Fig. 11 and 12, that the magnitudes of higher order terms were suppressed to become fairly small owing to the Pauli exclusion principle.

The healing distance λ discussed in § 2, ($\boldsymbol{c}$) actually is k-dependent, while we have employed its mean value in our wave function. This point will be improved in the forthcoming paper, in which we also wish to try to discuss the more realistic forms of the nuclear two body potentials. Evaluation of the surface energy and the symmetry energy is also in progress.

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Note added in proof As has been explained in this text, eq. (62) is the effective potential referred to the chosen nucleon pair being in nuclear matter and has been obtained from (55) by introducing the effect of interactions of the chosen pair with other nucleons. Thereby, we have made use of effective mass approximation and consequently phase shift δ_0, which is contained implicitly in the effective potential through the two-body correlation function $f_{\alpha\beta}(r)$, has been replaced by the effective phase shift $\delta_0{}^*$ and further, the nucleon mass M, which is contained explicitly in the kinetic part of the effective potential, has been replaced by the effective mass M^*. The justsfication of such procedure has been discussed by Weisskopf et al[9]. In this paper, however, regarding with the process of our treatment for the determination of a effective mass value, it would be more consistent for M in (55) to remain unchanged even in (52), though δ_0 should be replaced by $\delta_0{}^*$ in accordance with the translation of (55) to (62). Author is very grateful to Prof. S. Yoshida for critical discussion on this point.

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Nonlocal Interactions and Dispersion Relations

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When the microscopic causality is not satisfied we could not give the usual physical meaning to the dispersion relations proposed by Oehme.

The problems of ghost seem to be closely connected with the propagation characters in the cases of nonlocal interactions.

Recently it has been confirmed by the experimental data of electron-proton scattering at Stanford[1] that the proton has a structure. This fact may suggest that all "elementary particles" have some structures. It may lead us to the modification of the conventional concepts of elementary particles so that it may solve many difficulties lying in the present field theories. Under these circumstances it will be worth-while to reexamine the nonlocal interaction formalism.

On the other hand, the dispersion relations[2),3)] seem to be one of the important relations in physics as they use only observable quantities. They have been derived only in the cases of local interactions but from the above reason it will be an important problem to extend them to the cases of nonlocal interactions. We shall, however, face various difficult problems when we will follow the usual method in those cases, i. e., the problems of (i) the existence of S-matrix and of interaction representation (ii) causality condition, etc.

Oehme[4] has shown that the dispersion relations can be derived by suitable modifications, even if the microscopic causality is not satisfied:

If the commutator is such that

$$[j^*(0),\ j(\boldsymbol{y},\ y_0)]=0 \quad \text{for } \boldsymbol{y}^2-y_0{}^2>l_0{}^2 \tag{1}$$

for the forward scattering, the Fourier transform $\mathfrak{M}(\xi_0)$ of the scattering amplitude $M(\omega)$ is

$$\mathfrak{M}(\xi_0)=\int_{-\infty}^{\infty} d\omega \exp(-i\omega\xi_0)\, M(\omega)=0 \quad \text{for } \xi_0<-l_0 \tag{2}$$

and the dispersion relations hold between the real- and the imaginary-part of $M(\omega)\times\exp(i\omega l_0)$ because $M(\omega)\exp(i\omega l_0)$ is regular in the upper half plane of ω instead of $M(\omega)$.

But it is very doubtful that his result has a usual physical meaning: The

above result is based upon that $\mathfrak{M}(\xi_0)$ is given by his eq. (22), i. e.,

$$\mathfrak{M}(\xi_0)=(2\pi)^2\int_0^\infty dy_0\int_{y_0-\xi_0}^\infty ydy\,B(y^2-y_0^2,\ -y_0)\frac{i}{\mu}$$
$$\times[y^2-(y_0-\xi_0)^2]^{1/2}J_1(i\mu[y^2-(y_0-\xi_0)^2]^{1/2}), \tag{3}$$

where B is defined by

$$\left(\frac{\partial}{\partial\xi}\langle \boldsymbol{p},\ p_0|[j^*(0),\ j(\boldsymbol{y},\ y_0)]|0,\ M\rangle\right)_{\xi=0}=|\boldsymbol{q}|iB(y^2-y_0^2,\ y_0;\ 0,\ M)\sigma(\boldsymbol{y}\times\boldsymbol{l}) \tag{4}$$

and has the property of commutator vanishing for $\boldsymbol{y}^2-y_0^2>l_0^2$. (For other notations, see his paper.) In fact, if ξ_0 is less than $-l_0$, the nonvanishing region of $B(y^2-y_0^2,\ -y_0)$ has no common part with the integration region over $(y,\ y_0)$, hence $\mathfrak{M}(\xi_0)$ is equal to zero for $\xi_0<-l_0$. But when the microscopic acausality like eq. (1) holds it is very doubtful that $\mathfrak{M}(\xi_0)$ is given by eq. (3). In the following we shall examine eq. (3) in the case of nonlocal interaction.

Because the problem is very complicated in general cases, we here consider a simple nonlocal interaction corresponding to Feynman's cut-off for the photon propagator which has often been reconsidered recently in the modification of quantum electrodynamics at small distances, in connection with the nucleon structure.[5] Namely, we consider the following nonlocal interaction Hamiltonian density

$$H'(x)=-ie\int d^4x'\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x')F((x-x')^2), \tag{5}$$

where $F(x^2)$ is a suitable real form function.

First we calculate an S-matrix before we consider the dispersion relations for the scattering processes. Because the S-matrices are in general not unitary if we calculate them with the Bloch method[6] in the cases of nonlocal interactions, we will calculate them with Hayashi-Katayama's method.[7] Following them, we put

$$A_\mu(x)=A_\mu(x,\ \sigma)+V_\mu(x,\ \sigma)+ie\int d^4x'd^4x''\tfrac{1}{2}\{\epsilon(x-x'')$$
$$-\epsilon(\sigma,\ x'')\}D(x-x'')\bar{\psi}(x')\gamma_\mu\psi(x')F((x'-x'')^2), \tag{6}$$

$$\psi(x)=\psi(x,\ \sigma)+W(x,\ \sigma)+ie\int d^4x'd^4x''\tfrac{1}{2}\{\epsilon(x-x')$$
$$-\epsilon(\sigma,\ x')\}S(x-x')\gamma_\mu\psi(x')A_\mu(x'')F((x'-x'')^2), \tag{7}$$

where

$$\lim_{\sigma\to-\infty}V_\mu(x,\ \sigma)=\lim_{\sigma\to-\infty}W(x,\ \sigma)=0, \tag{8}$$

so that

$$\lim_{\sigma\to-\infty}A_\mu(x,\ \sigma)=A_\mu^{\rm in}(x),\quad \lim_{\sigma\to-\infty}\psi(x,\ \sigma)=\psi^{\rm in}(x), \tag{9}$$

and we define as

$$\lim_{\sigma\to+\infty} A_\mu(x,\sigma)=A_\mu^{\text{out}}(x),\quad \lim_{\sigma\to+\infty}\psi(x,\sigma)=\psi^{\text{out}}(x). \tag{10}$$

$A_\mu(x,\sigma)$, $\psi(x,\sigma)$ and $\bar{\psi}(x,\sigma)$ are assumed to satisfy the usual equations of motion and commutation relations for free fields in the case of local interaction formalism. In general,

$$\lim_{\sigma\to+\infty} V_\mu(x,\sigma)\neq 0,\quad \lim_{\sigma\to-\infty} W(x,\sigma)\neq 0 \tag{11}$$

and we will state later that the appearence of so-called ghost is closely related to eq. (11).

$V_\mu(x,\sigma)$ and $W(x,\sigma)$ shall be determined in such a way that we can have unitary matrix $U[\sigma,\sigma']$ which satisfies the relations

$$A_\mu(x,\sigma)=U^{-1}[\sigma,\sigma']A_\mu(x,\sigma')\,U[\sigma,\sigma']=U^{-1}[\sigma]A_\mu^{\text{in}}(x)\,U[\sigma], \tag{12}$$

$$\psi(x,\sigma)=U^{-1}[\sigma,\sigma']\psi(x,\sigma')\,U[\sigma,\sigma']=U^{-1}[\sigma]\psi^{\text{in}}(x)\,U[\sigma], \tag{13}$$

$$U[\sigma]\equiv U[\sigma,-\infty].$$

Then from the relations

$$A_\mu^{\text{out}}(x)=U^{-1}[\infty,-\infty]A_\mu^{\text{in}}(x)\,U[\infty,-\infty]\quad\text{etc.}, \tag{14}$$

the S-matrix is given by

$$S=U[\infty,-\infty]. \tag{15}$$

If we assume that

$$i\delta U[\sigma]/\delta\sigma(x)=U[\sigma]H(x/\sigma), \tag{16}$$

where $H(x/\sigma)$ is a Hermitian operator to be determined later, then we have

$$i\delta A_\mu(x,\sigma)/\delta\sigma(x')=[A_\mu(x,\sigma),\ H(x'/\sigma)], \tag{17}$$

$$i\delta\psi(x,\sigma)/\delta\sigma(x')=[\psi(x,\sigma),\ H(x'/\sigma)] \tag{18}$$

from eqs. (12) and (13) and

$$\delta V_\mu(x,\sigma)/\delta\sigma(x')=i[A_\mu(x,\sigma),\ H(x'/\sigma)]+ie\int d^4x''D(x-x')$$
$$\times\bar{\psi}(x'')\gamma_\mu\psi(x'')F((x'-x'')^2)|_{x'/\sigma}, \tag{19}$$

$$\delta W(x,\sigma)/\delta\sigma(x')=i[\psi(x,\sigma),\ H(x'/\sigma)]+ie\int d^4x''S(x-x')\gamma_\mu$$
$$\times\psi(x')A_\mu(x'')F((x'-x'')^2)|_{x'/\sigma} \tag{20}$$

from eqs. (6), (7), (17) and (18).

If we put

$$H(x/\sigma)=\sum_{n=1}^{\infty}e^n H_n(x/\sigma), \tag{21}$$

$$V_\mu(x,\sigma)=\sum_{n=1}^{\infty} e^n V_{\mu n}(x,\sigma) \tag{22}$$

and

$$W(x,\sigma)=\sum_{n=1}^{\infty} e^n W_n(x,\sigma), \tag{23}$$

each term in the expansions can be calculated perturbationally. For instance,

$$H_1(x/\sigma)=-i\int d^4y d^4z \bar{\psi}(y,\sigma)\gamma_\mu\psi(y,\sigma)A_\mu(z,\sigma)$$
$$\times\{2a\delta(x-y)+b\delta(x-z)\}F((y-z)^2)|_{x/\sigma}, \tag{24}$$

where a and b are constants satisfying the relation

$$2a+b=1 \tag{25}$$

from eq. (8). Especially when $b=0$ and $a=1/2$, then

$$H_1(x/\sigma)=-i\int d^4x' \bar{\psi}(x,\sigma)\gamma_\mu\psi(x,\sigma)A_\mu(x',\sigma)F((x-x')^2)|_{x/\sigma}$$
$$=-i\bar{\psi}(x,\sigma)\gamma_\mu\psi(x,\sigma)A_\mu(x,\sigma)|_{x/\sigma}, \tag{26}$$

which coincides with a usual local interaction Hamiltonian density. Here we have used the relations

$$\left.\begin{aligned} F((x-x')^2)&=1/(2\pi)^4\int d^4p f(p^2)\exp\{ip(x-x')\},\\ f(0)&=1, \end{aligned}\right\} \tag{27}$$

$$A_\mu(x,\sigma)=1/(2\pi)^4\int d^4k\delta(k^2)a_\mu(k,\sigma)\exp(ikx) \tag{28}$$

and

$$H_2(x/\sigma)=-1/2\int d^4x' d^4y d^4y' \bar{\psi}(x,\sigma)\gamma_\mu\psi(x,\sigma)\bar{\psi}(y,\sigma)\gamma_\mu$$
$$\times[\{\epsilon(x'-y')-\epsilon(\sigma,y)\}/2]D(x'-y')\psi(y,\sigma)F((x-x')^2)F((y-y')^2)|_{x/\sigma}, \tag{29}$$

etc.

The S-matrix is given by the equation

$$S=\cdots\left[1-i\int_{\sigma_{-1}}^{\sigma_0} d^4x H(x/\sigma)\right]\left[1-i\int_{\sigma_0}^{\sigma_1} d^4x H(x/\sigma)\right]\cdots$$
$$=\cdots\left[1-i\int_{\sigma_0}^{\sigma_1} d^4x H^{\mathrm{in}}(x)\right]\left[1-i\int_{\sigma_{-1}}^{\sigma_0} d^4x H^{\mathrm{in}}(x)\right]\cdots, \tag{30}$$

where

$$H^{\text{in}}(x)=\sum_{n=1}^{\infty} e^n H_n^{\text{in}}(x)$$
$$=-ie\bar{\psi}^{\text{in}}(x)\gamma_\mu\psi^{\text{in}}(x)A_\mu^{\text{in}}(x)$$
$$-\tfrac{1}{2}e^2\bar{\psi}^{\text{in}}(x)\gamma_\mu\psi^{\text{in}}(x)\int d^4x'd^4yd^4y'\bar{\psi}^{\text{in}}(y)\gamma_\mu$$
$$\times[\{\epsilon(x'-y')-\epsilon(x-y)\}/2]D(x'-y')\psi^{\text{in}}(y)F((x-x')^2)F((y-y')^2)$$
$$+\cdots\cdots. \tag{31}$$

That is, we have only to use eq. (31) as an interaction Hamiltonian density in the usual Feynman-Dyson formalism to calculate the S-matrix. Since this S-matrix is unitary, the commutation relations $[A_\mu^{\text{out}}(x),\ A_\nu^{\text{out}}(y)]$ have the same values as $[A_\mu^{\text{in}}(x),\ A_\nu^{\text{in}}(y)]=-i\delta_{\mu\nu}D(x-y)$. But $[A_\mu^{\text{out}}(x),\ A_\nu^{\text{in}}(y)]$, which are necessary for finding the dispersion relations following the usual manner, have different values as follows: From the relations

$$A_\mu^{\text{out}}(x)=S^{-1}A_\mu^{\text{in}}(x)S=\sum_{n=0}^{\infty} i^n\int d^4x_1\cdots d^4x_n[H^{\text{in}}(x_n)[\cdots[H^{\text{in}}(x_1),\ A_\mu^{\text{in}}(x)]\cdots]]$$
$$\times\eta(x_1-x_2)\cdots\eta(x_{n-1}-x_n), \tag{32}$$

$$\eta(x)=\begin{cases}1 & \\ 0 & \end{cases}\quad \text{for } x_0\gtrless 0,$$

the commutation relations are

$$[A_\mu^{\text{out}}(x),\ A_\nu^{\text{in}}(y)]$$
$$=[A_{\mu 1}^{\text{out}}(x),\ A_\nu^{\text{in}}(y)]+[A_{\mu 2}^{\text{out}}(x),\ A_\nu^{\text{in}}(y)], \tag{33}$$

where $A_{\mu 1}^{\text{out}}(x)$ are the parts of $A_\mu^{\text{out}}(x)$ which are obtained with the local interaction Hamiltonian density $eH_1^{\text{in}}(x_i)$ instead of $H^{\text{in}}(x_i)$ $(i=1,\cdots,n)$ in eq. (32) and $A_{\mu 2}^{\text{out}}(x)$ are the rest parts of $A_\mu^{\text{out}}(x)$. Therefore

$$[A_{\mu 1}^{\text{out}}(x),\ A_\nu^{\text{in}}(y)]=0 \quad \text{for } (x-y)^2>0 \tag{34}$$

but

$$[A_{\mu 2}^{\text{out}}(x),\ A_\nu^{\text{in}}(y)]\neq 0 \quad \text{for } (x-y)^2>0. \tag{35}$$

As $A_{\mu 2}^{\text{out}}(x)$ are of third order in the charge e, the microscopic causality for eq. (33) is satisfied only to the second order in e. Also the S-matrix elements for the scattering are the same as those in the case of usual local interaction to the second order and therefore, in this case, we find that the dispersion relations for the scattering are satisfied only to the second order. It is remarkable that the microscopic causality

$$[A_\mu^{\text{out}}(x),\ A_\nu^{\text{in}}(y)]=0 \quad \text{for } (x-y)^2>0 \tag{36}$$

is not satisfied while the interaction Hamiltonian density $H^{\text{in}}(x)$ satisfies the integrability condition. Even if the nonlocal interactions are such that eq. (36) is

satisfied, the conditions used by Goldberger,[3] especially the following conditions, could not be used for those interactions:

(i) The photon propagator is

$$\langle f|P(A_\mu(x),\ A_\nu(y))|i\rangle$$

and this is proportional to the S-matrix element since $\lim_{x_0\to+\infty} A_\mu(x)=A_\mu^{\text{out}}(x)$, $\lim_{y_0\to-\infty} A_\nu(y)=A_\nu^{\text{in}}(y)$ and $A_\mu^{\text{out}}(x)=S^{-1}A_\mu^{\text{in}}(x)S$.

(ii) The canonical commutation relations are satisfied.

Therefore, the exact dispersion relations in the cases of nonlocal interactions cannot be obtained by the usual manner.

Now, the regions of acausality are determined by the characters of form functions but, even if they are like eq. (1), we find, as in the above-mentioned example that the scattering amplitudes may have complicated forms including form functions and differ very much from the form of eq. (3). Moreover, even if the microscopic acausality is not due to nonlocal interactions, the integrability conditions would also be destroyed, and we could not look for the S-matrices following the usual manner, hence in these cases the scattering amplitudes also could not be written in the form like eq. (3). Consequently, when the microscopic causality is destroyed we could not give the usual physical meaning to the dispersion relations modified by Oehme. We should keep these circumstances in mind for the discussions of errors of dispersion relations.[8]

Finally, we mention the problem of "ghost". As was seen above, when the interactions are nonlocal the field quantities are not of outgoing wave characters in the limit t (time)$\to+\infty$ even if they are of incoming wave characters in the limit $t\to-\infty$; accordingly, in order to search for the unitary S-matrices, it is necessary to separate the outgoing waves from the field quantities in the limit $t\to+\infty$ and to combine them with the incoming waves. That is, in these cases the propagations of the systems due to the initially given interactions do not coincide with those due to the unitary S-matrices in general because the former do not propagate to the outgoing waves while the latter do. The former do not exhibit true propagation characters but the latter have relation to the true propagations of observables. The so-called ghost seems to express these circumstances. In the Lee model,[9] in practice, if we want to extend it relativistically by taking into account the recoils, the interaction becomes nonlocal, satisfying no integrability condition.[10] Namely, the Lee model is, by origin, of acausal, nonlocal character and therefore, when the S-matrices are looked for in the usual manner, they are nonunitary and the ghost appears. It is well known that if we use the cut-off functions to look for the unitary S-matrices, the "ghost" disappears, and for this purpose the above method in the cases of nonlocal interactions will be applicable. For instance, the cut-off formalism used by Kaellen-Pauli[11] in the Lee model seems to correspond to the nonlocal interaction having the form function of two point variables like eq. (5).

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Nucleon Exchange Effects in Deuteron Stripping Reactions and Angular Distributions of $Be^9(d,n)B^{10}$ Reaction

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Ordinary deuteron stripping theory has been extended to include the heavy particle stripping process by antisymmetrizing a final state wave function with respect to an ejected nucleon and a target nucleon. A formalism presented by Owen and Madansky is inconsistent for discussing an interference effect between the ordinary deuteron stripping process and the heavy particle stripping process. Experimental results of angular distributions of the Be^9 $(d,n)B^{10}$ ground state neutron are well reproduced by the present theory.

§ 1. Introduction

Characteristic forward peaks observed in angular distributions of (d, p) and (d, n) reactions have been successfully interpreted by the deuteron stripping theory[1]. In many cases, however, it has been observed that angular distributions have also a significant backward component. Madansky and Owen[2] have suggested the so-called heavy particle stripping mechanism to account for the backward component observed in the ground state $B^{11}(d, n)C^{12}$ reaction, and later they have extended the originary deuteron stripping theory to include the effects of heavy particle stripping by using a final state wave function which is antisymmetrized in the exchange of a neutron from the deuteron and an outer shell neutron of the target nucleus.[3] In this case, however, they referred to different directions of space quantization axes to calculate transition amplitude of the ordinary stripping process and the heavy particle stripping process. This procedure is inconsistent for discussing interference effects between the two.

Concerning the exchange effects in the nuclear stripping reactions, except Madansky and Owen,[2)-3)] French[4] has tried to evaluate the value of the stripping amplitude and of the exchange amplitude with a limited range of validity of the formalism, and Kammuri and his collaborators[5] have investigated the neutron exchange effect of the $C^{13}(d, n)N^{14}$ reaction. Recently, Ames, Owen and Swartz[6] have analyzed angular distributions of the neutron from the $B^{11}(d, n)C^{12}$ first excited state reaction, basing on the formalism of Owen and Madansky.[3]

The exchange effect discussed above will, however, be essential when the

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target nucleus has a loosely bound extra neutron. Furthermore, one may expect more pronounced manifestation of the exchange effect at lower incident energies of deuterons. Therefore, among the rest of all cases, the ground state $Be^9(d, n)B^{10}$ reaction will be the most suitable case for investigating the exchange effect ; the binding energy of the extra neutron of the Be^9 nucleus is known to be only 1.67 MeV. Fortunately, experimental data have been obtained by several investigators for the reaction at low bombarding energies of deuterons.[7)-9)] In section 3, we will examine this very case on the basis of a general formulation of the problem discussed in next section.

§ 2. General formulation

In what follows we will formulate the problem along the Born approximation stripping theory developed by Bhatia and others,[10)] assuming that the target nucleus consists of a core and one outside neutron, and that internal states of the core are not excited by the reaction. We shall make use of the notations of reference 10).

(2-i) Deuteron stripping term

The amplitude of the outgoing neutron through the deuteron stripping process is given by

$$I_{st}(\theta) = -(M_n^*/2\pi\hbar^2)\int \Psi^*_{\mu_f}(\boldsymbol{r}_p, \sigma_p, \boldsymbol{r}_{n'}, \sigma_{n'}, \tilde{\xi})\chi^*_{\mu_n}(\sigma_n)e^{-i\boldsymbol{k}_n\boldsymbol{r}'_n}$$
$$\times V(\boldsymbol{r}_p, \sigma_p, \boldsymbol{r}_n, \sigma_n, \boldsymbol{r}_{n'}, \sigma_{n'}, \tilde{\xi})\Psi_{\mu_i}(\boldsymbol{r}_{n'}, \sigma_{n'}, \tilde{\xi})e^{i\boldsymbol{k}_d(\boldsymbol{r}_n+\boldsymbol{r}_p)/2}$$
$$\times \Psi_{\mu_d}(\boldsymbol{r}_p-\boldsymbol{r}_n, \sigma_n, \sigma_p)d\boldsymbol{r}_p d\sigma_p d\boldsymbol{r}_n d\sigma_n d\boldsymbol{r}_{n'} d\sigma_{n'} d\tilde{\xi} \qquad (1)$$

with

$$\boldsymbol{r}'_n = \boldsymbol{r}_n - (M_p/M_f)\boldsymbol{r}_p\,, \qquad (2)$$

where

$$M_n^* = M_n M_f/M_n + M_f\,, \quad M_d^* = M_d M_i/M_d + M_i$$

and $\boldsymbol{k}_n = M_n^*\boldsymbol{V}_n/\hbar$, $\boldsymbol{V}_n$ being the velocity of the ejected neutron relative to the final nucleus, $\boldsymbol{k}_d = M_d^*\boldsymbol{V}_d/\hbar$, $\boldsymbol{V}_d$ being the velocity of the incident deuteron relative to the initial nucleus. The suffices n, n' and $\tilde{\xi}$ denote the neutron of the deuteron, the outside neutron and the internal coordinates of the core of the target nucleus, respectively. For the internal state of the deuteron, we take a pure S-state wave function.

Following the stripping approximation, we neglect the interaction between n and the target nucleus, and assume a surface interaction of radius R_1 for the interaction between the captured proton and the target nucleus. Then, by considering conservation of angular momentum, we obtain from eq. (1)

$$I_{st}(\theta) = \sum_{l_p, m_p, \mu_p} C_{1/2,1/2}(1, \mu_d\,; \mu_p, \mu_n)\langle \mu_f | V | \mu_i\,; l_p, m_p, \mu_p\rangle_{st} G_{st}{}^{l_p m_p}(\theta) \qquad (3)$$

where

$$\langle \mu_f | V | \mu_i ; l_p, m_p, \mu_p \rangle_{st} = -\frac{M_n^*}{2\pi\hbar^2} \int \Psi^*_{\mu_f}(\boldsymbol{r}_p, \sigma_p, \boldsymbol{r}_{n'}, \sigma_{n'}, \xi)$$

$$\times V(\boldsymbol{r}_p, \sigma_p, \boldsymbol{r}_{n'}, \sigma_{n'}, \xi) \Psi_{\mu_i}(\boldsymbol{r}_{n'}, \sigma_{n'}, \xi) Y_{l_p m_p}(\omega_p) \chi_{\mu_p}(\sigma_p) \tag{4}$$

$$\times d\boldsymbol{r}_p d\sigma_p d\boldsymbol{r}_{n'} d\sigma_{n'} d\xi ,$$

$$G_{st}{}^{l_p m_p}(\theta) = \int Y^*_{l_p m_p}(\Omega_p) e^{i\boldsymbol{k}_1 \boldsymbol{R}_1} e^{i\boldsymbol{K}_1(\boldsymbol{r}_n - \boldsymbol{R}_1)} \Phi_d(|\boldsymbol{r}_n - \boldsymbol{R}_1|) d\boldsymbol{r}_n d\Omega_p \tag{5}$$

with

$$\boldsymbol{k}_1 = \boldsymbol{k}_d - (M_i/M_f)\boldsymbol{k}_n, \quad \boldsymbol{K}_1 = \boldsymbol{k}_d/2 - \boldsymbol{k}_n . \tag{6}$$

l_p is an orbital angular momentum of the captured proton relative to the target nucleus.

Eq. (3) is rewritten as

$$I_{st}(\theta) = \sum_{l_p m_p \mu_p} \sum_{j_c(\mu_c)} C_{1/2,1/2}(1, \mu_d ; \mu_p, \mu_n) \; C_{l_p,1/2}(j_c, \mu_c ; m_p, \mu_p)$$

$$\times C_{j_i j_c}(j_f, \mu_f ; \mu_i, \mu_c) \langle j_f \| V \| j_i ; j_c(l_p, 1/2) \rangle_{st} G_{st}{}^{l_p m_p}(\theta) \tag{7}$$

where

$$\langle j_f \| V \| j_i ; j_c(l_p, 1/2) \rangle_{st} = \frac{2j_c+1}{2j_f+1} \sum_{m_p \mu_p} \sum_{\mu_i \mu_f} C_{l_p,1/2}(j_c, \mu_c, m_p, \mu_p)$$

$$\times C_{j_i j_c}(j_f, \mu_f ; \mu_i, \mu_c) \langle \mu_f | V | \mu_i ; l_p, m_p, \mu_p \rangle_{st} \tag{8}$$

and use has been made of the relation

$$\sum_{M m''} C_{jj'}(J, M ; m, m'') C_{j'j''}(J, M ; m', m'') = \frac{2J+1}{2j+1} \delta_{jj'} \delta_{mm'} . \tag{9}$$

To calculate the factor $G_{st}{}^{l_p m_p}(\theta)$, we take for the radial part $\Phi_d(r)$ of the deuteron internal wave function the approximate form

$$\Phi_d(r) = (\alpha/2\pi)^{1/2} e^{-\alpha r}/r ; \; \alpha = 0.23 \times 10^{13} \text{ cm}^{-1} \tag{10}$$

and take the axis of space quantization in the direction of $\boldsymbol{k}_1$.

Then, we obtain

$$G_{st}{}^{l_p m_p}(\theta) \equiv \delta_{m_p,0} G_{st}{}^{l_p}(\theta) = \delta_{m_p,0} (4\pi(2l_p+1))^{1/2} i^{l_p} j_{l_p}(k_1 R_1) G_1(K_1) \tag{11}$$

where

$$G_1(K_1) = (8\pi\alpha)^{1/2}/(\alpha^2 + K_1^2) \tag{12}$$

and $j_{l_p}(k_1 R_1)$ is an l_p-th order spherical Bessel function.

(2-ii) Heavy particle stripping term

The heavy particle stripping amplitude, corresponding to the exchange of the

neutron n of the deuteron and the neutron n' of the target nucleus, is given by

$$I_{ex}(\theta) = -\frac{M_n^*}{2\pi\hbar^2}\int \Psi^*_{\mu_f}(\boldsymbol{r}_p, \sigma_p, \boldsymbol{r}_n, \sigma_n, \xi)\chi^*_{\mu_n}(\sigma_{n'})e^{-i\boldsymbol{k}_n\boldsymbol{r}'_{n'}}$$
$$\times V(\boldsymbol{r}_p, \sigma_p, \boldsymbol{r}_n, \sigma_n, \boldsymbol{r}_{n'}, \sigma_{n'}, \xi)\Psi_{\mu_i}(\boldsymbol{r}_{n'}, \sigma_{n'}, \xi)$$
$$\times e^{i\boldsymbol{k}_d(\boldsymbol{r}_n+\boldsymbol{r}_p)/2}\Psi_{\mu_d}(\boldsymbol{r}_n-\boldsymbol{r}_p, \sigma_n, \sigma_p)\,d\boldsymbol{r}_p d\sigma_p d\boldsymbol{r}_n d\sigma_n d\boldsymbol{r}_{n'} d\sigma_{n'} d\xi\,. \tag{13}$$

This time, we take the origin of integration at the centre of mass of the deuteron and neglect the interaction between the neutron n' and the deuteron. Then, eq. (13) becomes

$$I_{ex}(\theta) = -\frac{M_n^*}{2\pi\hbar^2}\int \Psi^*_{\mu_f}(\boldsymbol{r}_p-\boldsymbol{r}_c, \sigma_p, \boldsymbol{r}_n-\boldsymbol{r}_c, \sigma_n, \xi)\chi^*_{\mu_n}(\sigma_{n'})e^{-i\boldsymbol{k}_n\boldsymbol{r}''_{n'}}$$
$$\times V(\boldsymbol{r}_p-\boldsymbol{r}_c, \sigma_p, \boldsymbol{r}_n-\boldsymbol{r}_c, \sigma_n, \xi)\Psi_{\mu_i}(\boldsymbol{r}_{n'}-\boldsymbol{r}_c, \sigma_{n'}, \xi)e^{-i\boldsymbol{K}_d(M_n\boldsymbol{r}_{n'}+M_c\boldsymbol{r}_c)/M_f}$$
$$\times \Psi_{\mu_d}(\boldsymbol{r}_n-\boldsymbol{r}_p, \sigma_n, \sigma_p)\,d\boldsymbol{r}_c d\boldsymbol{r}_p d\sigma_p d\boldsymbol{r}_n d\sigma_n d\boldsymbol{r}_{n'} d\sigma_{n'} d\xi\,, \tag{14}$$

where $\boldsymbol{r}''_{n'}=\boldsymbol{r}_{n'}-(M_c/M_f)\boldsymbol{r}_c$, $\boldsymbol{r}_c$ being the position vector of the centre of mass of the core.

We divide the internal wave function of the initial nucleus into two parts, the outside neutron part and the core part, making an approximation as follows,

$$\Psi_{\mu_i}(\boldsymbol{r}_{n'}, \sigma_{n'}, \xi) = \sum_{\mu_I\mu(m_i\mu_0)} C_{j_I j_0}(j_i, \mu_i; \mu_I, \mu_0)C_{l_i,1/2}(j_I, \mu_I; m_i, \mu)$$
$$\times \Psi_{l_i m_i}(\boldsymbol{r}_{n'})\chi_{\mu}(\sigma_{n'})\Phi_{j_0\mu_0}(\xi)\,, \tag{15}$$

j_0 being the internal angular momentum of the core. Then,

$$I_{ex}(\theta) = \sum_{l_c m_c}\sum_{m_i(\mu_I)} C_{j_I j_0}(j_i, \mu_i; \mu_I, \mu_0)C_{l_i,1/2}(j_I, \mu_I; m_i, \mu_n)$$
$$\times \langle \mu_f|V|\mu_d; l_c, m_c, \mu_0\rangle_{ex} G_{ex}^{l_c m_c m_i}(\theta), \tag{16}$$

where

$$\langle \mu_f|V|\mu_d; l_c, m_c, \mu_0\rangle_{ex} = -\frac{M_n^*}{2\pi\hbar^2}\int \Psi^*_{\mu_f}(\boldsymbol{r}_p-\boldsymbol{r}_c, \sigma_p, \boldsymbol{r}_n-\boldsymbol{r}_c, \sigma_n, \xi)$$
$$\times V(\boldsymbol{r}_p-\boldsymbol{r}_c, \sigma_p, \boldsymbol{r}_n-\boldsymbol{r}_c, \sigma_n, \xi)\Psi_{\mu_d}(\boldsymbol{r}_n-\boldsymbol{r}_p, \sigma_n, \sigma_p) \tag{17}$$
$$\times \Phi_{j_0\mu_0}(\xi)Y_{l_c m_c}(\Omega_c)\,d\boldsymbol{r}_c d\boldsymbol{r}_p d\sigma_p d\boldsymbol{r}_n d\sigma_n d\xi\,,$$

$$G_{ex}^{l_c m_c m_i}(\theta) = \int Y^*_{l_c m_c}(\Omega_c)e^{-i\boldsymbol{k}_2\boldsymbol{R}_2}e^{-i\boldsymbol{K}_2(\boldsymbol{r}_{n'}-\boldsymbol{R}_2)}\Psi_{l_i m_i}(\boldsymbol{r}_{n'}-\boldsymbol{R}_2)\,d\Omega_c d\boldsymbol{r}_{n'} \tag{18}$$

with

$$\boldsymbol{k}_2=\boldsymbol{k}_d+(M_d/M_f)\boldsymbol{k}_n,\quad \boldsymbol{K}_2=(M_n/M_i)\boldsymbol{k}_d+\boldsymbol{k}_n\,, \tag{19}$$

and R_2 is the interaction radius between the core and the deuteron.

Eq. (16) is, as in the case of eq. (3), rewritten as

$$I_{ex}(\theta)=\sum_{l_c m_c}\sum_{m_i(\mu_I)}\sum_{j_s(\mu_s)} C_{j_I j_0}(j_i,\mu_i;\mu_I,\mu_0)\,C_{l_i,1/2}(j_I,\mu_I;m_i,\mu_n)$$
$$\times C_{j_0 l_c}(j_s,\mu_s;\mu_0,m_c)C_{j_s j_d}(j_f,\mu_f;\mu_s,\mu_d) \qquad (20)$$
$$\times\langle j_f\|V\|j_d;j_s(l_c,j_0)\rangle_{ex}G_{ex}^{l_c m_c m_i}(\theta),$$

where

$$\langle j_f\|V\|j_d;j_s(l_c,j_0)\rangle_{ex}=\frac{2j_s+1}{2j_f+1}\sum_{\mu_0 m_c}\sum_{\mu_f\mu_d} C_{j_0 l_c}(j_s,\mu_s;\mu_0,m_c)$$
$$\times C_{j_s j_d}(j_f,\mu_f;\mu_s,\mu_d)\langle\mu_f|V|\mu_d;l_c,m_c,\mu_0\rangle_{ex}. \qquad (21)$$

By taking the axis of space quantization in the direction of $\boldsymbol{k}_1$, in accordance with the case of evaluating $I_{st}(\theta)$, and taking as the internal wave function of the outside neutron n' of the target nucleus, the wave function $\Psi_{l_i m_i}(\boldsymbol{r}_{n'})$ for a spherical potential well, then we get

$$G_{ex}^{l_c m_c m_i}(\theta)=16\pi^2 i^{-(l_c+l_i)} j_{l_c}(k_2 R_2)\,Y_{l_c m_c}(\boldsymbol{k}_2,\hat{}\,\boldsymbol{k}_1)$$
$$\times G_2(K_2)\,Y_{l_i m_i}(\boldsymbol{K}_2,\hat{}\,\boldsymbol{k}_1), \qquad (22)$$

where

$$G_2(K_2)=\int_0^\infty j_{l_i}(K_2 r)R(r)\,r^2 dr, \qquad (23)$$

$R(r)$ being the radial part of $\Psi_{l_i m_i}(\boldsymbol{r})$, l_c the relative orbital angular momentum between the deuteron and the core.

(2-iii) Differential cross section

The differential cross section is given by

$$d\sigma/d\Omega\propto\sum|I_{st}-I_{ex}|^2$$
$$=\sum\{|I_{st}|^2-2R_e(I_{st}\bar{I}_{ex})+|I_{ex}|^2\}, \qquad (24)$$

where $\sum$ means an average over spin components of the initial nucleus and of the deuteron and a sum over those of the final nucleus and of the outgoing neutron. Explicitly, using eq. (7), the first term is obtained as follows,

$$\frac{1}{3(2j_i+1)}\sum_{\mu_i\mu_f\mu_d\mu_n}|I_{st}|^2=\frac{1}{2(2j_i+1)}\sum_{l_p}\Lambda_{st}^{l_p}|G_{st}^{l_p}|^2 \qquad (25)$$

where

$$\Lambda_{st}^{l_p}=\frac{2j_f+1}{2l_p+1}|\langle j_f\|V\|j_i;j_c(l_p,1/2)\rangle_{st}|^2. \qquad (26)$$

Using eqs. (7) and (20), the second term is expressed as follows,

$$
\begin{aligned}
&-\frac{2}{3(2j_i+1)}R_e\sum_{\mu_i\mu_f\mu_d\mu_n}I_{st}\check{I}_{ex}\\
&=-\frac{2}{3(2j_i+1)}\sum_{\mu_i\mu_f\mu_d\mu_n}\sum_{l_pm_p\mu_pj_c(\mu_c)}\sum_{l_cm_c}\sum_{m_i(\mu_I)}\sum_{j_s(\mu_s)}\\
&\times C_{1/2,1/2}(1,\mu_d;\mu_p,\mu_n)C_{l_p,1/2}(j_c,\mu_c,m_p,\mu_p)C_{j_ij_c}(j_f,\mu_f;\mu_i,\mu_c)\\
&\times C_{j_Ij_0}(j_i,\mu_i;\mu_I,\mu_0)C_{l_i,1/2}(j_I,\mu_I;m_i,\mu_n)C_{j_0l_c}(j_s,\mu_s;\mu_0,m_c)\\
&\times C_{j_sj_d}(j_f,\mu_f;\mu_s,\mu_d)\delta_{m_p,0}\\
&\times R_e\langle j_f\|V\|j_i;j_c(l_p,1/2)\rangle_{st}\overline{\langle j_f\|V\|j_d;j_s(l_c,j_0)\rangle}_{ex}G_{st}^{l_p}\bar{G}_{ex}^{l_cm_cm_i}\,. \qquad (27)
\end{aligned}
$$

Considering the conservation of the components of angular momentum which appear in the Clebsch-Gordan coefficients, we find that the non-vanishing terms are only those with

$$m_i=m_c\,. \qquad (28)$$

The above expression is, therefore, reduced to

$$-\frac{2}{3(2j_i+1)}R_e\sum_{l_pl_cm_c}\Lambda_c^{l_pl_cm_c}G_{st}^{l_p}\bar{G}_{ex}^{l_cm_cm_c}\,, \qquad (29)$$

where

$$
\begin{aligned}
\Lambda_c^{l_pl_cm_c}=&\sum_{\mu_i\mu_f\mu_d\mu_n}\sum_{\mu_pm_p}\sum_{j_c(\mu_c)}\sum_{j_s(\mu_s)}C_{1/2,1/2}(1,\mu_d;\mu_p,\mu_n)\\
&\times C_{l_p,1/2}(j_c,\mu_c;m_p,\mu_p)C_{j_ij_c}(j_f,\mu_f;\mu_i,\mu_c)C_{j_cj_0}(j_i,\mu_i;\mu_I,\mu_0)\\
&\times C_{l_i,1/2}(j_I,\mu_I;m_c,\mu_n)C_{j_0l_c}(j_s,\mu_s;\mu_0,m_c)C_{j_sj_d}(j_f,\mu_f;\mu_s,\mu_d)\\
&\times\delta_{m_p,0}\langle j_f\|V\|j_i;j_c(l_p,1/2)\rangle_{st}\overline{\langle j_f\|V\|j_d;j_s(l_c,j_0)\rangle}_{ex}\,. \qquad (30)
\end{aligned}
$$

Finally, the third term is expressed by eq. (20) as

$$
\begin{aligned}
\frac{1}{3(2j_i+1)}\sum_{\mu_i\mu_f\mu_d\mu_n}|I_{ex}|^2=&\frac{1}{3(2j_i+1)}\sum_{\mu_i\mu_f\mu_d\mu_n}\Big|\sum_{l_cm_c}\sum_{m_i}\sum_{j_s(\mu_s)}C_{j_ij_0}(j_i,\mu_i;\mu_I,\mu_0)\\
&\times C_{l_i,1/2}(j_I,\mu_I;m_i,\mu_n)C_{j_0l_c}(j_s,\mu_s;\mu_0,m_c)\,C_{j_sj_d}(j_f,\mu_f;\mu_s,\mu_d)\\
&\times\langle j_f\|V\|j_d;j_s(l_c,j_0)\rangle_{ex}G_{ex}^{l_cm_cm_i}(\theta)\Big|^2\\
=&\frac{1}{3(2j_i+1)}\sum_{l_cm_c}\sum_{l_c'm_c'}\sum_{m_im_i'}\Lambda_{ex}^{l_cm_c;\,l_c'm_c'}\,G_{ex}^{l_cm_cm_i;\,l_c'm_c'm_i'}\,, \qquad (31)
\end{aligned}
$$

where

$$\Lambda_{ex}^{l_cm_c;\,l_c'm_c'}=\sum_{j_s(\mu_s)}\frac{1}{2j_s+1}C_{j_0l_c}(j_s,\mu_s;\mu_0,m_c)C_{j_0l_c'}(j_s,\mu_s;\mu_0',m_c')$$

$$\times\langle j_f\|V\|j_a\,;\,j_s(l_c,\,j_0)\rangle_{ex}\overline{\langle j_f\|V\|j_a\,;\,j_s(l_c',\,j_0)\rangle_{ex}}, \tag{32}$$

$$G_{ex}^{l_cm_cm_i\,;\,l_c'm_c'm_i'}=C_{j_Ij_0}(j_i,\,\mu_i\,;\,\mu_I,\,\mu_0)C'_{j_Ij_0}\overline{(j_i,\,\mu_i\,;\,\mu_I',\,\mu_0')} \tag{33}$$

$$\times C_{l_i,1/2}(j_I,\,\mu_I\,;\,m_i,\,\mu_n)C_{l_i,1/2}(j_I',\,\mu_I'\,;\,m_i',\,\mu_n)G_{ex}^{l_cm_cm_i}\bar{G}_{ex}^{l_c'm_c'm_i'}.$$

Here attention should be paid to the fact that, in the expression (31) for the heavy particle stripping term, the cross terms between different values of l_c and m_c do not vanish in general, differing from the case of the expression (25) for the deuteron stripping term.* In such special cases, however, in which either the intrinsic spin of the core of the target nucleus j_0 or the internal angular momentum of the extra nucleon l_i is equal to zero, these cross terms do vanish so that the expression (31) is greatly simplified.

Now, by the conservation of angular momentum, the angular momenta l_p, l_c of the captured particles are subject to conditions,

$$\boldsymbol{j}_f=\boldsymbol{j}_i+\boldsymbol{l}_p+\boldsymbol{s}_p\,, \tag{34}$$

$$\boldsymbol{j}_f=\boldsymbol{j}_0+\boldsymbol{l}_c+\boldsymbol{j}_a\,, \tag{35}$$

respectively. Furthermore, the conservation of parity restricts the values of l_p and l_c to either even or odd, and in the deuteron stripping process the value of l_p is restricted to a single value in almost all of the cases. Therefore, combining eqs. (11), (22), (25), (29) and (31), we obtain finally

$$\begin{aligned}\frac{d\sigma}{d\Omega}\propto&|\,j_{l_p}(k_1R_1)G_1(K_1)\\&+j_{l_p}(k_1R_1)G_1(K_1)\sum_{l_cm_c}A_{l_cm_c}j_{l_c}(k_2R_2)G_2(K_2)\,Y_{l_cm_c}(\boldsymbol{k}_2,\hat{\ }\boldsymbol{k}_1)\,Y_{l_im_c}(\boldsymbol{K}_2,\hat{\ }\boldsymbol{k}_1)\\&+\sum_{l_cm_c}\sum_{l_c'm_c'}\sum_{m_im_i'}B^{m_im_i'}_{l_cm_c\,;\,l_c'm_c'}[G_2(K_2)]^2j_{l_c}(k_2R_2)\,j_{l_c'}(k_2R_2)\\&\times Y_{l_cm_c}(\boldsymbol{k}_2,\hat{\ }\boldsymbol{k}_1)\,Y_{l_c'm_c'}(\boldsymbol{k}_2,\hat{\ }\boldsymbol{k}_1)\,Y_{l_im_i}(\boldsymbol{K}_2,\hat{\ }\boldsymbol{k}_1)\,Y_{l_im_i'}(\boldsymbol{K}_2,\hat{\ }\boldsymbol{k}_1),\end{aligned} \tag{36}$$

where

$$A_{l_cm_c}=-(16\pi/3)(4\pi/2l_p+1)^{1/2}R_c[i^{l_p+l_i+l_c}\Lambda_c^{l_cm_cm_c}/\Lambda_{st}^{l_p}], \tag{37}$$

$$\begin{aligned}B^{m_im_i'}_{l_cm_c\,;\,l_c'm_c'}=&256\pi^4C_{j_Ij_0}(j_i,\,\mu_i\,;\,\mu_I,\,\mu_0)C'_{j_Ij_0}(j_i,\,\mu_i\,;\,\mu_I',\,\mu_0')\\&\times C_{l_i,1/2}(j_I,\,\mu_I\,;\,m_i,\,\mu_n)C_{l_i,1/2}(j_I',\,\mu_I'\,;\,m_i',\,\mu_n)\\&\times\Lambda_{ex}^{l_cm_c\,;\,l_c'm_c'}/\Lambda_{st}^{l_p}.\end{aligned} \tag{38}$$

$A_{l_cm_c}$ and $B^{m_im_i'}_{l_cm_c\,;\,l_c'm_c'}$ are specific nuclear factors which are independent of the kinematics of the reaction concerned here.

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* We are obliged to Dr. S. Okai for clearly pointing out this fact to us and for helpful discussions during the preparation of the present article.

§ 3. Analysis of $Be^9(d,n)B^{10}$ ground state reaction

Basing on the discussion developed in the first section, we will discuss here the angular distribution of neutrons of $Be^9(d, n)B^{10}$ ground state reaction as a specific example of application of the general theory presented in section 2.

It is known that the ground state of Be^9 is of $P_{3/2}$, odd parity and that of B^{10} is of P_3, even parity, so l_p is restricted to values of

$$l_p=1,3 \quad \text{or} \quad 5.$$

We take here the value $l_p=1$. As for l_c, if we take the spin 0 of the Be^8 ground state for the intrinsic angular momentum j_0 of the core, we are restricted to only those values of

$$l_c=2 \quad \text{or} \quad 4,$$

while if we take for j_0 the spin 2 of the Be^8 first excited state, l_c are allowed to take those values of

$$l_c=0,\ 2,\ 4 \quad \text{or} \quad 6.$$

Since we are concerned with lower incident energy of the deuteron, we take the value of $l_c=0$, assuming that the core of Be^9 has an intrinsic angular momentum $j_0=2$.

Thus, eq. (36) is reduced to the following expression,

$$\begin{aligned} d\sigma/d\Omega \propto & |j_1(k_1R_1)G_1(K_1)|^2 \\ & + Aj_1(k_1R_1)G_1(K_1)\, j_0(k_2R_2)G_2(K_2)\cos(\boldsymbol{K}_2,\hat{}\boldsymbol{k}_1) \\ & + B|j_0(k_2R_2)G_2(K_2)|^2, \end{aligned} \tag{39}$$

where

$$A=(16\pi/3)Re\Lambda_c^{000}/\Lambda_{st}^{1} \tag{40}$$

$$B=(2\pi)^{-1}\Lambda_{ex}^{00;00}/\Lambda_{st}^{1}. \tag{41}$$

A, B are the nuclear factors and we take them to be adjustable parameters in the following analysis of experimental data. We notice that the parameter B is a positive definite number. Taking the square well wave function for the radial wave function $R(r)$ for the outside neutron n', we obtain

$$G_2(K_2)=-ar_0\sin k_0r_0\left[\frac{1}{k_0^2-K_2^2}+\frac{1}{\beta^2+K_2^2}\right]\left[\frac{1+\beta r_0}{\beta^2 r_0^2}\sin K_2r_0+j_1(K_2r_0)\right], \tag{42}$$

where

$$k_0=\sqrt{2M^*(E-V)}/\hbar=6.70\times10^{12}\,\mathrm{cm}^{-1},$$

$$\beta=\sqrt{2M^*|E|}/\hbar=2.66\times10^{12}\,\mathrm{cm}^{-1},$$

$$a=2(k_0^3/g)^{1/2},$$

$$g=k_0r_0+[(2+\beta r_0)(k_0/\beta)^4+(1+\beta r_0)(k_0/\beta)^2-1]\sin^2 k_0r_0/k_0r_0,$$

$$k_0r_0\cot k_0r_0=1+(1+\beta r_0)(k_0/\beta)^2.$$

Here we take for the well radius r_0 the following values,

$$r_0=5.0\times10^{-13}\text{ cm}.$$

Those numerical values presented above are determined through the analysis of the photo-disintegration of Be^9 by Guth and Mullin.[11]

Using the formula (39), the experimental data obtained by Pruitt, Swartz and Hanna[7] have been reproduced by the following values of parameters,

$$R_1=9.0\times10^{-13}\text{ cm}, \qquad R_2=4.5\times10^{-13}\text{cm};$$
$$A=-0.44, \qquad B=2.71.$$

Fig. 1 shows a comparison of the experimental data and the theoretical curve. Since the parameters A, B are independent of the kinematics of the reaction within the frame of the stripping approximation, we have analyzed other experimental data at different incident energies of the deuteron, using the above determined values of R_1, R_2, A and B. For the case of $E_d=0.8$ MeV observed by Green et al.[8] a better agreement between the experimental data and the theoretical curve is obtained by taking $R_2=5.2\times10^{-13}\,cm$. This is shown in Fig. 2 by a broken line. Agreement between the experimental data obtained by Shpetny[9] and the theoretical curve is rather poor at $E_d=1$ MeV, yet we believe that this is due to experimental defects, since Shpetny's result at $E_d=0.8$ MeV has a stronger asymmetric angular distribution than Green's result at the same incident energy of the deuteron, which is in good agreement with the theoretical curve as shown in Fig. 2. The result at $E_d=1.6$ MeV is consistent with the theoretical curve calculated from the formula (39) using the above values of the parameters. This is shown in Fig. 4. We have also

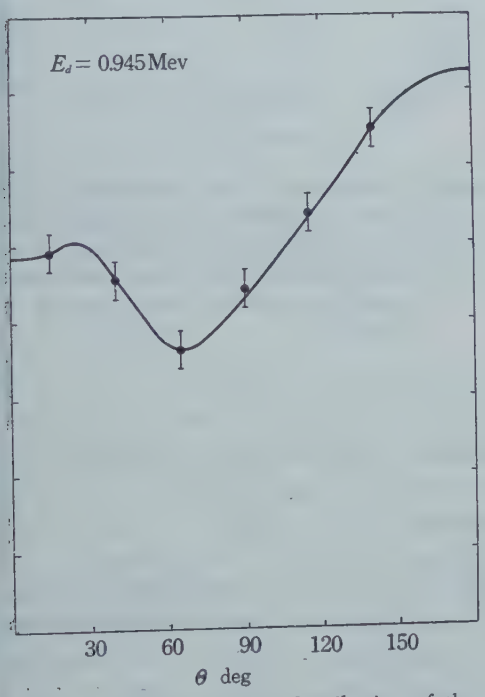

Fig. 1 Relative angular distribution of the $Be^9(d,n)B^{10}$ ground state neutron at $E_d=0.945$ MeV. The circles are the experimental data observed by Pruitt, Swartz and Hanna[7]. The solid curve is the theoretical angular distribution which has been calculated by using the parameters presented in the text.

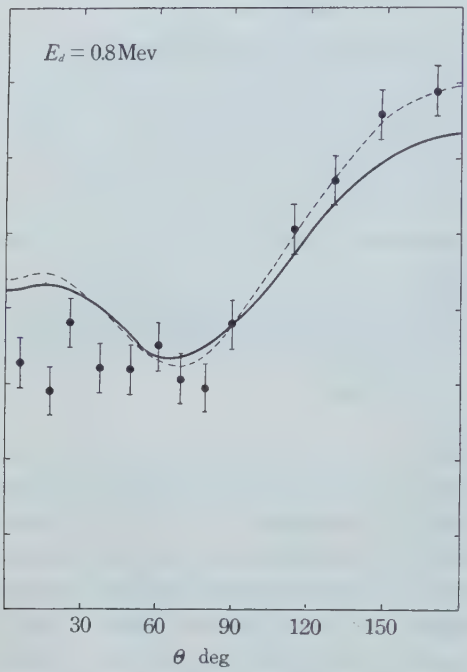

Fig. 2 Comparison of the experimental and theoretical relative angular distribution of the $Be^9(d, n)B^{10}$ ground state neutron at E_d=0.8 MeV. The circles are the experimental data observed by Green et al.[8] The broken line is the calculated curve by using the larger value of 5.2×10^{-13} *cm* for the interaction radius R_2.

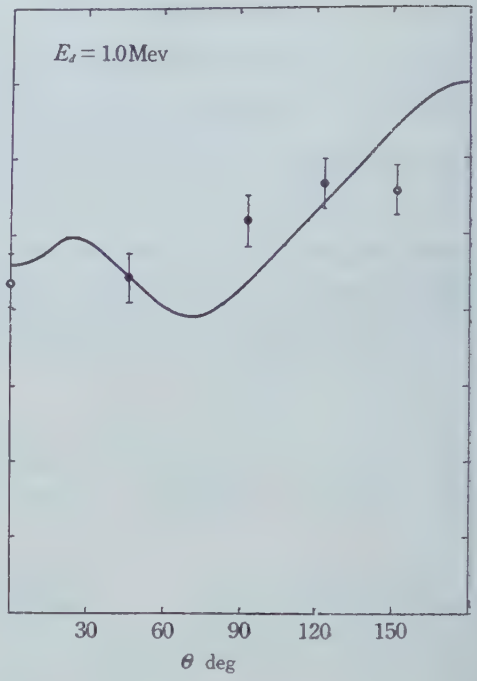

Fig. 3 Comparison of the experimental and theoretical relative angular distribution of the $Be^9(d, n)B^{10}$ ground state neutron at E_d=1.0 MeV. The circles are the experimental data observed by Shpetny[9].

calculated expected theoretical angular distributions at E_d=3 MeV and 5 MeV.

§ 4. Discussion

Applying the general formula (36) to the $Be^9(d, n)B^{10}$ ground state reaction, we have been able to reproduce overall features of the observed angular distributions at different deuteron incident energies. The interaction radius R_1 for the deuteron stripping has been found to be much larger than R_2, the interaction radius for the heavy particle stripping. This suggests that the deuteron stripping mainly takes place through interaction between the deuteron and the outside neutron of the Be^9 nucleus which spreads out to a large extension due to its weak binding energy.

As for the interference term, in the calculation of Owen and Madansky the axis of space quantization is taken in the direction of $\boldsymbol{k}_1$ for the deuteron stripping term, while for the heavy particle stripping term in the direction of $\boldsymbol{K}_2$, so the factor $\cos(\boldsymbol{K}_2, \hat{} \ \boldsymbol{k}_1)$ in the second term of eq. (39) is absent in their interference

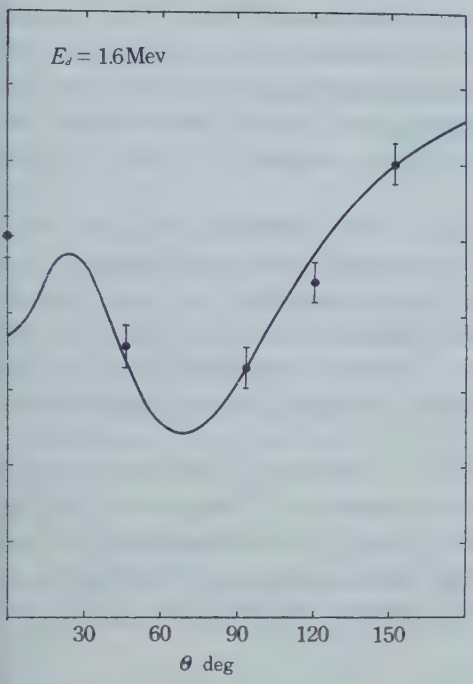

Fig. 4 Comparison of the experimental and theoretical relative angular distribution of the $Be^9(d, n)B^{10}$ ground state neutron at E_d=1.6 MeV. The circles are the experimentel data observed by Shpetny[9].

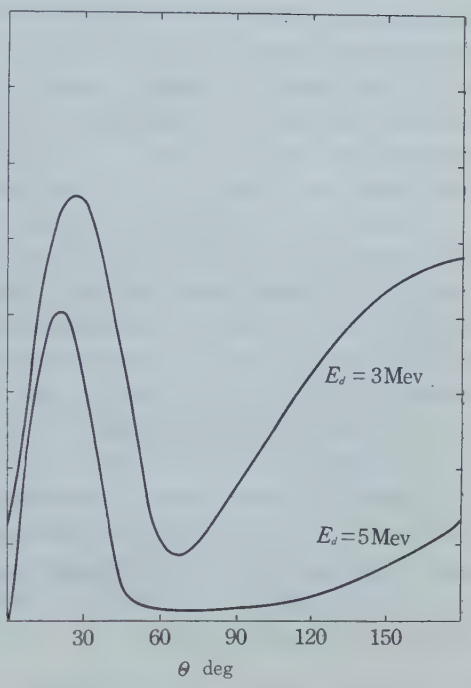

Fig. 5 Calculated relative angular distribution of the $Be^9(d, n)B^{10}$ ground state neutron at E_d=3 MeV, and E_d=5 MeV.

term. With the negative values of the parameter A, the interference term is destructive in the forward quadrant and constructive in the backward. The presence of factor $\cos(\boldsymbol{K}_2, \hat{}\ \boldsymbol{k}_1)$ is essential for obtaining such overall good agreements between the experimental data and the theoretical curve at various incident energies of the deuteron such as shown in Figs. 1–4.

Concerning the value of the angular momentum l_c of the captured deuteron, our choice of $l_c=0$, $j_0=2$ being assumed, may be justified since we have $k_2R_2<2$ for E_d=2 MeV, hence contributions of the deuteron wave with $l_c=2$ can be neglected at the lower deuteron energy. Furthermore, if we take $l_c=2$, assuming $j_0=0$, then the factor $j_2(k_2R_2)$ becomes larger at the forward angle and thus we fail to interpret the backward component of the angular distributions by the heavy particle stripping machanism.

Thus, the overall good agreement between the experimental data and the theoretical analysis obtained in the last section leads to the conclusion that the core, say $\langle Be^8 \rangle$, of the Be^9 nucleus has an intrinsic angular momentum $j_0=2$ in the ground state of Be^9. Then, it may seem that we have to take a spheroidal well wave function for the wave function of the outside neutron to calculate the factor $G_2(K_2)$.

But, since the main factor $j_0(k_2R_2)$ is insensitive to the detailed form of the outside neutron distribution, the general feature of our analysis will not be changed by the use of the spheroidal wave function. This may, however, give rise to a sizeable effect on theoretical determination of the excitation curve for the photo-neutron disintegration of the Be^9 nucleus. It will be interesting to investigate this effect in more detail along the theory of Guth and Mullin.

Finally, we will comment on the theoretical curves calculated at $E_d=3$ MeV and 5 MeV. For $E_d>3$ MeV, k_2R_2 becomes larger than 2, thus the deuteron wave with $l_c=2$ becomes much effective. According to our estimation, the contributions of the D-wave deuteron capture gives a rather isotropic angular distribution so as to fill in the valleys of the angular distributions. Further experimental investigations at those higher deuteron energies will be expected to make clear the contributions of the D-wave deuteron capture.

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Lorentz Covariance of Quantum Electrodynamics with the Indefinite Metric*

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It is shown that the norm of any state vector in quantum electrodynamics with the indefinite metric is Lorentz invariant. The treatment involves the use of certain Lorentz-covariant quantities called expansors.

§ 1. Introduction

The most general state vector for the photon field can be expressed as[1]

$$\Psi = \sum_{k,l,m,n=0}^{\infty} A_{k,l,m,n}\Psi(k, l, m, n), \tag{1}$$

where k, l, m and n are the numbers of the four types of photons in the state represented by the normalized vector $\Psi(k, l, m, n)$, the $A_{k,l,m,n}$ are arbitrary coefficients, and for simplicity we are confining our attention to a single Fourier component of the photon field. We shall investigate the behavior of the quantities $A_{k,l,m,n}$ under a Lorentz transformation. The main interest of the present paper lies in the fact that although the $A_{k,l,m,n}$ are neither tensors nor spinors, a Lorentz transformation for these quantities does exist, which keeps the norm of the state vector (1) invariant. The present investigation is similar to that of Dirac[2], but we shall follow our formalism[1] of quantum electrodynamics with the indefinite metric instead of the older unsatisfactory treatment.

Recently in a paper published in this journal Sunakawa[3] has questioned the Lorentz invariance of the state vector in quantum electrodynamics with the indefinite metric, but we shall show that there does not exist any difficulty of Lorentz invariance. Our treatment, of course, is in complete agreement with the earlier work of Belinfante.[4]

§ 2. State vector for the photon field

In our original treatment[1] of quantum electrodynamics with the indefinite metric, two sets of quantities were introduced (called the Hermitian conjugates and the adjoints), which made the resulting formalism unnecessarily complicated. Therefore,

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we shall follow a simplified formalism described in a recent paper[5], which involves only one set of quantities called the Hermitian conjugates*.

The normalized state vector $\Psi(k, l, m, n)$ for the photon field is given by

$$\Psi(k, l, m, n) = (k!\, l!\, m!\, n!)^{-1/2} a_1{}^k\, a_2{}^l\, a_3{}^m\, a_0{}^n\, \Psi(0, 0, 0, 0), \tag{2}$$

where a_1, a_2, a_3 and a_0 are the annihilation operators for the four types of photons. The Hermitian conjugate of (2) is

$$\Psi^*(k, l, m, n) = (k!\, l!\, m!\, n!)^{-1/2} \Psi^*(0, 0, 0, 0)\, a_1{}^{*k}\, a_2{}^{*l}\, a_3{}^{*m}\, a_0{}^{*n}, \tag{3}$$

where a_1^*, a_2^*, a_3^* and a_0^* are the creation operators. It follows from the commutation relations

$$[a_1,\ a_1^*]=[a_2,\ a_2^*]=[a_3,\ a_3^*]=1,\ [a_0,\ a_0^*]=-1, \tag{4}$$

that

$$\Psi^*(p, q, r, s)\, \Psi(k, l, m, n) = (-1)^n \delta_{kp}\, \delta_{lq}\, \delta_{mr}\, \delta_{ns}. \tag{5}$$

Using (2), we can also express the general state vector (1) for the photon field as

$$\Psi = \sum A_{k,l,m,n} (k!\, l!\, m!\, n!)^{-1/2} a_1{}^k\, a_2{}^l\, a_3{}^m\, a_0{}^n \Psi(0, 0, 0, 0), \tag{6}$$

and its norm as

$$\Psi^*\, \Psi = \sum (-1)^n\, A^*_{k,l,m,n}\, A_{k,l,m,n}. \tag{7}$$

§ 3. Lorentz invariance of the norm of state vector

We shall show that when the operators a_1, a_2, a_3 and a_0 undergo a Lorentz transformation, the coefficients $A_{k,l,m,n}$ in (6) transform in such a way that the norm (7) remains unchanged. Since (a_1, a_2, a_3, ia_0) is a four-vector, it will be sufficient for our purpose to consider the infinitesimal transformations

$$a_1 = a_1' + \epsilon a_2',\ a_2 = a_2' - \epsilon a_1',\ a_3 = a_3', a_0 = a_0', \tag{8}$$

and

$$a_1 = a_1',\ a_2 = a_2',\ a_3 = a_3' + \epsilon a_0',\ a_0 = a_0' + \epsilon a_3', \tag{9}$$

where ϵ is infinitesimal. The above transformations correspond to infinitesimal rotations in the (x_1, x_2) and the (x_3, x_4) planes respectively.

Case I. Carrying out the infinitesimal transformation (8), we can express the state vector (6) as

$$\Psi = \sum A_{k,l,m,n} (k!\, l!\, m!\, n!)^{-1/2} (a_1' + \epsilon a_2')^k (a_2' - \epsilon a_1')^l a_3'{}^m a_0'{}^n \Psi(0, 0, 0, 0)$$
$$= \sum A_{k,l,m,n} (k!\, l!\, m!\, n!)^{-1/2} (a_1'{}^k\, a_2'{}^l + k\epsilon a_1'{}^{k-1}\, a_2'{}^{l+1} - l\epsilon a_1'{}^{k+1}\, a_2'{}^{l-1})$$

* It is important to note that the Hermitian conjugate quantities, as defined in reference 5, correspond to the adjoint quantities in reference 1.

$$\times a_3'^m a_0'^n \Psi(0,0,0,0), \tag{10}$$

which can be written as

$$\Psi=\sum A'_{k,l,m,n}(k!\,l!\,m!\,n!)^{-1/2}\, a_1'^k\, a_2'^l\, a_3'^m\, a_0'^n \Psi(0,0,0,0) \tag{11}$$

with

$$\begin{aligned} A'_{k,l,m,n} &= A_{k,l,m,n}+\epsilon l^{1/2}(k+1)^{1/2}A_{k+1,l-1,m,n} \\ &\quad -\epsilon(l+1)^{1/2}k^{1/2}\,A_{k-1,l+1,m,n}, \end{aligned} \tag{12}$$

where $A_{k,l,m,n}$ with a negative suffix is to be taken as zero. We then find

$$\begin{aligned} &\sum(-1)^n\, A'^*_{k,l,m,n}\; A'_{k,l,m,n} \\ &=\sum(-1)^n[A^*_{k,l,m,n}\; A_{k,l,m,n} \\ &\qquad +\epsilon l^{1/2}(k+1)^{1/2}(A^*_{k,l,m,n}\; A_{k+1,l-1,m,n}+A^*_{k+1,l-1,m,n}\; A_{k,l,m,n}) \\ &\qquad -\epsilon(l+1)^{1/2}k^{1/2}(A^*_{k-1,l+1,m,n}\; A_{k,l,m,n}+A^*_{k,l,m,n}\; A_{k-1,l+1,m,n})] \\ &=\sum(-1)^n\; A^*_{k,l,m,n}\; A_{k,l,m,n}. \end{aligned} \tag{13}$$

Case II. Under the infinitesimal transformation (9) the state vector (6) becomes

$$\begin{aligned} \Psi&=\sum A_{k,l,m,n}\,(k!\,l!\,m!\,n!)^{-1/2}a_1'^k\, a_2'^l(a_3'+\epsilon a_0')^m \\ &\quad\times(a_0'+\epsilon a_3')^n\;\Psi(0,0,0,0) \\ &=\sum A_{k,l,m,n}(k!\,l!\,m!\,n!)^{-1/2}\, a_1'^k\, a_2'^l \\ &\times(a_3'^m\, a_0'^n+m\epsilon a_3'^{m-1}a_0'^{n+1}+n\epsilon a_3'^{m+1}\, a_0'^{n-1})\,\Psi(0,0,0,0), \end{aligned} \tag{14}$$

which can be expressed as

$$\Psi=\sum A'_{k,l,m,n}(k!\,l!\,m!\,n!)^{-1/2}a_1'^k\, a_2'^l\, a_3'^m\, a_0'^n\,\Psi(0,0,0,0) \tag{15}$$

with

$$\begin{aligned} A'_{k,l,m,n} &= A_{k,l,m,n}+\epsilon(m+1)^{1/2}n^{1/2}A_{k,l,m+1,n-1} \\ &\quad +\epsilon m^{1/2}(n+1)^{1/2}A_{k,l,m-1,n+1}, \end{aligned} \tag{16}$$

where again $A_{k,l,m,n}$ with a negative suffix is to be taken as zero. We then obtain

$$\begin{aligned} &\sum(-1)^n\; A'^*_{k,l,m,n}\; A'_{k,l,m,n} \\ &=\sum(-1)^n[A^*_{k,l,m,n}\; A_{k,l,m,n} \\ &\qquad +\epsilon(m+1)^{1/2}n^{1/2}(A^*_{k,l,m,n}\; A_{k,l,m+1,n-1}+A^*_{k,l,m+1,n-1}\; A_{k,l,m,n}) \\ &\qquad +\epsilon m^{1/2}(n+1)^{1/2}(A^*_{k,l,m-1,n+1}\; A_{k,l,m,n}+A^*_{k,l,m,n}\; A_{k,l,m-1,n+1})] \\ &=\sum(-1)^n A^*_{k,l,m,n}\; A_{k,l,m,n}. \end{aligned} \tag{17}$$

It should be noted that the factor $(-1)^n$ in the expression for the norm of the state vector is necessary to ensure the cancellation of the infinitesimal term in (17).

We may call the quantity $A_{k,l,m,n}$ which is neither a tensor nor a spinor, an *expansor*. However, our treatment of expansors is much more symmetrical with respect to the four types of photons than that of Dirac[2].

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Photodisintegration of the Deuteron at High Energies

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It is shown that the photodisintegration of the deuteron up to about 140 Mev can be well accounted for only by the nucleon parts, if many effects are correctly treated. It is pointed out that the influence of the nature of the triplet odd state (which is the final state of the electric dipole transition) is remarkable, and the necessity of using exact multipole formulas at high energies is emphasized.

§ 1. Introduction

The photodisintegration of the deuteron up to about 140 Mev (the threshold energy for pion production) can be well accounted for only by the nucleon parts (without including the virtual pion effect), if many elements which take part in the process are correctly treated. The important points are that the D-state probability of the deuteron should be large[1] (perhaps of the order $6\sim8\%$)[2], the tensor force of the triplet odd state (^3O) must be strong,[3),4)] and also be positive,[5)] the transition $^3D-^3F_2$ is to be included,[6),7)] the hard core radius of the singlet even state (^1E) would be large[4),8)] and, furthermore, the formulas of multipole radiation should be treated correctly.[8),9)]

In this paper, these points being considered, the computation of photodisintegration of the deuteron at high energies is carried out and it is indicated that the photodisintegration of the deuteron up to about 140 Mev would be mostly due to the nucleon parts and the contribution of the nucleon parts would be still large at $E_\gamma > 140$ Mev. Consequently it suggests that the virtual pion effect would be negligible at $E_\gamma < 140$ Mev, and its effect would not be so large even at $E_\gamma > 140$ Mev. Especially we place great emphasis on the influence of the nature of the triplet odd state, and shall show that the ambiguity about the natures of the triplet odd state in $p-p$ sacttering can be excluded. Also the necessity for using exact multipole formulas at high energies is emphasized.

In § 2 the formulas of the transitions are given, and our method of computation is explained. In § 3 the calculated results are shown. Finally some discussions are made in § 4. Also the exact treatment of multipole is explained in the Appendix.

§ 2. Preliminary

In this paper we compute only the isotropic part and the total cross section which are most important ones in the photodisintegration of the deuteron.[10)]

The transitions which contribute mainly to the isotropic part and the total cross section at $E_\gamma < 140$ Mev are the electric dipole (*e. d.*) and the magnetic dipole (*m. d.*). The transitions are induced by the $j_1(k_\gamma r)$ part of the incident γ ray* (writing in the spherical harmonics),[8),9)] and may be represented by[11)]

$$j_1(k_\gamma r) + ({}^3S_1 + {}^3D_1) \longrightarrow {}^3P_0,\ {}^3P_1,\ {}^3P_2 + {}^3F_2$$
$$j_1(k_\gamma r) + {}^3S_1 \longrightarrow {}^1S_0$$
$$+ {}^3D_1 \longrightarrow {}^1D_2.$$

The formulas for the *e. d.* and *m. d.* cross sections (we denote them by σ_{ed} and σ_{md}) and the isotropic cross sections given by *e. d.* and *m. d.* (σ_{ed}^{ip} and σ_{md}^{ip}) are as follows.[6),11)]**

$$\sigma_{ed} = \frac{\pi}{3}\frac{e^2}{\hbar c}\frac{ME_\gamma}{\hbar^2} k \frac{1}{9}\left(I_{10}^2 + 3I_{11}^2 + 5I_{12}^2 + \frac{15}{2} I_f^2\right) \tag{1}$$

$$\sigma_{md} = \frac{\pi}{3}(\mu_n - \mu_p)^2 \frac{e^2}{\hbar c}\left(\frac{\hbar}{Mc}\right)^2 \frac{ME_\gamma}{\hbar^2} k (I_{00}^2 + I_{02}^2) \tag{2}$$

$$\sigma_{ed}^{ip} = \frac{\pi}{3}\frac{e^2}{\hbar c}\frac{ME_\gamma}{\hbar^2} k (a_1 + a_2) \tag{3}$$

$$a_1 = \frac{1}{36}\left\{4I_{10}^2 + 9I_{11}^2 + 13I_{12}^2 - 8I_{10}I_{12}\cos(\delta_1^0 - \delta_1^2) - 18 I_{11} I_{12}\cos(\delta_1^1 - \delta_1^2)\right\} \tag{4}$$

$$a_2 = \frac{I_f}{6}\left\{3I_f - 2I_{10}\cos(\delta_1^0 - \delta_F) + 3I_{11}\cos(\delta_1^1 - \delta_F) - I_{12}\cos(\delta_1^2 - \delta_F)\right\} \tag{5}$$

in the approximation in which the coupling of ${}^3P_2$ with ${}^3F_2$ is neglected.

$$\sigma_{md}^{ip} = \frac{\pi}{3}(\mu_n - \mu_p)^2 \frac{e^2}{\hbar c}\left(\frac{\hbar}{Mc}\right)^2 \frac{ME_\gamma}{\hbar^2} k \left\{I_{00}^2 - \sqrt{2}\cos(\delta_0^0 - \delta_2^2) I_{00} I_{02} + \tfrac{1}{2} I_{02}^2\right\}. \tag{6}$$

Here I_{10}, I_{11}, I_{12} and I_F represent the matrices whose final states are ${}^3P_0$, ${}^3P_1$, ${}^3P_2$ and ${}^3F_2$ respectively; and I_{00}, I_{02} represent those whose final states are ${}^1S_0$ and ${}^1D_2$.

In the case in which we take $k_\gamma r/3$ as approximate one to $j_1(k_\gamma r)$, the formulas of I_{ij} are relatively simple. In this case they become

$$I_{10} = \int (U_d - \sqrt{2}\, W_d)\, r U_{10}\, dr \tag{7}$$

$$I_{11} = \int \left(U_d + \frac{1}{\sqrt{2}} W_d\right) r U_{11}\, dr \tag{8}$$

$$I_{12} = \int \left(U_d - \frac{1}{5\sqrt{2}} W_d\right) r U_{12}\, dr \tag{9}$$

---

* The expansion into power series, which is usually adopted, is not correct.[9)]

** Symbols used in this paper are same as those used in the previous papers.[3)]

$$I_f=\frac{3\sqrt{2}}{5}\int W_d r U_F dr \tag{10}$$

$$I_{00}=\int U_d U_0 dr \tag{11}$$

$$I_{02}=\int W_d U_2 dr, \tag{12}$$

if we neglect the contribution from virtual pions. Here U_d and W_d are the radial wave function of the deuteron of ${}^3S_1$ and ${}^3D_1$, and U_{ij} are the radial wave functions of the state, with $L=i$ and $J=j$.

The formulas for exact $j_1(k_\gamma r)$ are quite complicated. When we neglect the D-state of the deuteron, it becomes as

$$I_{1i}=\int\left\{1-\frac{1}{5}(k_\gamma r)^2+\frac{3}{280}(k_\gamma r)^4\cdots\right\} r U_d U_{1i} dr$$
$$+\frac{k_\gamma \hbar}{2Mc}\int\left\{1-\frac{1}{10}(k_\gamma r)^2+\cdots\right\} r^2 U_{1i}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)U_d dr \tag{13}$$

instead of

$$I_{1i}=\int U_d r U_{1i} dr, \tag{14}$$

and

$$I_{00}=\int U_d U_0\left\{1-\frac{1}{3!}(k_\gamma r)^2+\frac{1}{5!}(k_\gamma r)^4\cdots\right\} dr=\int U_d U_0 \frac{\sin(k_\gamma r)}{k_\gamma r} dr \tag{15}$$

instead of (11). (See the Appendix.)

The actual computation indicates that the last part of (13) including $1/M$ is very small, and it is sufficient only to take the first part (without $1/M$).

We have used the phase shifts of ${}^3O$ and ${}^1E$ obtained from the analysis of $p-p$ scattering data[12] for the computation of cross sections, and for the wave function of the deuteron we adopted that of $I-O-T-W$.[13]

In the computation of $e.d.$ we have taken the P wave so that it is shifted by the given phase shift[3] at $r>1/k$ $(k=mc/\hbar)$ and that it continues smoothly and falls to zero at $r=0.1/k$ in $r<1/k$. For the F wave we used free wave at all distances. In the computation of $m.d.$ we have estimated the magnitude by computing its upper limit.[3]

Because the contributions of σ_{eq} and σ_{mq} to σ_{ip} and σ_T are negligible, we do not touch upon them in this paper.

§ 3. The calculated results

I. *When the final interaction is neglected.*

At first, let us consider the case in which we neglect the final state interaction. In this case the computation is simpler. The calculated σ_{ed} are plotted in Fig. 1.

In Fig. 1 we show the three cases, $k_\tau r/3$ + deuteron $\longrightarrow {}^3P$, $k_\tau r/3$ + deuteron $\longrightarrow {}^3P + {}^3F$, $j_1(k_\tau r)$ + deuteron $\longrightarrow {}^3P + {}^3F$, from which we can see the contributions of ${}^3P$, ${}^3F$ and the higher order terms of $j_1(k_\tau r)$. For the last case, to avoid wearisome computation, we have only estimated its magnitude by the following computation. We have computed σ_{el} neglecting the D-state of the deuteron and the ratio σ_{el} (for $j_1(k_\tau r) + {}^3S_1 \longrightarrow {}^3P)/\sigma_{el}$ (for $k_\tau r/3 + {}^3S_1 \longrightarrow {}^3P$) at various energies. Then we multiplied the ratio to σ_{el} of the case $k_\tau r/3 + ({}^3S_1 + {}^3D_1) \longrightarrow {}^3P + {}^3F$. (Such estimation would not be very reliable at $E_\tau > 150$ Mev where ${}^3D_1 \longrightarrow {}^3F_2$ transition becomes the main one.)

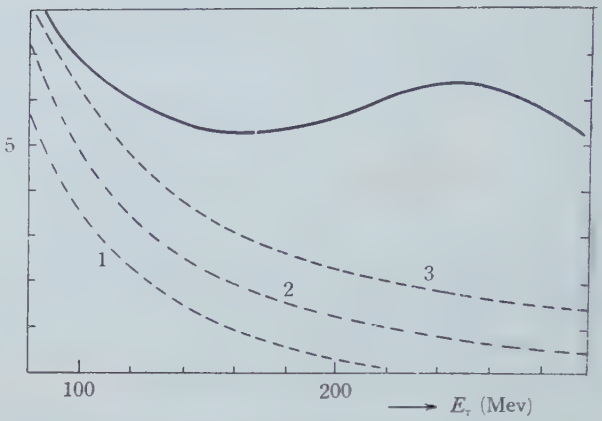

Fig. 1. The total electric dipole cross section of the case I and the experimental σ_T are shown in unit of 10^{-29} cm^2. If the magnetic dipole cross section is added, the difference between the experimental curve and the calculated curves is decreased more. The dotted curves, 1, 2, and 3 represent σ_{el} for the cases, 1) $k_\tau r/3 + ({}^3S_1 + {}^3D_1) \rightarrow {}^3P$, 2) $k_\tau r/3 + ({}^3S_1 + {}^3D_1) \rightarrow {}^3P + {}^3F$, 3) $j_1(k_\tau r) + ({}^3S_1 + {}^3D_1) \rightarrow {}^3P + {}^3F$ respectively. The solid curve represents the experimental σ_T.

II. *When the final interaction is included.*

Now let us consider the cases when the final state interactions are included. For example, we have chosen for the computation of the photodisintegration of the deuteron at 87 Mev and 132 Mev two sorts of phase shifts which can account for $p-p$ scattering data at 170 Mev and 260 Mev given by $C-J-V$[12]. Set A corresponds to positive tensor potential, and Set B corresponds to negative tensor potential.

The phase shifts are obtained by neglecting waves with $l \geq 2$. (It will be called $S-P$ approximation hereafter.)* In the two sets, only set A can be fitted to the polarization data.[12] But we have also shown the results for set B for comparison, because, if the phase shifts of D and F waves are also introduced in $p-p$ scattering analysis, there would be sufficient possibilities that a set belonging

Table 1. Phase shifts which are fitted to $p-p$ scattering data got by $C-J-V$.[12] (in unit of degree)

Energy		$\delta_0{}^0$	$\delta_1{}^0$	$\delta_1{}^1$	$\delta_1{}^2$
170 Mev	A	19	22	−21	10
	B	19	−19	24	− 7
260 Mev	A	−22	18	−25	15
	B	−22	−11	32	− 9

to class B fits the data.*

The phase shifts are shown in Table 1, and the computed results are presented in Table 2. We see that the differences between A and B are quite remarkable. Comparing with the experimental data,[10] we know at once that only set A is favourable, and set B seems to be excluded.[14] (It is to be noted that all the previous theories predict that the virtual pion effect is still small at least at $E_\gamma \simeq$ 90 Mev.[15])

The experimental data are[10]

$$\sigma_{ip}=4.5\sim5.5 \quad \sigma_T=7\sim9 \quad \text{at } E_\gamma=87 \text{ Mev}$$

$$\sigma_{ip}=4\sim5 \quad \sigma_T=5\sim6 \quad \text{at } E_\gamma=132 \text{ Mev}$$

in unit of $10^{-29}\,\text{cm}^2$.

Table 2. Cross sections corresponding to Table 1 are shown in unit of $10^{-29}\,\text{cm}^2$.

1) Cross sections for $k_\gamma r/3+({}^3S_1+{}^3D_1)\to{}^3P$

E_γ		$\sigma_{ed}{}^{ip}$	σ_{ed}	σ_{md}
87	A	1.51	5.05	0.6
	B	0.45	3.04	0.6
132	A	0.90	2.07	0.4
	B	0.36	1.05	0.4

2) Cross sections for $k_\gamma r/3+({}^3S_1+{}^3D_1)\to{}^3P+{}^3F$

E_γ		$\sigma_{ed}{}^{ip}$	σ_{ed}	σ_{md}
87	A	4.68	6.99	0.6
	B	1.69	4.99	0.6
132	A	2.53	3.09	0.4
	B	0.64	2.07	0.4

3) Cross sections for $j_1(k_\gamma r)+({}^3S+{}^3D)\to{}^3P+{}^3F$

E_γ		$\sigma_{ed}{}^{ip}$	σ_{ed}	σ_{md}
87	A	5.62	8.39	0.7
	B	2.03	5.99	0.7
132	A	3.54	4.30	0.6
	B	0.90	2.88	0.6

In these cases $\sigma_{md}{}^{ip}$ is negligible, and in the computation of σ_{md} the hard core radius of ${}^1E$ is taken to be $0.2/k$.[3] Also E_γ is in unit of Mev.

* The validity of the argument in $S-P$ approximation is referred in § 4.

Comparing with A of Table 2, we see that photodisintegration of the deuteron up to about 140 Mev would perhaps be accounted for almost only by the nucleon parts.

σ_{ed} of the case I is smaller than that of the case A (σ_{ed} of the former is $3.56\times10^{-29}\,\mathrm{cm}^2$ at $E_\gamma=132$ Mev, whereas σ_{ed} of the latter is $4.30\times10^{-29}\,\mathrm{cm}^2$ at $E_\gamma=132$ Mev), and the case I shows that σ_{ed} is not small even at $E_\gamma>140$ Mev. Therefore we may say that σ_T (Note that $\sigma_T>\sigma_{ed}$) of the nucleon parts would not be small even at $E_\gamma>140$ Mev.

§ 4. Discussion

I. *On the natures of photodisintegration of the deuteron*

1. From § 3 we see that the photodisintegration of the deuteron up to about 140 Mev would be well accounted for almost only by the nucleon parts, if many effects mentioned in § 1 are correctly treated. Furthermore, from § 3 we see that the contribution of the nucleon parts would perhaps be still large even at $E_\gamma>140$ Mev.

2. Consequently it suggests that the contribution of the virtual pion would yet be small at $E_\gamma<140$ Mev, and it may not be so large even at $E_\gamma>140$ Mev. Previously, several authors computed the contribution of virtual pions in the photodisintegration of the deuteron.[15] However, their treatments are very rough, and there are many questions on the validity of the initial state and the final states adopted by them. (In their computation the final states are put as free waves, but the phase shifts at high energies are large, and their influence is very important as we have seen in § 3. Also they neglected the ${}^3D_1$ state of the deuteron, whereas, as we have seen in § 3, the contribution of the ${}^3D_1$ state is quite remarkable at high energies.) Although our computation is not exact, our treatments on initial and final states are much better than those made in the previous theories.[15] (Also refer to II, 2 of this section.) Therefore our computation may present some suggestion on the magnitude of the contribution of virtual pions.

II. *On the properties of the nuclear forces*

1. Photodisintegration of the deuteron imposes some conditions on the properties of nuclear forces. We have indicated that the D-state probability of the deuteron would at least be larger than 4%, for if it were not, both A and B would have given less σ_{ip} and less σ_T than those given in § 3.[1] Furthermore, we have seen that photodisintegration of the deuteron suggests about the properties of ${}^1E$ and ${}^3O$ at lower energies.[3] Now in the present paper we see that photodisintegration of the deuteron also removes the ambiguities about the natures of the triplet odd state.

2. In § 3 we have shown that set A can well account for the data, but it is not for set B, in which we have only shown the computed results of two sets. However, the conclusion is more general.[14] By adopting many sets of phase shifts to the computation we can easily show that the set corresponding to strong positive tensor force (which is characterized by $\delta_1{}^0>0$ and $\delta_1{}^1<0$) can well account for the

data on the photodisintegration of the deuteron, but it is not for the set corresponding to negative tensor force (which is characterized by $\delta_1^0<0$ and $\delta_1^1>0$).

3. It is important to ask if our conclusions should be changed, when the phase shifts of $^1D_2$ and ^3F waves are introduced into the analysis of $p-p$ scattering and also into the computation of photodisintegration of the deuteron. We might say that our conclusions would not be so much changed owing to the following reasons. When higher waves are introduced in $p-p$ scattering analysis, we can still get classes A and B by the natures of the phase shifts of ^3P.[12] The important point is that: how would the phase shifts of $^1D_2$ and $^3F_2$ waves affect the results? We might say that the change by the phase shifts of these waves are very small at least at $E_\gamma \simeq 90$ Mev, for the $^1D_2$ wave which is concerned with $m.d.$ is not very important, and the phase shift of the $^3F_2$ wave is yet very small[16] so that the integral $\int W_d U_F dr$ is not so changed by the shift of $^3F_2$ wave at $E_\gamma \simeq 90$ Mev.

Therefore we may say that the tensor force of the triplet odd state should be positive and strong at least at $E_\gamma < 90$ Mev,[14] if the phase shifts of $^1D_2$ and ^3F waves are also introduced into the computation.

4. At last, let us discuss the applicability of pion potential (to nucleon-nucleon scattering and photodisintegration of the deuteron). The static pion potential gives $\delta_1^0 > \delta_1^2 > \delta_1^1$ at many energies, and the mutual separation of δ_1^i is large due to the strong tensor force of the triplet odd state.[17] Furthermore, it gives $20^\circ > \delta_0^0 > 0$ and predicts that δ_3^2 is very small at $E_\gamma \simeq 90$ Mev.[18] Such set belongs to category A, and can well account for the photodisintegration of the deuteron, and also can be fitted to the nucleon-nucleon scattering and the polarization data.[19]

Therefore we may say that the static pion potential is useful up to about 170 Mev (corresponding to $E_\gamma \simeq 90$ Mev in $d(\gamma, p)n$) in qualitative nature.

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Appendix

The exact amplitude of multipole radiation are to be given by[8]

$$a_E(l, m) = \frac{4\pi}{c}\int r^{-1} F_l(k_\gamma r) \boldsymbol{X}_{lm}^* \cdot (\boldsymbol{\nabla}\times\boldsymbol{j} + ck_\gamma^2 \boldsymbol{M}) dv \qquad (A\cdot 1)$$

$$a_M(l, m) = \frac{4\pi i k_\gamma^2}{c}\int r^{-1} F_l(k_\gamma r) \boldsymbol{X}_{lm}^* \cdot (\boldsymbol{j} + c\boldsymbol{\nabla}\times\boldsymbol{M}) dv. \qquad (A\cdot 2)$$

Applying (A·1) to the case $l=1$, $m=0$, which is needed in the computation of the electric dipole transition, we get

$$a_E(1, 0) = \frac{4\pi}{c}\cdot\frac{1}{\sqrt{2}}\int r^{-1} F_1(k_\gamma r) Y_{lm}^*(\theta, \phi)[\omega(\boldsymbol{r}\cdot\boldsymbol{\nabla}+2)\rho + i\boldsymbol{\nabla}^2(\boldsymbol{r}\cdot\boldsymbol{j}) + ick_\gamma^2\boldsymbol{\nabla}(\boldsymbol{r}\times\boldsymbol{M})] dv. \qquad (A\cdot 3)$$

Omitting the last term which is needless in $d(\gamma, p)n$, and substituting

$$F_1(k_\tau r) = (k_\tau r)\left[\frac{2}{3!}(k_\tau r) - \frac{4}{5!}(k_\tau r)^3 + \frac{6}{7!}(k_\tau r)^5 \cdots\right] \qquad (A\cdot 4)$$

in (A·3), we finally get

$$a_E(1,\ 0) = -\frac{4\pi k_\tau^2}{\sqrt{2}} \cdot \sqrt{\frac{3}{4\pi}} \cdot 2^2 \int\left[\left\{\frac{1}{3!}(k_\tau r) - \frac{2^2}{5!}(k_\tau r)^3 + \frac{3^2}{7!}(k_\tau r)^5 \cdots\right\}\right.$$

$$\left. \times e\varphi_a^* \varphi_b \cos\theta + \frac{e\hbar}{2Mc} k_\tau \left\{\frac{1}{3!}(k_\tau r) - \frac{2}{5!}(k_\tau r)^3 \cdots\right\} \varphi_a^* r \frac{\partial}{\partial r} \varphi_b \cos\theta\right] d\tau, \qquad (A\cdot 5)$$

from which (13) is obtained.

Similarly, from (A·2) we can get (15).

If we take only the first term of (A·5), we get (14), or (7)∼(9). But if we include the higher order terms of (A·5), σ_{ed} is increased by about 50% at $E_\tau = 140$ Mev, and becomes about twice at $E_\tau = 200$ Mev. σ_{md} at high energies is also increased by the higher order terms of $j_1(k_\tau r)$.

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Electron Interaction in Very Long Linear Conjugated Molecules. I

——*One-Dimensional Collective Oscillation of* π *Electron*——

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Low-lying electronic states of very long linear conjugated molecules with equal CC bond lengths is studied within the framework of Pariser and Parr's π electron approximation, by means of Tomonaga's one-dimensional sound wave method. It is found that as the number N of carbon atoms increases, the frequency of the sound wave with the lowest wave number tends to zero in proportion to $\sqrt{\log N/N}$. It is shown that Araki and Murai's treatment which gives a result contradicting with ours contains a difficulty.

§ 1. Introduction

There has been found an interesting experimental rule[1] on the wave-length of the first optical absorption band of linear conjugated molecules, such as polyenes, carotenoids, cyanine dyes. As the number N of carbon atoms increases, the wave-length grows longer but in cases of polyenes and asymmetric dyes the wave-length converges to a finite limit, whereas in the case of symmetric cyanines the wave-length seems to increase linearly with N. An explanation of this rule was given at first by Kuhn[2]: From the standpoint of the VB theory there exist two equivalent structures with the lowest energy for any symmetric cyanine, but in the case of a polyene or an asymmetric dye molecule there exists only a single structure with the lowest energy. Hence in the former molecules all $C-C$ bonds become equivalent on account of the resonance, while in the latter ones single and double bonds tend to localize due to the incomplete resonance, and the bond lengths alternate. This alternation has been concluded also from the MO theory.[3] As a result of this the Hartree field for π-electrons in polyenes and asymmetric dyes has the period of two bond lengths, and in the limit of $N\to\infty$ we get two energy bands, one completely filled and the other empty, which are separated by a Brilliouin gap of finite breadth, so that the wave-length of the absorption corresponding to the transition from the top of the filled band to the bottom of the empty band converges to a finite value.

Another explanation of this convergency was proposed by Araki and Murai.[4] They maintain that the correlation among π electrons is very strong. Then they adopted a one-dimensional free electron model, and calculated its excitation energy by using Tomonaga's one-dimensional sound-wave method.[5] In this method low excitations of this electron system may be interpreted as creations of bosons belonging

to sound-waves of these electrons. They have ascribed the first absorption of a carotenoid to the creation of the boson with the lowest wave number or the lowest energy, and they have also succeeded in getting the convergency.

As has been pointed by several authors, Araki and Murai's theory cannot be regarded as established because it is not clear why such a sound wave does not exist in symmetrical dye molecules. However, the above objection does not give a positive proof that the sound-waves do not exist or the collective excitation does not correspond to the observed first absorption of a carotenoid. Moreover, if any theoretical difficulty could not be found in Araki and Murai's treatment, it would be natural to accept their opinion that the correlation effect is very strong. Then the validity of Kuhn's explanation would also become doubtful, because Kuhn's treatment is based on the simple MO method in which the electron correlation is neglected without any reason. However, Araki and Murai's result also cannot be accepted straightforwardly, because in their treatment several drastic simplifications are made. Accordingly, it seems necessary and interesting to make a more careful theoretical study on low-lying electronic states of linear conjugated molecules at the limit of $N \to \infty$.

In this first paper we restrict ourselves to the atomic configuration where all the $C-C$ bond lengths and all the $C-C-C$ bond angles are equal, respectively (See Fig. 1). Throughout in this paper the straight line passing through the middle point of each $C-C$ bond is taken as the x-axis and a line which is perpendicular to this x-axis and lies in the molecular plane is taken as the y-axis (See Fig. 1).

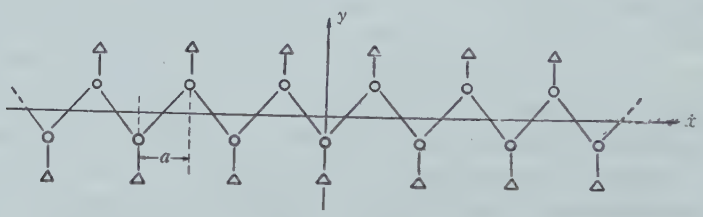

Fig. 1. The model considered in this paper: Equal bond lengths and equal bond angles. Small circles represent the carbon atoms. Triangles are hydrogen atoms.

In our treatment the effective Hamiltonian for the low-lying electronic states under consideration is constructed on the basis of Pariser and Parr's π electron approximation[6] which has gained great success in the interpretation of optical spectra of organic molecules. Then this Hamiltonian is diagonalized by means of the Tomonaga theory of one-dimensional sound waves.

§ 2. The Hamiltonian for the π electron system

In the usual theory of π electrons, one assumes that it will bring no serious error to treat the π electrons separately from the remaining electrons, that is, one

supposes that the role of the latter electrons is only to give a classical electrostatic field for the π electrons. In this approximation these π electrons form a closed system whose Hamiltonian is given by

$$H=H_0+H_I$$

$$H_0\equiv\int\psi^*(\boldsymbol{x})K\psi(\boldsymbol{x})\,d^4x^*$$

$$H_I\equiv\int\psi^*(\boldsymbol{x})U(\vec{x})\psi(\boldsymbol{x})\,d^4x$$

$$+\frac{1}{2}\iint\psi^*(\boldsymbol{x})\psi^*(\boldsymbol{x}')\frac{e^2}{|\vec{x}-\vec{x}'|}\psi(\boldsymbol{x}')\psi(\boldsymbol{x})\,d^4x\,d^4x'. \qquad (2\cdot1)$$

In the above, the quantized wave function of this π electron system is denoted by $\psi(\boldsymbol{x})$, in which $\boldsymbol{x}=(\vec{x},\ \zeta)$, $\vec{x}$ and ζ being respectively the space and the spin coordinate, and K is the kinetic energy operator $-(\hbar^2/2m)\Delta$, and further $U(\vec{x})$ is the electrostatic potential for a π electron due to the π-electronless molecular ion core. This $U(\vec{x})$ may be written as

$$U(\vec{x})=\sum_{-N/2\leq R\leq N/2}u_R(\vec{x})$$

where $u_R(\vec{x})$ is the potential due to the ion core belonging to the R-th atom.

Following Pariser and Parr, we simplify the Hamiltonian (2·1) by following procedures I, II and III.

(I) We adopt the formal procedure of the LCAO approximation, i. e. we approximate $\psi(\boldsymbol{x})$ as follows,

$$\psi(\boldsymbol{x})=\sum_{-N/2\leq R\leq N/2}^{\sigma}b(R,\ \sigma)\varphi_R(\vec{x})\sigma(\zeta) \qquad (2\cdot2)$$

where $\varphi_R(\vec{x})$ is the $2p\pi$ orbital of the R-th atom, and σ is a spin function α or β.

(II) We neglect formally differential overlaps of the form

$$\overline{\varphi_R(\vec{x})}\cdot\varphi_{R'}(\vec{x})\quad(R\neq R')$$

except in the following integral

$$B\equiv\int\varphi_R^*(\vec{x})K\varphi_{R+l}(\vec{x})\,d^3x \qquad (2\cdot3)$$

where l denotes 1 or -1. Then, the commutation relations of the operators $b(R,\ \sigma)$ become

$$b(R,\ \sigma)b^*(R',\ \sigma')+b^*(R',\ \sigma')b(R,\ \sigma)=\delta_{RR'}\delta_{\sigma\sigma'} \qquad (2\cdot4)$$

$$b(R,\ \sigma)b(R',\ \sigma')+b(R',\ \sigma')b(R,\ \sigma)=0$$

and, substituting (2·2) into (2·1), we get

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* $d^4x=dx\,dy\,dz\,d\zeta$, while $d^3x=dx\,dy\,dz$.

$$H_0=B\sum_{|R|,|R+l|\leq N/2}^{\sigma}\sum_{l} b^*(R,\ \sigma)b(R+l,\ \sigma) \tag{2·5}$$

where we have omitted a constant $N\int \bar{\varphi}_R K\varphi_R d^3x$. Similarly we obtain

$$\begin{aligned} H_I=&\sum_{-N/2\leq R_1,R_2\leq N/2}\rho(R_1)\cdot I(R_1-R_2)\\ &+\tfrac{1}{2}\sum_{-N/2\leq R_1,R_2\leq N/2}^{\sigma_1\sigma_2} b^*(R_1,\ \sigma_1)b^*(R_2,\ \sigma_2)b(R_2,\ \sigma_2)b(R_1,\ \sigma_1)\cdot J(R_1-R_2) \end{aligned} \tag{2·6}$$

where

$$\rho(R)\equiv b^*(R,\ \alpha)b(R,\ \alpha)+b^*(R,\ \beta)b(R,\ \beta) \tag{2·7}$$

$$I(R_1-R_2)\equiv\int |\varphi_{R_1}(\vec{x})|^2\cdot u_{R_2}(\vec{x})\,d^3x \tag{2·8}$$

and

$$J(R_1-R_2)=e^2\iint \frac{|\varphi_{R_1}(\vec{x}_1)|^2\cdot|\varphi_{R_2}(\vec{x}_2)|^2}{|\vec{x}_1-\vec{x}_2|}\,d^3x_1d^3x_2. \tag{2·9}$$

Further, omitting a constant $-\frac{1}{2}N\cdot J(0)$, we can write the second term of the right side of eq. (2·6) as

$$\begin{aligned} &\tfrac{1}{2}\sum_{-N/2\leq R_1,R_2\leq N/2}\rho(R_1)\rho(R_2)\cdot J(R_1-R_2)\\ &\quad=\tfrac{1}{2}\sum_{-N/2\leq R_1,R_2\leq N/2}\tau(R_1)\tau(R_2)\cdot J(R_1-R_2)\\ &\quad+\sum_{-N/2\leq R_1,R_2\leq N/2}\rho(R_1)\cdot J(R_1-R_2)+\text{const.} \end{aligned}$$

where we have introduced the notation

$$\tau(R)=\rho(R)-1. \tag{2·10}$$

Therefore, we obtain

$$\begin{aligned} H_I=&\tfrac{1}{2}\sum_{-N/2\leq R_1,R_2\leq N/2}\tau(R_1)\tau(R_2)J(R_1-R_2)\\ &+\sum_{-N/2\leq R_1,R_2\leq N/2}\rho(R_1)K(R_1-R_2) \end{aligned} \tag{2·11}$$

where

$$K(R_1-R_2)=J(R_1-R_2)+I(R_1-R_2) \tag{2·12}$$

and we have omitted the constants mentioned above. From (2·5) and (2·11) we get the following Hamiltonian

$$\begin{aligned} H=&H_0+H_I\\ =&B\sum_{|R|,|R+l|\leq N/2}\sum_{l} b^*(R,\ \sigma)b(R+l,\ \sigma)\\ &+\tfrac{1}{2}\sum_{-N/2\leq R_1,R_2\leq N/2}\tau(R_1)\tau(R_2)J(R_1-R_2)\\ &+\sum_{-N/2\leq R_1,R_2\leq N/2}\rho(R_1)K(R_1-R_2) \end{aligned} \tag{2·13}$$

(III) In the Hamiltonian (2·13) B and $J(0)$ are regarded as empirical parameters. Therefore they are not always the quantities restricted by (2·3) and (2·9) respectively, but more flexible ones which contain to some extent the effect of electron correlation of short range character formally neglected in the procedures to derive (2·13). Strictly speaking, the orbital φ_R corresponding to the operator $b(R, \sigma)$ appearing in the present Hamiltonian cannot be defined explicitly as the $2p\pi$ AO itself or the Wannier function. In any rate, ρ_R given by (2·7) may be regarded as an operator corresponding to the number of electrons belonging to the R-th atom.

Now we define $\rho_n^{(N)}$ $(n=0, \pm 1, \pm 2, \cdots ; |n| \leq N/2)$ by the following equation,

$$\rho_n^{(N)} = \sum_{-N/2 \leqq R \leqq N/2} \tau(R) \exp\left(\frac{2\pi i\, n}{N} R\right)$$

or

$$\tau(R) = N^{-1} \sum_{-N/2 \leqq n \leqq N/2} \rho_n^{(N)} \exp\left(-\frac{2\pi i\, n}{N} R\right). \tag{2·14}$$

Then $H_I^{(1)} \equiv \frac{1}{2} \sum_{-N/2 \leqq R_1, R_2 \leqq N/2} \tau(R_1)\tau(R_2) J(R_1 - R_2)$, i. e. the first term on the right side of (2·11) becomes

$$\frac{1}{2} \cdot \frac{1}{N^2} \sum_{(-N/2 \leqq n_1, n_2 \leqq N/2)} \sum_{(-N/2 \leqq R_1, R_2 \leqq N/2)} \exp\left[-\frac{2\pi i}{N}(n_1 R_1 + n_2 R_2)\right] \rho_{n_1}^{(N)} \rho_{n_2}^{(N)} J(R_1 - R_2)$$

$$= \frac{1}{2} \cdot \frac{1}{N^2} \sum_{-N/2 \leqq n_1, n_2 \leqq N/2} \rho_{n_1}^{(N)} \rho_{n_2}^{(N)} \sum_{-N/2 \leqq Q_1 \leqq N/2} \exp\left[-\frac{2\pi i}{N}(n_1 + n_2) Q_1\right]$$

$$\times \sum_{-(N-2|Q_1|) \leqq Q_2 \leqq N-2|Q_1|} \exp\left[-\frac{\pi i}{N}(n_1 - n_2) Q_2\right] J(Q_2).$$

Rearranging the sum $\sum_{-(N-2|Q_1|) \leqq Q_2 \leqq N-2|Q_1|}$ into $\sum_{-N \leqq Q_2 \leqq N} - \sum_{N-2|Q_1| < |Q_2| \leqq N}$, we get

$$H_I^{(1)} = \frac{1}{2} \sum_{-N/2 \leqq n \leqq N/2} \rho_n^{(N)} \rho_{-n}^{(N)} \cdot J_n^{(N)}$$

$$- \frac{1}{2} \cdot \frac{1}{N^2} \sum_{-N/2 \leqq n_1, n_2 \leqq N/2} \rho_{n_1}^{(N)} \rho_{n_2}^{(N)} \sum_{-N/2 \leqq Q_1 \leqq N/2} \exp\left[-\frac{2\pi i}{N}(n_1 + n_2) Q_1\right]$$

$$\times \sum_{N-2|Q_1| \leqq |Q_2| \leqq N} \exp\left[-\frac{\pi i}{N}(n_1 - n_2) Q_2\right] J(Q_2) \tag{2·15}$$

where

$$J_n^{(N)} \equiv N^{-1} \sum_{-N \leqq Q \leqq N} \exp\left(-\frac{2\pi i\, n}{N} Q\right) J(Q). \tag{2·16}$$

If N is infinitely large, the second term on the right side of (2·15) is negligibly small compared with the first term.* Accordingly we use the following approxima-

* This is owing to the fact that our $J(R)$ approaches to zero at a sufficiently rapid rate as R increases, as well as to the fact that $J(R)$ is a slowly varying function in the region $R \gg 1$.

tion:

$$H_I^{(1)}=\frac{1}{2}\sum_{-N/2\leq n\leq N/2}\rho_n^{(N)}\rho_{-n}^{(N)}\cdot J_n^{(N)}. \tag{2·15′}$$

We next consider the second term on the right side of (2·11), i.e.

$$H_I^{(2)}=\sum_{-N/2\leq R_1\leq N/2}\tau(R_1)C(R_1)$$

where

$$C(R_1)\equiv\sum_{-N/2\leq R_2\leq N/2}K(R_1-R_2).$$

Here it should be noted that $K(R)$ defined by (2·12) vanish almost completely if R is larger than a certain value, say R_0, irrespective of N. Therefore we may suppose that $C(R_1)=C$ unless $|R_1|>N/2-R_0$, C being a constant irrespective of R_1. Then, noting that $\sum_{-N/2\leq R_1\leq N/2}\tau(R_1)=0$, we get

$$H_I^{(2)}=\sum_{N/2-R_0<|R_1|\leq N/2}\tau(R_1)\,(C(R_1)-C).$$

From this one can easily derive the conclusion that $H_I^{(2)}$ is negligibly small compared with $H_I^{(1)}$. Therefore we obtain

$$H_I=\frac{1}{2}\sum_{-N/2\leq n\leq N/2}\rho_n^{(N)}\rho_{-n}^{(N)}\cdot J_n^{(N)}. \tag{2·17}$$

Now, $J(R)$ should satisfy the following properties,

$$\left.\begin{aligned}&J(R)=e^2/Ra,\ \text{if } R \text{ is infinitely large}\\&J(0)=\text{large but finite}\end{aligned}\right\} \tag{2·18}$$

where a is the length of the x-component of a vector combining two adjacent carbon atoms (cf Fig. 1). Here we assume for mathematical convenience that

$$N\cdot J_n^{(N)}\equiv\sum_{-N\leq R\leq N}\exp(ikR)\cdot J(R)=\frac{e^2}{a}\int_{-N}^{N}\frac{\exp(ikR)}{\sqrt{R^2+\gamma^2}}\,dR \tag{2·19}$$

where $k=2\pi n/N$ and γ is some dimensionless number. Then we obtain

$$J_n^{(N)}\leq J_0^{(N)}=\frac{2e^2}{Na}\log(\sqrt{N^2+\gamma^2}+N/\gamma) \tag{2·20}$$

and

$$\lim_{N\to\infty}N\cdot J^{(N)}[k]=(2e^2/a)K_0(\gamma|k|) \tag{2·21}$$

where $J^{(N)}[k]=J_n^{(N)}$, k and n being combined by the relation $k=2\pi n/N$, and k retains a finite value irrespective of N by the process $n\to\infty$ according as $N\to\infty$. In the above, K_0 is a modified Bessel function of the second kind.

In the kinetic part H_0 given by (2·5) we use the following approximation,

$$\sum_R\sum_l b^*(R,\sigma)b(R+l,\sigma)\to\sum_{|R|\leq N/2}\sum_l b^*(R,\sigma)b(R+l,\sigma),$$

which is valid in the case of sufficiently large N. Then H_0 can be easily diagonalized by the following transformation:

$$a_{n\sigma}=N^{-1/2}\sum_{-N/2\leq R\leq N/2} b(R,\ \sigma)\cdot\exp\left(-\frac{2\pi i n}{N}R\right). \tag{2·22}$$

Then we have

$$H_0=\sum_{-N/2\leq n\leq N/2}^{\sigma}\epsilon(n)\cdot a_{n\sigma}^{*}a_{n\sigma} \tag{2·23}$$

where

$$\epsilon(n)\equiv 2B\cos(2\pi n/N). \tag{2·24}$$

By means of eq. (2·4) we also get

$$\left.\begin{aligned} &a_{n\sigma}^{*}a_{n'\sigma'}+a_{n'\sigma'}a_{n\sigma}^{*}=\delta_{nn'}\delta_{\sigma\sigma'}\\ &a_{n\sigma}a_{n'\sigma'}+a_{n'\sigma'}a_{n\sigma}=0. \end{aligned}\right\} \tag{2·25}$$

Here it should be noted that

$$\rho_n^{(N)}=\sum_{|n'|,\,|n'+n|\leq N/2}^{\sigma,n'} a_{n'+n\sigma}^{*}a_{n'\sigma} \tag{2·26}$$

for $|n|\ll N$.

§ 3. Eigenvalues of the Hamiltonian

Here we seek for excitation energies of low excited states of our system whose Hamiltonian is given by (2·17) and (2·23), i. e.

$$\left.\begin{aligned} &H=H_0+H_I\\ &H_0\equiv\sum_n^{\sigma}\epsilon(n)N_{n\sigma},\quad H_I\equiv\tfrac{1}{2}\sum_n J_n\rho_n\rho_{-n}\\ &N_{n\sigma}\equiv a_{n\sigma}^{*}a_{n\sigma},\quad \rho_n\equiv\sum_{n'}^{\sigma}a_{n'+n\sigma}^{*}a_{n'\sigma} \end{aligned}\right\} \tag{3·1}$$

where J_n is the abbreviation of $J_n^{(N)}$ and so on. According to Tomonaga,[5] sufficiently low excited states given by the above Hamiltonian may be interpreted as the oscillating states of density waves, if for these low excited states the Hilbert space may be replaced by the sub-space in which holes and excited particles exist only in the neighbourhood of the Fermi levels. Here we assume that our H_I, i. e. the Coulomb interaction, allows this replacement in a good approximation, though we cannot say that there is no question about this assumption.

At first, ρ_n is splitted as

$$\rho_n=\sum_{l}^{\sigma}(\rho_{n\sigma}^{+}+\rho_{n\sigma}^{-})$$

where

$$\left.\begin{aligned} &\rho_{n\sigma}^{+}\equiv\sum_{n'>0}a_{n'+(n/2)\sigma}^{*}a_{n'-(n/2)\sigma}\\ &\rho_{n\sigma}^{-}\equiv\sum_{n'<0}a_{n'+(n/2)\sigma}^{*}a_{n'-(n/2)\sigma} \end{aligned}\right\}. \tag{3·2}$$

Then, in our sub-space the following commutation relations hold,

$$\left.\begin{aligned} &[\rho_{n\sigma}^{+},\ \rho_{n'\sigma'}^{+}]=-n\delta_{n,-n'}\delta_{\sigma,\sigma'}\\ &[\rho_{n\sigma}^{-},\ \rho_{n'\sigma'}^{-}]=n\delta_{n,-n'}\delta_{\sigma,\sigma'}\\ &[\rho_{n\sigma}^{+},\ \rho_{n'\sigma'}^{-}]=0, \end{aligned}\right\} \tag{3·3}$$

for sufficiently small values of $|n|$ and $|n'|$ compared with n_F ($\pm n_F$ are the values of n corresponding to the Fermi levels; $N=2(2n_F+1)$).

The frequency of $a_{n\sigma}$ is $(1/i\hbar)\epsilon(n)$. Further, for $a^*_{n'+(n/2)\sigma}$ and $a_{n'-(n/2)\sigma}$ appearing in the expression of ρ^+_n in (3·2), we may use the following approximation,

$$\epsilon\left(n'\pm\frac{n}{2}\right)=\epsilon(n_F)+\left(n'\pm\frac{n}{2}-n_F\right)\epsilon'(n_F),$$

in our subspace, where $\epsilon'(n)=d\epsilon(n)/dn$. Therefore the time dependency of $\rho^+_{n\sigma}$ becomes

$$\rho^+_{n\sigma}\propto\exp\left[\frac{i}{\hbar}\epsilon'(n_F)nt\right].$$

Similarly we get $\rho^-_{n\sigma}\propto\exp\left[-\frac{i}{\hbar}\epsilon'(n_F)nt\right]$. Therefore the operator

$$\mathfrak{H}=\epsilon'(n_F)\sum_{n>0}^{\sigma}(\rho^+_{n\sigma}\rho^+_{-n\sigma}+\rho^-_{-n\sigma}\rho^-_{n\sigma})$$

gives the equation

$$i\hbar\dot{\rho}^{\pm}_{n\sigma}=[\rho^{\pm}_{n\sigma},\ \mathfrak{H}].$$

This is easily verified by means of eq. (3·3).

Now, in $H_0=\sum\epsilon(n)N_{n\sigma}$, $\epsilon(n)$ may be approximated as

$$\epsilon(n)=\epsilon(n_F)+\epsilon'(n_F)(|n|-n_F)$$

in our sub-space. Then we obtain

$$H_0-E_0=\epsilon'(n_F)\{\sum_n^{\sigma}|n|N_{n\sigma}-2n_F(n_F+1)\}$$

where $E_0\equiv 2\sum_{|n|\leq n_F}\epsilon(n)$. Tomonaga tried to show that this H_0-E_0 is equal to $\mathfrak{H}$ in the sub-space mentioned above. This is true if the following relation holds,

$$\sum_{n>0}(\rho^+_{n\sigma}\rho^+_{-n\sigma}+\rho^-_{-n\sigma}\rho^-_{n\sigma})=\sum_n|n|N_{n\sigma}-n_F(n_F+1). \tag{3·4}$$

The procedure necessary for the proof of eq. (3·4) in our case is same as that in the case of free particles. For diagonal elements in the number representation there are no doubts about the validity of eq. (3·4), while for off-diagonal elements eq. (3·4) does not rigorously hold. Here we suppose that this discrepancy between both sides of eq. (3·4) is trivial. Namely we here assume that

$$H_0-E_0=\mathfrak{H}.$$

This may be transformed as

$$\begin{aligned}H_0-E_0=&\tfrac{1}{2}\epsilon'(n_F)\sum_{n>0}(\rho^+_n\rho^+_{-n}+\rho^-_{-n}\rho^-_n)\\&+\tfrac{1}{2}\epsilon'(n_F)\sum_{n>0}(\tilde{\rho}^+_n\tilde{\rho}^+_{-n}+\tilde{\rho}^-_{-n}\tilde{\rho}^-_n)\end{aligned} \tag{3·4′}$$

where

$$\rho_n^{\pm}=\rho_{n\alpha}^{\pm}+\rho_{n\beta}^{\pm} \quad \text{and} \quad \tilde{\rho}_n^{\pm}=\rho_{n\alpha}^{\pm}-\rho_{n\beta}^{\pm}.$$

Commmutation relations for $\rho_n^{\mp}$ and $\rho_n^{\pm}$ can be easily obtained from (3·3) and are as follows

$$[\rho_n^+,\ \rho_{n'}^+]=-2n\delta_{n,-n'}, \qquad [\tilde{\rho}_n^+,\ \tilde{\rho}_{n'}^+]=-2n\delta_{n,-n'}$$
$$[\rho_n^-,\ \rho_{n'}^-]=2n\delta_{n,-n'}, \qquad [\tilde{\rho}_n^-,\ \tilde{\rho}_{n'}^-]=2n\delta_{n,-n},$$
$$[\rho_n^+,\ \rho_{n'}^-]=0, \qquad [\tilde{\rho}_n^+,\ \tilde{\rho}_{n'}^-]=0 \qquad [\rho_n^{\pm},\ \tilde{\rho}_{n'}^{\pm}]=0.$$

Thus, noting that

$$\mathrm{H}_J=\sum_{n>0} J_n(\rho_n^+\rho_{-n}^+ +\rho_{-n}^-\rho_n^- +\rho_{-n}^+\rho_n^- +\rho_{-n}^-\rho_n^+)+\sum_{n>0} nJ_n,$$

we obtain

$$\begin{aligned}\mathscr{H} &\equiv \mathrm{H}-E_0=\mathfrak{H}+\mathrm{H}_J \\ &=\sum_{n>0} A_n(\rho_n^+\rho_{-n}^+ +\rho_{-n}^-\rho_n^-)+\tfrac{1}{2}\epsilon'(n_F)\sum_{n>0}(\tilde{\rho}_n^+\tilde{\rho}_{-n}^+ +\tilde{\rho}_{-n}^-\tilde{\rho}_n^-) \\ &+\sum_{n>0} J_n(\rho_{-n}^+\rho_n^- +\rho_n^+\rho_{-n}^-)+\sum_{n>0} nJ_n\end{aligned}$$

where $A_n \equiv \frac{1}{2}\epsilon'(n_F)+J_n$.

Further, introducing the following transformation,

$$\left.\begin{aligned}\rho_n^+ &=(\sqrt{n}/2)\{(\alpha_n+\alpha_n^{-1})\eta_n^*-(\alpha_n-\alpha_n^{-1})\eta_{-n}\} \\ \rho_n^- &=(\sqrt{n}/2)\{-(\alpha_n-\alpha_n^{-1})\eta_n^*+(\alpha_n+\alpha_n^{-1})\eta_{-n}\} \\ \rho_{-n}^+ &=(\sqrt{n}/2)\{(\alpha_n+\alpha_n^{-1})\eta_n-(\alpha_n-\alpha_n^{-1})\eta_{-n}^*\} \\ \rho_{-n}^- &=(\sqrt{n}/2)\{-(\alpha_n-\alpha_n^{-1})\eta_n+(\alpha_n+\alpha_n^{-1})\eta_{-n}^*\}\end{aligned}\right\} \tag{3·5}$$

$$\left.\begin{aligned}\tilde{\rho}_n^+ &=\sqrt{2n}\,\tilde{\eta}_n^*, & \tilde{\rho}_n^- &=\sqrt{2n}\,\tilde{\eta}_{-n}, \\ \tilde{\rho}_{-n}^+ &=\sqrt{2n}\,\tilde{\eta}_n & \tilde{\rho}_{-n}^- &=\sqrt{2n}\,\tilde{\eta}_{-n}^*,\end{aligned}\right\} \tag{3·5′}$$

where $n>0$ and $\alpha_n \equiv \left(\dfrac{\epsilon'(n_F)+4J_n}{\epsilon'(n_F)}\right)^{1/4}$, we get

$$\left.\begin{aligned}&[\eta_n,\ \eta_{n'}]=[\tilde{\eta}_n,\ \tilde{\eta}_{n'}]=[\tilde{\eta}_n,\ \eta_{n'}]=[\tilde{\eta}_n,\ \eta_{n'}^*]=0 \\ &[\eta_n,\ \eta_{n'}^*]=\delta_{nn'},\ [\tilde{\eta}_n,\ \tilde{\eta}_{n'}^*]=\delta_{nn'} \\ &\text{for negative as well as positive } n\end{aligned}\right\} \tag{3·6}$$

and

$$\begin{aligned}\mathscr{H}=\sum_n |n|\sqrt{\epsilon'(n_F)}\{\sqrt{\epsilon'(n_F)+4J_n}(\eta_n^*\eta_n+\tfrac{1}{2})-\sqrt{\epsilon'(n_F)}\}. \\ +\sum |n|\sqrt{\epsilon'(n_F)}\,\tilde{\eta}_n^*\,\tilde{\eta}_n.\end{aligned} \tag{3·7}$$

Though (3·6) is valid only in a restricted sense, we suppose, according to Tomo-

naga, that the operators $\eta_n^*\eta_n$ and $\tilde{\eta}_n^*\tilde{\eta}_n$ are usual number operators of some independent Bose particles. Then our assemblage of interacting π electrons becomes equivalent to an assemblage of non-interacting bosons belonging to two different groups. One of these groups corresponds to usual charge-density waves (plasma waves). Their energy quanta $\hbar\omega_n^{(N)}$ $(n=\pm1, \pm2, \cdots)$ are given by

$$\hbar\omega_n^{(N)}=|n|\sqrt{\epsilon'(n_F)\{\epsilon'(n_F)+4J_n^{(N)}\}}.$$

Thus, the energy quantum of the plasmon with wave number $k/a=2\pi n/Na$ is

$$\hbar\omega^{(N)}[k]=|k|\sqrt{b\left\{b+\frac{2N}{\pi}J^{(N)}[k]\right\}} \tag{3·8}$$

where $b\equiv(d\epsilon[k]/dk)_{k_F}=(N/2\pi)\epsilon'(n_F)$ is independent of N (in our present treatment $b=-\mathrm{B}$). Therefore, using eq. (2·21), we obtain

$$\lim_{N\to\infty}\hbar\omega^{(N)}[k]=|k|\cdot\sqrt{\frac{2e^2}{\pi}b\cdot\sqrt{K_0(\gamma|k|)}}. \tag{3·9}$$

It should be noted that eq. (2·20) gives the relation

$$\hbar\omega_n^{(N)}\leqq|n|\sqrt{\epsilon'(n_F)\{\epsilon'(n_F)+4J_0^{(N)}\}}$$

$$\therefore\quad\lim_{N\to\infty}\hbar\omega_n^{(N)}=0 \tag{3·10}$$

for fixed values of n, because, in the case of infinitely large N, $J_0^{(N)}\propto 1/N\cdot\log N$ as easily shown from eq. (2·20) and $\epsilon'(n_F)\propto 1/N$. Another group of the bosons corresponds to spin-density waves whose energy spectrum is equal to the one given by a non-interacting particle model:

$$\hbar\tilde{\omega}_n^{(N)}=|n|\,\epsilon'(n_F)\propto\frac{|n|}{N} \tag{3·11}$$

Evidently $\hbar\tilde{\omega}_n^{(N)}\ll\hbar\omega_n^{(N)}$ for sufficiently small n.

(3·10) and (3·11) tell us that the lowest excited state is given by creating one boson corresponding to the spin density wave with the lowest frequency $\tilde{\omega}_1^{(N)}$. However, a simple consideration on the spin symmetry leads to the conclusion that the absorption of a photon by our spin density wave is forbidden. Then we assume, according to Araki and Murai, that the first optical absorption band corresponds to the creation of one plasmon with the lowest energy quantum $\hbar\omega_1^{(N)}$.

Then the result shown in eq. (3·10) does not give the convergency of the absorption wave-length. This disappearence of the convergency will be inevitable as long as we adopt the model shown in Fig. 1, because the use of the Tomonaga theory seems to bring no serious error into the plasma frequencies. This fact is favorable to those who suggest the existence of the bond alternation or a Brilliouin gap in the spectra of H_0.

It is obvious that the Tomonaga theory cannot be directly applied to the system

in which there is a finite Brillioun gap between the Fermi level and the lowest excited level, because the Taylor expansion of $\epsilon(n)$ in the neighbourhood of the Fermi level breaks down in this case.

§ 4. Criticism on Araki-Murai's theory

According to Araki and Murai $\lim_{N\to\infty}(1/\hbar\omega_n^{(N)})$ has a finite value. This contradicts with our eq. (3·10). Therefore we here examine their theory.

According to them we adopt the following picture: the π electrons are kept in the rectangular parallelopiped shown in Fig. 2 in which L, the longitudinal length of this pipe, is equal to the chain length of the conjugated double bonds, and the transverse motions of these electrons are fixed. In this case one should write the quantized wave function as

$$\left.\begin{aligned}\psi(\boldsymbol{x}) &= \sum_{\kappa}{}^{\sigma}\, a(\kappa,\sigma)\varphi_\kappa(\vec{x})\sigma(\zeta)\\ \varphi_\kappa(\vec{x}) &= L^{-1/2}\exp(i\kappa x)f(y,z)\end{aligned}\right\},$$

if the periodic boundary condition at both ends of this pipe is allowed. In the above

$$\kappa = 2\pi n/L \quad (n=0,\ \pm1,\ \pm2,\ \cdots)$$

and $f(y, z)$ describes the transverse motion and is normalized on the transverse section of this pipe whose area is denoted by $A = L_y L_z$ and is a constant irrespective of L.

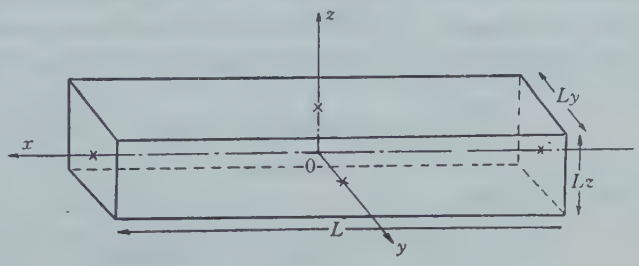

Fig. 2

Then the Hamiltonian for the π electron system becomes of the same form as that given in (3·1), provided that

$$a_{n\sigma} = a(\kappa,\ \sigma),$$

$$\epsilon(n) = ((2\pi\hbar)^2/2mL^2)\cdot n^2,$$

and J_n is an abbreviation of

$$J_n^{(L)} \equiv J^{(L)}[\kappa] = L^{-1}\int_{-L}^{L}\exp(i\kappa x)\overline{J(x)}\,dx$$

where $$\overline{J(x)}=e^2\int_A dy_1 dz_1 \int_A dy_2 dz_2 \frac{|f(y_1,\ z_1)|^2\cdot|f(y_2,\ z_2)|^2}{\sqrt{x^2+(y_1-y_2)^2+(z_1-z_2)^2}}.$$

In the above, κ and n are combined by the relation $\kappa=2\pi n/L$. Therefore the excitation energies of the plasma oscillations become

$$\begin{aligned}\hbar\omega_n^{(L)}&=|n|\sqrt{\epsilon'(n_F)\{\epsilon'(n_F)+2J_n^{(L)}\}}\quad(n=\pm1,\ \pm2,\ \cdots)\\&\leqq|n|\sqrt{\epsilon'(n_F)\{\epsilon'(n_F)+2J_0^{(L)}\}},\end{aligned}\tag{4·1}$$

where we have used the relation $J_n^{(L)}\leqq J_0^{(L)}$. Now $\overline{J(x)}$ should satisfy the properties similar to those of $J(R)$ stated in (2·18), i. e.

$$\overline{J(x)}=e^2/|x|\quad\text{if } |x| \text{ is infinitely large,}\tag{4·2}$$

and $$\overline{J(0)}=\text{large but finite.}$$

Therefore in qualitative arguments one may assume that

$$\overline{J(x)}=\frac{e^2}{\sqrt{x^2+\delta^2}}.$$

Then $$J_0^{(L)}=\frac{e^2}{L}\int_{-L}^{L}\frac{dx}{\sqrt{x^2+\delta^2}}=\frac{2e^2}{L}\log(\sqrt{L^2+\delta^2}+L/\delta).$$

Consequently, $\lim_{L\to\infty}\hbar\omega_n^{(L)}=0$, ($L/N\equiv a'$ being kept constant) for fixed values of n, because $\epsilon'(n_F)=\pi^2/ma'\cdot(1/L)$ and $J_0^{(L)}\to(2e^2/L)\log L$ as $L\to\infty$. Namely, the free electron model adopted by Araki-Murai should also lead to the disappearence of the convergence of the wave length. The reason why they have got the convergency is as follows: They replaced the interaction potential $e^2/|\vec{x}_1-\vec{x}_2|$ by

$$\frac{4\pi e^2}{V}\sum_{\vec{K}\neq0}\frac{1}{K^2}\exp[i\vec{K}\cdot(\vec{x}_1-\vec{x}_2)],\tag{4·3}$$

where $\vec{K}=(K_x,\ K_y,\ K_z)$ and

$$K_x=\kappa\equiv\frac{2\pi n}{L},\quad K_{y,z}=\frac{2\pi}{L_{y,z}}\times(\text{integer}),$$

and, by assuming $f(y,\ z)=1/\sqrt{A}$, they got

$$J^{(L)}[\kappa]=\frac{4\pi e^2}{V}\cdot\frac{1}{\kappa^2}\quad\text{or}\quad J_n^{(L)}=\frac{e^2}{\pi A}\cdot L\cdot\frac{1}{n^2}\tag{4·4}$$

and inserted this into (4·1).

Now (4·3) corresponds to the periodic solution of Poisson's equation,

$$\begin{aligned}&\Delta\varphi(\vec{x})=-4\pi\rho(\vec{x})\\&\rho(\vec{x})=e\left\{\frac{1}{V}-\sum_{\vec{M}}\delta(\vec{x}+\vec{M})\right\}\end{aligned}\tag{4·5}$$

where $$\vec{M}=(m_xL,\ m_yL_y,\ m_zL_z),$$

m_x, m_y and m_z being integers extending from $-\infty$ to $+\infty$. Namely, (4·3) is the electrostatic potential induced by the negative charge of an assemblage of electrons which are arranged so as to form the ortho-rhombic crystal with lattice constants (L, L_y, L_z), each lattice point being occupied by one electron, plus the uniform positive charge with density (e/LL_yL_z) which can neutralize the negative charge of the electrons as a whole. Therefore (4·3) resembles the true interaction potential only in the region $|\vec{x}_1-\vec{x}_2| \ll \text{Min}\,(L, L_y, L_z)$. Then the use of (4·3) will be valid only for the sound-waves whose wave-length is very small compared with Min (L, L_y, L_z). In the present problem L_y and L_z should not be very large, hence it is very difficult to justify the use of (4·4).

As can be easily expected from the above explanation, Araki-Murai's one-dimensional interaction potential obtained by averaging (4·3) over the yz-plane is nothing but the electrostatic potential due to a group of negatively charged and infinitely extensive metal sheets which are placed parallel to each other at regular intervals of L plus the uniform positive charge distributed three-dimensionally. One can easily obtain this potential by means of the Gauss theorem. The results is

$$\frac{2\pi e^2}{A}\left(\frac{1}{L}x^2-|x|\right)+\text{const.},$$

where x is measured from a sheet. This coincides with

$$\frac{4\pi e^2}{V}\sum_{\kappa\neq 0}\frac{1}{\kappa^2}\exp(-i\kappa x)=\frac{2\pi e^2}{A}\left(\frac{1}{L}x^2-|x|+\frac{L}{6}\right).$$

The long-range part of this potential is unreasonably large.

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Hydromagnetic Plane Steady Flow in Compressible Ionized Gases

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The magnetohydrodynamics in compressible gases of infinite conductivity is discussed under the assumptions that the flow is isentropic and steady and that the directions of flow velocity and of magnetic field are in the same plane. The hyperbolicity conditions for the starting equations, under which the discontinuity surfaces can appear, are investigated. The special case in which the flow is parallel to the direction of magnetic field is analyzed in detail.

§ 1. Introduction

The magnetohydrodynamics of compressible ionized gases of infinite conductivity has been investigated in connection with the hydromagnetic shock waves appearing in various branches of plasma physics, and it has been shown that there exist many remarkable features essentially due to the effect of magnetic field. It has especially been pointed out by a few authors[1)2)] that the equation for steady hydromagnetic flow can be hyperbolic even for the subsonic fluid velocity if the magnetic field is of moderate strength. In this paper, we shall deal with the plane isentropic steady flow, which is specified by the condition that the flow and the magnetic field are in the same plane, and discuss the conditions for the discontinuity surfaces to occur, or under which the characteristic equation of the original system has real roots. In the next section, we shall write down the characteristic equation explicitly and show that in our case this equation is the fourth order algebraic equation for the local characteristic direction and that the above conditions can be expressed not only by the usual sonic Mach number but also by the magnetic Mach number, the flow speed *versus* the Alfvén velocity. We shall also give a graphical method convenient for the calculation of the real roots of this equation under arbitrary configurations of flow and magnetic field and it will then be shown that there exist, at least, two real roots when one double root is counted as two single roots, moreover, that the stream lines cannot, in general, be the characteristics unless the flow is parallel to the magnetic field.* In the parallel flow, the stream line corresponds to one double root of the characteristic equation and may be a separation or a free surface, as is usual in hydrodynamics. It seems interesting to note that as the

* This special configuration of the flow and the magnetic field will hereafter be called the "parallel flow".

direction of magnetic field begins to deviate from the flow direction, this double root tends to split up into two single roots or, in other words, the contact discontinuity is transformed into the shock-like discontinuity. In § 3, we shall investigate the parallel flow in detail, showing that the usual Bernoulli law is still valid, the strong form of which does not, however, lead to the irrotational flow. The analytical expressions of the real roots of the characteristic equation and the characteristic equation in the hodograph plane will also be given.

§ 2. Characteristic equation

Under the configuration described in the previous section, the fundamental equations can be given by

$$v_x H_y - v_y H_x = a, \tag{2·1}$$

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+\frac{\mu_0}{\rho}\left(H_y\frac{\partial H_y}{\partial x}-H_y\frac{\partial H_x}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2·2}$$

$$v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+\frac{\mu_0}{\rho}\left(-H_x\frac{\partial H_y}{\partial x}+H_x\frac{\partial H_x}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2·3}$$

$$v_x\frac{\partial \rho}{\partial x}+v_y\frac{\partial \rho}{\partial y}+\rho\frac{\partial v_x}{\partial x}+\rho\frac{\partial v_y}{\partial y}=0, \tag{2·4}$$

$$\partial H_x/\partial x+\partial H_y/\partial y=0, \tag{2·5}$$

in which all quantities are independent of z and t, the z components of the flow velocity and of the magnetic field are put equal to zero, μ_0 is the magnetic susceptibility multiplied by 4π[3] and a is a constant.

Using the relation (2·1), one can eliminate $\partial H_y/\partial x$ and $\partial H_y/\partial y$ from eqs. (2·2)$\sim$(2·5), and we have

$$Dv_x+\frac{\mu_0}{\rho}\frac{H_y}{v_x}\left(v_y\frac{\partial H_x}{\partial x}-v_x\frac{\partial H_x}{\partial y}+H_x\frac{\partial v_y}{\partial x}-H_y\frac{\partial v_x}{\partial x}\right)+\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial x}=0, \tag{2·2}'$$

$$Dv_y-\frac{\mu_0}{\rho}\frac{H_x}{v_x}\left(v_y\frac{\partial H_x}{\partial x}-v_x\frac{\partial H_x}{\partial y}+H_x\frac{\partial v_y}{\partial x}-H_y\frac{\partial v_x}{\partial x}\right)+\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial y}=0, \tag{2·3}'$$

$$D\rho+\rho\cdot\partial v_x/\partial x+\rho\cdot\partial v_y/\partial y=0, \tag{2·4}'$$

$$v_x\cdot\partial H_x/\partial x+v_y\cdot\partial H_x/\partial y+H_x\cdot\partial v_y/\partial y-H_y\cdot\partial v_x/\partial y=0, \tag{2·5}'$$

in which D stands for $v_x\cdot\partial/\partial x+v_y\cdot\partial/\partial y$, and c_s is the sound velocity. Introducing the vector $\boldsymbol{h}$ defined by

$$\boldsymbol{h}=\sqrt{\frac{\mu_0}{\rho}}\boldsymbol{H}, \tag{2·6}$$

and denoting the line-element of the characteristics as $d\sigma$, we have from eqs. (2·2)$'\sim$(2·5)$'$ the following characteristic equation for x_σ and y_σ,

$$\Lambda^4-\Lambda^2\Omega(c_a^2+c_s^2)+c_s^2\Omega(x_\sigma h_y-y_\sigma h_x)^2=0,* \tag{2·9}$$

in which Λ and Ω are defined by

$$\Lambda=v_x y_\sigma-v_y x_\sigma, \tag{2·7}$$

$$\Omega=x_\sigma^2+y_\sigma^2, \tag{2·8}$$

and c_a is the Alfvén velocity equal to $|\boldsymbol{h}|$.

Let us introduce the local polar representation:

$$\left.\begin{aligned} h_x&=h\cos\theta, \quad h_y=h\sin\theta,\\ x_\sigma&=r\cos\chi, \quad y_\sigma=r\sin\chi,\end{aligned}\right\} \tag{2·10}$$

in which the direction of the polar axis coincides with that of the flow velocity. The equation (2·9) then reduces to the following equation for $\sin\chi$,

$$\beta^2\alpha^2\sin^4\chi-(\alpha^2+\beta^2)\sin^2\chi=-\sin^2(\theta-\chi), \tag{2·11}$$

where α and β are equal to the magnetic Mach number v/c_a and the sonic one v/c_s, respectively. It should be noted that $-\chi$ is not a root of (2·11) when χ is a root unless $\theta=0$ or $\pi/2$, or the characteristic direction is, in general, not symmetric with respect to the flow direction. Our purpose is, now, to investigate the number of the real roots of (2·11). It is, however, quite tremendous though not quite impossible, to give their general analytical expressions. Hence, we have recourse to the following graphical method. Let us put $x=\sin^2\chi$, then eq. (2·11) becomes a system of two equations:

$$y=(\alpha^2x-1)(\beta^2x-1), \tag{2·11a}$$

$$y=\cos^2(\theta-\chi). \tag{2·11b}$$

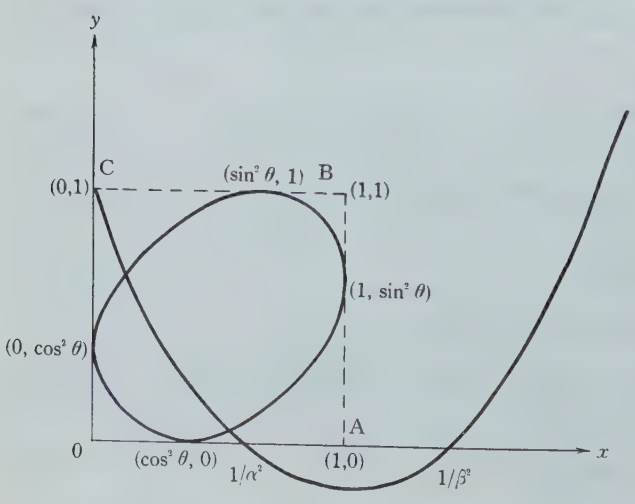

Fig. 1

The latter equation represents the ellipse tangent to the square OABC at the points $(\cos^2\theta,\ 0)$ $(0,\ \cos^2\theta)$ $(1,\ \sin^2\theta)$ and $(\sin^2\theta,\ 1)$ (see Fig. 1). As can obviously be seen from Fig. 1, the parabola given by (2·11a) surely intersects the ellipse. Hence we have at least two real roots of (2·11). Of course, in the special case, $\theta=0$, the ellipse shrinks to the straight line $\overline{AC}$ and in this case the root $\chi=0$ should be regarded as the double one.

* This equation can also be obtained directly from eq. (2·2) in (I) (3).

Fig. 1 also indicates that there exist four real roots if $\alpha>1$, $\beta>1$ and $(\alpha^2-1)\times(\beta^2-1)>1$, and two real roots if $\alpha>1>\beta$ or $\alpha<1<\beta$.

By virtue of Fig. 1 the number of the roots can be calculated graphically case by case, if the values of α, β and θ are given. However, in the special cases specified by $\theta=0$ and $\theta=\pi/2$, the analytical expressions of these roots can be easily obtained.

(i) $\theta=0$[1][2] $(\boldsymbol{v}\parallel\boldsymbol{H})$

In this case, eq. (2·11) splits up into two equations:

$$\sin^2\chi=0, \tag{2·12a}$$

and

$$\alpha^2\beta^2\sin^2\chi-(\alpha^2+\beta^2)+1=0. \tag{2·12b}$$

The latter equation, (2·12b), gives two real roots:

$$\sin\chi=\pm\sqrt{(\alpha^2+\beta^2-1)/\beta^2\alpha^2},$$

if and only if the following conditions are satisfied:

$$\alpha>1 \quad \text{and} \quad \beta>1, \tag{2·13a}$$

or

$$\beta<1 \quad \text{and} \quad \sqrt{1-\beta^2}<\alpha<1. \tag{2·13b}$$

These conditions are illustrated in Fig. 2. They can also be expressed in terms of the sound velocity and the Alfvén velocity, c_s and c_a, respectively (c.f. Fig. 3).

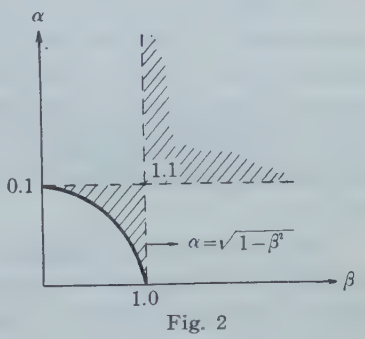

Fig. 2

(a) If $c_s>c_a$,

$$v>c_s \text{ or } c_a[c_s^2/(c_s^2+c_a^2)]^{1/2}<v<c_a;$$

(b) If $c_a>c_s$,

$$v>c_a \text{ or } c_s[c_a^2/(c_s^2+c_a^2)]^{1/2}<v<c_s.$$

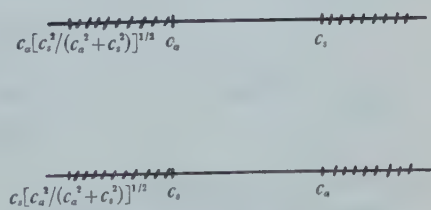

Fig. 3

On the closer analogy of ordinary hydrodynamics, the effective sound speed c^* may be defined by

$$1/\sin\chi=v/c^*, \tag{2·14a}$$

or

$$c^{*2}=c_s^2+c_a^2(1-1/\beta^2). \tag{2·14b}$$

Then, one sees that if β is larger than unity, the magnetic field increases c^*, and on the contrary if β is smaller than unity, it decreases c^* and as a result it turns out to be that even in the subsonic flow the discontinuity surfaces may appear.

From (2·12a) it follows that the stream line can be a discontinuity surface. As is well known in ordinary hydrodynamics, the discontinuity of this kind corresponds to the separation surface of fluid. However, as can be seen from (2·11a·b),

$\chi=0$ cannot be a root except in the case of the parallel flow ($\boldsymbol{v}\parallel\boldsymbol{H}$), and hence the stream lines are not characteristics if the flow is not parallel to the magnetic field.

(ii) $\theta=\pi/2$, ($\boldsymbol{v}\perp\boldsymbol{H}$)

In this case, eq. (2·11) reduces to

$$\alpha^2\beta^2\sin^4\chi-(\alpha^2+\beta^2+1)\sin^2\chi+1=0,$$

and we have four real roots:

$$\sin^2\chi=[(\alpha^2+\beta^2+1)\pm\{(\alpha^2+\beta^2+1)^2-4\alpha^2\beta^2\}^{1/2}]/2\alpha^2\beta^2, \tag{2·15}$$

if and only if the condition

$$v^2>c_s^2+c_a^2 \tag{2·16}$$

is satisfied.

In the limiting case of vanishing magnetic field, the roots corresponding to the plus sign in (2·15) tend to $1/\beta^2$ and those corresponding to the minus sign tend to zero, that is to say, as the strength of the magnetic field approaches zero under this configuration ($\theta=\pi/2$), the former tends to the discontinuity corresponding to a shock but the latter tends to the separation surface; this implies that in the magnetohydrodynamics the contact discontinuity can be transformed into the shock.[4]

It should be noted, however, that all the relations except for those in the parallel flow have only local significance and are not valid in the large. Hence, it seems to be worth-while to investigate the parallel flow in detail.

§ 3. The parallel flow

The equations (2·1), (2·4) and (2·5) are satisfied automatically by the relation

$$\boldsymbol{H}=\kappa^{1/2}\rho\boldsymbol{v} \tag{3·1}$$

in which κ is a constant.

Substituting (3·1) into (2·2), (2·3) and (2·4), we have the following equations for $\boldsymbol{v}$ and ρ,

$$v_x\cdot\partial v_x/\partial x+v_y\cdot\partial v_x/\partial y+\bar{\mu}_0 v_y(\partial\rho v_y/\partial x-\partial\rho v_x/\partial y)=-(1/\rho)\partial p/\partial x, \tag{3·2}$$

$$v_x\cdot\partial v_y/\partial x+v_y\cdot\partial v_y/\partial y+\bar{\mu}_0 v_x(-\partial\rho v_y/\partial x+\partial\rho v_x/\partial y)=-(1/\rho)\partial p/\partial y, \tag{3·3}$$

$$v_x\cdot\partial\rho/\partial x+v_y\cdot\partial\rho/\partial y+\rho\cdot\partial v_x/\partial x+\rho\cdot\partial v_y/\partial y=0, \tag{3·4}$$

in which $\bar{\mu}_0=\mu_0\kappa$.

Since in this case the stream line is one branch of the C-characteristics, there should exist, along each stream line, a relation between $\boldsymbol{v}$, ρ and their derivatives with respect to the parameter specifying this line. This relation can be easily obtained as follows:

Multiplying (3·2) and (3·3) respectively by v_x and v_y and adding, we have

$$v_x(v_x\cdot\partial/\partial x+v_y\cdot\partial/\partial y)v_x+v_y(v_x\cdot\partial/\partial x+v_y\cdot\partial/\partial y)v_y$$
$$=-(1/\rho)(v_x\cdot\partial/\partial x+v_y\cdot\partial/\partial y)p,$$

from which we obtain the relation

$$\tfrac{1}{2}(v^2)_{,\sigma}+p_{,\sigma}/\rho=0, \tag{3·5b}$$

or $\frac{1}{2}v^2+i=$const* along each stream line, in which i is the enthalpy and $d\sigma$ is the line-element of the stream line and is defined by $x_\sigma/y_\sigma=v_x/v_y$. Eq. (3·5b) is nothing but Bernoulli's law.

In the following discussion we assume the strong form of Bernoulli's law, namely

$$\tfrac{1}{2}v^2+i=\text{constant everywhere.} \tag{3·6}$$

Under this condition, eqs. (3·2) and (3·3) reduce to

$$\mathrm{rot}(1-\bar{\mu}_0\rho)\boldsymbol{v}=0. \tag{3·7}$$

It may be interesting to note that, contrary to the circumstances in ordinary hydrodynamics, the strong Bernoulli Law does not lead to the irrotational velocity field. From (3·6), (3·7) and (3·4), we finally have

$$\left(c^{*2}-\frac{1}{\alpha^2}v_x^2\right)\partial v_y/\partial x-\frac{1}{\alpha^2}v_xv_y(-\partial v_x/\partial x+\partial v_y/\partial y)$$
$$-\left(c^{*2}-\frac{1}{\alpha^2}v_y^2\right)\partial v_x/\partial y=0, \tag{3·8a}$$

$$(c_s^2-v_x^2)\partial v_x/\partial x-v_xv_y(\partial v_x/\partial x+\partial v_y/\partial y)+(c_s^2-v_y^2)\partial v_y/\partial y$$
$$=0, \tag{3·8b}$$

in which c_a is equal to

$$\sqrt{\bar{\mu}_0\rho}\,v.$$

According to the usual method,[5] we obtain the following equations for the C-and the Γ-characteristics:

$$(c^{*2}-v_x^2)y_\sigma^2+2v_xv_yx_\sigma y_\sigma+(c^{*2}-v_y^2)x_\sigma^2$$
$$=0, \tag{3·9a}$$

$$c^{*2}c_s^2\{(dv_x)^2+(dv_y)^2\}-c^{*2}(v_xdv_x+v_ydv_y)^2$$
$$-\left(\frac{c_s}{\alpha}\right)^2(v_ydv_x-v_xdv_y)^2=0, \tag{3·9b}$$

* We assume that the flow is isentropic.

in which $d\sigma$ is the line-element of the C-characteristics given by (3·9a). Eq. (3·9a) is the same as that in ordinary hydrodynamics if c^* is replaced by c_s. The integration of these equations will be discussed elsewhere.

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Note added in proof

Some results similar to those in this paper have been obtained also by W. R. Sears and E. L. Resler, JR, in more general cases including the three dimensional flow. We are very grateful to Professor W. R. Sears for his kind information.

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On the Wave Functions of Higher Spin Particles*

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The $2s+1$ mutually orthogonal wave functions of a particle with spin s are explicitly worked out in the tensor formalism for a boson and in the Rarita-Schwinger formalism for a fermion, respectively. As an example of application, a method is described according to which an angular correlation function is calculated relativistically, in the case of an associated production of a hyperon and a kaon and their ensuing decays, assming that they have higher spins.

§ 1. Introduction

So far we have had no clear evidence for the existence of a particle with spin higher than one. However, the possibility that some of strange particles may have higher spin values would not be excluded from the present experimental data.

One of the orthodox methods for determining the spin of a particle consists in comparing a theoretically calculated angular correlation function with the corresponding experimental one. If we wish to calculate the angular correlation function relativistically, it is desirable to work out the $2s+1$ mutually independent wave functions of a particle with arbitrary spin s. In such calculations, relativistic effects have been taken into consideration, up to the present, only in the case of the electron and the electromagnetic field. Of course we can do almost the same thing using the knowledge only of the projection operators, as R. E. Behrends and C. Fronsdal did[1]. However the advantage of using the wave functions explicitly is obvious.

We choose these $2s+1$ wave functions in such a way that they are the eigenfunctions of the z-component of the spin of the particle in the rest system of the particle. Therefore these functions are orthogonal to each other. The principle we adopted to obtain these functions is quite simple. In the rest system of the particle, a set of the $2s+1$ wave functions of spin s can be constructed with the help of two sets of wave functions of spin s_1 and spin s_2, respectively $(s_1,\ s_2 < s)$:

$$\psi(s,\ m)=\sum_{m_1 m_2}(s_1\ s_2\ sm|s_1\ s_2\ m_1\ m_2)\psi(s_1\ m_1)\psi(s_2\ m_2), \qquad (1)$$

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* The most part of this paper was translated from S. Hori, Soryusiron Kenkyu **12** (1956), 339. Although the content of this paper may not be up-to-date, it will not spoil the main context of this paper.

where $(s_1\ s_2\ sm|s_1\ s_2\ m_1\ m_2)$ is the Clebsh-Gordan coefficient. The successive application of eq. (1) shows that the wave functions of arbitrary spin can be expressed in terms of spin $\frac{1}{2}$ and/or spin 1 wave functions. The wave functions thus obtained will larter be transformed into a general coordinate system by a Lorentz transformation.

The tensor formalism for a boson and the Rarita-Schwinger formalism for a fermion are most convenient for our purpose, in which the equations of motion are

$$(p_\lambda{}^2+\kappa^2)\phi_{\mu_1\cdots\mu_s}=0 \tag{2a}$$

$$p_\mu\phi_{\mu\mu_2\cdots\mu_s}=0 \tag{2b}$$

$$\phi_{\mu\mu\mu_3\cdots\mu_S}=0 \tag{2c}$$

$$\phi_{\cdots\mu_i\cdots\mu_j\cdots}=\phi_{\cdots\mu_j\cdots\mu_i\cdots}, \tag{2d}$$

for a boson and

$$(i\gamma_\lambda p_\lambda+\kappa)\Psi_{\mu_1\cdots\mu_K}=0 \qquad (K=S-\tfrac{1}{2}) \tag{3a}$$

$$\gamma_\mu\Psi_{\mu\mu_2\cdots\mu_K}=0 \tag{3b}$$

$$p_\mu\Psi_{\mu\mu_2\cdots\mu_K}=0 \tag{3b'}$$

$$\Psi_{\mu\mu\mu_3\cdots\mu_K}=0 \tag{3c}$$

$$\Psi_{\cdots\mu_i\cdots\mu_j\cdots}=\Psi_{\cdots\mu_j\cdots\mu_i\cdots}, \tag{3d}$$

for a fermion respectively.* Eq. (3b') can be derived from eqs. (3a) and (3b) and eq. (3c) from eq. (3b).

We shall not here consider the case in which the mass κ of the particle vanishes.

§ 2. Construction of the wave functions

Our work will be carried out most conveniently in the momentum space. In the rest system of the particle, spin 1 wave functions are shown to be as follows :**

$$\psi_\mu(1,m)=e_\mu^{(m)}(0)=\mp\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \pm i\\ 0\\ 0\end{bmatrix}, \text{ for } m=\pm 1$$

* Through out this paper we use the units $\hbar=c=1$ and Dirac representation $\gamma_i=\rho_2\sigma_i$, $\gamma_4=\rho_3=\beta$, $\alpha_k=-i\gamma_k\gamma_4$ and the summation convention $a_\mu b_\mu=a_ib_i+a_4b_4=a_ib_i-a_0b_0$.

** Phase factors in eq. (4) are chosen so that we can make use of the tables of the C-G coefficients in Condon-Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935).

$$=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ for } m=0 \tag{4'}$$

and it is well known that the positive energy solutions for spin $\frac{1}{2}$ particle are

$$\psi_\rho(\tfrac{1}{2}\, m)=u_\rho^{(m)}(0)=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{for } m=\frac{1}{2};\ =\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ for } m=-\frac{1}{2} \tag{5}$$

and the negative energy solutions are

$$\psi_\rho(\tfrac{1}{2}\, m)=v_\rho^{(m)}(0)=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ for } m=\frac{1}{2};\ =\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } m=-\frac{1}{2}. \tag{6}$$

In eq. (4) each row corresponds to the vector index μ and in eqs. (5) and (6) it corresponds to the spinor index ρ. The latter will often be suppressed.

The principle in the previous section leads to the following expression of the spin s wave functions in terms of these functions;

$$\phi^{(m)}_{\mu_1\cdots\mu_s}(0)=\sum_{q_1\cdots q_s} b(sm|q_1\cdots q_s)e^{(q_1)}_{\mu_1}(0)\cdots e^{(q_s)}_{\mu_s}(0),\quad (m=-s\cdots s) \tag{7}$$

and

$$\Psi^{(m)}_{\mu_1\cdots\mu_K}(0)=\sum_{q_1\cdots q_K\zeta} a(sm|q_1\cdots q_K\zeta)e^{(q_1)}_{\mu_1}(0)\cdots e^{(q_K)}_{\mu_K}(0)u^{(\zeta)}(0) \tag{8}$$

where

$$b(sm|q_1\cdots q_s)=(s-1\ \ 1sm|s-1\ \ 1\ m-q_K\, q_K)\cdots$$
$$\cdots(2\ 1\ 3\ q_1+q_2+q_3|2\ \ 1\ q_1+q_2\, q_3)(1\ 1\ 2\ q_1+q_2|1\ 1\ q_1\, q_2), \tag{9}$$

and

$$a(sm|q_1\cdots q_K\zeta)=\sum_Q (K\tfrac{1}{2}\ sm|K\tfrac{1}{2}\ Q\,\zeta)b(KQ|q_1\cdots q_K). \tag{10}*$$

The corresponding negative energy wave function can be obtained by replacing $u^{(\zeta)}(0)$ in eq. (8) by $v^{(\zeta)}(0)$.

We carried out computations to obtain the Lorentz transformation which transforms the rest system of the particle into the system in which the particle is moving with momentum $\boldsymbol{p}$. The result is

$$L=1/\kappa(\kappa+p_0)\begin{bmatrix} \kappa(\kappa+p_0)+p_1^2, & p_1p_2, & p_1p_3, & -i(\kappa+p_0)p_1 \\ p_2p_1, & \kappa(\kappa+p_0)+p_2^2, & p_2p_3, & -i(\kappa+p_0)p_2 \\ p_3p_1, & p_3p_2, & \kappa(\kappa+p_0)+p_3^2, & -i(\kappa+p_0)p_3 \\ i(\kappa+p_0)p_1, & i(\kappa+p_0)p_2, & i(\kappa+p_0)p_3, & (\kappa+p_0)p_0 \end{bmatrix} \tag{11}$$

or

$$L_{\mu\nu}=\delta_{\mu\nu}-\frac{i}{\kappa}(p_\mu\delta_{\nu 4}-p_\nu\delta_{\mu 4})-\frac{p_\mu p_\nu}{\kappa(\kappa+p_0)}-\frac{ip_0}{\kappa(\kappa+p_0)}(p_\mu\delta_{\nu 4}+p_\nu\delta_{\mu 4})$$
$$-\frac{\kappa}{\kappa+p_0}\delta_{\mu 4}\delta_{\nu 4} \qquad (11')$$

where, as usual, $p_4=ip_0=i(\boldsymbol{p}^2+\kappa^2)^{1/2}$. We can easily verify that

$$L_{\lambda\mu}L_{\lambda\nu}=L_{\mu\lambda}L_{\nu\lambda}=\delta_{\mu\nu}$$

and

$$L_{\mu\nu}0_\nu=p_\mu$$

where we put $0_\nu=(0, 0, 0, i\kappa)$.*

By virtue of this Lorentz tranformation and the definition of $p^{(q)}=p_\mu e_\mu^{(q)}(0)$, we have

$$e_\mu^{(q)}(\boldsymbol{p})=e_\mu^{(q)}(0)+[1/\kappa(\kappa+p_0)]\ (p_\mu+i\kappa\delta_{\mu 4})p^{(q)}. \qquad (12)$$

which is to be inserted in place of $e_\mu^{(q)}(0)$ after the Lorentz transformation has been made. On the other hand, $u^{(\xi)}(0)$ and $v^{(\xi)}(0)$ should be replaced by $u^{(\xi)}(\boldsymbol{p})=Su^{(\xi)}(0)$ and $v^{(\xi)}(\boldsymbol{p})=Sv^{(\xi)}(0)$, respectively, where the spinor transformation S must satisfy the relation

$$S^{-1}\gamma_\mu S=L_{\mu\nu}\gamma_\nu \qquad (13)$$

from which we obtain

$$S=[1/2\kappa(\kappa+p_0)]^{1/2}(p_0+\kappa+\alpha\boldsymbol{p})$$
$$S^{-1}=[1/2\kappa(\kappa+p_0)]^{1/2}(p_0+\kappa-\alpha\boldsymbol{p}). \qquad (14)$$

Thus $u^{(\xi)}(\boldsymbol{p})$ and $v^{(\xi)}(\boldsymbol{p})$ can be expressed as

$$u^{(\xi)}(\boldsymbol{p})=-N(\boldsymbol{p})(i\gamma_\mu p_\mu-\kappa)u^{(\xi)}(0)$$
$$v^{(\xi)}(\boldsymbol{p})=+N(\boldsymbol{p})(i\gamma_\mu p_\mu+\kappa)v^{(\xi)}(0) \qquad (15)$$
$$N(\boldsymbol{p})=[1/2\kappa(\kappa+p_0)]^{1/2},$$

where the use has been made of the relations, $\gamma_4 u^{(\xi)}(0)=u^{(\xi)}(0)$ and $\gamma_4 v^{(\xi)}(0)=-v^{(\xi)}(0)$. We shall list below the properties of these functions $e_\mu^{(q)}(\boldsymbol{p})$, $u^{(\xi)}(\boldsymbol{p})$ and $v^{(\xi)}(\boldsymbol{p})$, which we can verify straightway since we wrote down these functions explicitly.

$$p_\mu e_\mu^{(q)}(\boldsymbol{p})=0 \qquad (16a)$$
$$e_\mu^{(q)*}(\boldsymbol{p})=(-1)^q e_\mu^{(-q)}(\boldsymbol{p}) \qquad (16b)$$
$$e_\mu^{(q)}(\boldsymbol{p})e_\mu^{(r)*}(\boldsymbol{p})=\delta(q, r) \qquad (16c)$$

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* It will sometimes be convenient if the coefficients a and b are understood to contain δ-symbols, $\delta(m, q_1+q_2+\cdots q_K+\zeta)$ and $\delta(m, q_1+\cdots q_s)$ respectively, as in the case of C-G coefficients.

$$\sum_{q=-1}^{1} e_{\mu}^{(q)}(\boldsymbol{p}) e_{\nu}^{(q)*}(\boldsymbol{p}) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{\kappa^2} \tag{16d}$$

$$\bar{u}_{\rho}^{(\zeta)}(\boldsymbol{p}) u_{\rho}^{(\zeta')}(\boldsymbol{p}) = \delta(\zeta, \zeta'), \quad \bar{v}_{\rho}^{(\zeta)}(\boldsymbol{p}) v_{\rho}^{(\zeta')}(\boldsymbol{p}) = -\delta(\zeta, \zeta') \tag{17a}$$

$$\bar{u}_{\rho}^{(\zeta)}(\boldsymbol{p}) v_{\rho}^{(\zeta')}(\boldsymbol{p}) = \bar{v}_{\rho}^{(\zeta)}(\boldsymbol{p}) u_{\rho}^{(\zeta')}(\boldsymbol{p}) = 0 \tag{17b}$$

$$\sum_{\zeta} u_{\rho}^{(\zeta)}(\boldsymbol{p}) \bar{u}_{\sigma}^{(\zeta)}(\boldsymbol{p}) = (1/2\kappa)(-i\gamma p + \kappa)_{\rho\sigma} \tag{17c}$$

$$\sum_{\zeta} v_{\rho}^{(\zeta)}(\boldsymbol{p}) \bar{v}_{\sigma}^{(\zeta)}(\boldsymbol{p}) = (1/2\kappa)(-i\gamma p - \kappa)_{\rho\sigma} \tag{17d}$$

$$\sum_{\zeta} [u_{\rho}^{(\zeta)}(\boldsymbol{p}) u_{\sigma}^{(\zeta)}(\boldsymbol{p}) - v_{\rho}^{(\zeta)}(p) \bar{v}_{\sigma}^{(\zeta)}(p)] = \delta_{\rho\sigma} \tag{17e}$$

where $\bar{u} = u^* \gamma_4$, etc. It should be noted that when we take the complex conjugate of $e^{(q)}(\boldsymbol{p})$, we should not change the sign of i (the relativistic imaginary unit) which occurs at the fourth component of $e^{(q)}(\boldsymbol{p})$, except in the case of $\gamma_\mu\, e_{\mu}^{(q)}(\boldsymbol{p})^*$.

Next, we shall show that the spin s wave functions

$$\phi_{\mu_1 \cdots \mu_s}^{(m)}(\boldsymbol{p}) = \sum_{q_1 \cdots q_s} b(sm|q_1 \cdots q_s) e_{\mu_1}^{(q_1)}(\boldsymbol{p}) \cdots e_{\mu_s}^{(q_s)}(\boldsymbol{p}) \tag{18}$$

and

$$\Psi_{\mu_1 \cdots \mu_K}^{(m)}(\boldsymbol{p}) = \sum_{q_1 \cdots q_K \zeta} a(sm|q_1 \cdots q_K \zeta) e_{\mu_1}^{(q_1)}(\boldsymbol{p}) \cdots e_{\mu_K}^{(q_K)}(\boldsymbol{p}) u^{(\zeta)}(\boldsymbol{p}), \tag{19}$$

really satisfy eqs. (2) and (3). Eq. (2a) is already satisfied since $\phi_{\mu_1 \cdots \mu_s}$ is defined on the energy shell $p_0 = (p^2 + \kappa^2)$. That eq. (2b) holds is obvious from eq. (16a). Using eqs. (16b) and (16c) we see that eq. (2c) follows if

$$\sum_{q q_3 \cdots q_s} b(sm|q, -q\, q_3 \cdots q_s)(-1)^q e_{\mu_3}^{(q_3)}(\boldsymbol{p}) \cdots e_{\mu_s}^{(q_s)}(\boldsymbol{p}) = 0.$$

This is the case since $\sum_{q} (-1)^q (11\,2\;0|11\,q\;-q) = 0$. Eq. (2d) is valid if the coefficient b is symmetric under the permutation of $q_1, \cdots, q_s$. We can easily prove

$$(l\,1\,l+1\,Q+q+q'|l\,1\,Q+q'\,q)(l\,1\,l\,Q+q'|l\,1\,Q\,q') =$$
$$(l\,1\,l+1\,Q+q+q'|l\,1\,Q+q\,q')(l\,1\,l\,Q+q|l\,1\,Q\,q),$$

inserting the explicit expression of the *C-G* coefficients, and writing $q = q_l$, $q' = q_{l+1}$ and $Q = \sum_{r=1}^{l-1} q_r$, this ensures the invariance of b under the exchange of q_l and q_{l+1}, $(1 \leqq l \leqq s-1)$.

Eqs. (3b′), (3c), and (3d) can be verified in the similar way. It is clear that eq. (3a) is satisfied, since $u^{(\zeta)}(\boldsymbol{p})$ is a solution of Dirac equation. It should be,

---

* Let χ_μ be a complex vector field. Then $\chi_\mu^* \chi_\mu$ does not mean $\chi_k^* \chi_k + \chi_0^* \chi_0$ bnt $\chi_k^* \chi_k - \chi_0^* \chi_0$. The only exceptional case is $\gamma_\mu \chi_\mu^*$ $(\gamma_\mu \chi_\mu^* = \gamma_k \chi_k^* - i\gamma_4 \chi_0^*)$. Otherwise we could not have both γ_k and γ_4 hermitian at the same time. The situation would be clearer if we had used, e.g., the Feynman representation.

however, that for the negative energy solution we must put $p_4=-ip_0$ in eq. (3a) (and also in eq. (3b')), whereas we have put $p_4=ip_0(p_0>0)$ in eq. (15), so that it is not

$$\Phi_{\mu_1\cdots\mu_K}(\boldsymbol{p})=\sum a(sm|q_1\cdots q_K\zeta)e^{(q_1)}_{\mu_1}(\boldsymbol{p})\cdots e^{(q_K)}_{\mu_K}(\boldsymbol{p})v^{(\zeta)}(\boldsymbol{p}) \tag{20}$$

but $\Phi_{\mu_1\cdots\mu_K}(-\boldsymbol{p})$ that satisfies the set of eqs. (3). To prove eq. (3b) we make use of the relations

$$\gamma_\mu e^{(q)}_\mu(\boldsymbol{p})u^{(\zeta)}(\boldsymbol{p})=S\gamma_\mu e^{(q)}_\mu(0)u^{(\zeta)}(0)=S\rho_2\sigma^{(q)}q^{(\zeta)}(0)$$

where

$$\sigma^{(q)}=\sigma_k e^{(q)}_k(0)$$

and

$$\sigma^{(\pm1)}u^{(\zeta)}(0)=\mp 2u^{(\zeta\pm1)}(0)$$
$$\sigma^{(0)}u^{(\zeta)}(0)=2\zeta u^{(\zeta)}(0),$$

and we may prove

$$\sum a(sm|q_1\cdots q_K\zeta)e^{(q_1)}_{\mu_1}(\boldsymbol{p})\cdots e^{(q_{K-1})}_{\mu_{K-1}}(\boldsymbol{p})\sigma^{(q_K)}u^{(\zeta)}(0)=0$$

or

$$\sum_Q[\sqrt{2}\,(K\tfrac{1}{2}\,sm|K\tfrac{1}{2}\,Q\,\tfrac{1}{2})(K-1\,1KQ|K-1\,1\,Q'-1)$$
$$-(K\tfrac{1}{2}\,sm|K\tfrac{1}{2}\,Q\,-\tfrac{1}{2})(K-1\,1KQ|K-1\,1\,Q\,0)]=0$$
$$\sum_Q[(K\tfrac{1}{2}\,sm|K\tfrac{1}{2}\,Q\,\tfrac{1}{2})(K-1\,1\,KQ|K-1\,1\,Q'\,0)$$
$$+\sqrt{2}\,(K\tfrac{1}{2}\,sm|K\tfrac{1}{2}\,Q\,-\tfrac{1}{2})(K-1\,1\,KQ|K-1\,1\,Q'\,1)]=0$$

where Q' is equal to $\sum_{r=1}^{K-1}q_r$. The validity of these equations is exhibited by the insertion of the explicit expressions of the *C-G* coefficients.

It is not difficult to show the orthonormality of these functions directly making use of eqs. (16c) and (17a, b) and the orthonormality of the *C-G* coefficients :

$$\phi^{(m)*}_{\mu_1\cdots\mu_s}(\boldsymbol{p})\phi^{(m')}_{\mu_1\cdots\mu_s}(\boldsymbol{p})=\delta(m,m'), \tag{21}$$

$$\overline{\Psi}^{(m)}_{\mu_1\cdots\mu_{K^\rho}}(\boldsymbol{p})\Psi^{(m')}_{\mu_1\cdots\mu_{K^\rho}}(\boldsymbol{p})=\delta(m,m') \tag{22a}$$

$$\overline{\Phi}^{(m)}_{\mu_1\cdots\mu_{K^\rho}}(\boldsymbol{p})\Phi^{(m')}_{\mu_1\cdots\mu_{K^\rho}}(\boldsymbol{p})=\delta(m,m') \tag{22b}$$

$$\overline{\Phi}^{(m)}_{\mu_1\cdots\mu_{K^\rho}}(\boldsymbol{p})\Psi^{(m')}_{\mu_1\cdots\mu_{K^\rho}}(\boldsymbol{p})=0 \tag{22c}$$

Also we can show that

$$\Psi^{(m)*}_{\mu\cdots}(\boldsymbol{p})\Psi^{(m')}_{\mu\cdots}(\boldsymbol{p})=(p_0/\kappa)\delta(m,m') \tag{23a}$$

$$\Phi^{(m)*}_{\mu\cdots}(-\boldsymbol{p})\Phi^{(m')}_{\mu\cdots}(-\boldsymbol{p})=(p_0/\kappa)\delta(m,m') \tag{23b}$$

$$\Phi^{(m)*}_{\mu\cdots}(-\boldsymbol{p})\,\Psi^{(m')}_{\mu\cdots}(\boldsymbol{p})=0. \tag{23c}$$

§ 3. Quantization

The quantized wave functions of spin s which will be denoted by bold letters can be expanded in terms of the corresponding c-number wave functions:

$$\boldsymbol{\phi}_{\mu_1\cdots\mu_s}(x)=(2\pi)^{-3/2}\int\sum_m\Big[c_m(\boldsymbol{k})\phi^{(m)}_{\mu_1\cdots\mu_s}(\boldsymbol{k})e^{ik_\mu x_\mu}+$$

$$+d^*_m(\boldsymbol{k})\phi^{(m)*}_{\mu_1\cdots\mu_s}(\boldsymbol{k})e^{-ik_\mu x_\mu}\Big](2k_0)^{-1/2}\,d^3\boldsymbol{k}. \tag{24}$$

$$\boldsymbol{\Psi}_{\mu_1\cdots\mu_K}(x)=(2\pi)^{-3/2}\int\sum_m\Big[A_m(\boldsymbol{p})\Psi^{(m)}_{\mu_1\cdots\mu_K}(\boldsymbol{p})e^{ip_\mu x_\mu}+$$

$$+B^*_m(+\boldsymbol{p})\Phi^{(m)}_{\mu_1\cdots\mu_K}(\boldsymbol{p})e^{-ip_\mu x_\mu}\Big](\kappa/p_0)^{1/2}d^3\boldsymbol{p}, \tag{25}$$

where c, d, A and B (c^*, d^*, A^* and B^*) and annihilation (creation) operators, and the commutation rules for them read

$$[c_m(\boldsymbol{k}),\ c^*_{m'}(\boldsymbol{k}')]=[d_m(\boldsymbol{k}),\ d^*_{m'}(\boldsymbol{k}')]=\delta(m,m')\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{26}$$

other commutators $=0$,

$$[A_m(\boldsymbol{k}),\ A^*_{m'}(\boldsymbol{k}')]_+=[B_m(\boldsymbol{k}),\ B^*_{m'}(\boldsymbol{k}')]_+=\delta(m,m')\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{27}$$

other anticommutators $=0$.

The commutation relations for the quantized wave functions are now easily written down, by using the commutation rules (26) and (27),

$$[\boldsymbol{\phi}_{\mu_1\cdots\mu_s}(x),\ \boldsymbol{\phi}^*_{\nu_1\cdots\nu_s}(x')]=iP_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}(\partial)\Delta(x-x') \tag{28}$$

$$[\boldsymbol{\Psi}_{\mu_1\cdots\mu_K}(x),\ \overline{\boldsymbol{\Psi}}_{\nu_1\cdots\nu_K}(x')]_+=-i\mathfrak{P}_{\mu_1\cdots\mu_K,\ \nu_1\cdots\nu_K}(\partial)\Delta(x-x'). \tag{29}$$

where Δ is the usual invariant Δ-function

$$\Delta(x)=-i(2\pi)^{-3}\int(1/2k_0)\,(e^{ik_\mu x_\mu}-e^{-ik_\mu x_\mu})\,d^3\boldsymbol{k}, \tag{30}$$

and it can readily be demonstrated that the projection operators defined by

$$P_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}(ik)=\sum_m\phi^{(m)}_{\mu_1\cdots\mu_s}(\boldsymbol{k})\phi^{(m)*}_{\nu_1\cdots\nu_s}(\boldsymbol{k}), \tag{31}$$

$$\mathfrak{P}_{\mu_1\cdots\mu_K,\nu_1\cdots\nu_K}(ip)=-2\kappa\sum_m\Psi^{(m)}_{\mu_1\cdots\mu_K}(\boldsymbol{p})\overline{\Psi}^{(m)}_{\nu_1\cdots\nu_K}(\boldsymbol{p}), \tag{32}$$

or equivalently

$$P_{\mu\cdots,\nu\cdots}(-ik)=\sum_m\phi^{(m)*}_{\mu\cdots}(\boldsymbol{k})\,\phi^{(m)*}_{\mu\cdots}(\boldsymbol{k}), \tag{31'}$$

$$\mathfrak{P}_{\mu\cdots,\nu\cdots}(-ip)=2\kappa\sum_m\Phi^{(m)}_{\mu\cdots}(\boldsymbol{p})\bar{\Phi}^{(m)}_{\nu\cdots}(\boldsymbol{p}), \tag{32'}$$

are identical with those obtained by M. Fierz[2].

§ 4. An example of application

The utility of this formalism has hitherto been rather obscure. Here we shall calculate an angular correlation function as an illustration of application of the above formulation in the case of an associated production of a hyperon and a K-meson and their ensuing decays,

$$\pi^- + p \to Y + K\,;\ Y \to p + \pi^- \text{ and } K \to \pi^+ + \pi^-, \tag{33}$$

assuming that both Y and K have higher spins, s_Y and s_K respectively.

Let us choose as the polar axis the emission direction of the hyperon in the center-of-mass system of the production reaction and we use the following notations,

$\boldsymbol{k}$ = momentum of the incident pion
$\boldsymbol{l}$ = relative momentum of p and π^- (decay secondaries of the hyperon)
$\boldsymbol{n}$ = relative momentum of π^+ and π^- (decay secondaries of the kaon)

and

$$\hat{\boldsymbol{k}} = \boldsymbol{k}/k, \text{ etc.}$$

Here we concern ourselves with the angular correlation function $W(\hat{\boldsymbol{k}}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{n}})$. The function W is expressed as a product of three density matrices: that of the production, $E_{prod}(m_Y, m_Y'; m_K, m_K'; \hat{\boldsymbol{k}})$ and that of the decay of the hyperon, $E_{dec}^{(Y)}(m_Y, m_Y'; \hat{\boldsymbol{l}})$ and that of the decay of the kaon, $E_{dec}^{(K)}(m_K, m_K'; \hat{\boldsymbol{n}})$:

$$W(\hat{\boldsymbol{k}}, \hat{\boldsymbol{l}}, \hat{\boldsymbol{n}}) = \sum_{m_Y m_Y' m_K m_K'} E_{prod}(m_Y, m_K'; m_K, m_K'; \hat{\boldsymbol{k}}) E_{dec}^{(Y)}(m_Y, m_Y'; \hat{\boldsymbol{l}}) E_{dec}^{(K)}(m_K, m_K'; \hat{\boldsymbol{n}}), \tag{34}$$

where m_Y and m_K are z-components of the spins of Y and K, respectively.

E_{prod} depends on the detailed structure of the interaction and cannot be determined only from geometrical consideration. So, we shall here confine our calculation to the density matrices of the hyperon and kaon decays.

i) $E_{dec}^{(Y)}(m_Y, m_Y; \hat{\boldsymbol{l}})$

The element of the S-matrix relevant to the process in consideration reads $\langle \zeta, \boldsymbol{p}, \boldsymbol{q}|S|m_Y, \boldsymbol{P}\rangle$ where

$$S \sim \iiint d^4x_1 d^4x_2 d^4x_3\ \bar{\psi}_\rho(x_1)(\mathscr{F}_{\mu_1 \cdots \mu_K})_{\rho\sigma} \Psi_{\mu_1 \cdots \mu_K \sigma}(x_2)\phi(x_2). \tag{35}$$

and $\mathscr{F}$ is a certain function with the indicated transformation property and invariant under translation, and ψ, Ψ and ϕ are the quantized wave function of the proton, the hyperon and the pion, respectively. The initial state $|m_Y, \boldsymbol{P}>$ and the final state $|\zeta, \boldsymbol{p}; \boldsymbol{q}>$ are expressed as

$$|m_Y, \boldsymbol{P}> = A_{m_Y}^*(\boldsymbol{P})|0> \tag{36}$$

$$|\zeta,\ \boldsymbol{p};\ \boldsymbol{q}>=a_\zeta^*(\boldsymbol{p})c^*(\boldsymbol{q})|0> \tag{37}$$

where $A_{m_Y}(\boldsymbol{P})$, $a_\zeta(\boldsymbol{p})$ $c^*(\boldsymbol{q})$ are the creation operators of the hyperon with the momentum $\boldsymbol{P}$ and the spin component m_Y, of the proton with the momentum $\boldsymbol{p}$ and the spin component ζ and of the pion with the momentum $\boldsymbol{q}$, respectively, and $|0>$ is the state vector of the vacuum normalized as $\langle 0|0\rangle=1$.

With all these relations together we can write

$$\langle\zeta,\ \boldsymbol{p};\ \boldsymbol{q}|S|m_Y,\ \boldsymbol{P}\rangle=(2\pi)^4\,\delta^4(P-p-q)\langle\zeta,\ \boldsymbol{p};\ \boldsymbol{q}|m_Y,\ \boldsymbol{P}\rangle \tag{38}$$

and the transition matrix $\langle\zeta,\ \boldsymbol{p};\ \boldsymbol{q}|m_Y,\ \boldsymbol{P}\rangle$ is given by

$$\langle\zeta,\ \boldsymbol{p};\ \boldsymbol{q}|m_Y,\ \boldsymbol{P}\rangle=\bar{\psi}_\rho^{(\zeta)}(\boldsymbol{p})\,(F_{\mu_1\cdots\mu_K}(\boldsymbol{p},\ \boldsymbol{P}))_{\rho\sigma}\Psi_{\mu_1\cdots\mu_K\sigma}^{(m_Y)}(\boldsymbol{p})\phi^*(\boldsymbol{q}). \tag{39}$$

As $\phi^*(\boldsymbol{q})=(2q_0)^{-1/2}$ have no connection with angles, so we shall omit it hereafter. The consideration of the transformation property suggests that the function F is expressible as the sum of products of $\gamma_\lambda p_\lambda$, $\gamma_\lambda P_\lambda$, p_μ, P_μ, $\delta(\mu_i,\ \mu_j)$, γ_5 and scalar functions of p_μ and P_μ. We can assume without loss of generality that $\gamma_\lambda p_\lambda$ or $\gamma_\lambda P_\lambda$ stands on the left or right end of each term, so that we can replace it by the proton or hyperon mass. Finally we have, by virtue of the eqs. (2) and (3)*,

$$F_{\mu_1\cdots\mu_K}(\boldsymbol{p},\ \boldsymbol{P})=(f+g\gamma_5)p_{\mu_1}\cdots p_{\mu_K}. \tag{40}$$

where f and g are certain invariant function of p_μ and P_μ but $-p_\mu{}^2$ and $-P_\mu{}^2$ can be replaced by m^2 and κ^2, and $p_\mu P_\mu$ becomes $-p_0\kappa$ in the rest system of the hyperon which depends only on the Q-value, so we may regard them as mere numerical constants. Therefore the transition matrix can be written as

$$\langle\zeta,\ \boldsymbol{p}|m_Y\rangle=\bar{\psi}^{(\zeta)}(\boldsymbol{p})\,(f+g\gamma_5)\,p_{\mu_1}\cdots p_{\mu_K}\Psi_{\mu_1\cdots\mu_K}{}^{(m_Y)}(0). \quad (\boldsymbol{P}=0) \tag{41}$$

Next we shall try to rewrite $p_{\mu_1}\cdots p_{\mu_K}\Psi_{\mu_1\cdots\mu_K}{}^{(m_Y)}(0)$. Making use of the relation

$$\begin{aligned} &p_\mu e_\mu^{(q)}(0)\equiv p^{(q)}=(4\pi/3)^{1/2}pY_1^q(\theta,\ \varphi)\\ &p=|\boldsymbol{p}|,\quad \theta,\varphi=\text{polar angles of } \hat{\boldsymbol{p}}=\hat{\boldsymbol{l}}, \end{aligned} \tag{42}$$

and applying the addition theorem of spherical harmonics

$$\begin{aligned} &\sum_{q_1q_2}(k_1k_2kq|k_1k_2q_1q_2)\ \ Y_{k_1}^{q_1}(\theta,\ \varphi)\,Y_{k_2}^{q_2}(\theta,\ \varphi)\\ &=[(2k_1+1)(2k_2+1)/4\pi(2k+1)]^{1/2}(k_1k_20k|k_1k_200)\,Y_k^q(\theta,\ \varphi) \end{aligned} \tag{43}$$

successively, we have

$$\sum_{q_1\cdots q_K} b(KQ|q_1\cdots q)\ p^{(q_1)}\cdots p^{(q_K)}=p^K\left(\frac{4\pi}{2K+1}\right)^{1/2}b(K0|0\cdots0)\,Y_K^q(\theta,\varphi) \tag{44}$$

and

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* The effect of parity nonconservation was not considered in the original paper (see the footnote of the first page). Here we supplemented it.

$$\langle\zeta,\ \boldsymbol{P}|m_Y\rangle=\sum\bar{\psi}^{(\zeta)}(p)\,(f+g\gamma_5)u^{(\eta)}(0)\,(K\,\tfrac{1}{2}\ s_Y\ m_Y|K\,\tfrac{1}{2}\ Q\ \eta)\,Y_K^Q(\theta,\ \varphi),\tag{45}$$

again omitting a factor independent of the angles.

The density matrix is defined according to the equation

$$E_{dec}^{(Y)}(m_Y,\ m_Y';\ \hat{\boldsymbol{p}})=\sum_{\zeta}\langle\zeta,\ \hat{\boldsymbol{p}}|m_Y\rangle\langle\zeta,\ \hat{\boldsymbol{p}}|m_Y'\rangle^*\tag{46}$$

which, upon inserting the above expression (45), yields

$$\begin{aligned}E_{dec}^{(Y)}(m_Y,\ m_Y',\ \hat{\boldsymbol{p}})\cong\sum_{\xi\eta QR}&Y_K^{R*}(\theta,\ \varphi)\,(K\,\tfrac{1}{2}\ s_Y\ m_Y'|K\,\tfrac{1}{2}\ R\ \xi)\times\\&u^{(\xi)}(0)\,(f^*-g^*\gamma_5)[(-i\gamma p+m)/2m](f+g\gamma_5)u^{(\eta)}(0)\\&(K\,\tfrac{1}{2}\ s_Y\ m_Y|K\,\tfrac{1}{2}\ Q\ \eta)\,Y_K^Q(\theta,\ \varphi)\end{aligned}\tag{47}$$

where the use has been made of the relation

$$\sum_{\zeta}\psi^{(\zeta)}(\boldsymbol{p})\bar{\psi}^{(\zeta)}(\boldsymbol{p})=(-i\gamma p+m)/2m.$$

By the help of the relations

$$\begin{aligned}&u^{(\xi)}(0)\,(f^*-g^*\gamma_5)[(-i\gamma p+m)/2m](f+g\gamma_5)u^{(\eta)}(0)\\&\quad=A(\xi|1+\lambda p^{-1}\,\boldsymbol{\sigma p}|\eta)\\&\quad=A[\delta(\xi,\ \eta)-(4\pi)^{1/2}\lambda\sum_q(1\,\tfrac{1}{2}\ q\xi|1\,\tfrac{1}{2}\,\tfrac{1}{2}\ \eta)\,Y_1^q(\theta,\ \varphi)]\end{aligned}\tag{48}$$

where

$$\begin{aligned}&A=|f|^2[(p_0+m)/2m]+|g|^2[(p_0-m)/2m]\\&\lambda=(p/m)Re(fg^*)A^{-1},\quad(|\lambda|\leqq 1)\end{aligned}\tag{49}$$

and the inverse relation of eq. (43)

$$\begin{aligned}Y_{k_1}^{q_1}(\theta,\ \varphi)\,Y_{k_2}^{q_2}(\theta,\ \varphi)=\sum_{k\ q}&[(2k_1+1)\,(2k_2+1)/4\pi(2k+1)]^{1/2}\\&(k_1\ k_2\ k\ 0|k_1\ k_2\ 0\ 0)\,(k_1\ k_2\ k\ q|k_1\ k_2\ q_1\ q_2)\ \ Y_k^q(\theta,\ \varphi),\end{aligned}\tag{50}$$

we obtain

$$\begin{aligned}&E_{dec}^{(Y)}(m,\ m';\ \hat{\boldsymbol{p}})=(-1)^{s_Y-1/2}(2K+1)[(2s_Y+1)/4\pi]^{1/2}\\&\times\sum_{LM}[(K\,K\,0\,0|K\,K\,L\,0)\,(s_Y\ Lm_Y'\ M|s_Y\ Ls_Y\ m_Y)\,W(KKs_Y\ s_Y;\ L\,\tfrac{1}{2})+\\&+\lambda[(2K+2)/(2K+1)]^{1/2}(KK+100|KK+1L0)\,W(KK+1s_Y\ s_Y;\ L\,\tfrac{1}{2})]Y_L^M(\theta,\varphi).\end{aligned}\tag{51}$$

The density matrix is normalized in the sense,

$$\sum_{m_Y}\int E_{dec}^{(Y)}(m_Y,\ m_Y;\ \hat{\boldsymbol{p}})\sin\,\theta\ \,d\theta\ \,d\varphi=1,\tag{52}$$

and the product of the factors we discarded would give, if retained, the reciprocal

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* Note that $Y_{K^R}{}^*=(-1)^K Y_{K^{-R}}$.

lifetime of the hyperon. The expression (51) is entirely equivalent to what we obtained in the 3-dimensional calculation[3]*.

If the hyperon is polarized along the z-axis and the z-component of its spin is m_Y, then the average value of cos θ is evaluated according to the equation,

$$\langle\cos\theta\rangle=\int\cos\theta\, E_{dec}^{(Y)}(m_Y, m_Y; \hat{\boldsymbol{p}})\sin\theta\, d\theta\, d\varphi.$$

The result is

$$\langle\cos\theta\rangle=\lambda(m_Y/s_Y)(2s_Y+2)^{-1} \tag{53}$$

or we have

$$|\langle\cos\theta\rangle|\leqq 1/(2s_Y+2). \tag{54}$$

This result was used by Lee and Yang as an indication of low spin value of Λ-particle[4].

ii) $E_{dec}^{(K)}(m_K, m_K', \hat{\boldsymbol{p}})$

$E_{dec}^{(K)}(m_K, m_K', \hat{\boldsymbol{p}})$ can be calculated in a similar and simpler manner. The transition matrix is

$$\begin{aligned}\langle\boldsymbol{p}|m_K\rangle&\cong p_{\mu_1}\cdots p_{\mu_s}\chi_{\mu_1\cdots\mu_s}^{(m_K)}(0)\\&=b(sm_K|q_1\cdots q_s)p^{(q_1)}\cdots p^{(q_s)}.\end{aligned} \tag{55}$$

This becomes, owing to the equation (44),

$$\langle\boldsymbol{p}|m_K\rangle\cong Y_s^{m_K}(\theta, \varphi). \tag{56}$$

Therefore we obtain

$$E_{dec}^{(Y)}(m_K, m_K'; \hat{\boldsymbol{p}})\cong Y_s^{m'K}(\theta, \varphi)\, Y_s^{m_K}(\theta, \varphi). \tag{57}$$

Using eq. (50), we get the following expression for the density matrix,

$$\begin{aligned}E_{dec}^{(K)}(m_K, m_K', \hat{\boldsymbol{p}})=(-1)^s(4\pi)^{-1/2}\sum_{L,M}(-1)^L(ss\,00|ss\,L0)\\(sLm_K'\,M|sLsm_K)\,Y_L^M(\theta, \varphi).\end{aligned} \tag{58}$$

This is entirely equivalent to the result obtained in the 3-dimensional calculation[3].

§ 4. Discussion

It is well known that the conservation of an orbital angular momentum holds

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* When we compare the two expressions with each other, we should note that here K is not an orbital angular momentum which is by no means a good quantum number in 4-dimensional calculation and always K is equal to $s_Y-\frac{1}{2}$, whereas K stands for the orbital angular momentum in the corresponding 3-dimensional result and $K=s_Y\pm\frac{1}{2}$ for parity-conserving and -anticonserving parts respectively. However, we can show that the two values of $(2K+1)$ $(KK00|KKL0)$ $W(KKs_Ys_Y; L\frac{1}{2})$ for $K=s_Y\pm\frac{1}{2}$ are equal to each other.

only in nonrelativistic approximation and, moreover, the usual 3-dimensional calculation is based upon the conservation of the space components [(i, j)-components $(i, j=1,2,3)$] of the total angular momentum. According to our method here described, angular correlation functions can be calculated without recourse to orbital angular momenta and also the conservation of the space-time components [$(i, 4)$--components] of the total angular momentum is properly taken into consideration. Though in the two-body decay of a particle, however, only the conservation of the space components of the total angular momentum completely fixes the geometrical relations of the decay, so that the 3-dimensional calculation yields the result not different from the 4-dimensional one, the calculation according to our method will give a more restrictive result than the 3-dimensional one in the case of a reaction in which four or more particles participate in the initial and final states.

In conclusion the author would like to express his cordial thanks to Prof. T. Tati for his encouragement and to Dr. S. Chiba, Dr. H. Segawa and Dr. A. Wakasa for their valuable discussions and reading of the manuscript.

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On the Relation between the Phase Shift and the Number of Bound States

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Some features of the Low equation are studied for Dyson's model. To select a unique solution the relation between the phase shift and the number of bound states is important. (§ 1) This relation is investigated more generally. It is connected with the energy spectra of the free Hamiltonian. (§ 2) The relation between the phase shift and the number of resonant states is also shown. Resonances are classified into two types and some differences in their properties are stated. (§ 3)

Introduction*

The Hamilton formalism with which we have been most acquainted has some unsatisfactory points in its fundamental character. The basic equations and the commutation relations for the renormalized operators involve renormalization constants. This fact seems to be closely connected with the well-known difficulty of divergences in the theory of quantized fields. Moreover, it is not so suited to the systems the Hamiltonians of which we do not know *a priori*. It requires us from the beginning to have a complete knowledge about the system. Unfortunately, we have no substitute for it at present. However, some recent attempts, namely Low equations,[1] dispersion relations[2] and Lehmann-Symanzik-Zimmermann equations,[3] have different features from those of the Hamilton formalism in these respects. They are already renormalized, and they do not determine the solutions uniquely because they require us, at first, only an incomplete knowledge about the system. Even if they could not replace the conventional theory, a further study of their new features may give us some insight to those of future complete theory. Among them the Low equation is comparatively easier to investigate these questions, and will be studied in § 1.

It was first shown by Castillejo, Dalitz and Dyson[4] in the charged and neutral theories that the Low equation, at least in the one-meson approximation, does not determine the scattering uniquely. Later investigations have gradually revealed the meaning of the so-called redundant solutions. Haag[5] attempted to choose one solution by means of Levinson's relation[6] between the phase shift and the number of

* A preliminary report of the present work has been published in Prog. Theor. Phys. **20** (1958), 979.

bound states. He also showed that all but the simplest solution lead to eigenstates of the total Hamiltonian that are not orthogonal. On the other hand, Dyson[7] presented an example in which all the solutions of the Low equation can be accepted as physical ones. Haag's argument can be reconciled with that of Dyson if one admits the existence of discrete-spectrum eigenstates for the free Hamiltonian, as noticed by Haag himself* and explicitly shown by Fairlie and Polkinghorne.[8]

In § 1 we take up the Low equation for the $N-\theta$ scattering in Dyson's model and study how to select a unique solution. We shall see that the difference between the phase shifts for zero and infinite momenta of an incident θ-particle and the number of bound states** (or stable V-particle states) determine the form of the solution. Next, renormalized coupling constants for bound and resonant states are given and the condition to prohibit the appearance of ghost state is shown, without using renormalization constants. In § 2 the relation between the phase shift and the number of bound states is studied more generally. Under some assumptions the relation is proved using Jauch's operator method[9] that

$$\delta(\infty)-\delta(0)=(m_0-m)\pi,$$

where $\delta(p)$ is the scattering phase shift for a given angular momentum, m and m_0 are the numbers of discrete-spectrum eigenstates of H and H_0 for the same angular momentum in the center-of-mass system. For $m_0=0$ the relation reduces to the theorem obtained by Levinson.[6),9] In § 3 resonances are classified into two types, positive and negative, then we have the relation

$$\delta(\infty)-\delta(0)=(r_p-r_n)\pi,$$

where r_p and r_n are the numbers of the resonances of respective types. Two types of resonances are shown to have contrasting properties in some respects.

§ 1. Dyson's model and the Low equation

To study the characters of the Low equation mentioned above we take Dyson's model[7] as an example, because this model has probably the simplest theoretical structure in quantum field theories. The Hamiltonian for this model is given by

$$H=H_0+H_1,$$

$$H_0=M_N\psi_N^\dagger\psi_N+\sum_{l=1}^{m_0}M_{V_l}\psi_{V_l}^\dagger\psi_{V_l}+\int d^3k\,\omega_k a_k^\dagger a_k, \tag{1·1}$$

$$H_1=\frac{1}{(2\pi)^{3/2}}\int d^3k\frac{v(k)}{\sqrt{2\omega_k}}(Qa_k+Q^\dagger a_k^\dagger),$$

where ψ_N and ψ_{V_l} are annihilation operators of N- and V_l-particles, a_k is that of

* See the note added in proof of reference 5).

** The term "bound state" is often used in the sense of a discrete-spectrum eigenstate of the total Hamiltonian in the center-of-mass system, although not so appropriately.

the θ-particle with momentum $\boldsymbol{k}$. Other notations are

$$\omega_k=(\mu^2+k^2)^{1/2},$$

$$Q=\sum_{i=1}^{m_0} g_i\psi_{V_i}^{\dagger}\psi_N,$$

and $v(k)$ is the cut-off factor normalized to unity for $k=0$. The Low equation for the $N-\theta$ scattering is

$$\frac{f(p)}{pv^2(p)}=\sum_{i=1}^{m}\frac{\lambda_{B_i}}{\omega_{B_i}-\omega_p}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega_q}{qv^2(q)}\frac{|f(q)|^2}{\omega_q-\omega_p-i\epsilon}, \tag{1·2}$$

where

$$f(p)=e^{i\delta(p)}\sin\delta(p), \tag{1·3}$$

$$\lambda_{B_i}=\frac{1}{4\pi}|<B_i|Q|N>|^2>0, \tag{1·4}$$

and

$$H|B_i>=(M_N+\omega_{B_i})|B_i>$$

with $\omega_{B_i}<\mu$. Apart from the original Schrödinger equation, we take the Low equation (1·2) as a starting point.

We first consider the function of a complex variable z defined by

$$h(z)=\sum_{i=1}^{m}\frac{\lambda_{B_i}}{\omega_{B_i}-z}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega_q}{qv^2(q)}\frac{|f(q)|^2}{\omega_q-z}. \tag{1·5}$$

It is a meromorphic function of z in the complex z plane with a cut along the real axis from μ to $+\infty$. From (1·2), (1·3) and (1·5) it follows that

$$\lim_{z\to\omega_p\pm i\epsilon}h(z)=e^{\pm i\delta(p)}\sin\delta(p)/pv^2(p). \tag{1·6}$$

As is well known, $h(z)$ and the function

$$H(z)=-1/h(z) \tag{1·7}$$

are generalized R-functions. According to the Herglotz theorem,[10),4)] the general form of $H(z)$ is

$$H(z)=Az+B+\sum_{l=1}^{m}\frac{R_l}{\omega_l-z}+\frac{1}{\pi}\int_{\mu}^{\infty}d\omega_q\, qv^2(q)\frac{1}{\omega_q-z}, \tag{1·8}$$

where A, B, R_l, ω_l are all real and $A\geqq 0$, $R_l>0$ (if $n\neq 0$). Although n may be mathematically infinite, we restrict ourselves to cases of finite n from the viewpoint of physical interest. The bound-state energy and the effective coupling to the $N-\theta$ system of the i-th bound state B_i are given by

$$H(\omega_{B_i})=0, \tag{1·9}$$

$$H'(\omega_{B_i}) = 1/\lambda_{B_i}. \tag{1·10}$$

When $v^2(q)$ is of the order of $1/q^{1+s}$, $s>0$, as q approaches to infinity, the integral on the right-hand side of (1·8) converges for arbitrary z not on the cut. Comparing the behavior of the two functions, $h(z)$ and $H(z)$, for infinitely large z, we see that A must not vanish, hence it is positive.

Next, we must determine the number n of (simple) poles of $H(z)$ in the whole z plane from physical quantities. For this purpose we consider the function

$$F(z) = H(z)/z, \tag{1·11}$$

which is meromorphic in the cut z plane, and

$$F(z) = A + o(1), \tag{1·12}$$

$$F'(z)/F(z) = o(1/z), \tag{1·13}$$

when z approaches infinity in any direction away from the real positive axis. We also see from (1·6), (1·7) and (1·11) that

$$\lim_{z\to\omega_p\pm i\epsilon} F(z) = -(pv^2(p)/\omega_p)(e^{\mp i\delta(p)}/\sin\delta(p)). \tag{1·14}$$

Integrating $F'(z)/F(z)$ over z along the curve indicated in Fig. 1, we get by the Cauchy theorem

$$\int_C \frac{F'(z)}{F(z)}\,dz = 2\pi i(m-n_1-1), \tag{1·15}$$

where n_1 is the number of the poles of $H(z)$ in the interval $-\infty<z<\mu$. The contribution from the infinite circle vanishes because of (1·13). When $H(\mu)$ does not vanish, we get from (1·14)

$$Im\int_C \frac{F'(z)}{F(z)}\,dz = -2\{\delta(\infty)-\delta(0)\}+2\pi n_2, \tag{1·16}$$

where n_2 denotes the number of the poles of $H(z)$ in the interval $\mu\leqq z<+\infty$. Thus we have obtained a relation

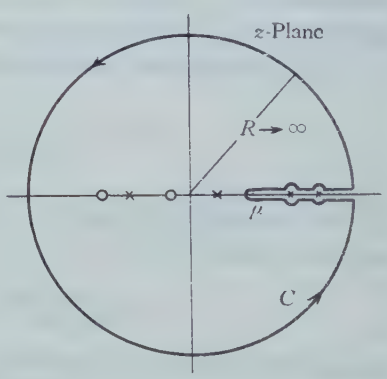

Fig. 1. The zeros (○) and the poles (×) of $H(z)$ and the integration path C for $F'(z)/F(z)$.

$$\delta(\infty)-\delta(0) = (n+1-m)\pi. \tag{1·17}$$

Here $n=n_1+n_2$ denotes the number of the poles of $H(z)$ in the whole z plane. When $H(\mu)$ vanishes, we get in a similar manner

$$\delta(\infty)-\delta(0) = (n+\tfrac{1}{2}-m)\pi. \tag{1·18}$$

Resonance occurs at zero kinetic energy in this case. For simplicity we do not consider this case throughout the present paper. The relation (1·17) means that Levinson's theorem does not hold for any solution of the Low equation for this

model. We only mention here that $n+1$ is equal to the number m_0 of bare V-particles.

The number of poles of $H(z)$ determines the form of the solutions of the Low equation, but there yet remain $2(n+1)$ parameters A, B, R_l, ω_l $(l=1, 2, \cdots, n)$ to be determined. We have $2m$ relations (1·9) and (1·10) for bound states. Similar relations for resonant states are necessary. Corresponding to the relations for bound states we define the resonant energy and the effective coupling to the $N-\theta$ system of the j-th resonant state as follows:

$$Re\,H(\omega_{R_j})=0, \tag{1·19}$$

$$\frac{d}{d\omega}Re\,H(\omega)|_{\omega=\omega_{R_j}}=\frac{1}{\lambda_{R_j}}. \tag{1·20}$$

$2(n+1)$ constants are then determined by giving ω_{B_i}, λ_{R_i} $(i=1, \cdots, m)$ and ω_{R_j}, λ_{R_j} $(j=1, \cdots, n+1-m)$. In this definition for resonant energies the scattering phase shift takes the value of half-integral multiple of π at these energies. In Lee's unstable case $(m=n=0)$ these definitions coincide with those given by Araki *et al.*[11] in terms of the K-matrix. They have also been obtained by Duimio.[12] Near a resonant energy, $H(\omega)$ takes an approximate form

$$H(\omega)\sim\frac{1}{\lambda_R}(\omega-\omega_R)+ip_Rv^2(p_R),$$

and so the width of the resonance is given by

$$\Gamma=2\lambda_R p_R v^2(p_R). \tag{1·21}$$

That the parameters A and R_l's thus determined must be positive gives restrictions for λ_B's and λ_R's. As an example, consider Lee's stable case $(m=1, n=0)$.[13] From (1·8) and (1·10) we can show that

$$A=-\frac{1}{\lambda_B}-J'(\omega_B)>0, \tag{1·22}$$

where $J(z)$ stands for the last term on the right-hand side of (1·8). Since both λ_B and $J'(\omega_B)$ must be positive, we get, as the critical value for λ_B,

$$\lambda_B^{crit}=\frac{1}{J'(\omega_B)}=\left[\frac{1}{\pi}\int_{\mu}^{\infty}\frac{qv^2(q)d\omega_q}{(\omega_q-\omega_B)^2}\right]^{-1}. \tag{1·23}$$

This coincides with the value obtained from the requirement that the renormalization constant Z_2 must be non-negative. In our method we can dispense with the renormalization constant. The appearance of the ghost state[14] is excluded by the condition that given parameters should assure the properties which any solution of the Low equation for this model must have.

§ 2. Generalization of Levinson's theorem

Jauch[9] has considerably enlarged the range of applicability of Levinson's theorem with the operator method. Now, following his method we attempt to generalize this theorem a little further.*

We firstly assume that the total Hamiltonian H can be separated into two parts

$$H=H_0+H_1, \tag{2·1}$$

so that no scattering occurs by the free Hamiltonian H_0 alone and that two sets of all the eigenstates of H and of H_0 form the complete orthogonal systems of the same Hilbert space. Among the states belonging to the discrete spectra of H in the center-of-mass system** those which have the corresponding discrete-spectrum states of H_0 will be called elementary particles and those which are not elementary will be called composite ones. We consider the scattering problem of two elementary particles in a given channel (angular momentum, iso-spin, etc.). We secondly assume that the two colliding particles need no renormalization and, moreover, do not induce any inelastic scattering. This is an unpleasant assumption, but it is satisfied for potential scattering or the $N-\theta$ scattering in Dyson's model. We thirdly assume a restriction for the contribution from the intermediate states of extremely high energy, which assures the uniform convergence of the integral (A·8) appearing in Appendix A.

We denote the continuous and discrete orthonormal eigenstates of H_0 by Φ_q and Φ_s, which satisfy

$$\begin{aligned} H_0\Phi_q&=E_q\Phi_q, \\ H_0\Phi_s&=E_s\Phi_s, \end{aligned} \tag{2·2}$$

and those of H by Ψ_q and Ψ_σ, which satisfy

$$\begin{aligned} H\Psi_q&=E_q\Psi_q, \\ H\Psi_\sigma&=E_\sigma\Psi_\sigma. \end{aligned} \tag{2·3}$$

The wave operator[15] Ω is defined as follows:

$$\begin{aligned} \langle\Phi_{q'}|\Omega|\Phi_q\rangle&=\langle\Phi_{q'}|\Psi_q^{(+)}\rangle, \\ \langle\Phi_{s'}|\Omega|\Phi_q\rangle&=\langle\Phi_{s'}|\Psi_q^{(+)}\rangle, \end{aligned} \tag{2·4}$$

where $\Psi_q^{(+)}$'s are outgoing scattering states, and

$$\langle\Phi_{q'}|\Omega|\Phi_s\rangle=\langle\Phi_{s'}|\Omega|\Phi_s\rangle=0. \tag{2·5}$$

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* After the completion of this work the author was informed by Dr. H. Kita that Polkinghorne independently obtained the relation (2·18) of this section using the same method: J. C. Polkinghorne, Proc. Cambr. Phil. Soc. **54** (1958), 560.

** In this section the center-of-mass system is used throughout and we omit the reference to this system hereafter.

The above definition (2·5) is rather artificial, but is convenient for our present purpose. From (2·4) and (2·5) we see that

$$\Omega\Omega^{\dagger}=1-P, \tag{2·6}$$

and

$$\Omega^{\dagger}\Omega=1-P_0, \tag{2·7}$$

where P and P_0 are the projection operators on the subspaces spanned by the discrete-spectrum eigenstates of H and H_0, respectively. Then we get the operator equation

$$[\Omega,\ \Omega^{\dagger}]=P_0-P. \tag{2·8}$$

With the use of the scattering wave-operator T defined by

$$T=\Omega-1, \tag{2·9}$$

the equation (2·8) can be rewritten as

$$[T,\ T^{\dagger}]=P_0-P. \tag{2·10}$$

Quantities such as the angular momentum and iso-spin commute with H and H_0, and so with Ω. Then we can separate the total Hilbert space into the subspaces spanned by the simultaneous eigenstates of these quantities. For clarity we restrict ourselves to the case of particles without ordinary and isotopic spin, then the channel index α is specified by the two quantum numbers l and $m(-l\leqq m\leqq l)$ of the orbital angular momentum. Taking the trace of both sides of eq. (2·10) in a given channel with $\alpha=(l,\ m)$, we get

$$\mathrm{Tr}([T,\ T^{\dagger}])=\mathrm{Tr}(P_0-P)=m_0-m, \tag{2·11}$$

where m and m_0 are the numbers of discrete-spectrum states of H and H_0 in the given channel, respectively. For simplicity m and m_0 are assumed to be finite. On the other hand,

$$\begin{aligned}\mathrm{Tr}([T,\ T^{\dagger}]) &= \iint \frac{1}{kk'}\delta(k-k')k^2\,dk\,k'^2\,dk'\,\langle\Phi_{k'lm}|[T,\ T^{\dagger}]|\Phi_{klm}\rangle \\ &\quad+\sum_{s,s'}\delta_{ss'}\langle\Phi_{s'lm}|[T,\ T^{\dagger}]|\Phi_{slm}\rangle \\ &=\int_0^{\infty}k^2\,dk\int_0^{\infty}\delta(k-k')\,dk'\,\langle\Phi_{k'lm}|[T,\ T^{\dagger}]|\Phi_{klm}\rangle \\ &\quad+\sum_{s=1}^{m_0}\langle\Phi_{slm}|[T,\ T^{\dagger}]|\Phi_{slm}\rangle.\end{aligned} \tag{2·12}$$

Hereafter we write simply $|k\rangle$ and $|s\rangle$ for Φ_{klm} and Φ_{slm}. Inserting the complete orthonormal set of states $\Phi_{k'l}$ and $\Phi_{s'l}$ between T and $T^{\dagger}$, we obtain after some calculations

$$\langle k'|[T,\ T^\dagger]|k\rangle=\int_0^\infty k''^2 dk''\{\langle k'|T|k''\rangle\langle k|T|k''\rangle^*-\langle k''|T|k'\rangle^*\langle k''|T|k\rangle\}$$

$$-\sum_{s''=1}^{m_0}\langle s''|T|k'\rangle^*\langle s''|T|k\rangle, \tag{2·13}$$

$$\langle s'|[T,\ T^\dagger]|s\rangle=\int_0^\infty k''^2 dk''\langle s'|T|k''\rangle\langle s|T|k''\rangle^*. \tag{2·14}$$

From (2·12), (2·13) and (2·14) follows

$$\mathrm{Tr}([T,\ T^\dagger])=\int_0^\infty k^2 dk\int_0^\infty \delta(k-k')dk'\{\langle k'|T|k''\rangle\langle k|T|k''\rangle^*-\langle k''|T|k'\rangle^*\langle k''|T|k\rangle\}. \tag{2·15}$$

We notice that in the above equation the contributions from the discrete-spectrum states of H_0 have disappeared completely. Now we need the relation between the wave operator and the phase shift, which is given below:

$$\langle k'|\Omega|k\rangle=\langle k'|1|k\rangle+\langle k'|T|k\rangle=\frac{1}{kk'}\delta(k-k')+\frac{1}{E'-E+i\epsilon}\langle k'|R|k\rangle, \tag{2·16}$$

and the diagonal element of the R-matrix is connected with the phase shift by the relation

$$f(k)\equiv e^{i\delta(k)}\sin\delta(k)=-\pi\frac{k^2}{dE/dk}\langle k|R|k\rangle. \tag{2·17}$$

The right-hand side of eq. (2·15) has the same form as in the case of potential scattering. For convenience of reference the method of its calculation used by Jauch[9] will be given with slight modifications in Appendix A. As is shown there, we finally obtain the relation

$$\delta(\infty)-\delta(0)=(m_0-m)\pi. \tag{2·18}$$

For potential scattering m_0 equals zero and we come back to Levinson's theorem.[16] For the $N-\theta$ scattering in Dyson's model m_0 is equal to the number of bare V-particles. The relation obtained excludes a possibility of "composite Lee model", which regards the V-particle not as an elementary but as a composite one of N- and θ-particles, and yet gives the same phase shift for the $N-\theta$ scattering as those given by Lee's model.*

§ 3. Positive and negative resonances

In § 2 we have obtained the relation between the phase shift and the number of bound states. Now we show the relation between the phase shift and the

* The investigation into this possibility was one of the motives for the present work.

number of resonant states. We classify resonances into two types, positive and negative, according to the sign of $d\delta(p)/dp$ at the resonant energies. If we denote the number of resonances of respective types in a given channel by r_p and r_n, then we get

$$\delta(\infty)-\delta(0)=(r_p-r_n)\pi, \tag{3·1}$$

under the elastic approximation and the assumption that the scattering cross section becomes sufficiently small for high energy. (Otherwise $\delta(p)$ does not approach an integral multiple of π as p goes to infinity.)

From (2·18) and (3·1) it follows that

$$m_0=m+r_p-r_n. \tag{3·2}$$

This relation suggests us that a discrete spectrum of H_0 goes over either to a discrete-spectrum state of H (elementary particle) or to a positive resonant state and that a negative resonance always appears either with a discrete-spectrum state of H (composite particle) or with a positive resonant state, when the strength of the interaction increases gradually from zero to the experimental value. For potential scattering the number of the negative resonances in a channel is not smaller than that of the bound states in the same channel. But this is no more the case for field theoretical scattering. Negative resonances appear more seldom than in the former case. As an example of a negative resonance in meson scattering, we take the static charged scalar meson theory.[4]

When we denote the phase shifts for $\pi^{\pm}-p$ scatterings by $\delta_{\pm}(p)$, then the Low equation for this theory is, in one-meson approximation, given by

$$\frac{f_{\pm}(p)}{pv^2(p)}=\pm\frac{\lambda_N}{\omega_p}+\sum_i\frac{\lambda_{B_i}}{\omega_{B_i}\mp\varepsilon_{B_i}\omega_p}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega_q}{qv^2(q)}\left\{\frac{|f_{\pm}(q)|^2}{\omega_q-\omega_p-i\epsilon}+\frac{|f_{\mp}(q)|^2}{\omega_q+\omega_p}\right\}, \tag{3·3}$$

where

$$f_{\pm}(p)=e^{i\delta_{\pm}(p)}\sin\delta_{\pm}(p), \tag{3·4}$$

and $\varepsilon_{B_i}=+1$, -1 for states B_i with charge 2 or 0, respectively. The functions defined by

$$h_{\pm}(z)=\pm\lambda_N+\sum_i\frac{\lambda_{B_i}z}{\omega_{B_i}\mp\varepsilon_{B_i}z}+\frac{z}{\pi}\int_{\mu}^{\infty}\frac{d\omega_q}{qv^2(q)}\left\{\frac{|f_{\pm}(q)|^2}{\omega_q-z}+\frac{|f_{\mp}(q)|^2}{\omega_q+z}\right\} \tag{3·5}$$

are generalized R-functions. So are the functions defined by

$$H_{\pm}(z)=-1/h_{\pm}(z), \tag{3·6}$$

and they can be expressed in the following form:

$$H_{\pm}(z)=\mp\left[\frac{1}{\lambda_N}-R(\mp z)\right]+\frac{2z}{\pi}\int_{\mu}^{\infty}\frac{qv^2(q)\,d\omega_q}{\omega_q(\omega_q^2-z^2)}, \tag{3·7}$$

where $R(z)$ is an R-function and vanishes for $z=0$. Owing to the crossing symmetry

$$H_+(-z)=-H_-(z), \tag{3·8}$$

we can restrict our considerations to the right-half of the complex z plane. The bound-state energy ω_{B_i} and the coupling-strength λ_{B_i} are given by

$$\begin{aligned} &H_\pm(\omega_{B_i})=0, \\ &H_\pm'(\omega_{B_i})=1/\lambda_{B_i}\omega_{B_i}, \end{aligned} \tag{3·9}$$

and the resonant energy ω_{R_j} and the coupling-strength λ_{R_j} by

$$\begin{aligned} &Re\, H_\pm(\omega_{R_j})=0, \\ &\frac{d}{d\omega} Re\, H_\pm(\omega)\,|_{\omega=\omega_{R_j}}=\frac{1}{\lambda_{R_j}\omega_{R_j}}. \end{aligned} \tag{3·10}$$

λ_{B_i} is always positive, but λ_{R_j} is positive or negative according to the type of the resonance. For $R(z)\equiv 0$ and $v(q)\equiv 1$ we obtain Serber-Lee's solution

$$H_+(z)=-\frac{1}{\lambda_N}+\frac{\mu+i\sqrt{z^2-\mu^2}}{z}. \tag{3·11}$$

When λ_N is larger than unity, a negative resonance appears in company with a bound state and we see that $\omega_R=\lambda_N\mu$ and $\lambda_R=-\lambda_N(<0)$. The cross section for π^+-p scattering

$$\sigma_+(p)=\frac{4\pi}{p^2}\frac{(\lambda_N p)^2}{(\omega_p-\lambda_N\mu)^2+(\lambda_N p)^2} \tag{3·12}$$

takes the typical form of resonance scattering.

Next we shall show that resonances of the two types have mutually contrasting properties in some respects.

a) Analytic properties of the S-matrix

We consider the S-matrix in a channel as a function of the complex energy ω by analytic continuation indicated in Fig. 2. For a positive resonance there usually appears a zero of $S(\omega)$ in the upper half and a pole in the lower half of the ω plane.[17),11)] The width of the resonance is given by the distance between them. It may be difficult to prove in general cases that $S(\omega)$ has neither a zero nor a pole near the resonant energy ($|\omega-\omega_R|\lesssim\Gamma/2$) of a negative resonance, but we shall give a plausible reason for it in Appendix B. Exactly soluble examples in the problem of potential scattering, the exponential, Hulthén and Eckart potentials,[17)]

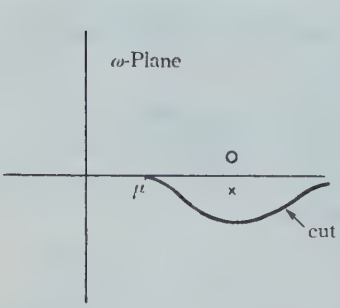

Fig. 2. The zero (○) and the pole (×) of the analytically continued S-matrix, $S(\omega)$, for a positive resonance.

and also Serber-Lee's solution mentioned above support this conjecture.

b) Retardation and advance of a wave packet by resonances

In non-relativistic cases we have a probability amplitude called wave function, which depends on the position of a particle and develops with time according to the Schrödinger equation. In relativistic cases, however, it is not an easy task to have a physical picture in configuration space. There exist many questions to be solved. Nevertheless, we tentatively assume here the existence of the amplitude which is related to the position probability density, when a wave packet of the particle is far from the target. As is well known, a positive resonance retards the motion of a packet by $1/\Gamma$, where Γ is the width of the resonance. A negative resonance, on the contrary, advances the motion of a packet by $1/\Gamma$, but the time of advance is usually too small to be observed. (See the next subsection.)

c) Causality condition and negative resonance

When a target has comparatively distinct edge, causality condition requires that the time of advance of the motion of a packet should not exceed $2a/v=2a\omega/p$,[18] where a denotes the radius of the target and v the mean velocity of the incident packet. In the case of nucleon target a is supposed to be about the Compton wave length of the π-meson, so we get the inequality $1/\Gamma \lesssim 2\omega/\mu p$ or

$$\Gamma \gtrsim \mu p/2\omega. \tag{3·13}$$

That is, the width of a negative resonance has a lower bound in order to be consistent with the requirement of causality. In the charged scalar theory, substituting $\Gamma/2=|\lambda_R|p_R$ into (3·13), we have

$$|\lambda_R| \gtrsim \mu/4\omega_R. \tag{3·14}$$

The negative resonance appearing in Serber-Lee's solution with $\lambda_N>1$ satisfies the inequality (3·14) because of $|\lambda_R|=\lambda_N>1$. In the static P-wave theory we get in a similar manner

$$|\lambda_R| \gtrsim \mu^3/4p_R{}^2\omega_R. \tag{3·15}$$

For a positive resonance there is no such restriction.

d) Composite particle and negative resonance

A negative resonance appears in company with a composite particle, which may be stable or unstable.

§ 4. Discussion

We have seen that the non-uniqueness of the solutions of the Low equation arises from the fact that the Low equation involves little knowledge about the free Hamiltonian of the system to be studied. We do not have any fundamental principle to choose a unique free Hamiltonian. But the structure of its energy spectra is restricted if we know the scattering phases and the numbers of the bound states. Such restriction may be related to the problem of the elementarity of, say, the π-meson.

The second assumption we used for the proof of the relation (2·18) (that is, two colliding particles need no renormalization and, moreover, do not induce any inelastic scattering) is very restrictive. Although some examples suggest that the relation with appropriate modification has a wider range of applicability, further investigations must be done before anything definite could be said. Yet, it may be somewhat interesting to apply the relation to the nucleon-nucleon scattering problem. The experimental ${}^3S_1$ phase shift for $n-p$ collision implies that $\delta(\infty)-\delta(0)=-\pi$, provided no appreciable change occurs for this phase shift at higher energies. Then, from (2·18) the free Hamiltonian has no discrete spectrum in the ${}^3S_1$ state, so the deuteron must be a composite particle. A defect in this argument is that the quantity $\delta(\infty)$ cannot be determined experimentally without any theoretical assumptions. However, that the deuteron state accompanies a negative resonance near the threshold favours its composite nature.

In meson-nucleon scattering we get from (2·18)

$$\delta_{IJ}(\infty)-\delta_{IJ}(0)=0$$

for all the states, provided that there are no discrete-spectrum states of H_0 other than the bare nucleon states and that inelastic scatterings do not appreciably change the behaviors of phase shifts. In particular, the $3/2-3/2$ P-wave phase shift, $\delta_{33}(p)$, must approach $\delta_{33}(0)$, which is usually taken to be zero, as p approaches to infinity. Experimentally it reaches $\pi/2$ at about 180 MeV of incident meson kinetic energy and continues to increase at least up to 400 MeV. Some possibilities to explain this apparent discrepancy are: 1) The relation (2·18) needs modification for the meson-nucleon scattering. For instance, inelastic scatterings will become appreciable in the BeV region. 2) The experimental phase shift turns to decrease at higher energies, and passing through a negative resonance approaches zero. The so-called third maximum in the π^+-p cross section may correspond to this negative resonance. 3) Discrete-spectrum states of H_0 other than the bare nucleon states must be introduced.

I should like to express my sincere thanks to Professor T. Inoue for his continual encouragement. I am also much indebted to the members of Yukawa Laboratory for their interesting discusstions, especially to Mr. N. Nakanishi for his valuable comments.

Appendix A. Derivation of equation (2·18)

We follow Jauch's argument with slight modifications. Using the definition for the R-matrix,

$$\langle k'|T|k\rangle=\frac{1}{E-E'+i\epsilon}\langle k'|R|k\rangle, \qquad (A\cdot 1)$$

we get

$$\int_0^\infty k''^2 dk'' \left\{ \langle k'|T|k''\rangle \langle k|T|k''\rangle^* - \langle k''|T|k'\rangle^* \langle k''|T|k\rangle \right\}$$

$$= \frac{1}{E'-E-i(\epsilon+\epsilon')} \int_0^\infty k''^2 dk'' \left\{ \frac{1}{E''-E'+i\epsilon'} - \frac{1}{E''-E-i\epsilon} \right\}$$

$$\times \left\{ \langle k'|R|k''\rangle \langle k|R|k''\rangle^* - \langle k''|R|k'\rangle^* \langle k''|R|k\rangle \right\} \qquad (A\cdot 2)$$

$$= \frac{1}{E'-E-i\epsilon} \left\{ D(E',\ E) + iA(E',\ E) \right\}, \qquad (A\cdot 3)$$

$D(E',\ E)$ and $A(E',\ E)$ being defined by

$$D(E',\ E) = \int_0^\infty k''^2 dk'' \left\{ \frac{\mathcal{P}}{E''-E'} - \frac{\mathcal{P}}{E''-E} \right\} F(E',\ E;\ E''), \qquad (A\cdot 4)$$

$$A(E',\ E) = -\pi \int_0^\infty k''^2 dk'' \left\{ \delta(E''-E') + \delta(E''-E) \right\} F(E',\ E;\ E''), \qquad (A\cdot 5)$$

and

$$F(E',\ E;\ E'') = \langle k'|R|k''\rangle \langle k|R|k''\rangle^* - \langle k''|R|k'\rangle^* \langle k''|R|k\rangle. \qquad (A\cdot 6)$$

From $D(E,\ E)=0$ follows

$$\int_0^\infty k^2 dk \int_0^\infty \delta(k-k') \frac{D(E',\ E)}{E'-E-i\epsilon} dk' = \int_0^\infty k^2 dk \left[\frac{\partial}{\partial E'} E(E',\ E) \right]_{E'=E}. \qquad (A\cdot 7)$$

Assuming the uniform convergence of the integral

$$\int_0^\infty k''^2 dk'' \left[\frac{\partial}{\partial E'} \left\{ \left(\frac{\mathcal{P}}{E''-E'} - \frac{\mathcal{P}}{E''-E} \right) F(E',\ E;\ E'') \right\} \right]_{E'=E}$$

$$= \int_0^\infty k''^2 dk'' \frac{1}{2} \left\{ \frac{1}{(E''-E+i\epsilon)^2} + \frac{1}{(E''-E-i\epsilon)^2} \right\} F(E;\ E''), \qquad (A\cdot 8)$$

where $F(E;\ E'')$ stands for $F(E,\ E;\ E'')$, we can equate it to $\left[\frac{\partial}{\partial E'} D(E',\ E) \right]_{E'=E}$. Then we get

$$\int_0^\infty k^2 dk \left[\frac{\partial}{\partial E'} D(E',\ E) \right]_{E'=E}$$

$$= \int_0^\infty k^2 dk \int_0^\infty k''^2 dk'' \frac{1}{2} \left\{ \frac{1}{(E''-E+i\epsilon)^2} + \frac{1}{(E''-E-i\epsilon)^2} \right\} F(E;\ E''). \qquad (A\cdot 9)$$

Because of the assumed uniform convergency of the integral (A·8) we can exchange the order of the two integrals in eq. (A·9), and the expression (A·9) vanishes on account of the antisymmetry of $F(E; E'')$. Next, from (A·5), (A·6) and (2·17) follows

$$A(E', E)=\langle k|R|k'\rangle^*\{f(k')-f(k)\}-\{f(k')^*-f(k)^*\}\langle k'|R|k\rangle. \quad (A\cdot 10)$$

Then we have

$$\begin{aligned} i\int_0^\infty k^2dk\int_0^\infty \delta(k-k')\frac{A(E', E)}{E'-E-i\epsilon}dk' &= -\frac{i}{\pi}\int_0^\infty \{f(k)^*f'(k)-f'(k)^*f(k)\}dk \\ &= \frac{2}{\pi}\int_0^\infty \sin^2\delta(k)\frac{d\delta(k)}{dk}dk \\ &= \frac{1}{\pi}\int_0^\infty \frac{d_l\rho(k)}{dk}dk \\ &= \frac{1}{\pi}\{\rho(\infty)-\rho(0)\}, \end{aligned} \quad (A\cdot 11)$$

where $\rho(k)=\delta(k)-\frac{1}{2}\sin 2\delta(k)$. We have thus obtained the relation

$$m_0-m=\mathrm{Tr}([T, T^\dagger])=(1/\pi)\{\rho(\infty)-\rho(0)\}, \quad (A\cdot 12)$$

which is equivalent to the relation (2·18). To show this we put $\delta(0)=\alpha$ and $\delta(\infty)=(m_0-m)\pi+\alpha'$, and obtain from (A·12)

$$\alpha'-\alpha=\tfrac{1}{2}(\sin 2\alpha'-\sin 2\alpha)=\sin(\alpha'-\alpha)\cos(\alpha'+\alpha).$$

From this we see that $|\alpha'-\alpha|\leqq|\sin(\alpha'-\alpha)|$, hence $\alpha'-\alpha=0$. Q. E. D.

Appendix B. Resonances and singularities of the S-matrix

Here we shall give a plausible reason for that the analytically continued S-matrix, $S(\omega)$, has no pole near the resonant energy ($|\omega-\omega_R|\lesssim\Gamma/2$) of a negative resonance. Consider the case in which a resonance occurs at $\omega=\omega_R$ and $S(\omega)$ has a zero at $\omega=\omega_1+i\omega_2$ and a pole at $\omega=\omega_1-i\omega_2$ in the cut ω plane indicated in Fig. 2. We know that ω_2 must be positive from the causality condition. We now assume that $\omega_1\sim\omega_R$ and $\omega_1\gg\omega_2$. When we separate the zero and the pole of $S(\omega)$ out of it, it can be written as follows:

$$S(\omega)=e^{2i\alpha(\omega)}\frac{\omega-(\omega_1+i\omega_2)}{\omega-(\omega_1-i\omega_2)}. \quad (B\cdot 1)$$

If $S(\omega)$ has neither zeros nor poles other than $\omega=\omega_1\pm i\omega_2$ near $\omega=\omega_R$, then $\alpha(\omega)$ may be considered as a slowly varying function of ω. That is, the variation of $\alpha(\omega)$ in the interval of $|\omega-\omega_1|\lesssim\omega_2$ may be regarded to be much smaller than unity. Differentiating the logarithm of $S(\omega)$ by ω, we get

$$\frac{d\delta(\omega)}{d\omega}=\frac{d\alpha(\omega)}{d\omega}+\frac{\omega_2}{(\omega-\omega_1)^2+\omega_2^2}. \qquad \text{(B·2)}$$

The first term on the right-hand side of the above equation is negligible compared with the second term. Therefore, $d\delta(\omega)/d\omega$ is positive near $\omega=\omega_1\sim\omega_R$, and so the resonance must be of positive type.

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Note added in proof Mr. H. Araki kindly informed the author of Dr. Haag's comment that Schwinger (unpublished) had proved the relation (2·18) for one-channel cases in a different way.

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On the Transformation Properties of the Dirac Equation

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In connection with two types of 3-dimensional rotation groups, which are termed in this paper as the helicity and the pseudo-helicity group respectively, a synthesis and generalization of the Foldy-Wouthuysen-Tani transformation and the Cini-Toushek transformation for the Dirac Hamiltonian is discussed. One is thus enabled to comprehend in one scope the extreme-non-relativistic and the extreme-relativisitic limits of the behavior of the free Dirac particles. Moreover, the parity-conserving and the parity-non-conserving interactions of the V-A form are derived on the basis of the principle of invariance of the interaction Hamiltonian for the continuous group of transformations corresponding to the extreme-relativistic case.

§ 1. Introduction

A spin 1/2 particle in interactions with various types of external fields is described by a four-component spinor ψ satisfying the Dirac type equation. For various purposes it is of interest to convert this equation into a two-component equation of the Pauli-type or the Weyl-type. It is shown by Foldy and Wouthuysen[1] and by Tani[2] that, by means of a canonical transformation** of the equation, a representation is found in which the Hamiltonian contains an even Dirac matrix alone. By means of a similar canonical transformation,*** another representation is found by Cini and Toushek,[3] where the Hamiltonian contains an odd Dirac matrix alone. The former representation is very convenient to interpret the physical quantities in conformity with the classical dynamics. The latter representation seems to be useful to contemplate the extreme-relativistic limit, where the value of the mass m can be taken to tend to zero. In both cases, the Dirac equations split into two uncoupled equations of the Pauli-type or the Weyl-type. It is worth-while to note that in the former limit the Hamiltonian is isotropic in the ordinary 3-dimensional space and in the latter limit the Hamiltonian has a kind of polarity. Needless to say, the usual Dirac Hamiltonian has no such properties. The usual Dirac Hamiltonian is thus contemplated to be in a kind of intermediary state between both limits. In these connections, it seems to be very important to throw a light upon the transformation properties of the Dirac equation. In the present paper

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** This will be referred to as F-W-T.

*** This will be referred to as C-T.

we introduce two kinds of 3-dimensional rotation groups and with the aid of them reformulate in a synthetical fashion the F-W-T and the C-T transformations, which shed respectively a light on the extreme-non-relativistic and extreme-relativistic limits of the Dirac equation.

Especially, we are interested in the extreme-relativistic limit in connection with the V-A universality of weak interactions. The recent experimental evidences on the asymmetry in the angular distribution of electron emitted from polarized nuclei or muons, on the electron polarization, on the nuclear recoil, and the ρ value of μ-decays tend to indicate that in both β- and μ-decays the four-fermion interactions are of the same strength of couplings and of the same V-A form. A theoretical basis for this V-A form of interaction has recently been suggested by many authors.[4] The vectorical nature of the weak Fermi- and electromagnetic-interactions suggests that the nature might probably be favourable to the V-A form of couplings. And then, the couplings of the strong interactions may also be expected to be of the V-A form.*

§ 2. Helicity group and pseudo-helicity group

First of all, let us introduce the following two kinds of 3-dimensional rotation groups.

i) The helicity rotation group.

We define:

$$K_1 \equiv -i\gamma_4\gamma_5\alpha_p\Omega = i\gamma_4\sigma_p\Omega, \tag{1a}$$

$$K_2 \equiv \gamma_4\Omega, \tag{1b}$$

$$K_3 \equiv -\gamma_5\alpha_p = \sigma_p, \tag{1c}$$

where K_3 is the usual helicity operator and K_2 the parity operator for the spin 1/2 Dirac field,[5] and σ_p and α_p are defined with $p=|\boldsymbol{p}|$ by

$$\sigma_p = (\boldsymbol{\sigma}\cdot\boldsymbol{p})/p, \quad \alpha_p = (\boldsymbol{\alpha}\cdot\boldsymbol{p})/p, \tag{2}$$

so that

$$\sigma_p^2 = \alpha_p^2 = 1, \tag{3}$$

and the operator $\Omega(=\Omega^{-1}=\Omega^T=\bar{\Omega})$ induces the transformation

$$\Omega\boldsymbol{p}\Omega = -\boldsymbol{p}. \tag{4}$$

Then,

$$K_1^2 = K_2^2 = K_3^2 = 1, \tag{5}$$

and there hold the commutation relations:

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* The author is indebted to Prof. S. Hayakawa for showing him, after completion of the present research, the preprint of the work due to S. K. Bose, A. Gamba and E. C. G. Sudarshan. The subject treated is similar but the intention and the method differ from the present one.

$$[K_i, K_j]_+=0 \quad \text{for} \quad i\neq j, \ (i, j=1, 2, 3) \tag{6}$$

$$K_1K_2=iK_3, \quad K_2K_3=iK_1, \quad K_3K_1=iK_2. \tag{7}$$

Thus, it is obvious that these operators generate a 3-dimensional rotation group.

ii) The pseudo-helicity rotation group.

Similarly to the above case, we define:

$$J_1\equiv -i\epsilon\sigma_p\gamma_4\Omega, \tag{8a}$$

$$J_2\equiv\gamma_4\Omega=K_2, \tag{8b}$$

$$J_3\equiv -\epsilon\sigma_p, \tag{8c}$$

where

$$\epsilon=\frac{m\beta+(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{E}, \tag{9}$$

$$\epsilon^2=1, \quad E^2=m^2+p^2, \tag{10}$$

and

$$[J_2, \epsilon]_-=0, \tag{11}$$

then, these operators satisfy the following relations:

$$J_1^2=J_2^2=J_3^2=1, \tag{12}$$

and

$$[J_i, J_k]_+=0 \quad \text{for} \quad i\neq k, \ (i, k=1, 2, 3) \tag{13}$$

$$J_1J_2=iJ_3, \quad J_2J_3=iJ_1, \quad J_3J_1=iJ_2. \tag{14}$$

Thus, it is clear that these operators also generate a 3-dimensional rotation group. It is to be noted that the K-rotation group is not independent of the J-rotation group, since

$$[K_i, J_k]_-\neq 0 \quad \text{for} \quad i\neq k, \tag{15}$$

and also that the J_3 is a kind of chirality operator, since

$$[J_3, K_2]_+=0.$$

Now, we consider the usual Dirac Hamiltonian

$$H=m\beta+(\boldsymbol{\alpha}\cdot\boldsymbol{p}), \tag{16}$$

then, we find immediately

$$[H, K_i]_-=0, \tag{17}$$

and

$$[H, J_k]_-=0, \tag{18}$$

i.e., the Hamiltonian is invariant under the rotations and the inversions in the J- and/or K-rotation groups. It is well known that K_3 is used to characterize eigenfunctions. And J_3 is very convenient to characterize eigenfunctions in connection

with the extreme-relativistic limit. (See Table I). In the following, it will be found that K_i is very useful in discussing the transformation properties of the Dirac equation, and that J_i is very convenient in contemplating the extreme-relativistic limit of the Dirac equation.

Now, as is usually done, we shall expand the field operator in the form:

$$\psi(\boldsymbol{r}, t)=V^{-1/2}\sum_{\boldsymbol{k},\alpha,\beta}\{a_\alpha(\boldsymbol{k})\,u_\alpha(\boldsymbol{k})\exp[i\boldsymbol{k}\cdot\boldsymbol{r}-iEt]+\bar{b}_\beta(\boldsymbol{k})\,u_\beta(-\boldsymbol{k})\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}+iEt]\},$$
$$(\alpha=1,\ 2 \text{ and } \beta=3,\ 4) \tag{19}*$$

where $\boldsymbol{k}$ is an eigenvalue of $\boldsymbol{p}$. We obtain the following expressions for K_3 and J_3:

$$\int(\psi,\ K_3\psi)\,dV=\sum_{\boldsymbol{k}}[A_1(\boldsymbol{k})-A_2(\boldsymbol{k})+B_3(\boldsymbol{k})-B_4(\boldsymbol{k})], \tag{20a}$$

and

$$\int(\psi,\ J_3\psi)\,dV=\sum_{\boldsymbol{k}}[-A_1(\boldsymbol{k})+A_2(\boldsymbol{k})+B_3(\boldsymbol{k})-B_4(\boldsymbol{k})], \tag{20b}$$

where

$$A_\alpha(\boldsymbol{k})=\bar{a}_\alpha(\boldsymbol{k})\,a_\alpha(\boldsymbol{k}),$$
$$B_\beta(\boldsymbol{k})=\bar{b}_\beta(\boldsymbol{k})\,b_\beta(\boldsymbol{k}). \tag{21}$$

Here, we investigate the behaviour of J_i in the extreme-non-relativistic and -relativistic limits. First, in the extreme-non-relativistic limit, when the momentum $\boldsymbol{p}$ tends to zero, the Hamiltonian tends to

$$H\to m\beta. \tag{22}$$

And then the characteristic feature of these rotation groups is destroyed, excepting that $K_2=J_2$ only survives as a good quantum number. This suggests that in this limit the Hamiltonian is perfectly isotropic in the ordinary 3-dimensional space.

Secondly, in the extreme-relativistic limit, (when the mass m tends to zero), the Hamiltonian tends to

$$H\to(\boldsymbol{\alpha}\cdot\boldsymbol{p}), \tag{23}$$

and the J_i tends to a new 3-dimensional rotation group isomorphic to the original one:

$$J_1\to L_1\equiv-i\gamma_4\gamma_5\Omega, \tag{24a}$$
$$J_2\to L_2\equiv\gamma_4\Omega, \tag{24b}$$
$$J_3\to L_3\equiv\gamma_5. \tag{24c}$$

In contrast with the former case, the particle has a kind of polarity in its propagating direction and a kind of chirality. This L_i-rotation group is of interest in

* The bar means the hermitian conjugate.

connection with the so-called V-A universality of interaction forms. This problem will be discussed in the later section.

In this case, we obtain the following expressions, analogous to J_3:

$$\int (\psi_L,\ L_3\psi_L)\,dV=\sum_k [-A_{L1}(\boldsymbol{k})+A_{L2}(\boldsymbol{k})+B_{L3}(\boldsymbol{k})-B_{L4}(\boldsymbol{k})], \tag{25a}$$

where

$$A_{L\alpha}(\boldsymbol{k})=\bar{a}_{L\alpha}(\boldsymbol{k})\,a_{L\alpha}(\boldsymbol{k}),$$

$$B_{L\beta}(\boldsymbol{k})=\bar{b}_{L\beta}(\boldsymbol{k})\,b_{L\beta}(\boldsymbol{k}). \tag{25b}$$

Table I. The adopted assignment for the u's corresponding to the sign of ϵ, K_3, J_3 and L_3.

	ϵ	K_3	J_3	L_3
$u_1(\boldsymbol{k})$	+	+	−	−
$u_2(\boldsymbol{k})$	+	−	+	+
$u_3(\boldsymbol{k})$	−	+	+	+
$u_4(\boldsymbol{k})$	−	−	−	−

From eqs. (24c), (25a) and Table I, we see immediately that the third component of J_i perfectly corresponds to the one of L_i.

§ 3. The deduction of new representations for the Dirac equation

As we have seen in the previous section, K_2, J_2 and L_2 have the same form in any case. By reference to the helicity rotation group, this property is found to be useful to investigate the transition feature from the usual Dirac equation to the both limits.

Now, we consider the operators which commute with K_i:

$$[O_\rho,\ K_i]_-=0. \tag{26}$$

Let us take such operators that satisfy the above condition as follows:

$$O_\rho;\ \boldsymbol{1},\ i\beta,\ i\alpha_p,\ i\epsilon,\ i\beta\alpha_p,\ \beta\epsilon,\ \alpha_p\epsilon,\ \text{and}\ i\beta\alpha_p\epsilon. \tag{27}$$

Furthermore, we define a set of canonical transformations made of K_2 and O_ρ:

$$T_\rho\equiv\frac{1+K_2}{2}+\frac{1-K_2}{2}O_\rho, \tag{28}$$

$$T_\rho'\equiv\frac{1-K_2}{2}+\frac{1+K_2}{2}O_\rho, \tag{29}$$

where T_ρ and T_ρ' satisfy the relations:

$$\bar{T}_\rho T_\rho=1,\quad \bar{T}_\rho=T_\rho^{-1}, \tag{30}$$

and

$$\overline{T}_p' T_p' = 1, \quad \overline{T}_p' = T_p'^{-1}. \tag{31}$$

Then, we find immediately

$$T_p' T_p = T_p T_p' = O_p, \tag{32}$$

and

$$T_p + T_p' = 1 + O_p. \tag{33}$$

Moreover, taking

$$U_p \equiv (1+O_p)/\sqrt{N_p}, \tag{34}$$

then, we obtain

$$\overline{U}_p U_p = 1, \quad \overline{U}_p = U_p^{-1}, \tag{35}$$

where N_p's are normalization constants.

For example, we take

$$O_f = \beta\epsilon, \tag{36}$$

then, we obtain

$$U_f = (1+\beta\epsilon)/\sqrt{N_f}, \tag{37}$$

where

$$N_f^{-1} = E/2(E+m). \tag{38}$$

Eq. (37) can be rewritten as follows:

$$U_f = \exp[\tfrac{1}{2}\beta\alpha_p \tan^{-1} p/m]. \tag{39}$$

Such U_f is the so-called F-W-T transformation for the spin 1/2 Dirac field. Furthermore, we take

$$O_c = \alpha_p \epsilon, \tag{40}$$

then we obtain

$$U_c = (1+\alpha_p \epsilon)/\sqrt{N_c}, \tag{41}$$

where

$$N_c^{-1} = E/2(E+p). \tag{42}$$

Eq. (41) can also be rewritten as

$$U_c = \exp[-\tfrac{1}{2}\beta\alpha_p \tan^{-1} m/p]. \tag{43}$$

This is the very C-T canonical transformation which has recently been found by Cini-Touschek. It is well known that the former transformation converts the usual Dirac Hamiltonian to

$$\begin{aligned} H_f &= U_f H U_f^{-1} \\ &= \beta E, \end{aligned} \tag{44}$$

and by the latter transformation to

$$H_c = U_c H U_c^{-1}$$
$$= \alpha_p E. \tag{45}$$

Now, it is of interest to consider the following canonical transformation, if we choose O_p as

$$O = i_l \beta \alpha_p \epsilon, \tag{46}$$

we obtain

$$U = (1 + i_l \beta \alpha_p \epsilon)/\sqrt{N}, \tag{47}$$

where $N=2$.
Then, the usual Dirac Hamiltonian is transformed to:

$$H' = UHU^{-1}$$
$$= i\beta \alpha_p E. \tag{48}$$

This transformed Hamiltonian is invariant under the K-rotation group, but not invariant under the J- and L-rotation groups. Starting from eq. (48), we obtain the generalized F-W-T and C-T transformations in the following way. First, we take

$$O_1 = i\alpha_p, \tag{49}$$

and

$$U_1 = (1 + i\alpha_p)/\sqrt{N_1}, \tag{50}$$

where $N_1=2$. By this transformation eq. (48) is transformed to

$$H_1 = U_1 H' U_1^{-1}$$
$$= \beta E. \tag{51}$$

Thus, we obtain the resulting transformation:

$$U_{f1} = \tfrac{1}{2}[(1+\beta\epsilon) + i\alpha_p(1-\beta\epsilon)]. \tag{52}$$

Excepting the normalization constants, the first parenthesized term in eq. (52) is the usual F-W-T transformation, and the second one is another F-W-T transformation. Eq. (52) can also be rewritten as

$$U_{f1} = \exp[+i/2\,\beta \tan^{-1} p/m] \exp[1/2\,\beta\alpha_p \tan^{-1} p/m]. \tag{53}$$

Secondly, we take

$$O_2 = -i_l \beta, \tag{54}$$

and

$$U_2 = (1 - i\beta)/\sqrt{N_2}, \tag{55}$$

where $N_2=2$. By this transformation, eq. (48) is converted to:

$$H_2=U_2H'U_2^{-1}$$
$$=\alpha_pE. \tag{56}$$

Hence, we obtain the resulting transformation:

$$U_{c2}=\tfrac{1}{2}[(1+\alpha_p\epsilon)-i\beta(1-\alpha_p\epsilon)]. \tag{57}$$

Similarly to the former case, the above first parenthesized term in eq. (57) corresponds to the usual C-T-transformation and the remaining term to another one. Eq. (57) can be rewritten as

$$U_{c2}=\exp[-i/2\cdot\alpha_p\tan^{-1}m/p]\exp[-1/2\,\beta\alpha_p\tan^{-1}m/p]. \tag{58}$$

Thus, it is to be noted that the usual F-W-T and C-T transformations are not sufficiently general.

From eqs. (53) and (58), we see that the transformed dynamical variables are better convergent in higher orders than those by the old- transformations. This fact is of importance for seeing the behaviour of the transitions from the usual Dirac case to the extreme-non-relativistic and extreme-relativistic limits.

Now, by eq. (53) the position operator, for example, becomes in new representation:

$$\boldsymbol{x}'=\boldsymbol{x}_1-\frac{i\beta\boldsymbol{\alpha}_1}{2E}+\frac{i\beta(\boldsymbol{\alpha}_1\cdot\boldsymbol{p})\boldsymbol{p}-(\boldsymbol{\sigma}\times\boldsymbol{p})E}{2E^2(E+m)}, \tag{59}$$

where

$$\boldsymbol{\alpha}_1=\frac{m}{E}\boldsymbol{\alpha}+i\frac{\boldsymbol{p}}{E}\beta\boldsymbol{\alpha}, \tag{60}$$

and

$$\boldsymbol{x}_1=\boldsymbol{x}+\frac{m\boldsymbol{p}}{2E^2p}\beta. \tag{61}$$

Other operators in this new representation are immediately obtained by replacing $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}_1$ in the corresponding operators in the old representation. (See Table I

Table II. Table of operator-representatives of Dirac operators in usual and new representations.

Usual repesentation	New representation
$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}'=\frac{\boldsymbol{p}}{E}\boldsymbol{\alpha}+\frac{m}{Ep}(\boldsymbol{\sigma}\times\boldsymbol{p})-\frac{m}{Ep}\boldsymbol{p}\beta_2$
β	$\beta'=\frac{p}{E}\beta_2+\frac{m(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{Ep}$
$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}'=\boldsymbol{\sigma}_2-\frac{im\beta_2(\boldsymbol{\alpha}_2\times\boldsymbol{p})}{Ep}-\frac{m^2(\boldsymbol{p}\times(\boldsymbol{\sigma}_2\times\boldsymbol{p}))}{E(E+p)p^2}$
γ_5	$\gamma_5'=\frac{-(\boldsymbol{\alpha}\cdot\boldsymbol{p})+\beta_2m}{E}\cdot\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{p}$

of F-W). We add γ_5' to the Table I of F-W:[1]

$$\gamma_5' = -\frac{m(\alpha_1\cdot\boldsymbol{p})+p^2\beta}{Ep}\cdot\frac{(\sigma\cdot\boldsymbol{p})}{p} \tag{62}$$

Similarly, the dynamical variables transformed by the new C-T transformation are tabulated in Tables II and III.

In Tables II and III, the following abbreviations are used:

$$\beta_2 \equiv \frac{p}{E}\beta+\frac{im}{Ep}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p}), \tag{63}$$

$$\alpha_2 \equiv \frac{p}{E}\alpha+\frac{m}{Ep}(\sigma\times\boldsymbol{p})+\frac{m^2(\alpha\cdot\boldsymbol{p})\boldsymbol{p}}{p^2(E+p)E}, \tag{64}$$

$$\boldsymbol{x}_2 \equiv \boldsymbol{x}-\frac{m}{2pE}\alpha+\frac{m(E+p)^2(\alpha\cdot\boldsymbol{p})\boldsymbol{p}-m^2pE(\sigma\times\boldsymbol{p})}{2p^3E^2(E+p)}. \tag{65}$$

The corresponding operators in the old C-T representation are obtained by replac

Table III. Table of operator-representatives of dynamical variables in usual and new representations.

Usual representation	New representation
position	
$\boldsymbol{x}$	$\boldsymbol{x}'=\boldsymbol{x}_2+\frac{im\beta_2\boldsymbol{\alpha}_2}{2Ep}-\frac{im(E+p)\beta_2(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}}{2p^3E^2}-\frac{m^2(\boldsymbol{\sigma}_2\times\boldsymbol{p})}{2E(E+p)p^2}$
Angular momentum	
$(\boldsymbol{x}\times\boldsymbol{p})$	$(\boldsymbol{x}'\times\boldsymbol{p})$
Mean position	
$\boldsymbol{X}=\boldsymbol{x}_2-\frac{im\beta_2\boldsymbol{\alpha}_2}{2Ep}+\frac{im(E+p)\beta_2(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}}{2p^3E^2}-\frac{m^2(\boldsymbol{\sigma}_2\times\boldsymbol{p})}{2E(E+p)p^2}$	$\boldsymbol{X}'=\boldsymbol{x}$
Mean velocity	
$\dot{\boldsymbol{X}}=\frac{E}{p}\boldsymbol{\alpha}_2-\frac{m\beta_2\boldsymbol{p}}{Ep^2}-\frac{m^2(m^2+Ep)(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}}{E^2(E+p)p^3}$	$\dot{\boldsymbol{X}}'=\frac{E}{p}\boldsymbol{\alpha}_2-\frac{m^2(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}}{Ep^3}$ $=\boldsymbol{\alpha}+\frac{m}{p}\cdot\frac{(\boldsymbol{\sigma}\times\boldsymbol{p})}{p}-\frac{m^2}{p^2}\cdot\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}}{E(E+p)}$
Mean orbital angular momentum	
$(\boldsymbol{X}\times\boldsymbol{p})$	$(\boldsymbol{x}\times\boldsymbol{p})$
Mean spin angular momentum	
$\Sigma=\boldsymbol{\sigma}_2+\frac{im\beta_2(\boldsymbol{\alpha}_2\times\boldsymbol{p})}{Ep}-\frac{m^2(\boldsymbol{p}\times(\boldsymbol{\sigma}_2\times\boldsymbol{p}))}{p^2E(E+p)}$	$\Sigma'=\sigma$
$\Lambda=\frac{\beta m+(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{E}$	$\Lambda'=\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{p}$

ing β_2, α_2 and x_2 with β, α, and x respectively in Tables II and III.

In these representations, the J-rotation group is converted to K' and/or L', respectively:

$$K_1' \equiv U_{(\,)} J_1 U_{(\,)}^{-1} = -\beta K_1, \tag{66a}$$

$$K_2' \equiv U_{(\,)} J_2 U_{(\,)}^{-1} = K_2, \tag{66b}$$

$$K_3' \equiv U_{(\,)} J_3 U_{(\,)}^{-1} = -\beta K_3, \tag{66c}$$

and/or

$$L_1' \equiv U_{(\,)} J_1 U_{(\,)}^{-1} = -i\gamma_4\gamma_5\Omega, \tag{67a}$$

$$L_2' \equiv U_{(\,)} J_2 U_{(\,)}^{-1} = \gamma_4\Omega, \tag{67b}$$

$$L_3' \equiv U_{(\,)} J_3 U_{(\,)}^{-1} = \gamma_5. \tag{67c}$$

Here $U_{(\,)}$, for example, means the U_f or U_{f1} and/or U_c or U_{c2}. Though this L'-rotation group, in new representation, has the same form as the L-rotation group, in the extreme-relativistic limit, the former must be distinguished from the latter from the field theoretical points of view. The former approaches to the latter in powers of m/p.

The corresponding dynamical variables in the K'-representation is obtained from the ones in the L'-representation by the following canonical transformation:

$$U_0 O_{L'} U_0^{-1} = O_{K'}, \tag{68}$$

where

$$U_0 = \tfrac{1}{2}[1 + i(\beta + \alpha_p) + \beta\alpha_p]. \tag{69}$$

§ 4. Discussions of J_3, L_3 and the V-A form of interactions

As pointed out in section 2, J_3 and L_3 have the similar properties when characterizing the eigenfunctions. (See Table I). Furthermore, it is to indicate that there exist the following similarities. We define:

$$\psi_{J\pm} = \frac{1 \pm J_3}{2}\psi_J, \tag{70}$$

and

$$\psi_{L\pm} = \frac{1 \pm L_3}{2}\psi_L, \tag{71}$$

then, we immediately see

$$J_2\psi_{J\pm} = \psi'_{J\mp}, \tag{72}$$

$$L_2\psi_{L\pm} = \psi'_{L\mp}. \tag{73}$$

Eq. (72) shows that J_2 plays a role of chiral conjugation as well as in the L-representation. In both representations, the Dirac Hamiltonians split into two

uncoupled parts, describing particles in chiral conjugate states respectively. It is worth-while to note that, even in the J-representation, the Dirac spinor is separable into two uncoupled parity-conjugate two component spinors.

Moreover, we obtain

$$\psi'_{J\pm}=e^{iJ_3\alpha}\psi_{J\pm}, \tag{74}$$

and

$$\psi'_{L\pm}=e^{iL_3\beta}\psi_{L\pm} \tag{75}$$

under the rotations about the third axis in the space J_i and L_i. These transformations correspond to the gauge transformations:

$$\psi_J'=e^{\pm i\alpha}\psi_J, \tag{76}$$

and

$$\psi_L'=e^{\pm i\beta}\psi_L \tag{77}$$

expressed in terms of the original four component spinors.

In addition, there exist the following Pauli transformations[6] in each representation. In the J-representation:

$$\text{a)}\quad \psi_J'=e^{iJ_3\alpha}\psi_J\text{: one parameter group,} \tag{78}$$

$$\text{b)}\quad \psi_J'=a\psi_J+bJ_3\psi_J^c\text{: three parameters group.} \tag{79}$$

In the L-representation:

$$\text{a')}\quad \psi_L'=e^{iL_3\beta}\psi_L, \tag{80}$$

$$\text{b')}\quad \psi_L'=a\psi_L+bL_3\psi_L^c, \tag{81}$$

$$|a|^2+|b|^2=1, \tag{82}$$

here a and b are complex numbers and $\psi_J^c=C^{-1}\overline{\psi}_J$, $C^{-1}\gamma_\mu^TC=-\gamma_\mu$ and $C^{-1}J_3^TC=J_3$.

The transformations (a') and (b') are the so-called Pauli transformations. For example, the transformation (b) can be rewritten in a matrix form:

$$\begin{pmatrix} a & bJ_3 \\ -b^*J_3 & a^* \end{pmatrix}=\begin{pmatrix} a_1+ia_2 & (b_1+ib_2)J_2 \\ -(b_1-ib_2)J_3 & a_1-ia_2 \end{pmatrix}=a_1E+ib_2J_3\tau_1+ib_1J_3\tau_2+ia_2\tau_3, \tag{83}$$

where E and τ_i are the 2×2 unit and Pauli matrices respectively, and a_1, a_2, b_1, b_2 are real numbers. Hence, the arguments concerning the well-known isomerism hold in both representations.

Here it must be alluded to that there exist the characteristic differences between the J-and L-rotation groups in connection with the V-A form of interactions established recently. They are:

i) The J-rotation group is commutable with the usual free Hamiltonian, but the L-rotation group is not commutable with it, except when the mass m equals to zero.

ii) By the transformation in the J-rotation group, interactions are necessarily cast into non-covariant and non-local forms, and on the other hand, by the transformation in the L-rotation group, interactions are necessarily restricted to the V-A form. In view of these facts, it is to be noted that the usual free Hamiltonian is invariant for the J-rotation group but not for the L-rotation group, while the interaction Hamiltonian of the V-A form is invariant for the L-rotation group but not for the J-rotation group. As has been seen in the previous section, it is to be pointed out that only when $\boldsymbol{L}' \simeq \boldsymbol{L}$, i. e. $m/p \ll 1$, the total Hamiltonian is approximately invariant for the L-rotation group. The requirement of the chiral invariance, by Sudarshan and Marshak, may be valid only in the above sense.

At present, we hope that the interactions of elementary particles can be deduced from two forms of interactions:

i) Strong and electromagnetic interactions:

These are approximately invariant under the rotations and inversion in the L-rotation group.

ii) Weak interactions:

These are approximately invariant under the rotation and inversions with respect to the third component of the L-rotation group.

These circumstances are analogous to that the charge independence symmetry about strong interactions is destroyed by the presence of electromagnetic interactions. Also, it may be alluded to that weak interactions contain the projection operator $(1+L_3)/2$, i. e. $(1+\gamma_5)/2$, analogously to the case of electromagnetic interactions, where the current contains the projection operator $(1+\tau_3)/2$.

§ 5. Conclusion

The transformation properties of the Dirac equation have been discussed from various points of view. Two kinds of 3-dimensional rotation groups, K_i and J_i, have been introduced. And it has been found that the former group is very useful to reformulate the new F-W-T and C-T transformations. From discussion of these canonical transformations, it has been pointed out that two kinds of the transformed Hamiltonians have the contrasting properties and the usual Dirac Hamiltonian is in the intermediary state between the extreme-non-relativistic and -relativistic limits. Especially, it is stressed that the transformations, which make the Hamiltonian an odd matrix, are of importance for contemplating the extreme-relativistic limit of the usual Dirac equation and at the same time the transformation properties with respect to the J-rotation group.

Moreover, in the extreme-relativistic approximation ($p \gg m$), it is of interest to investigate the behaviour of the Dirac field in interactions with various types of external fields. Finally, the similarities and the differences between the J-and L-rotation groups, and a kind of conjecture about the origin of the V-A form of interactions have been discussed, though in a somewhat qualitative fashion. It is

to be noted that the J- and L-rotation groups have the complementary property for the total Hamiltonian of the system.

The interpretation of operators and dynamical variables in new representations is left for a later communication.

Acknowledgment

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

A Note on the Fundamental Symmetry of Elementary Particles

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December 22, 1958

Recent experiments[1] seem to support the hypothesis[2] that the type of weak interactions is universally a $V-A$ combination, not only in the reactions where neutrino takes part but also in the others. On the other hand, the so-called T-invariance (or PC-invariance) of the law of nature may also be considered to hold at least in the zeroth approximation.[3] Experiments have already shown us many characteristic features of the interactions of elementary particles, and we may try to unify such features and give a clear picture for deeper understanding of them. In this note, we shall try to form a conjecture upon a more symmetrical world from the two characteristic features mentioned above.

The bilinear products of two spinor fields, $\bar{\psi}_a O_i \psi_b$, are transformed by the time-reversal into $\epsilon_i \bar{\psi}_b O_i \psi_a$ in the rule of Schwinger,[4] where the matrices O_i's are such that $\bar{\psi}_a O_i \psi_b$ transforms either like a scalar, pseudo-scalar, vector, pseudo-vector, or like a tensor. The constants ϵ_i's are sign factors and listed in Table I.[5] We can at once see from the table that *the only possible Yukawa type interaction, which involves both parity conserving and parity non-conserving interactions and at the same time is invariant under the time reversal, is a combination of vector and axial-vector interactions if* $\psi_a=\psi_b$. Further the transformation changes the Fermi interaction $(\bar{\psi}_a O_i \psi_b)(\bar{\psi}_c O_i \psi_d)$ into $(\bar{\psi}_b O_i \psi_a)(\bar{\psi}_d O_i \psi_c)$, and the pseudo-scalar quantity $(\bar{\psi}_a O_i \psi_b)(\bar{\psi}_c \gamma_5 O_i \psi_d)$ into $\eta_i (\bar{\psi}_b O_i \psi_a)(\bar{\psi}_d \gamma_5 O_i \psi_c)$. The sign factors η_i's are also listed in Table I. We have the T-invariant and hermitian Fermi interactions,

$$g_i \{(\bar{\psi}_a O_i \psi_b)(\bar{\psi}_c O_i \psi_d) + (\bar{\psi}_b O_i \psi_a)(\bar{\psi}_d O_i \psi_c)\} \qquad (1)$$

and

Table I

O_i	1	$i\gamma_5$	$i\gamma_k$	$i\gamma_4$	$i\gamma_5\gamma_k$	$i\gamma_5\gamma_4$	$i\gamma_{[i,k]}$	$i\gamma_{[4,k]}$	$\gamma_5\gamma_{[i,k]}$	$\gamma_5\gamma_{[4,k]}$
ϵ_i	+	−	−	+	−	+	−	+	+	−
η_i	−		+		+		−		−	

$$g_i'\{(\bar{\psi}_a O_i \psi_b)(\bar{\psi}_c \gamma_5 O_i \psi_d) + \eta_i (\bar{\psi}_b O_i \psi_a)(\bar{\psi}_d \gamma_5 O_i \psi_c)\}, \quad (2)$$

for parity conserving and parity non-conserving cases, respectively. (Here the coupling constants g_i and g_i' are restricted to be real, as is well known.) Therefore, we get a similar rule as for the Yukawa type weak interactions: *only $V-A$ combination can have both properties, the T-invariance and the indefiniteness of parity, if $\psi_a=\psi_b$ and $\psi_c=\psi_d$.*

Taking into account these situations and the universality of coupling constants of Fermi's interactions, we might make a speculation about a more symmetrical world and the behaviour of its transition to our real physical world as follows. In a certain stage, we may consider such a symmetrical world in which there is no more detailed hierarchy of elementary particles except the discrimination between baryons and leptons, and that baryon and lepton numbers are conserved separately,[6] and we assume that the T-invariance holds exactly. In such symmetrical world, the concerned spinor biproducts have the characters $\psi_a=\psi_b$ and $\psi_c=\psi_d$ mentioned above. The weak interactions are first introduced in this world and there the $V-A$ combination is picked up as a *secondary* effect. Then in the next less symmetrical stage strong interactions, which give rise to the real variety of elementary particles, are added in such a way that the T-invariance is not disturbed. It may be emphasized that this image is in striking constrast to the theories[2),7)] in which new vector fields are introduced to treat the $V-A$ combination separately from (or in preference to) the T-invariance.

If such an image is to reflect the truth, in a future theory the weak interactions will play a more fundamental role than the strong ones, and the T-invariance, the discrimination between baryons and leptons and the conservation of heaviness will have to be accounted much of. At this point, it is very interesting that the time-reversal is an anti-linear transformation, and that the anti-character is due to the special role of the fourth axis in Minkowski's space or in our *quantum* field theory.[8]

Further, judging from our recent experiments,[1] there is every indication that the relative sign between A and V and the ratio of their coupling constants are nearly $V-A$ universally. Such a tendency can be understood as the result of the spirality of neutrino, when it takes part in the concerned interactions. But, concerning the other kinds of weak interactions and the specialization of the spirality, some mechanism must be deviced. In this connection, the recent work[9] of the present author would partly be helpful for this purpose. (There the type of the weak interactions is not taken into account, but it is easy to improve the theory to meet the present requirement.) Further, the spirality of neutrino and the anti-linear character of the time-reversal might be useful for understanding the situation, which brings about the parity violation, in a similar way as the character of photon defined the structure of Minkowski's space.

Before concluding this note, we wish to point out that the weak interactions

of K_1^0 with baryons, i. e. $\bar{\Lambda}\Lambda K_1^0$, $\bar{\Sigma}\Sigma K_1^0$, $\bar{N}NK_1^0$ and $\bar{\Xi}\Xi K_1^0$, have the characteristic features mentioned above, if we neglect the selection rules concerning the isotopic spin. We may consider that such interactions retain the character of the symmetrical world.

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5) In this table, i, k=1, 2, 3, and $\gamma_{[\mu,\nu]}=\frac{1}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$.
6) Such a discrimination can survive in this symmetrical world, as ϕ_a and ϕ_c couple with ϕ_b and ϕ_d respectively, in both terms of the exuations (1) and (2).
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In this occasion, the author want to add a comment to the work. In the page 589 of the article, we assumed that "$\psi^{(-)}$ has the same intrinsic parity as $\psi^{(+)}$". This assumption is automatically satisfied when the parameter β is not zero. Because $\psi^{(-)}$ has the same character as $\gamma_5\gamma_2\psi^{(+)*}$, that is, the latter satisfies the same equation as the one for $\psi^{(-)}$.

Isotope Shift and Fermi-Segré Formula

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January 22, 1959

The applicability of the Fermi-Sergè formula for the estimation of the I. S.* is considered a questionable matter,[1] because of the rough approximation used in its derivation. In this paper, we present a plausible justification for this applicability which results from the I. S. itself. For this purpose we derive formulae for the I. S. by Broch's[2] as well as by Sommerfeld's method,[3] and we identify the two final formulae.

In Broch's method we choose the variant in which the integration of Darwin-Gordon's radial wave-functions is carried out over all space outside the nucleus, and we use the wave-functions from Breit's paper.[4] Finally, we reach the following expression:

$$\frac{\delta E}{\hbar c}\approx aC_+^2\left[\frac{C_-}{C_+}\right]\frac{2\rho}{\Gamma(1+2\rho)\Gamma(1-2\rho)}, \tag{1}$$

where the constants C_+ and C_- are those involved in the linear combination of singular and nonsingular solutions of Darwin-Gordon's system, for an electron in the Coulombian unscreened field of a point nucleus. The quantities ρ and a which appear in (1) are linked with each other by the relation $\rho^2=1-a^2$, $a=Z/137$.

* We abbreviate "isotope shift" as I. S.

Now, we introduce explicitly the C_+ and C_- coefficients as determined by the condition of continuity on the nuclear surface of the interior and exterior wave-functions, and we make some approximations allowed by the inequality $y_0 = 2Zr_0/a_H \ll 1$. In this way, we obtain for an $S_{1/2}$ level the final formula

$$\frac{\delta E}{\hbar c} \approx 2\pi R_\infty \frac{a_H^3}{Z_i} \psi_n^2(0) \cdot \frac{2_l\rho}{\Gamma^2(1+2_l\rho)} \times \frac{1-(1+\rho)\zeta}{1-(1-\rho)\zeta} y_0^{2\rho} \qquad (2)$$

where ζ was determined by us by solving Darwin-Gordon's system of differential equations, within the nucleus for a homogeneous charge distribution and for heavy atoms, where the inequality $[-U(r_0)/mc^2] \gg 1$ is fulfilled as

$$\zeta(a) = -\frac{F}{aG}\Big|_{r=r_0} = \frac{2}{5}\{1+0.106233a^2 + 0.016649a^4 + 0.002788a^6 + 0.000479a^8 + \cdots\}. \qquad (3)$$

In Sommerfeld's method we consider only an undetermined constant for the electronic wave-functions outside the nucleus, but supposing a slow shift of principal quantum number from an integer. After some intricate transformations concerning the gamma and hypergeometric functions, Sommerfeld's wave functions may be obtained in the form

$$G_e \approx -(-1)^n C_e (n-2)!\, n \frac{(1+\rho)}{\Gamma(1+2\rho)} \times \left\{1 - \frac{1-\rho}{1+\rho} \frac{\Gamma^2(1+2_l\rho)}{2_l\rho} y_0^{-2\rho}\, \delta n\right\} \qquad (4)$$

$$F_e \approx +(-1)^n C_e (n-2)!\, n \frac{a^2}{\Gamma(1+2_l\rho)} \times \left\{1 - \frac{\Gamma^2(1+2_l\rho)}{2_l\rho} y_0^{-2\rho}\, \delta n\right\}.$$

The I. S. is expressed now by the formula

$$\delta E = \frac{\partial E}{\partial n} \delta n$$

where $E = -\frac{1}{2} mc^2 \left[\frac{Z_0}{137}\right]^2 \frac{1}{n_*^2}$. (5)

The amount δn is determined by the condition of continuity on the nuclear surface of the interior and exterior wave-functions. Hence, the second formula for the I. S. is derived as

$$\frac{\delta E}{\hbar c} \approx 2\pi R_\infty \frac{a_H^3}{Z_i} \left\{\frac{Z_i Z_0^2}{\pi a_H^3 n_*^3}\left(1 - \frac{d\sigma}{dn}\right)\right\} \times \frac{2_l\rho}{\Gamma^2(1+2_l\rho)} \cdot \frac{1-(1+\rho)\zeta}{1-(1-\rho)\zeta} y_0^{2\rho} \qquad (6)$$

The formulae (2) and (6) coincide with each other only if the Fermi-Segrè formula is supposed to be correct.

An attempt to verify the above formula was made by computing the nuclear radius of Hg^{200} from the I. S. of the following transitions 4057 Å ($6s6p\,^3P_0^0 \rightarrow 6s7s\,^3S_1$), 4358 Å ($6s6p\,^3P_1^0 \rightarrow 6s7s\,^3S_1$) 5461 Å ($6s6p\,^3P_2 \rightarrow 6s7s\,^3S_1$), and 4078 Å ($6s6p\,^3P_1^0 \rightarrow 6s7s\,^1S_0$). The mean value

$$r_0(Hg^{200}) = (6.27 \pm 0.12) \times 10^{-13}\ \text{cm}$$

falls in good agreement with the value $r_0 = 6{,}30.10^{-13}$ cm as determined from Coulombian scattering data[5].

We believe that the present paper scatters at least some of doubts about the Fermi-Segrè formula and is a positive contribution to the subject of I. S.

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Effect of Covalency on the Spin-Orbit Coupling Constant of Transition Metal Ions

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February 16, 1959

It is generally observed that the spin orbit coupling constant, λ, of transition metal ions is reduced from the value of free ionic state when they are in complex salts or admixed in non-magnetic crystals as impurities. This phenomenon has been explained by the effect of covalency. Owen[1] and Tinkham[2] considered that the reduction of λ is proportional to the diminution of electron charge which is accomodated in d-orbital of antibonding orbit. Tanabe and Sugano[3] investigated the overlapping effect in $[Cr(H_2O)_6]^{3+}$ complex by a molecular calculation.

Confining our attention to the type of $[M(H_2O)_6]^{n+}$ complex, where M stands for transition metal ion, we can describe the general trends of experimental results as follows: the relative reduction of λ is greater for metal ion of smaller atomic number and larger valancy. The purpose of the present paper is to explain this observed trend using the Owen-Tinkham's model.

Let ψ_n be the wave function of n-th d-orbital, which may be any one of $d\gamma$- or $d\varepsilon$-orbitals, and ϕ_n be one of ligand orbitals, comprising the wave functions of neighbouring water molecules and having the same symmetry as ψ_n. Then, antibonding orbital can be expressed by

$$\bar{\psi}_n = N_n(\psi_n - C_n\phi_n), \qquad (1)$$

where $N_n = 1/(1+C_n^2-2C_nS_n)^{-1/2}$ and S_n is the overlap integral between ψ_n and ϕ_n which is considered to be small. Though the problem is not clear-cut on account of the overlapping effect, if we share the electron charge of $\bar{\psi}_n$-orbital to ψ_n- and ϕ_n-orbitals proportionally to the ratio of the square of their coefficients in $\bar{\psi}_n$, the charge contained in each orbital can be written as in Table I, where P_n means the number of electron in ψ_n-orbital when complex is in pure ionic state. This alteration of electron charge in each orbital can be easily understood from Fig. 1, where the solid curve and arrow indicate the wave function and the electron spin belonging to the antibonding orbital, respectively, and the dotted curve and arrow those belonging to the bonding orbital. As there is the increment of the d-electron number given by

$$\Delta n_d = \sum_n (2-P_n)C_n^2/(1+C_n^2), \qquad (2)$$

the effective nuclear charge for a d-electron is reduced by the $d-d$ electron screening to the value, $Z = Z_0 - \alpha\Delta n_d$, where Z_0 is the effective nuclear charge

Table I. The electron charge in each orbital.

	antibonding orbital	bonding orbital	Total
ψ_n	$P_n/(1+C_n^2)$	$2C_n^2/(1+C_n^2)$	$(2C_n^2+P_n)/(1+C_n^2)$
ϕ_n	$P_n C_n^2/(1+C_n^2)$	$2/(1+C_n^2)$	$(2+P_n C_n^2)/(1+C_n^2)$

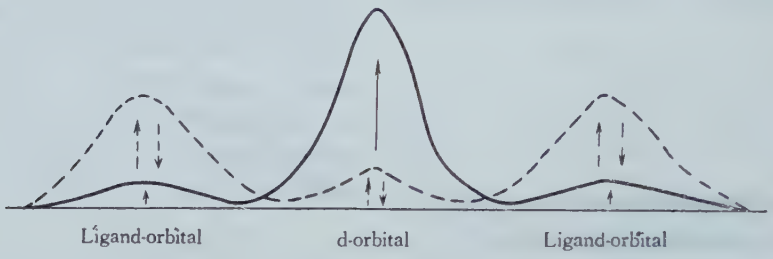

Fig. 1. Distribution of electron charge in each orbital. ($P_n=1$)

Table II. The values of λ/λ_0. (in %) $C=0.4$ and $S=0.106$ are assumed in our calculation.

	Ti^{2+}	V^{2+}	Cr^{2+}	Mn^{2+}	Fe^{2+}	Co^{2+}	Ni^{2+}	Cu^{2+}	V^{3+}	Cr^{3+}	Mn^{3+}
Δn_d	1.10	0.966	0.828	0.690	0.552	0.414	0.276	0.138	1.10	0.966	0.828
$(\lambda/\lambda_0)_{calc.}$	55	62	68	72	76	79	82	84	61	66	70
$(\lambda/\lambda_0)_{exp.}$		~80				(81)	83	84	62	63	

in free ion and α the $d-d$ screening constant. If we take a hydrogen model, this change of Z leads us to the following reduction of λ;

$$\lambda/\lambda_0 \propto [1-(\alpha \Delta n_d/Z_0)]^4, \qquad (3)$$

where λ_0 is the spin-orbit coupling constant in free ion. When the effect given by eq. (3) is combined with that proposed by Owen, the whole expression for λ/λ_0 can be obtained.

Expression (3) has the form desirable for explaining the general trend of λ/λ_0, because Δn_d is larger and Z_0 is smaller for the ions of smaller atomic number and larger valency. (cf. Table II) In order to roughly estimate the effect from eq. (3), we regard C_n and S_n as a constant independent of n and put them as C and S. Then,

$$\frac{\lambda}{\lambda_0} = \frac{1}{(1+C^2-2CS)^2} \times \left[1-\frac{\alpha}{Z_0}\frac{C^2}{1+C^2}\sum_n (2-P_n)\right]^4, \qquad (4)$$

where the first factor is that proposed by Owen, which should be proportional to N_n^4 in our approximation[3] because we compare our result with λ/λ_0 estimated from the observed g-values. For α and Z_0, we apply Slater's rule. ($\alpha=0.35$). Furthermore, we determine the value of C and S so as to make the value of λ/λ_0 of Cu^{2+} fit to the experimental value, 0.84. As a typical example, we show in Table II the case in which $C=0.4$ and $S=0.106$, which corresponds to the covalency of about 16%. In Table II, experimental values are taken from reference 1) and, for

Co^{2+}, $(\lambda/\lambda_0)_{exp.}$ is estimated from the value of E'/E[1] by noting that the value of E'/E is approximately proportional to λ/λ_0.

The agreement between experiment and theory is qualitatively satisfactory except V^{2+}. Our calculation may, however, include overestimation of the reduction of λ/λ_0 for the ions of smaller atomic number, because C and S may change so as to suppress the increase of Δn_d in those ions. When we have enough data to determine the coefficients C_n more precisely, we may be able to decide whether the electronic configuration in some compounds is $(d\varepsilon)^n(d\gamma)^m$ or $(d\varepsilon)^{n+m}$, because, in a series of similar compounds containing different metal ions, the values of λ/λ_0 in the two cases may change in a different way if there is a large difference between the values of $C_{d\varepsilon}$ and $C_{d\gamma}$.

The author thanks his colleagues at Tôhoku University for valuable discussions.

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On Quantum Theory of Transport Phenomena

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February 17, 1959

Concerning my paper[1] that has appeared in this journal under the above title, I should like first to add the following note to the part where the Einstein relation is discussed. It was Kasuya[2] who first not only pointed out, but also explicitly showed how the Einstein relation breaks down in the case of one-electron problems. I regret that my description and references were not enough to make his priority quite clear.

Secondly I should like to make a remark as regards the general formulation of the theory. Kubo et al.[3] and also Kasuya have started from a certain initial fluctuation from equilibrium and then calculated its decay by integrating the Liouville equation. In doing so, it is essential to discard the time variation of a macroscopic parameter such as the chemical potential, which is certainly much slower compared with microscopic relaxation times if the wave length of its space variation is very long Mathematically, as mentioned first by Kubo et al., the limit of infinite system and infinite wave length should be taken before integrating with respect to time. Now, in my paper in which a steady solution is directly sought for, one particular solution among others should be chosen either by taking account of the boundary condition explicitly or by considering how the steady state has been established. Actually I took the latter point of view; then the same sort of argument as mentioned above becomes necessary.[4] In view of this, two ways of approach, namely the fluctuation theory of Kubo et al. on the one hand and the kinetic theory of mine on the other are not essentially different from each other and, indeed, lead to the same result.

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4) This has been pointed out to me by Prof. R. Kubo.

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An Analysis of the Charged Particles emitted following Negative Muon Absorptions in Photographic Emulsions

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(Received December 4, 1958)

Calculations have been performed for the energy distribution of alpha particles and protons following the negative muon absorption in photographic emulsions. Three shapes, (I) Chew-Goldberger distribution, (II) Fermi gas at $kT=0$ and (III) Fermi gas at $kT=9$ MeV, are assumed for the proton momentum distributions in the nucleus and the muon absorption probability is calculated for the single particle process. Assuming that the total kinetic energy of produced neutron is imparted into the compound nucleus, we apply the statistical theory to emission processes.

With the assumption of (III) for Ag and Br, the energy distributions of emitted alpha particles have been well explained by the calculations using the statistical theory, but the rate of proton emission has come out to be too small. If it is assumed that most of produced neutrons were directly emitted, (I) may explain the energy spectrum of emitted particles. In (II), the emission rate is too small. As for the absorption probability, all three shapes show reasonable agreement with experiment.

§ 1. Introduction

The elementary process of negative muon absorption by nucleus is given by

$$\mu^- + p \rightarrow n + \nu. \tag{1}$$

When a stopped muon is absorbed by a free proton, the produced neutron has the energy of only about 5 MeV due to the energy and the momentum conservation laws. However, when the muon is absorbed in the nucleus, the produced neutron receives much higher energy, since the proton in the nucleus has a finite momentum. Therefore, the information about the proton momentum distribution in the nucleus can be obtained from the investigation of transferred energies to the nucleus or of nuclear excitations following the nuclear absorption of the muon.

There are many data concerning the nuclear absorptions of the muons[1], involving the data of the mean energies transferred to the nucleus and the emitted neutron multiplicities from the excited nucleus. But these are not very suitable for testing the proton momentum distribution, especially the feature of higher tails in the momentum distribution.

Morinaga and Fry[2] measured the energy spectra of emitted charged particles following the muon absorption in photographic emulsions. This is one of the most

sensitive experiments about the momentum distribution, since the energy spectra of the charged particles may be considerably affected by the features of the higher tails of the momentum distribution. The attempt of this paper is to obtain the knowledge about the momentum distribution of the proton in the medium weight nuclei (Ag and Br) by means of the analysis of emitted charged particles followed by the muon absorption.

For protons inside nuclei, three types of momentum distributions have been assumed;

(I)* As a typical example of momentum distribution with strong correlation, we used the Chew-Goldberger distribution[3], which was postulated by Chew and Goldberger from the analysis of high energy pick-up processes on carbon[4] in the Born approximation.

(II) We used the momentum distribution of free particles restricted only by the Pauli principle, that is the Fermi gas distribution at absolute zero temperature.

(III) The Fermi gas distribution at the finite temperature of $kT=9$ MeV was assumed in order to take into account moderate correlation effects.

If we calculate the momentum distribution on the basis of the single particle wave functions taking into account the two-body correlations by perturbation, a distribution similar to the last one is obtained[5]**. These momentum distributions are shown in Fig. 1 for Ag^{107}.

In the course of calculations it is necessary to know the mechanism of the charged particle emission. Recently, it has been established that the reactions involving alpha particles are well described by statistical theory[9]. Therefore, the energy spectra of the emitted alpha particles following the muon absorption may be well explained by the use of the statistical theory. However, it is doubtful if proton emissions may be explained by the statistical theory.

In § 2, the muon absorption probability is calculated, the single particle process (1) being assumed. In case (I), it is assumed that the produced neutron can receive the sufficiently high energy and the restriction due to the Pauli principle is omitted. In cases (II) and (III), the effect of neutron excess in the nucleus and the Pauli correlation are taken into account. The emitted charged particle spectra are calculated in § 3, assuming that the total kinetic energy of produced neutron is imparted together with the other nucleons in the nucleus by multiple collisions before it is ejected from the nucleus.

§ 2. Nuclear absorption cross sections

The interaction Hamiltonian describing the elementary process (1) is assumed

* Hereafter, (I), (II) and (III), are used for the distributions of Chew-Goldberger, Fermi gas at $kT=0$ and $kT=9$ MeV, respectively.

** Assuming the oscillator potential, we can obtain a similar distribution to (III) for the medium weight nuclei[10].

as

$$H=g\tau_{np}\,\tau_{\nu\mu}\,\delta(\boldsymbol{r}_{np}-\boldsymbol{r}_{\nu\mu})+\text{comp. conj.}, \tag{2}$$

where τ_{np} and $\tau_{\nu\mu}$ are operators which change a proton to a neutron and a muon to a neutrino, respectively, g is coupling constant, $\boldsymbol{r}_{np}$ and $\boldsymbol{r}_{\nu\mu}$ are nucleon and lepton coordinates, and the delta function indicates that a neutrino is produced at the point where a proton changes to a neutron by absorbing a muon.

Since the muon is absorbed by the proton not in flight but after the capture on to the K-orbit[7], the momentum of the absorbed muon is assumed to be small. Then the conservation law of momentum for the elementary process (1) is given by

$$\boldsymbol{p}_p=\boldsymbol{p}_n+\boldsymbol{p}_\nu, \tag{3}$$

where $\boldsymbol{p}_p$, $\boldsymbol{p}_n$ and $\boldsymbol{p}_\nu$ are the momentum of proton, neutron and neutrino, respectively.

For the calculation of the nuclear absorption probability Λ of the muon, the following approximations are used. The nuclear wave functions are given from the Hartree model, while the produced neutron is assumed to be expressed by a plane wave. The absorbed muon is expressed by the K-orbit wave function and the neutrino is assumed to be a plane wave.

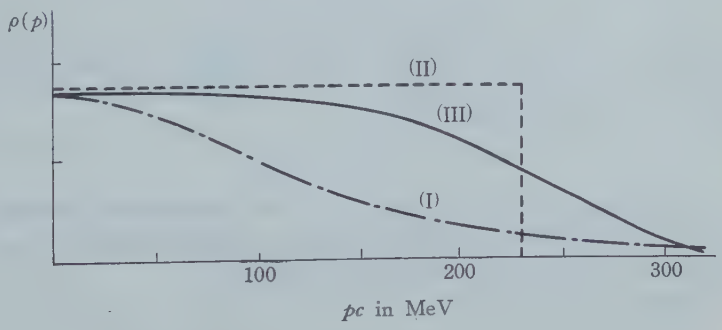

Fig. 1. Proton momentum distribution in Ag^{107}. It is normalized that $\int\rho_p(\boldsymbol{p})\,d\boldsymbol{p}=1$.

Then Λ is given by

$$\Lambda=\frac{2\pi}{\hbar}g^2\frac{4}{(2\pi\hbar)^3}\left(\frac{M_\mu e^2}{\hbar^2}\right)^3 Z_{eff}^4\int p_\nu^2\,dp_\nu\,d\boldsymbol{p}_n\,d\boldsymbol{p}_p\,\rho_p(\boldsymbol{p}_p)\,\delta(\boldsymbol{p}_p-\boldsymbol{p}_n-\boldsymbol{p}_\nu)\,\delta(E_f-E_i), \tag{4}$$

where δ-functions indicate the momentum and energy conservations. M_μ is the muon rest mass and Z_{eff} is the quantity used instead of Z (atomic number of initial nucleus), if we take into account the effect of the finite extension of nuclear charges[8]. $\rho_p(\boldsymbol{p}_p)$ is the momentum distribution of the proton in the initial nucleus and is defined by

$$Z\rho_p(\boldsymbol{p}) = \sum_{protons} \left| (2\pi\hbar)^{-3/2} \int \exp(-i\boldsymbol{pr}/\hbar)\, \varphi_p(\boldsymbol{r})\, d\boldsymbol{r} \right|^2, \tag{5}$$

where $\varphi_p(\boldsymbol{r})$ is the proton wave function.

A. Chew-Goldberger distribution

The energy conservation law in this case is given by

$$(2M)^{-1}\{p_n^2 - p_p^2\} = E_0 - p_\nu c \tag{6}$$

with

$$E_0 = M_\mu c^2 - B_\mu + B_n - B_p,$$

where B_μ, B_n and B_p are the binding energies of the muon in the K-orbit, the last neutron in the final nucleus, and the last proton in the initial nucleus.

Furthermore, the restriction due to the Pauli principle is omitted. In this approximation, one may overestimate the number of the produced neutrons with small momenta, i.e. the low energy part in the nuclear excitation functions. Hence the muon absorption probability may be overestimated. However, since the energy spectra of the emitted particles are determined esssentially by the higher tail of the momentum distribution, it is believed that the above approximation is not so bad for calculating the energy spectra of the emitted charged particles.* Thus the absorption probability is calculated by (4) with

$$\rho_p(\boldsymbol{p}) = (\hbar\alpha/\pi^2)\,(\hbar^2\alpha^2 + \boldsymbol{p}^2)^{-2}. \tag{7}$$

For the parameter α, a value obtained from the analysis of $C^{12}(n, d)\ B^{11}$ reaction at high energies[3] is adopted as follows:

$$\hbar^2\alpha^2/(2M) = 18\ \text{MeV}. \tag{8}$$

This assumption is probably not drastic since the momentum distributions showing the strong correlation between a pair of nucleons are essentially determined by the local density of nucleons but not by the nuclear size.

The result is

$$\Lambda = \frac{1}{\pi^2}\left(\frac{e^2}{\hbar c}\right)^3 \left(\frac{M_\mu}{m_e}\right)^3 \left(\frac{m_e c}{\hbar}\right)^6 \frac{g^2}{\hbar m_e c^2} \frac{Z_{eff}^4}{(m_e c^2)^2} \times \frac{\hbar\alpha c \cdot Mc^2}{\pi} \int p_\nu c\, d(p_\nu c) \left[(\hbar\alpha c)^2 + \left\{ \frac{Mc^2}{p_\nu c}(E_0 - p_\nu c - \frac{p_\nu^2}{2M}) \right\}^2 \right]^{-1}, \tag{9}$$

where m_e is the electron rest mass.

B. Fermi gas distribution at $kT=0$

The conservation of energy is given by

$$(2M)^{-1}\{(p_n^2 - Q^2) - (p_p^2 - P^2)\} = E_0 - p_\nu c. \tag{10}$$

---

* The Pauli correlation effect is estimated for the case (III). See also ref. 9).

When the effect of neutron excess is considered,* the maximum momenta of the proton and neutron are respectively given by

$$P=\left(\frac{9\pi}{4}\right)^{1/3}\frac{\hbar}{r_0}\left(\frac{Z}{A}\right)^{1/3},\quad Q=\left(\frac{9\pi}{4}\right)^{1/3}\frac{\hbar}{r_0}\left(\frac{N+1}{A}\right)^{1/3} \tag{11}$$

with

$$r_0=1.2\times10^{-13}\,\text{cm}$$

for the nucleus initially composed of Z protons and N neutrons. Furthermore, the Pauli correlation is taken into account. Then

$$\rho_p(\boldsymbol{p})=\begin{cases} 3/(4\pi P^3), & \text{for } |\boldsymbol{p}|\leq P, \\ 0, & \text{for } |\boldsymbol{p}|>P, \end{cases} \tag{12}$$

and the integration over neutron momentum $\boldsymbol{p}_n$ is restricted to $|\boldsymbol{p}_n|>Q$. In order to calculate (4), it is convenient to introduce the cylindrical coordinates, taking $\boldsymbol{p}_\nu$ as z direction. The result is

$$\begin{aligned} \Lambda=&\frac{1}{\pi^2}\left(\frac{e^2}{\hbar c}\right)^3\left(\frac{M_\mu}{m_e}\right)^3\left(\frac{m_e c}{\hbar}\right)^6\frac{g^2}{\hbar m_e c^2}\,\frac{Z_{eff}^4}{(m_e c^2)^2} \\ &\times\frac{3Mc^2}{4(Pc)^3}\int p_\nu c\,d(p_\nu c) \\ &\times\begin{cases} (Pc)^2-\left\{\dfrac{p_\nu c}{2}-\dfrac{(Qc)^2-(Pc)^2}{2p_\nu c}-\dfrac{Mc^2(E_0-p_\nu c)}{p_\nu c}\right\}^2, & \text{for } E'_{ex}\leq E_0-p_\nu c\leq E_{ex}^{max}, \\ 2Mc^2(E_0-p_\nu c), & \text{for } 0\leq E_0-p_\nu c\leq E'_{ex}, \end{cases} \end{aligned} \tag{13}$$

where

$$E_{ex}^{max}=E_0+Pc+Mc^2-\sqrt{(Mc^2)^2+(Qc)^2+2Mc^2(Pc+E_0)},$$

$$E'_{ex}=E_0-Qc-Mc^2+\sqrt{(Mc^2)^2+(Pc)^2+2Mc^2(Qc-E_0)}.$$

C. Fermi gas distribution at $kT=9$ MeV

In this case, the energy conservation law is given by (10), and the neutron excess are also considered. Taking the Pauli correlation into consideration, the function $f_n(\boldsymbol{p}_n)$ is introduced for the neutrons in (4). Strictly speaking, the temperature of the final nucleus differs from the initial one, but for the sake of simplicity the difference were omitted. It is assumed that

$$f_n(\boldsymbol{p})=1-\left[\exp\left\{\frac{1}{kT}\left(\frac{\boldsymbol{p}^2}{2M}-\mu_n\right)\right\}+1\right]^{-1}, \tag{14}$$

and

* In ref. 11) this effect has also been considered.

and

$$\rho_p(\boldsymbol{p}) = (3/4\pi N^3)\left[\exp\left\{\frac{1}{kT}\left(\frac{\boldsymbol{p}^2}{2M}-\mu_p\right)\right\}+1\right]^{-1} \tag{15}$$

where

$$N=(2M\mu_p)^{1/2}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu_p}\right)^2\right]^{1/3}.$$

μ_n and μ_p are the values of $\boldsymbol{p}^2/2M$ at the point where the values of momentum distribution reduce to one half of values at $\boldsymbol{p}=0$. They are approximately given by

$$\mu_n=\frac{Q^2}{2M}\left\{1-\frac{\pi^2}{12}\left(\frac{kT}{Q^2/2M}\right)^2\right\},\ \mu_p=\frac{P^2}{2M}\left\{1-\frac{\pi^2}{12}\left(\frac{kT}{P^2/2M}\right)^2\right\}.$$

The integration with $p_p{}^2$ in (4) has a finite lower limit $p_pc=Mc^2(p_\nu c)^{-1}\{E_0-p_\nu c+(2M)^{-1}(Q^2-P^2-p_\nu{}^2)\}$ because of the energy conservation. Then

$$\Lambda=\frac{1}{\pi^2}\left(\frac{e^2}{\hbar c}\right)^3\left(\frac{M_\mu}{m_e}\right)^3\left(\frac{m_ec}{\hbar}\right)^6\frac{g^2}{\hbar m_ec^2}\,\frac{Z_{eff}^4}{(m_ec^2)^2}$$

$$\times\frac{3Mc^2}{4(Nc)^3}\cdot 2Mc^2\,kT\int p_\nu c\,d(p_\nu c)\ \ \{I_1-I_2\}, \tag{16}$$

where

$$I_1=\log|1+\exp(\eta+u)|-(\eta+u), \tag{16a}$$

$$I_2=(e^v-e^\eta)^{-1}\{\eta e^\eta-ve^v-e^\eta\log|1+\exp(\eta+u)|+e^v\log|1+\exp(v+u)|\}-u, \tag{16b}$$

with

$$u=\frac{Mc^2}{2kT(p_\nu c)^2}\left\{E_0-p_\nu c-\frac{p_\nu{}^2}{2M}+\frac{Q^2-P^2}{2M}\right\}^2,$$

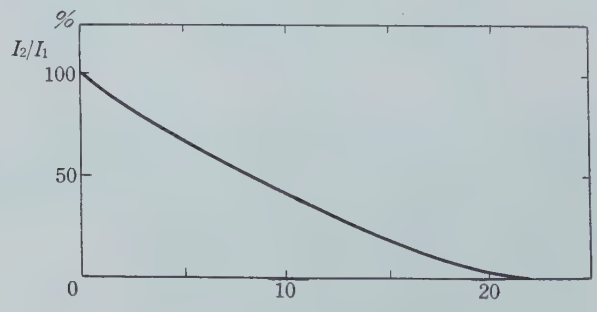

Fig. 2. The Pauli correlation effects for the nuclear excitation function in Pd^{107} for the case (III). The ordinate is I_2/I_1 of eqs. (16a) and (16b).

$$v=\frac{1}{kT}\left\{E_0-p_\nu c+\frac{Q^2-P^2}{2M}-\mu_n\right\},\ \eta=-\mu_p/kT.$$

I_2 is the term due to the Pauli correlation. The effect of the Pauli correlation for the nuclear excitation function is shown by Fig. 2 for Pd^{107}. Fig. 2 indicates that the Pauli correlation reduces the low excitation parts strongly as is expected. The estimates show, however, that the effects of the Pauli correlation is not important for the charged particle emission probabilities.

D. Numerical results

Since the reactions due to muon absorptions in Ag and Br are separated from other cases[2] and treatment using statistical theory is more valid for heavier nuclei, considerations are made only for the reactions

$$\mu^- + Ag \rightarrow Pd^* + \nu,$$

$$\mu^- + Br \rightarrow Se^* + \nu,$$

and the absorption probabilities for the above reactions are calculated using the values given in Table 1.[12]

Table 1

nucleus	B_n (MeV)	B_p (MeV)	$B_\mu{}^*$ (MeV)
Ag^{109}	6.0	6.2	4.6
Ag^{107}	6.3	5.8	4.6
Br^{81}	6.9	7.3	2.9
Br^{79}	7.2	6.4	2.9

* B_μ are determined according to ref. 13).

Table 2

	Ag^{107}		Br^{79}	
	Λ^* ($\times 10^7$ sec^{-1})	$\langle E_{ex}\rangle$ (MeV)	Λ^* ($\times 10^7$ sec^{-1})	$\langle E_{ex}\rangle$ (MeV)
(I) Chew-Goldberger	0.91	20.6	0.53	21.1
(II) Fermi gas ($kT=0$)	0.30	11.7	0.21	11.6
(III) Fermi gas ($kT=9$ MeV)	0.42	12.0	0.30	12.0
Experimental Results[14]	1.04			

* The values of Λ are estimated for $g=1\times10^{-49}$ erg cm^3.

The universal Fermi interactions being presupposed, the coupling constant g is the strength of about 10^{-49} erg cm^3, then g is taken to be 1×10^{-49} erg cm^3, and Z_{eff} are the same as those adopted by Wheeler[8]. The calculated result of the nuclear excitation function following muon absorption by Ag^{107} is shown in Fig. 3.

The calculated muon absorption probabilities are given in Table 2 with the experimental results.[14] In Table 2 the calculated nuclear mean excitation energies $\langle E_{ex} \rangle$ are also given. The experimentally predicted values of $\langle E_{ex} \rangle$ are about 20 MeV for Pb, Bi and Sn.[15]

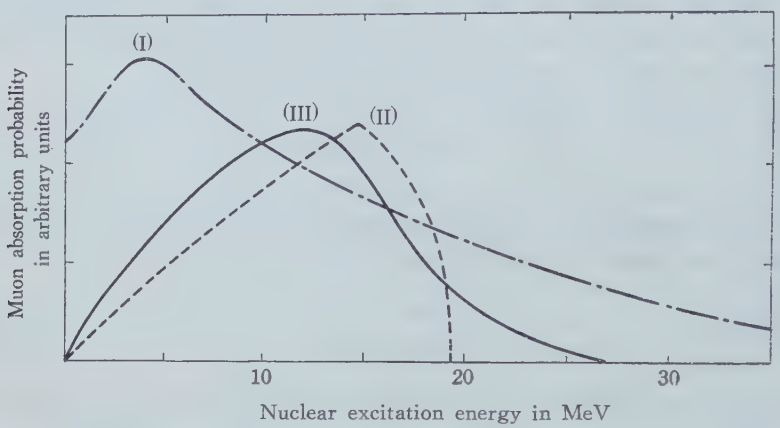

Fig. 3. Nuclear excitation function following muon absorption by Ag^{107}.

§ 3. Decay of the compound nuclei

We calculate energy spectra of emitted alpha particles and protons, assuming the statistical theory. Then, the emission probability σ of particle b is given by

$$\sigma = \int \frac{\sigma_c(E_{ex}) F_b(E_{ex} - B_b)}{\sum_i F_i(E_{ex} - B_i)} dE_{ex}, \tag{17}$$

where

$$F_b = (2M_b/\hbar^2) \int \sigma_b(\epsilon) w(E_{ex} - B_b - \epsilon) \epsilon d\epsilon. \tag{18}$$

σ_c is the compound nucleus formation cross section by muon absorption, $\sigma_b(\epsilon)$ the cross section of the inverse reaction which is estimated following ref. 16), $w(E)$ the level density at excitation E of the final nucleus, E_{ex} the excitation energy of the compound nucleus, and B_b the separation energy of the particle b from the compound nucleus. The value of B_b are shown in Table 3.[12]

Table 3

Compound Nucleus	B_α (MeV)	B_p (MeV)	B_n (MeV)
Pd^{109}	3.9		6.0
Pd^{107}	3.3	8.7	6.3
Se^{81}	7.7		6.9
Se^{79}	6.4	10.3	7.2

The level density is assumed to be

$$w(E)=C \exp\ (\sqrt{aE}\ ) \qquad (19)^*$$

with

$$4C_{even\ even}=2C_{even\ odd}=C_{odd\ odd}. \qquad (20)$$

The parameters a are determined by interpolation of the values from the analysis of (α, p) reactions of Cu, Ag and Au with 40 MeV alpha particles.[17] The values obtained are

$$\begin{aligned} a&=4.7\ \mathrm{MeV}^{-1}, \text{ for Ru,} \\ a&=5.4\ \mathrm{MeV}^{-1}, \text{ for Ge.} \end{aligned} \qquad (21)$$

Assuming $E_{ex}=20$ MeV, we perform the preliminary evaluation of the energy spectra of emitted alpha particles for four isotopes Pd^{107}, Pd^{109}, Se^{79} and Se^{81}. The mean weighted spectra obtained from the above calculations were almost the same in shape with those obtained from the calculations in which only Pd^{107} and Se^{79} were considered. As to the emission rate, the latter spectra is only about 20% larger than the former. Furthermore, the separation energies from these two isotopes of the same element are nearly equal to each other. Due to these two reasons, we calculate the energy spectra considering only Pd^{107} and Se^{79}.

The comparison between the calculated and experimental energy spectra is given in Figs. 4 and 5,** where the calculated results are normalized so that the areas under the curves are proportional to the emission rate. The calculated emission rates of alpha particles and protons are given in Table 4 in comparison with the experimental results.[2]

Table 4. Charged particle emission rate normalized to the total reaction rate as 100%

	Emitted Particles	
	alpha(%)	proton(%)
(I) Chew-Goldberger	7.9	2.3
(II) Fermi gas ($kT=0$)	~0.1	~0.02
(III) Fermi gas ($kT=9$MeV)	0.45	0.23
Experimental results[2]	0.5	2.2

* In the above density formula (19), E is the total nuclear excitation energy measured from the ground state at $kT=0$, therefore, for the Fermi gas with finite temperature, (19) is not quite consistent. Moreover, the excitation energy is about 650 MeV at $kT=9$ MeV for Ag, i.e. under these temperatures the Fermi gas model loses its physical meaning. In this paper it is assumed that level density is determined by (19), (20) and (21) for all assumed momentum distributions, and considering the emission processes it is assumed that the term, "Fermi gas at $kT=9$ MeV", means only the condition corresponding to the state that has the momentum distribution given by (15).

** In ref. 2, Fig. 4 and Fig. 5 must be interchanged.

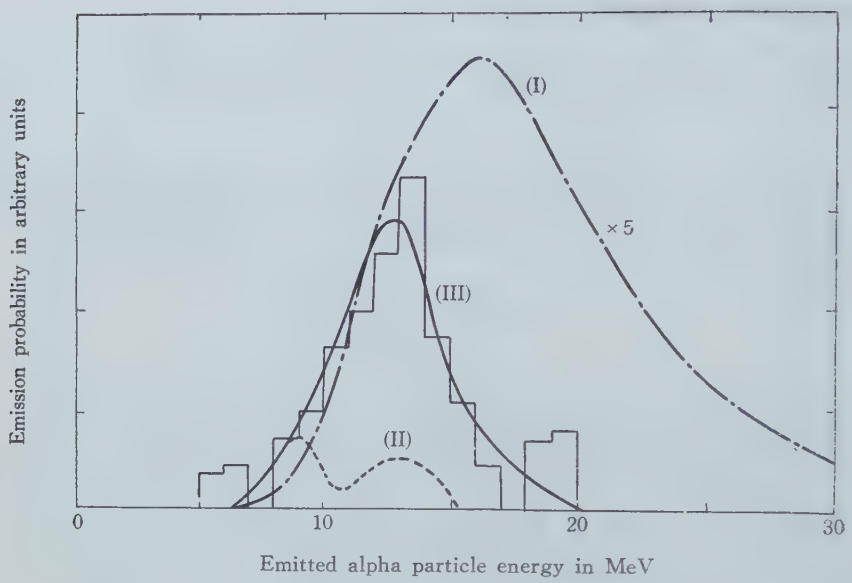

Fig. 4. Energy distribution of emitted alpha particles following muon absorption in AgBr. Experimental histogram is taken from ref. 2).

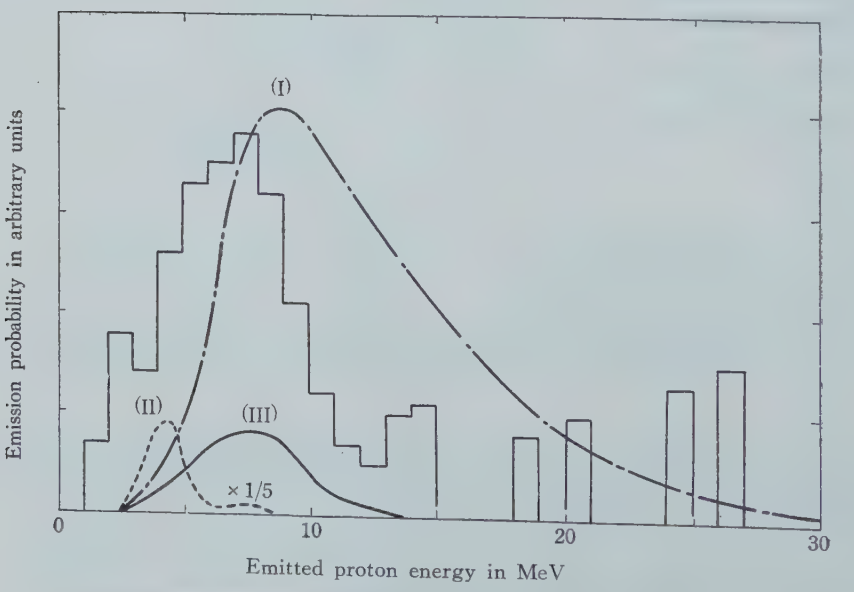

Fig. 5. Energy distribution of emitted protons following muon absorption in AgBr. Experimental histogram is taken from ref. 2).

§ 4. Discussion

It seems that the calculated values of absorption probability Λ are too small for the assumed value of $g=1\times10^{-49}$ erg cm^3. In order to obtain the agreement with experimental results of Ag^{107}, it is sufficient to assume that $g=1.07$, 1.86 and 1.57×10^{-49} erg cm^3 for (I), (II) and (III), respectively. And all these g values are not unreasonable. Thus from the calculation of absorption probability, the conclusion regarding momentum distribution of the proton in the nucleus cannot be obtained.

Experimentally predicted values of $\langle E_{ex}\rangle$ are about 20 MeV for heavy elements[15], and a few of the cases of light elements indicate excitation energies of some 40 MeV[2)18)]. These predictions are well reproduced by (I). However, in (I) the high energy tail in muon absorption probability seems to be too large, because the alpha particle emission probability is too large. In (II), both of $\langle E_{ex}\rangle$ and the high energy tail in muon absorption probability are too small.

It has been assumed that the level density of the residual nucleus is given by (19). The values of parameter a were determined from (α, p) reactions, investigated by Eisberg et al.[17]. They studied only the protons emitted with large angles in order to exclude the effects of the direct process, and yet it shows that (21) may contain the effect due to direct nuclear reactions[17]. Therefore, although the statistical theory gives good agreement with experiments for alpha emissions when (21) is used, it may not be concluded that all the alpha particles are emitted by evaporation.

In fact the statistical theory may not give good agreement with proton experiments when (19), (20) and (21) are used. In order to investigate this disagreement, the data from low energy $Ag(\gamma, p)Pd$ experiments have been compared with the theoretical values. This experiment was performed by Diven and Almy[19] using 20.8 MeV bremsstrahlung, and it was shown that emitted protons were explained by statistical theory. The agreement between the experimental and the theoretical values used (19), (20) and (21) is good for emitted proton spectrum and $(\gamma, p)/(\gamma, n)$ yield ratio. Therefore, (21) may not be so bad for the reactions with proton emission, too.

Because (21) was obtained by the (α, p) experiment and the spectra of emitted alpha particles are generally explained by the statistical theory,* it may be concluded that the momentum distribution which gives the measured alpha energy spectra is the most correct one for explaining muon absorption phenomena. Thus (III) is the most reasonable among three. However, for this distribution the proton emission rate is too small compared with experiment. This fact suggests that appreciable part of protons are emitted due to the direct process. Actually, in the

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* There is an evidence for direct emission[20], inferred from the fact that the angular distribution of emitted alpha particles with energies larger than 6 MeV shows strongly forward peaking in $Zr(n,\alpha)Sr$ reactions by 14 MeV neutrons.

reaction (n, p) by 14 MeV neutrons with moderately heavy nuclei, there are many evidences that most of protons are directly emitted.[21]

The lower part of the calculated proton spectrum is too small. Most of this discrepancy may be due to the contributions from light nuclei in geratin[2].

For (II), the energy spectrum has two peaks, the first one being due to Br and the second due to Ag. And the emission rate is too low to explain the experiments.

The reason why the emission rate is too large in (I) may be as follows: It is incorrect to calculate the muon absorption probabilities with single particle approximation, i.e. it is necessary to take into account the two-body correlations[22,23], and to consider that a partner of the pair is also ejected. Furthermore, there are evidences that the tail in (I) is too long even for light nuclei[3,9,24] and that at high energies appreciable amount of emitted particles may be due to the direct process. Actually, at energies higher than about 30 MeV of an excited nucleus, about 20% of produced neutrons are emitted without colliding with other nucleons in the nucleus. Then if most of the neutrons are assumed to be emitted by direct reactions, (I) may fit to experiments for the emission rate.

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On the Absorptive Range in the Pion-Nucleon Collision at High Energies*

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The absorptive range in the pion-nucleon collision at high energies with an additional pion production is investigated on the standpoint of the $\pi-\pi$ interaction model. For this purpose, the "root mean square of the impact parameter" $(\bar{b}^2)^{1/2}$ is introduced and is calculated under the local $\pi-\pi$ interaction and some assumptions. $(\bar{b}^2)^{1/2}$ thus obtained is somewhat smaller compared with the experimentally inferred one.

§ 1. Introduction

According to the recent experiments on the pion-nucleon collision at high energies with an additional pion production[1]; (1) the angular distribution of the recoil nucleon has a peak in the backward direction in the center of mass system, (2) the angular distribution relative to the recoil nucleon of the faster pion has a strong peak in the forward direction and that of the slower is nearly isotropic.

It has been shown by Kovacs[2] and more extensively by Ito et al.[3] that these evidences seem to suggest the knock-out process in which the incident pion interacts directly with the pion field surrounding the nucleon core, and knocks out the pion, rather than to suggest the thermal process in which the incident pion is absorbed directly by the core and the system goes to the thermal equilibrium. This may also be suggested by the fact that the elastic scattering in this energy range is consistent with the diffraction scattering by the absorptive disk with its radius about $(1.0\sim1.2)\times10^{-13}$ cm,[1] because this large radius (which is comparable with the pion Compton wave length) does not seem to be due to the direct absorption by the core. Here, we shall then investigate the absorptive range from the standpoint of the $\pi-\pi$ interaction model. In Sec. 2, we introduce the "root mean square of the impact parameter" to characterize the absorptive range, and give a simple formula to calculate this quantity. In Sec. 3, we then estimate the absorptive range applying this formula to the $\pi-\pi$ interaction model. In Sec. 4, the results are discussed.

§ 2. The root mean square of the impact parameter[4]

In the low energy nuclear reaction, the absorptive phenomena may be described

* A brief report on this work has been published in Prog. Theor. Phys. **20** (1958), 774.

by the imaginary potential $iW(r)$ as the function of the relative coordinate of the interacting nuclei. In the interaction between the elementary particles, especially in the relativistic energy region, however, such a simple static picture may not hold, so it may be necessary to introduce the new physical measure to characterize the absorptive range.

Now, suppose that the elastic scattering is only the shadow scattering due to the absorption of the incident pion from the nucleon. The diffraction pattern then can be predicted if one knows the opacity $\alpha(b)$ as a function of the impact parameter b. As $b=l/p$† (where p is the momentum of the incident pion, l is the angular momentum,) and

$$\alpha(b)=\alpha(l/p)=\sigma_{r,l}/\pi(2l+1)\lambda\!\!\!^{-2}, \tag{1}$$

$\alpha(b)$ is obtained if one can calculate the reaction cross section $\sigma_{r,l}$ for each partial wave. However, it may be difficult to calculate the $\sigma_{r,l}$ generally, so we give up to calculate $\alpha(b)$, and intend to estimate only the absorptive range. For this purpose, we introduce a simple quantity to characterize the transversal absorptive range, e. g. the root mean square of the impact parameter $(\bar{b}^2)^{1/2}$ defined as

$$(\bar{b}^2)^{1/2}=1/p\{\Sigma\, l(l+1)\sigma_{r,l}/\Sigma\,\sigma_{r,l}\}^{1/2}. \tag{2}$$

This quantity is simplified as follows:

$$p^2\bar{b}^2=\{L'^2\langle \boldsymbol{p}'|T_r^{\dagger}\delta(E-H_0)\,T_r|\boldsymbol{p}\rangle/\langle \boldsymbol{p}|T_r^{\dagger}\delta(E-H_0)\,T_r|\boldsymbol{p}\rangle\}_{p'\to p}, \tag{3}$$

where T_r is reaction matrix, $|\boldsymbol{p}'\rangle$ and $|\boldsymbol{p}\rangle$ are both the initial states with the relative momenta $\boldsymbol{p}'$ and $\boldsymbol{p}$, respectively, $(p'=p)$, E is the total energy of the system, H_0 is the free Hamiltonian, and L'^2 is the square of the angular momentum operator acting on $\boldsymbol{p}'$ expressed as

$$L'^2=-\left(\frac{\partial}{\partial \boldsymbol{p}'}\times\boldsymbol{p}'\right)^2=\left[\left(\boldsymbol{p}'\cdot\frac{\partial}{\partial \boldsymbol{p}'}\right)^2+\left(\boldsymbol{p}'\cdot\frac{\partial}{\partial \boldsymbol{p}'}\right)-\boldsymbol{p}'^2\left(\frac{\partial}{\partial \boldsymbol{p}'}\right)^2\right]. \tag{4}$$

Eq. (3) may be easily verified taking into account that the operator $T_r^{\dagger}\delta(E-H_0)\,T_r$ is diagonal with respect to the angular momentum, averaging over the nucleon spin directions. $(b^2)^{1/2}$ thus introduced is the simplest quantity, because it is unnecessary to calculate each $\sigma_{r,l}$.

§ 3. The calculation of $(\bar{b}^2)^{1/2}$

We then calculate the $(\bar{b}^2)^{1/2}$ under the following assumptions. The pion number surrounding the core is only one, which may be supported by the fixed source theory, at least, in the outer region in which the knock-out process may be dominant; the pion-core interaction Hamiltonian in the momentum representation is

† Hereafter we take the units as $\hbar=c=1$.

$$H=\frac{ig}{\mu}\sum_{k}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k})}{\sqrt{2\omega_k}}[a_k+a^*_{-k}],* \tag{5}$$

where the nucleon is assumed to be fixed in the laboratory system, which may be natural, since the nucleon recoil is small for the knock-out process; the $\pi-\pi$ interaction "Hamiltonian" is simply

$$H_{\pi\pi}=\lambda\int\phi^4(\boldsymbol{x})\,d^3\boldsymbol{x}\,; \tag{6}$$

finally the final state interaction between the outgoing particles can be neglected.

Then, the transition matrix element for an additional pion production from the pion-nucleon collision is written as follows;

$$\langle\boldsymbol{k}_1,\ \boldsymbol{k}_2|T_r|\boldsymbol{p}\rangle=-i6\lambda\frac{g}{\mu}\frac{1}{\sqrt{2\omega_p\omega_{k_1}\omega_{k_2}}}\frac{\sigma(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})}{(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})^2+\mu^2}, \tag{7}$$

where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the momenta of the outgoing pion. $T_r^\dagger\delta(E-H_0)T_r$ thus can be written as follows:

$$\begin{aligned}&\langle\boldsymbol{p}'|T_r^\dagger\delta(E-H_0)T_r|\boldsymbol{p}\rangle\\&=\iint\frac{d\boldsymbol{k}_1}{(2\pi)^3}\frac{d\boldsymbol{k}_2}{(2\pi)^3}\frac{1}{2}\langle\boldsymbol{p}'|T_r^\dagger|\boldsymbol{k}_1,\ \boldsymbol{k}_2\rangle\delta(\omega_{k_1}+\omega_{k_2}-\omega_p)\langle\boldsymbol{k}_1,\ \boldsymbol{k}_2|T_r|\boldsymbol{p}\rangle\\&=C(p)\iint\frac{d\boldsymbol{k}_1d\boldsymbol{k}_2}{\omega_{k_1}\omega_{k_2}}\delta(\omega_{k_1}+\omega_{k_2}-\omega_p)\frac{\sigma(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p}')\sigma(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})}{[(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p}')^2+\mu^2][(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})^2+\mu^2]},\end{aligned} \tag{8}$$

where we write the factor independent of the direction of $\boldsymbol{p}'$ and $\boldsymbol{p}$ as $C(p)$, which we omit hereafter, because it is cancelled out for the calculation of the right-hand side of eq. (3). Transforming the integral variables $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ to $\boldsymbol{k}_1$ and $\boldsymbol{q}=\boldsymbol{k}_1+\boldsymbol{k}_2$, and performing the integration with $\boldsymbol{k}_1$, we obtain

$$\int\frac{d^3\boldsymbol{k}_1}{\omega_{k_1}\omega_{q-k_1}}\delta(\omega_{k_1}+\omega_{q-k_1}-\omega_p)=\begin{cases}2\pi & \text{for } p>q,\\ 0 & \text{for } p<q\end{cases} \tag{9}$$

in high energy limit, $p\gg\mu$. Averaging with the spin variable, we can simplify $T^{r\dagger}\delta(E-H_0)T_r$ as

$$\langle\boldsymbol{p}'|T_r^\dagger\delta(E-H_0)T_r|\boldsymbol{p}\rangle=\int_{q<p}d^3\boldsymbol{q}\frac{(\boldsymbol{q}-\boldsymbol{p}',\ \boldsymbol{q}-\boldsymbol{p})}{[(\boldsymbol{q}-\boldsymbol{p}')^2+\mu^2][(\boldsymbol{q}-\boldsymbol{p})^2+\mu^2]}. \tag{10}$$

Next, carrying out the operation L'^2, (see eq. (4)), we obtain the following expression for the numerator of eq. (3),

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* For simplicity, the neutral meson interaction is considered here, but the result is as same as in the case of the actual symmetrical meson interaction.

$$\{L'^2\langle \boldsymbol{p}'|T_r^\dagger \delta(E-H_0)T_r|\boldsymbol{p}\rangle\}_{p'\to p}$$
$$=2\int_{q<p} d^3\boldsymbol{q}\frac{1}{[(\boldsymbol{q}-\boldsymbol{p})^2+\mu^2]^2}\left\{\boldsymbol{p}^2+(\boldsymbol{p}\cdot\boldsymbol{q})-2\frac{\boldsymbol{q}^2\boldsymbol{p}^2-(\boldsymbol{q}\cdot\boldsymbol{p})^2}{[(\boldsymbol{q}-\boldsymbol{p})^2+\mu^2]}\right\}, \tag{11}$$

where we neglect the terms of the order μ/p and of higher orders. The denominator is

$$\langle \boldsymbol{p}|T_r^\dagger \delta(E-H_0)T_r|\boldsymbol{p}\rangle=\int_{q<p} d^3\boldsymbol{q}\frac{1}{[(\boldsymbol{q}-\boldsymbol{p})^2+\mu^2]} \tag{12}$$

in this approximation. These integrals may be easily carried out expressing the variable $\boldsymbol{q}$ by the cyrindrical co-ordinate, in which the z-axis is taken in the direction of $\boldsymbol{p}$. Results are as follows:

the R. H. S. of eq. (11)

$$=4\pi p\left\{\frac{p}{2\mu}\tan^{-1}\frac{2p}{\mu}+\frac{1}{4}-\frac{1}{16}\left(4+\frac{\mu^2}{p^2}\right)\left(1-\frac{\mu^2/p^2}{4+\mu^2/p^2}\right)-\frac{1}{4}\log\left(1+\frac{4p^2}{\mu^2}\right)\right\} \tag{11'}$$

and

the R. H. S. of eq. (12)

$$=2\pi p\left\{1-\frac{\mu}{p}\tan^{-1}\frac{2p}{\mu}+\frac{1}{4}\frac{\mu^2}{p^2}\log\left(1+\frac{4p^2}{\mu^2}\right)\right\}. \tag{12'}$$

$(\bar{b}^2)^{1/2}$ thus obtained is simplified in the high energy limit as follows:

$$\bar{b}^2=\pi/2p\mu, \quad \text{for } p\gg\mu. \tag{13}$$

The numerical values are listed in Table 1.

Table 1. The root mean square of the impact parameter $(\bar{b}^2)^{1/2}$ and the corresponding radius of the uniform absorptive disk R $(\bar{b}^2=R^2/2)$.

Energy of the incident pion	1.0 Bev	1.4 Bev	2.0 Bev
$(\bar{b}^2)^{1/2}\times 10^{13}$ cm	0.67	0.56	0.48
$R\times 10^{13}$ cm	0.94	0.78	0.67

§ 4. Discussion

In Sec. 2, we introduced the quantity $(\bar{b}^2)^{1/2}$ to generally estimate the transversal absorptive range in the diffraction scattering. Next, in Sec. 3, using the formula derived in Sec. 2, we tried to calculate the $(b^2)^{1/2}$ in the pion-nucleon collision at high energies from the standpoint of the $\pi-\pi$ interaction model (in which the incident pion knocks out the pion cloud surrounding the core to produce an additional pion), with some assumptions. In actual collisions, an additional pion

production may also occur by different mechanisms from the knock-out process, even if this might be the most important one. Also, many other processes such as $\pi+N\to N+3\pi$, $N+4\pi$, ……, and potential scattering can occur. However, it can be inferred that the $(\bar{b}^2)^{1/2}$ calculated with the assumption of the knock-out process gives the maximum one, since the most spread-out interaction may be caused by the interaction between the incident pion and one pion surrounding the core. Thus, the collision processes is considered to occur within the transversal range of radius $R=\sqrt{2}\,(\bar{b}^2)^{1/2}$ on the average, if one assumes the uniformly distributed absorptive matter, as has been done in the analysis of the experiments. Thus calculated R (which is also listed in Table 1) is somewhat smaller compared with the experimentally inferred one (which is about $(1.0\sim1.2)\times10^{-13}$ cm over the Bev energy range). This discrepancy, however, may be overcome within the frame of the $\pi-\pi$ interaction model, if one takes into account the correction for the pion-nucleon vertex function and the spread-out $\pi-\pi$ interaction. The former may be necessary at the high energies, because the correct vertex function $\Gamma_{5i}(p',\ p)\equiv\tau_i\gamma_5 f((p'-p)^2)$ (where p and p' are 4-momenta of the incident and recoil nucleon respectively), may not be simple $\tau_i\gamma_5$ for the large momentum transfer $(p'-p)^2$. The latter seems to be natural, since the local $\lambda(\phi^2)^2$ interaction which acts only between the isotopic spin even states may not be consistent with the experimental charge ratios of the outgoing pions, (and the most important interacting state may be in the isotopic spin 1),[3] the *local* $\lambda(\phi^2)^2$ may thus play little role except to cancell the divergences in the renormalization theory, and the correct "Hamiltonian" may be expressed as the spread-out interactions including the space-time derivatives.[5]

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The Imaginary Part of the Optical Potential

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We derive the form of the imaginary part of the optical potential as an operator for the finite nucleus. This is done by reducing the Schroedinger equation approximately. It is essentially a second order calculation but this is supported by the intermediate coupling model and the assumption that the non-diagonal elements of the interaction matrix have random signs. The imaginary part thus obtained is a non-local but almost separable potential. The form allows the physical interpretation that the imaginary part corresponds to the process in which the incident nucleon jumps down to an unoccupied single particle state below the incident energy and excites the target nucleus making the total energy nearly conserved. It is suggested that the imaginary part may become abruptly stronger as the shape resonance is passed since in this case there is a single particle state just below the incident energy and the corresponding excited states of the target are near the ground state. The relation to Bloch's theory is also discussed.

§ 1. Introduction

It is now a well established fact that the nucleon-nucleus interaction cross section averaged over many resonances can be reproduced by an optical potential with a small imaginary part. The derivation of the imaginary part of the optical potential from the theory of the Fermion many body problem with strong interaction is however, a little diverse. First, there was a semi-classical approach from the mean free path of a nucleon in the nuclear matter[1]. It was pointed out that the smallness of the absorption probability was due to the fact that many possible interaction processes are prohibited by the Pauli principle. Then, there were more fundamental quantum mechanical theories using the dispersion formula and the sum-rules for the reduced widths, the so-called intermediate coupling model[2]. In these works the Pauli principle was partly neglected since the incident nucleon was distinguished from those in the target nucleus. Finally, there are the t-matrix theory[3] and a perturbation theory including higher order terms[2a]. These are essentially rigorous quantum mechanical formulations of the above mentioned semi-classical picture.

In this paper we first incorporate the Pauli principle in the dispersion formula. Before doing so we show in section 2 that the antisymmetrization with respect to the incident particle is actually important in reducing the imaginary part of the optical potential by the semi-classical method. In section 3 the totally antisymmet-

rized dispersion formula will be derived. It is seen that the result is formally the same as that for which the incident nucleon is distinguished. In section 4 we develop the intermediate coupling model taking into account the proper antisymmetrization. These sections are the preparation for the justification of the second order calculation which is done in the next section. Those who take the second order calculation for granted can skip sections 3 and 4. In section 5 the imaginary part of the optical potential is derived by reducing the Schroedinger equation approximately with the help of the intermediate model and the assumption of the randomness of the signs of the non-diagonal interaction matrix elements. The imaginary part thus obtained is similar to that of Bloch[2a)] but the present one is a non-local but almost separable potential. In section 6 the correspondence to the semi-classical picture will be discussed. It will be shown that our result seems to indicate a systematic change of the imaginary part before and after the shape resonance for medium weight nuclei at low energy. The relation to Bloch's theory is also discussed.

§ 2. The importance of the Pauli principle

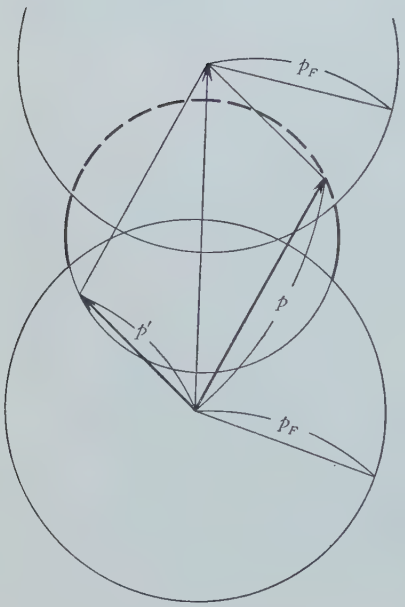

Fig. 1 Permissible region of the momentum space for the final state
p: momentum of the incident nucleon in the nucleus
p_F: the Fermi momentum
— : indistinguishable
-- —: distinguishable

Before formulating the dispersion formula of the nuclear reaction with completely antisymmetrized wave functions we shall see if the antisymmetrization with respect to the incident nucleon is really important. For this purpose we calculate the effective cross section in the nuclear matter for a nucleon, assuming the incident nucleon to be indistinguishable and distinguishable from those in the target. The region of the momentum space which is permissible for the final state for each case is illustrated in figure 1. The effective cross sections $\bar{\sigma}$ become, for a constant free two-body cross section σ,

$$\bar{\sigma}/\sigma=1-(7/5)(E_F/E_i)+(2/5)(E_F/E_i)\times(2-E_i/E_F)^{5/2} \quad \text{(indistinguishable)} \quad (1)$$

$$\bar{\sigma}/\sigma=1-(3/5)(E_F/E_i) \quad \text{(distinguishable)}$$

for $E_i<2E_F$ where E_i and E_F are the kinetic energy of the incident nucleon in the nuclear matter and the Fermi energy respectively. $\bar{\sigma}/\sigma$ is shown in figure 2. It is apparent that the antisymmetrization

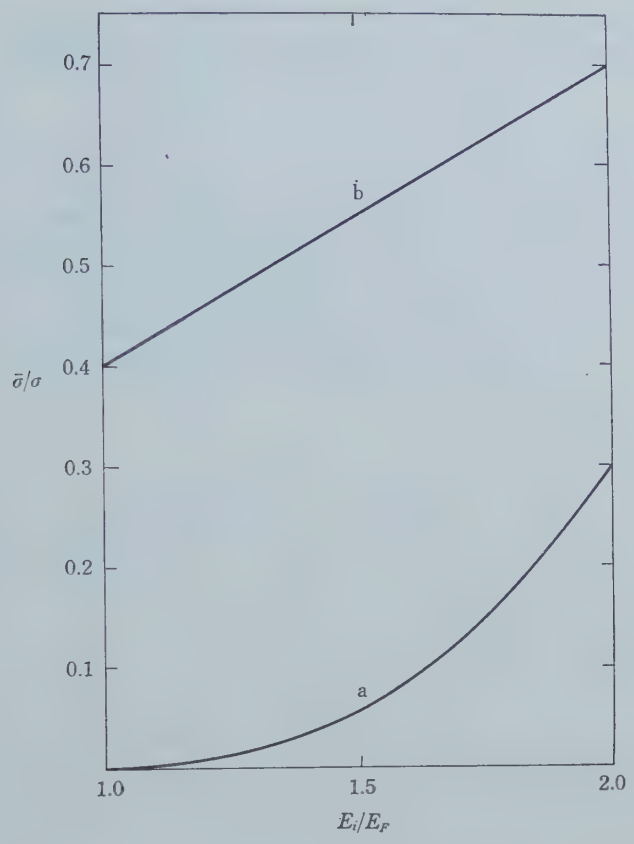

Fig. 2 The ratio of the effective cross section $\bar{\sigma}$ to the free particle cross section σ versus the kinetic energy of a nucleon in the nuclear matter in the unit of the Fermi energy E_i/E_F.
a: indistinguishable
b: distinguishable

with respect to the incident nucleon plays an important role in reducing the effective cross section, and hence the imaginary part of the optical potential.

§ 3. Antisymmetrized dispersion formula

We start from the $A+1$ body Schroedinger equation

$$(\sum_i T_i + \sum_{i<j} V_{ij})\,\Psi(\boldsymbol{r}_0, \cdots, \boldsymbol{r}_A) = E\Psi. \tag{2}$$

The $3(A+1)$ dimensional configuration space is divided into two parts, the internal region where all nucleons are interacting and the external region where one nucleon is removed far from the others and is free. We do not consider the case where more than one nucleon are removed far from the rest, such as (d, p) or $(n, 2n)$ reaction. The surface of the internal region is $3A+2$ dimensional.

In the internal region, where all r_i's are smaller than a certain value R, eigenfunctions and eigenvalues of the compound nucleus are obtained as

$$(\sum T_i+\sum V_{ij})\Psi_n(\boldsymbol{r}_0,\cdots,\boldsymbol{r}_A)=E_n\Psi_n \tag{3}$$

with the outgoing wave boundary condition[4]. R is to be taken sufficiently large so that at R the wave functions of the target nucleus are almost zero. Both Ψ and Ψ_n are antisymmetric, otherwise they do not give the correct answer. Because of the antisymmetrization there is a little difficulty regarding the boundary condition for Ψ_n. To make it unambiguous we develop Ψ_n in terms of the target nucleus wave functions ψ_s, where s denotes the state of the target nucleus,

$$\Psi_n=\sum_{s,i}\frac{(-)^{P_{0i}}}{\sqrt{A+1}}P_{0i}\,\psi_s(\boldsymbol{r}_1,\cdots,\boldsymbol{r}_A)f_n^{A}(\boldsymbol{r}_0). \tag{4}$$

In equation (4) P_{oi} is the exchange operator between 0 and i and $(-)^{P_{oi}}$ is $+1$ for $i=0$ and -1 otherwise. Writing $u_n^s(\boldsymbol{r})/r$ for $f_n^s(\boldsymbol{r})$, and considering only the s-state for the sake of simplicity, the outgoing wave boundary condition is

$$\frac{du_n^s(r)}{dr}\Big|_{r=R}=ik_s\,u_n^s(R),\qquad \frac{\hbar^2}{2m}k_s^2=E-\mathcal{E}_s=\varepsilon-\mathcal{E}_s^{ex} \tag{5}$$

where $\mathcal{E}_s$, is the energy of the state s, ε the kinetic energy of the incident nucleon and $\mathcal{E}_s^{ex}$ the excitation energy $\mathcal{E}_s-\mathcal{E}_0$.

The wave function Ψ is developed in terms of the wave function of the compound state Ψ_n

$$\Psi=\sum_n c_n(E)\Psi_n. \tag{6}$$

The coefficient $c_n(E)$ is evaluated as follows. We operate the total Hamiltonian on equation (6) and note that the convergence of the series is not uniform at the surface of the internal region. This means that the kinetic energy operators cannot be operated term by term. So we have, by multiplying Ψ_n* and integrating

$$(E_n-E)c_n(E)+\int\Psi_n(\sum T_i-\sum T_i')\Psi d\tau$$

$$=(E_n-E)c_n(E)-\frac{\hbar^2}{2m}(A+1)R\int\Psi_n\Big(\frac{d}{dr_0}r_0-\frac{d'}{d'r_0}r_0\Big)_{r_0=R}\Psi d\tau_{-0}=0 \tag{7}$$

where the prime means that the operation is done term by term. We further

* This is the cojugate complex of the original Ψ_n regarding the spin-angle part but the radial part is the same as the original Ψ_n.

develop the wave function Ψ in terms of the target nucleus wave function

$$\Psi=\sum_{s,i}\frac{(-)^{P_{0i}}}{\sqrt{A+1}}P_{0i}\,\psi_s(\boldsymbol{r}_1,\cdots,\boldsymbol{r}_A)f^s(\boldsymbol{r}_0). \tag{8}$$

The non-uniform convergence occurs only for the incident channel wave function

$$f^0(\boldsymbol{r})=\sum_n c_n(E)f_n^0(\boldsymbol{r}),$$

for other channels the boundary conditions ensure smooth connections. For $f^0(r)$ we have, from

$$Rf^0(R)=\frac{\sin kR}{k}+S_{00}\,e^{ikR}\qquad(k=k_0)$$

and

$$\frac{d}{dr}rf^0(r)|_{r=R}=\cos kR+ik\,S_{00}\,e^{ikR},$$

the relation

$$\frac{d}{dr}rf^0(r)|_{r=R}=e^{-ikR}+ikRf^0(R).$$

On the other hand, the term by term operation gives

$$\frac{d'}{d'r}rf^0(r)|_{r=R}=\sum c_n(E)\frac{d}{dr}rf_n^0(r)|_{r=R}=ikR\sum c_n(E)f_n^0(R)=ikRf^0(r).$$

Therefore we have

$$\begin{aligned}(E_n-E)c_n(E)&=\frac{\hbar^2}{2m}e^{-ikR}\sqrt{A+1}\,R\int\Psi_n(\boldsymbol{R},\boldsymbol{r}_1,\cdots,\boldsymbol{r}_A)\psi_0(\boldsymbol{r}_1,\cdots,\boldsymbol{r}_A)\,d\tau_{-0}\\&=\frac{\hbar^2}{2m}e^{-ikR}R\Big\{f^0(R)-A\sum_s\int\psi_s(\boldsymbol{R},\boldsymbol{r}_2,\cdots,\boldsymbol{r}_A)f_n^s(\boldsymbol{r}_1)\psi_0(\boldsymbol{r}_1,\cdots,\boldsymbol{r}_A)\,d\tau_{-0}\Big\}.\end{aligned} \tag{9}$$

The second term in the last line of equation (9) is nearly equal to zero since $\psi_s(\boldsymbol{R},\boldsymbol{r}_2,\cdots,\boldsymbol{r}_A)\simeq 0$. There is another reason to assume that the second term is negligible but this point will be discussed later in section 4. The factor A is only apparent since even when f_n^s is identical with one of the functions forming ψ_0 the last term just cancels the former one. In any case the neglect of the second term is reasonable in view of the reciprocity theorem which will be seen just below. So we have

$$c_n(E)=\frac{1}{E_n-E}\,\frac{\hbar^2}{2m}e^{-ikR}u_n^0(R),\qquad(u_n^s(r)=rf_n^s(r)). \tag{9'}$$

In the external region Ψ is written as

$$\Psi=\sum_{s,i}\frac{(-)^{P_{oi}}}{\sqrt{A+1}}P_{oi}\psi_0(\boldsymbol{r}_1,\cdots,\boldsymbol{r}_A)\left\{\delta_{0s}\frac{\sin kr_0}{kr_0}+S_{0s}\frac{e^{ik_s r_0}}{r_0}\right\}. \tag{10}$$

The S-matrix can be obtained by connecting the internal wave function (6) and (4) with the external one (10) term by term,

$$S_{0s}=\sum_n c_n(E)e^{-ik_s R}u_n^s(R)-\delta_{0s}e^{-ikR}\frac{\sin kR}{k}.$$

By using equation (9′) for $c_n(E)$ we have

$$S_{0s}=\frac{\hbar^2}{2m}e^{-ik_0R}\ e^{-ik_sR}\sum_n\frac{u_n^0(R)u_n^s(R)}{E_n-E}-\delta_{0s}\,e^{-ikR}\frac{\sin kR}{k}. \tag{11}$$

It is apparent that the form (9′) is reasonable since otherwise each resonance term is not symmetric with respect to 0 and s.

§ 4. The intermediate coupling model with antisymmetrization

In this section we prepare to derive the imaginary part of the optical potential by reducing the Schroedinger equation approximately. This is done with the help of the intermediate coupling model. For this purpose we need the development coefficient $c_n(E)$ given by equation (9′), but not the S-matrix.

The success of the optical potential means that the incident nucleon moves inside the nucleus nearly according to the single particle model. So we take single particle wave function as the basic stone. We keep of the wave function of the target nucleus exact so that the inelastic processes leading to discrete final states can be treated. From the single particle wave function

$$\psi_s(\boldsymbol{r}_1\cdots,\ \boldsymbol{r}_A)\varphi^{sp}(\boldsymbol{r}_0) \tag{12}$$

we form the antisymmetrized function

$$\Phi^{sp}(\boldsymbol{r}_0,\cdots,\ \boldsymbol{r}_A)=\frac{1}{\sqrt{A+1}}\sum_i(-)^{Poi}\psi(\boldsymbol{r}_1,\cdots,\boldsymbol{r}_A)\varphi^{sp}(\boldsymbol{r}_0). \tag{13}$$

φ^{sp} is determined from a suitable average potential, which will be defined later, with the outgoing wave boundary condition (5).

There is a little complication regarding Φ^{sp}'s since, strictly speaking, they are not orthonormal*. We can see, however, that the orthonormality holds approximately in general. In fact the non-orthogonality occurs through the exchange overlap integral

* This is related to the fact that the Hamiltonian which on operating on Φ^{sp} gives the energy eigenvalue $E^{sp}=\varepsilon_s+\varepsilon^{sp}$

$$\bar{H}=\sum_i T_i+\sum_{i>j}V_{ij}+\sum_k(U_k-\sum_i V_{ki})Q_k$$

is not exactly Hermitic, where $Q_k=1$ if it operates on $\psi_s(r_0,\cdots r_{k+1},\ r_{k-1},\cdots,r_A)\varphi^{sp}(r_k)$ and zero otherwise.

$$\int \psi_{s'}(\boldsymbol{r}_0, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_A)\varphi^{s'p'}(\boldsymbol{r}_1)\psi_s(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_A)\varphi^{sp}(\boldsymbol{r}_0)\,d\tau.$$

The ψ_s's can be developed in terms of the Slater determinants,

$$\psi_s = \sum_m a_s^m \det(A)_m,$$

where m indicates the configuration. The single particle wave functions in the Slater determinants are obtained from an average potential (which is assumed to be independent of the states) by the boundary condition that they decay exponentially for large r. On the other hand φ^{sp} is derived from approximately the same potential with the outgoing boundary condition (5) (note in equation (5) that $k_s = i\sqrt{(2m/\hbar^2)(\xi_s^{ex} - \varepsilon)}$ if $\varepsilon - \xi_s^{ex} < 0$ and the boundary condition is real in this case). Therefore the two eigenvalues do not coincide generally. (They coincide if $\varepsilon - \xi_s^{ex} \simeq \varepsilon^{sp}$ where ε^{sp} is the eigenvalue of φ^{sp}.) If they do not coincide, these two wave functions are shown to be orthogonal since we assumed that at $r = R$ the wave functions in ψ_s are negligibly small. Even if both combinations (s, p) and (s', p') satisfy the above mentioned equality for the boundary condition and so the two kinds of single particle wave functions coincide, the integral is generally small unless $E^{sp} \simeq E^{s'p'}$ (where $E^{sp} = \xi_s + \varepsilon^{sp}$) since for the integral to be non-zero the $\det(A)_m$ in the states s and s' must be identical except for one function in each determinant and they must be $\varphi^{s'p'}$ and φ^{sp} respectively. So we assume that Φ^{sp}'s are approximately orthonormal in general. In the above argument the motion of the center of mass was ignored and so it does not hold for light nuclei. A similar argument applies to the second term of equation (9) since

$$f_n^s = \sum_p c_n^{sp}\,\varphi^{sp}$$

where c_n^{sp} is defined now.

When the compound nucleus wave function is developed in terms of the single particle wave function

$$\Psi_n = \sum_{s,p} c_n^{sp}\,\Phi^{sp} \tag{14}$$

the coefficient $|c_n^{sp}|$ is large if the eigenvalue E_n is nearly equal to the single particle energy E^{sp}. Otherwise it is assumed to be small. This is the intermediate model. In the single particle model Ψ_n and Φ^{sp} become identical and in the strong coupling model the development coefficients $|c_n^{sp}|$ are distributed uniformly.

The actual wave function can be written in terms of the single particle wave function by using equations (6), (9′) and (14), apart from a constant factor,

$$\Psi(\boldsymbol{r}_0, \cdots, \boldsymbol{r}_A) = \frac{1}{\sqrt{A+1}} \sum_n \sum_i \sum_{p,s,p} \frac{1}{E_n - E} c_n^{0p}\, c_n^{sp'} (-)^{P_{0i}} P_{0i} \psi_s(\tilde{\boldsymbol{\xi}}_{-0})\varphi^{0p}(\boldsymbol{R})\varphi^{sp'}(\boldsymbol{r}_0), \tag{15}$$

where we write $\tilde{\boldsymbol{\xi}}_{-0}$ for $\boldsymbol{r}_1, \cdots, \boldsymbol{r}_A$ for brevity. Since we are interested in the

behaviour of the wave function when it is averaged over the small energy interval containing many resonances, we divide the sum over n into two parts:

$$\sum_n \frac{1}{E_n - E} c_n^{0p}\, c_n^{sp'} = \sum_{n \subset I} \frac{1}{E_n - E} c_n^{0p}\, c_n^{sp'} + \sum_{n \supset I} \frac{1}{E_n - E} c_n^{0p}\, c_n^{sp'}, \tag{16}$$

where I is a small interval for which $E_n \simeq E$. For the first term averaging gives

$$\left\langle \sum_{n \subset I} \frac{1}{E_n - E} c_n^{0p}\, c_n^{sp'} \right\rangle_{AV} = \frac{\pi i}{D(E)} \langle c_n^{0p}\, c_n^{sp'} \rangle I, \tag{16a}$$

where $D(E)$ is the average level spacing of the compound nucleus, and for the second term the averaged result is almost the same as the unaveraged. In accordance with the intermediate model the latter can be developed as

$$\sum_{n \supset I} \frac{1}{E_n - E} c_n^{0p}\, c_n^{sp'} = \sum_{n \supset I} \frac{1}{E^{0p} - E} c_n^{0p}\, c_n^{sp'} + \sum_{n \supset I} \frac{(E^{0p} - E_n)}{(E^{0p} - E)(E^{sp'} - E)} c_n^{0p}\, c_n^{sp'} + \sum_{n \supset I} \frac{(E^{0p} - E_n)(E^{sp'} - E_n)}{(E^{0p} - E)(E^{sp'} - E)(E_n - E)} c_n^{0p}\, c_n^{sp'}. \tag{16b}$$

In the third term of the above development we have factors in the numerator which are small when $|c_n^{0p}|$ or $|c_n^{sp'}|$ is large. Therefore the third term will be smaller than the former two and it is neglected below.

For the first and the second term we make use of the sum rules which are derived from the completely antisymmetrized wave functions. They are, under the approximation that Φ^{sp}'s are orthonormal,

$$\sum_n c_n^{sp}\, c_n^{s'p'} = \delta_{ss'}\, \delta_{pp'}, \tag{17a}$$

$$\sum_n (E_{sp} - E_n) c_n^{sp}\, c_n^{sp'} = \langle sp(0) | U - V_0 | s'p'(0) \rangle - A \langle sp(1) | U - V_0 | s'p'(0) \rangle \tag{17b}$$

where U is the average interaction to be determined later and

$$V_0 = \sum_{i=1}^{A} V_{0i}.$$

$|sp(0)\rangle$ is the wave function of the form (12), the number in the parenthesis indicating the particle occupying the single particle state. Equation (17a) follows from the orthonormality of the wave functions Φ_{sp}. In (17b) the factor A in the second term is only apparent since, in the Slater determinant approximation, the contributing nuclear force is only V_{01}, for the nuclear forces for the other pairs the

integrals go to zero because of the exchange overlap integrals. So the second term is at most of the same order of magnitude as the first. When the Pauli principle is violated the second term just cancells the first term. Strictly speaking the second term of (17b) must be symmetrized with respect to 0 and 1 but we omit it for brevity.

Inserting (16a,b) and (17a,b) in (15) the wave function Ψ becomes approximately

$$\Psi=\frac{1}{\sqrt{A+1}}\sum_i(-)^{p_{0i}}P_{0i}\sum_s\Big\{\pi i\sum_{p\,p'}\frac{\langle c_n^{0p}\,c_n^{sp'}\rangle_I}{D(E)}\varphi^{0p}(\boldsymbol{R})\varphi^{sp'}(\boldsymbol{r}_0)+G_0(\boldsymbol{R},\boldsymbol{r}_0)\delta_{0s}$$
$$+\int G_0(\boldsymbol{R},\boldsymbol{r}')G_s(\boldsymbol{r}_0,\boldsymbol{r}')\langle 0(0)|U-V_0|s(0)\rangle_{r'}d\boldsymbol{r}' \qquad (18)$$
$$-A\iint G_0(\boldsymbol{R},\boldsymbol{r}')G_s(\boldsymbol{r}_0\,\boldsymbol{r}'')\langle 0(1)|U-V_0|s(0)\rangle_{r',r''}d\boldsymbol{r}'d\boldsymbol{r}''\Big\}\psi_s(\xi_{-0})$$

where $G_s(\boldsymbol{r},\boldsymbol{r}')$ is the Green function of the single particle Schroedinger equation with the average interaction,

$$G_s(\boldsymbol{r},\boldsymbol{r}')=\sum_p\frac{\varphi^{sp}(\boldsymbol{r})\varphi^{sp}(\boldsymbol{r}')}{E^{sp}-E}.$$

In deriving equation (18) it is assumed that the number of levels in $\sum_{n\subset I}$ is rather small and $\sum_{n\supset I}$ is replaced by $\sum_{all\ n}$. This is valid if E is far away form any E^{sp}. Even if E is near one of E^{sp}'s this will still be permissible unless $|c_n^{sp}|$'s have a very narrow distribution.

As the average interaction we now take, for the target nucleus in the state s,

$$U_s=\langle s(0)|V_0|s(0)\rangle_r-A\int\langle s(1)|V_0|s(0)\rangle_{rr'}\,d\boldsymbol{r}' \text{ (to be operated)}$$
$$\equiv U_s^c(\boldsymbol{r})+\int K_s(\boldsymbol{r},\boldsymbol{r}')d\boldsymbol{r}' \text{ (to be operated)}. \qquad (19)$$

c in U_s^c will be omitted below. Then it can be shown that in equation (18) the integral terms cancel each other for the incident channel provided the exchange integral for U is zero. The last assumption is reasonable for large nuclei for which U will be nearly constant in the nucleus.

§ 5 The derivation of the imaginary part

The first imaginary term in equation (18) must be determined by solving the Schroedinger equation (2) approximately. Writing the wave function in the form (8) we notice a similarity with Wheeler's resonating group method[5]. The only difference is that here we take only one group and consider all the excited states of the group. The equation for the incident channel becomes, by multiplying $\psi_0(\boldsymbol{r}_0,\cdots\boldsymbol{r}_A)$ on equation (2) from the left and intergrating over all r's except $\boldsymbol{r}_0$,

$$[T-\varepsilon+\int\{U_0(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')+K_0(\boldsymbol{r},\boldsymbol{r}')\}d\boldsymbol{r}']f^0(\boldsymbol{r}')$$

$$=-\sum_{s\neq 0}\int\{U_{0s}(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')+K_{0s}(\boldsymbol{r},\boldsymbol{r}')\}f^s(\boldsymbol{r}')d\boldsymbol{r}' \qquad (20)$$

where $U_{0s}(\boldsymbol{r})+K_{0s}(\boldsymbol{r},\boldsymbol{r}')$ is defined by

$$U_{0s}(\boldsymbol{r})+\int K_{0s}(\boldsymbol{r},\boldsymbol{r}')d\boldsymbol{r}' \text{ (to be operated)}$$

$$=\langle 0(0)|V_0|s(0)\rangle_r-A\int\langle 0(1)|V_0|s(0)\rangle_{rr'}d\boldsymbol{r}' \text{ (to be opeated).} \qquad (19')$$

In deriving (20) we neglected the exchange overlap integral and the exchange kinetic energy integral. These are small for large nuclei. The equations for other channels become

$$[T-\varepsilon+\mathfrak{E}_s^{ex}+\int\{U_s(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')+K_s(\boldsymbol{r},\boldsymbol{r}')\}d\boldsymbol{r}']f^s(\boldsymbol{r}')$$

$$=-\int\{U_{0s}(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')+K_{0s}(\boldsymbol{r},\boldsymbol{r}')\}d\boldsymbol{r}'f^0(\boldsymbol{r}') \qquad (21)$$

$$-\sum_{s'\neq 0,s}\int\{U_{ss'}(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')+K_{ss'}(\boldsymbol{r},\boldsymbol{r}')\}d\boldsymbol{r}'f^{s'}(\boldsymbol{r}').$$

Equations (20) and (21) are exact apart from small exchange overlap integrals.

In the weak coupling limit all coupling terms are small. In the strong coupling limit, on the other hand, all coupling terms will be equally important. In the intermediate case we substitute for $f^{s'}(r)$ in equation (21) the expression in brackets of equation (18). Then the right-hand side becomes (U in equation (18) is assumed to be diagonal)

$$-\int(U_{0s}\delta+K_{0s})d\boldsymbol{r}'(\pi i\,g^{00}(\boldsymbol{r}')+G_0(\boldsymbol{R},\boldsymbol{r}'))$$

$$+\sum_{s'\neq 0,s}\int(U_{ss'}\delta+K_{ss'})d\boldsymbol{r}'\cdot\{\pi i g^{0s'}(\boldsymbol{r}')+\iint(U_{0s'}\delta+K_{0s'})G_0(\boldsymbol{R},\boldsymbol{r}'')G_{s'}(\boldsymbol{r}',\boldsymbol{r}''')d\boldsymbol{r}''d\boldsymbol{r}'''\} \qquad (22)$$

where we put

$$\sum_{p,p'}\frac{\langle c_n^{0p}\,c_n^{sp'}\rangle_1}{D(E)}\varphi^{0p}(\boldsymbol{R})\varphi^{sp'}(\boldsymbol{r})=g^{0s}(\boldsymbol{r}).$$

In the last term of the expression (22) we have, apart from the small imaginary term $\pi i g^{0s'}$* the product of the matrix elements of the nuclear force between states 0-s' and s-s' and the Green function $G_{s'}(\boldsymbol{r}',\boldsymbol{r}''')$. Here we make a random sign

* This is also related to $U_{0s'}$.

approximation to the effect that the last term in equation (22) is summed to zero when suitable energy average is taken. If the signs of $U_{ss'}$ and $U_{0s'}$ are strongly correlated the correction from the second term of equation (22) will not be negligible, but we assume this is not the case.

With the aid of the intermediate model and the assumption that the non-diagonal elements $U_{ss'}$'s have random signs the wave function of the channel s couples only to the incident channel on the average.* Then we can write $f^s(\boldsymbol{r})$ in terms of $f^0(\boldsymbol{r})$ by equation (21),

$$f^s(\boldsymbol{r}) = -\int G_s(\boldsymbol{r}, \boldsymbol{r}')\,d\boldsymbol{r}' \int \{U_{0s}\,\delta + K_{0s}\} f^0(\boldsymbol{r}'')\,d\boldsymbol{r}''. \tag{23}$$

We substitute this in equation (20) and the equation for the incident channel becomes

$$\{T-\varepsilon+\int (U_0\,\delta+K_0)\}f^0$$

$$=\sum_{s\neq 0}\iiint (U_{0s}\,\delta+K_{0s})G_s(U_{0s}\,\delta+K_{0s})f^0\,. \tag{24}$$

We expect to have the imaginary part from the right-hand side of the above equation, but if we regard $G_s(\boldsymbol{r}, \boldsymbol{r}')$ as the pure single particle Green function we do not get the physically meaningful imaginary part. It is true $G_s(\boldsymbol{r}, \boldsymbol{r}')$ is comlpex if the channel s is open but the imaginary part from this goes to zero as the incident energy decreases. Since the imaginary part is expected to be finite at zero energy, this is too crude an approximation. As the better approximation we regard $G_s(\boldsymbol{r}, \boldsymbol{r}')$ as the Green function when the channel s has infinitely weak coupling with the incident channel. Then the energy denominators in the expression

$$G_s(\boldsymbol{r}, \boldsymbol{r}') = \sum_p \frac{\varphi^{sp}(\boldsymbol{r})\varphi^{sp}(\boldsymbol{r}')}{E^{sp}-E} = \sum_p \frac{\varphi^{sp}(\boldsymbol{r})\varphi^{sp}(\boldsymbol{r}')}{\varepsilon^{sp}+\xi_s^{ex}-\varepsilon} \tag{25}$$

have infinitely small imaginary parts the exact form of which are irrelevant apart from the sign which is determined from the outgoing wave boundary condition. Because of this infinitely small imaginary part we get a finite imaginary part on

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* If we retain the last term of equation (21) for the channel s' and express the wave function $f^{s'}$ as equation (23) below in terms of $f^{s''}$, there is a contribution from f^s in this expression. If we substitute this $f^{s'}$ in the equation for the channel s we can pick up the correction term for the average interaction in the channel s. This correction makes the average interaction complex. This procedure, however, looks a little artificial once the random sign approximation is made. The form (23) is reasonable in view of the general theory $\bar{S}_{ab} = (\Phi_b, H_1 \Psi_a^{(+)})$[6] (the bar means the average over energy) with the intermediate model for $\Psi_a^{(+)}$ and the randomness of the signs of the non-diagonal interaction matrix elements. According to Bloch's theory for the direct interaction[10] G_s in equation (23) must be the single particle Green function for the complex optical potential. It is not clear to the author which is the more reasonable form.

averaging over energy ε^*,

$$<\sum_{s\neq 0}\iiint(U_{0s}\,\delta+K_{0s})G_s(U_{0s}\,\delta+K_{0s})>_{Av}$$

$$=P\int\frac{d\mathcal{E}_s^{ex}}{D'(\mathcal{E}_s^{ex})}\iiint(U_{0s}\,\delta+K_{0s})\sum_p\frac{\varphi^{sp}\cdot\varphi^{sp}}{\varepsilon^{sp}+\mathcal{E}_s^{ex}-\varepsilon}(U_{0s}\delta+K_{0s})$$

$$+\pi i\sum_{\varepsilon^{sp}+\mathcal{E}_s^{ex}\simeq\varepsilon}\frac{1}{D'(\mathcal{E}_s^{ex})}\iiint(U_{0s}\,\delta+K_{0s})\varphi^{sp}\varphi^{sp}(U_{0s}\,\delta+K_{0s}) \qquad (26)$$

where $D'(\mathcal{E}_s^{ex})$ is the level spacing of the target nucleus at the excitation energy $\mathcal{E}_s^{ex}$. This procedure will be the same as the inclusion of $i\varepsilon$ in the energy denominator in the Lippmann-Schwinger's operator formalism[6] but less formal. The first term of the above expression is the second order correction to the average potential and the second term is the imaginary potential we have been seeking.

A similar procedure was taken by Cini and Fubini[7] for an infinitely large Fermi gas.

§ 6. Discussion of the result

The average interaction obtained above (equation (19)) is non-local as it should be[8] and makes it appear as energy dependent when it is regarded as local[9]. The second order correction given by the first term of equation (26) will not be very large unless the nuclear force has very odd radial dependence[8]. The energy independent non-local interaction has the advantage that the eigenfunctions corresponding to different eigenvalues become orthogonal to each other. If the average interaction is assumed to be energy dependent, eigenfunctions are not orthogonal to esch other for the finite nucleus.

The imaginary interaction is, apart from being non-local, essentially energy dependent since the contribution to it comes from the single particle states *below* the incident energy. This is so because the imaginary part appears if

$$\varepsilon^{sp}+\mathcal{E}_s^{ex}-\varepsilon\simeq 0$$

and

$$\mathcal{E}_s^{ex}>0.$$

Further, the exchange term ensures the Pauli principle and it cancels the local term when the Pauli principle is violated. This can be most easily seen from the second sum rule (17b) by taking Slater determinants for the target wave functions. Since

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* Here it is assumed that

$$\langle G_s f^0\rangle_{Av}=\langle G_s\rangle_{Av}\cdot\langle f^0\rangle_{Av}.$$

This will be permissible since the poles of f^0, which are the actual resonances, are approximately independent of the poles of G_s.

the nuclear temperature corresponding to the low incident energy is very low compared to the average kinetic energy of nucleons in the nucleus, in the excited states contributing to the imaginary potential only few nucleons are excited from the ground state. Therefore, the contribution to the imaginary potential will come essentially from *the unoccupied states below the incident energy.*

Thus the correspondence to the semi-classical argument is apparent. The only difference is that in the semi-classical case only one particle in the target nucleus is excited whereas in the present case the target nucleus is excited as a whole. However, since the relevant quantity is the matrix element of the two-body nuclear force, the excited states which are very different from the ground state will contribute very little. If the corresponding excited state is nearly a single particle state the contribution from it will be fairly large. From this we can expect that the imaginary potential will increase abruptly when the shape resonance is passed, since in this case we have a single particle state just below the incident energy and the corresponding excited state of the target nucleus is near the ground state. For heavy nuclei this increase will not be appreciable since there are many unoccupied single particle states, but for medium weight nuclei near $A=60$ this effect may be important. The single particle state which is occupied by the incident particle and the corresponding nearly single particle state of the target nucleus must satisfy some selection rules regarding the spin and parity and this may cause some fluctuation of the imaginary potential.

It is interesting to note that the imaginary potential obtained above

$$\pi i \sum_{\varepsilon^{sp}+\varepsilon_s^{ex}\simeq\varepsilon} \frac{1}{D'(\Xi_s^{ex})} \int \{U_{0s}(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}') + K_{0s}(\boldsymbol{r},\boldsymbol{r}')\varphi^{sp}(\boldsymbol{r}')\}\, d\boldsymbol{r}' \cdot$$

$$\times \iint \varphi^{sp}(\boldsymbol{r}'')\{U_{0s}(\boldsymbol{r})\delta(\boldsymbol{r}''-\boldsymbol{r}''') + K_{0s}(\boldsymbol{r}'',\boldsymbol{r}''')\}\, d\boldsymbol{r}''d\boldsymbol{r}'''f(\boldsymbol{r}''')$$

has a peculiar non-locality. When the exchange interaction is neglected it is still non-local but separable

$$\pi i \sum_{\varepsilon^{sp}+\varepsilon_s^{ex}\simeq\varepsilon} \frac{1}{D'(\Xi_s^{ex})} U_{0s}(\boldsymbol{r})\varphi^{sp}(\boldsymbol{r}) \int \varphi^{sp}(\boldsymbol{r}')U_{0s}(\boldsymbol{r}')f(\boldsymbol{r}')\, d\boldsymbol{r}'. \tag{27}$$

This becomes local if and only if the average potential determining the wave function φ^{sp} is independent of s, and

$$\varphi^{sp}(\boldsymbol{r})\varphi^{sp}(\boldsymbol{r}')$$

is summed over all single particle states with equal weight. As the sum is taken only over the unoccupied states below the incident energy, the non-locality of the order of the wave length of the inter-nuclear particles always remains. At first sight the sum looks a little complicated. However, if we assume that the average interaction U_s defined by equation (19) is nearly independent of s, we can first take the unoccupied single particle state and seek for the excited state of the target

nucleus for which the total energy becomes almost conserved.

If we calculate the single particle strength function from equation (24) by using equation (27), with the proper restriction for the sum, for the right-hand side we will get a result similar to that of Bloch[2a),10)] including the effect of the Pauli principle approximately. Bloch's method of collecting the "leading parts" of the expansion of the S-matrix similar to the perturbation theory is equivalent to keeping the lowest term in the Schroedinger equation and solving the equation exactly.

To derive the imaginary part explicitly the single particle wave functions φ^{ν} do not cause any difficulty but the nuclear matrix elements and the level density are very difficult to estimate. This is so even in the approximation (27) since for the target wave function we must take the exact one. It may become manageable if we make a suitable independent particle approximation for the target nucleus but even in this case the problem looks extremely complicated for the finite nucleus.

Frantz et al.[11)] proposed an optical potential which is essentially of the form

$$(T+U)f(x)+i\sum_{p}{}'u_p(x)\int W(x')u_p(x')f(x')dx'=\varepsilon f(x)$$

where the sum over p is to be done for *all* unoccupied states. The result obtained in the present paper can also be written in the above form with suitable approximations (particularly for the one dimensional problem) but the sum over p is to be done for the unoccupied states *below* the incident energy. Frantz et al. proposed the above form so that the solution for the scattering state becomes orthogonal to the bound occupied states, but the physical nature of the imaginary potential was not considered. In our case we can clearly see the physical picture that the imaginary potential is due to the process in which the incident particle jumps down to an unoccupied single particle state and excites the target nucleus so that the total energy is approximalely conserved.

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The Electric and Magnetic Response of a Thermodynamic System

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When a system in thermal equilibrium is exposed to arbitrary but weak electromagnetic fields, currents are induced by the fields, proportional to the field strengths. The Kubo formalism allows a general discussion of this "linear response", and the calculation has been carried out by Nakajima. However, the separation of the induced current into an "electric" and a "magnetization" current requires additional physical assumptions. This separation is carried out here. The "electric" current is then further separated into in phase and out of phase contributions. This yields an explicit formula for the complex dielectric constant as a function of the wave number and frequency of the impressed electric field. These formulas have the same generality as the formulas of equilibrium statistical mechanics.

§ 1. Introduction

We shall consider a thermodynamic system which, in the absence of electromagnetic fields, is governed by the Hamiltonian H_0:

$$H_0=\sum_{i=1}^{N}(\boldsymbol{p}_i)^2/2m_i+V(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N). \tag{1·1}$$

This system is assumed to be in equilibrium at time $t=-\infty$. Its statistical matrix at that time is

$$U_0=\exp[\beta(F_0-H_0)], \tag{1·2}$$

where $\beta=1/kT$, and F_0 is the free energy in the absence of electromagnetic fields.

The system is then exposed to the influence of arbitrary space and time dependent electric and magnetic fields; these fields can be derived from vector and scalar potentials through the usual equations;

$$\boldsymbol{E}(\boldsymbol{x}, t)=-\text{grad}\,\phi-(1/c)\partial\boldsymbol{A}/\partial t, \tag{1·3}$$

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$$\boldsymbol{B}(\boldsymbol{x},\ t)=\text{curl}\ \boldsymbol{A}. \tag{1·4}$$

When these potentials are introduced into the Hamiltonian in the usual fashion, the Hamiltonian becomes time-dependent, and so does the statistical matrix U. The latter satisfies the differential equation

$$dU/dt=(i\hbar)^{-1}[H(t),\ U(t)]_- \tag{1·5}$$

with the initial condition

$$U(-\infty)=U_0\,. \tag{1·6}$$

Kubo[1] has pointed out that (1·5) and (1·6) can be solved explicitly to terms linear in the potentials, and that this solution suffices to get expressions for the linear part of the induced current, of the same generality as the expressions of equilibrium statistical mechanics. If we let $\boldsymbol{J}(\boldsymbol{x},\ t)$ be the expectation value of the current density, at point $\boldsymbol{x}$ and time t, over the statistical distribution $U(t)$, then to terms linear in the fields we expect to find the following form:

$$\boldsymbol{J}(\boldsymbol{x},\ t)=\boldsymbol{J}_{el}(\boldsymbol{x},\ t)+c\ \text{curl}\ \boldsymbol{M}(\boldsymbol{x},\ t), \tag{1·7}$$

where the "electric" current is*

$$\boldsymbol{J}_{el}(\boldsymbol{x},\ t)=\int_{-\infty}^{t}dt'\int d^3x'\ Y(\boldsymbol{x}-\boldsymbol{x}',\ t-t')\boldsymbol{E}(\boldsymbol{x}',\ t') \tag{1·8}$$

and the magnetization density is

$$\boldsymbol{M}(\boldsymbol{x},\ t)=\int_{-\infty}^{t}dt'\int d^3x'\ K(\boldsymbol{x}-\boldsymbol{x}',\ t-t')\boldsymbol{B}(\boldsymbol{x}',\ t')\,. \tag{1·9}$$

The functions $Y(\boldsymbol{x},\ t)$ and $K(\boldsymbol{x},\ t)$ defined by (1·8) and (1·9), respectively, will be called electric and magnetic response functions of the medium. The electric part of the current, (1·8), is usually written as a sum of two terms, the "conduction" and "dielectric polarization" contributions:[2]

$$\boldsymbol{J}_{el}(\boldsymbol{x},\ t)=\boldsymbol{J}_{con}(\boldsymbol{x},\ t)+\partial\boldsymbol{P}(\boldsymbol{x},\ t)/\partial t\,. \tag{1·10}$$

We therefore have one physical quantity, the induced current $\boldsymbol{J}(\boldsymbol{x},\ t)$, which is to be written as a sum of three parts:

$$\boldsymbol{J}(\boldsymbol{x},\ t)=\boldsymbol{J}_{con}(\boldsymbol{x},\ t)+\partial\boldsymbol{P}(\boldsymbol{x},\ t)/\partial t+c\ \text{curl}\ \boldsymbol{M}(\boldsymbol{x},\ t). \tag{1·11}$$

The usual way of making this separation is by considering a special model of the substance. In this model there are free charges which can move about, electric dipoles which can have their permanent moments aligned with an electric field, or can obtain an induced electric moment, and there are magnetic dipoles with similar properties with respect to a magnetic field. In (1·11), the part $\boldsymbol{J}_{con}$ is said to be due to the moving free charges, the second term is due the electric dipoles, and the last term due to the magnetic dipoles.

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* We shall assume throughout that the system has cubic symmetry, so that the response function Y and K are scalars.

This physical picture works well for a dilute gas, but it suffers from the same defect as the (very closely related) kinetic theory of gases: Serious troubles, not merely of calculation but of principle, are encountered as soon as the interaction between the molecules becomes important. As long as the separation of terms in (1·11) is based upon a specific model of the substance, it is in principle impossible to decide how to perform the separation in cases where the model breaks down.

In this paper, we shall perform the separation (1·11) in a very general fashion based on postulates independent of specific models of the substance. The result is then a set of equations for the coefficients which enter into the linear response of the medium (the dielectric constant, the conductivity, and the magnetic susceptibility) of the same generality as the equations of equilibrium statistical mechanics.

The separation (1·11) will be carried out in two stages: first we separate the magnetic and electric contributions, i.e., obtain equations (1·7)–(1·9) in explicit form; second, we separate the electric current into the two parts (1.10). This latter separation, carried through in section 3 of this paper, is quite simple. The former separation, however, is not altogether trivial. Since the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are introduced into the formalism not as such, but rather through the potentials $\boldsymbol{A}$ and ϕ, it follows that two of the four Maxwell equations are satisfied automatically by any fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which can be introduced. These Maxwell equations are

$$\operatorname{div} \boldsymbol{B}=0, \tag{1·12}$$

$$c \operatorname{curl} \boldsymbol{E}=-\partial \boldsymbol{B}/\partial t. \tag{1·13}$$

Equation (1·13) relates $\boldsymbol{B}$ to the "transverse" part of $\boldsymbol{E}$; in fact, since by assumption both fields go to zero for $t\to-\infty$, (1·13) allows us to express the field $\boldsymbol{B}$ entirely in terms of a time integral over the electric field $\boldsymbol{E}$; thus, purely formally, we could express the entire response of the medium as an integral of type (1·8). Or, conversely, we could use (1·13) to rewrite the contribution to (1·8) coming from the "transverse" part of $\boldsymbol{E}$ as a "magnetic" response.

It follows that the separation (1·7) requires additional physical assumptions. Two reasonable assumptions are:

1) *Localizability assumption:*

We split the electric field into a "parallel" and a "transverse" part:

$$\boldsymbol{E}=\boldsymbol{E}_p+\boldsymbol{E}_t \quad \operatorname{curl} \boldsymbol{E}_p=0 \qquad \operatorname{div} \boldsymbol{E}_t=0. \tag{1·14}$$

The electric part of the current would then be, in general,

$$\boldsymbol{J}_{el}(\boldsymbol{x}, t)=\int_{-\infty}^{t} dt' \int d^3x' \left[Y_p(\boldsymbol{x}-\boldsymbol{x}', t-t') \boldsymbol{E}_p(\boldsymbol{x}', t') + Y_t(\boldsymbol{x}-\boldsymbol{x}', t-t') \boldsymbol{E}_t(\boldsymbol{x}', t') \right]. \tag{1·15}$$

However, the splitting (1·14) is "non-local", i.e., it requires knowledge of the field $\boldsymbol{E}$ over the entire specimen. Hence, if (1·15) is to be localizable, we

must have

$$Y_p = Y_t. \tag{1·16}$$

Furthermore, if as a result of some calculation Y_p and Y_t are apparently different, then the electric response must be taken to be Y_p, and the difference $Y_t - Y_p$ must be interpreted as a magnetic term. The reason for this is that one possible electric field is a purely parallel one, obtained from the choice of potentials $\phi \neq 0$, $\boldsymbol{A} = 0$.

2) *Causality assumption*:

The response of the system to the impressed fields $\boldsymbol{E}$ and $\boldsymbol{B}$ cannot be instantaneous. An instantaneous and memory-less response would be indicated in (1·8) or (1·9) by terms which involve no integration over t', i.e., by terms in either $Y(\boldsymbol{x}, t)$ or $K(\boldsymbol{x}, t)$ proportional to $\delta(t-t')$. The causality assumption leads to the requirement that there should not be any such terms.

The calculation of $\boldsymbol{J}(\boldsymbol{x}, t)$ on the basis of the Kubo formalism has been carried out by Nakajima,[3] who arrives at the following expression:

$$J_\mu(\boldsymbol{x}, t) = \int d^3x' \, J_{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}') A_\nu(\boldsymbol{x}', t) + c^2 \int_{-\infty}^{t} dt' \int d^3x' \, S_{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}', t-t') E_\nu(\boldsymbol{x}', t') \tag{1·17}$$

where $J_{\mu\nu}$ and $S_{\mu\nu}$ are definite averages over the field free ensemble U_0. Nakajima interprets the first integal as the magnetic response, the second as the electric response of the system, However, the first term of (1·17) contains no integration over previous times, and thus indicates an instantaneous and memory-less magnetic response. As Nakajima remarks: "Since the kernel $J_{\mu\nu}$ is independent of time, there is no magnetic time-relaxation."

In the present paper we shall start with Nakajima's expression, and carry out further transformations on it so as to satisfy our assumptions (1) and (2). We shall see that localizability is enough to make the separation into electric and magnetic response unique; this unique expression turns out to satisfy the causality condition automatically. This is done in section 2.*

In Section 3 we then separate the electric part of the current into conduction and dielectric polarization currents, and thereby obtain explicit, general expressions for the frequency and wave-number-dependent conductivity and dielectric constant. As an example of the use of these expressions, we compute, in section 4, the dielect-

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* It is possible to construct systems in which this method of separation gives unreasonable results. For example, if we imagine an ideal gas of objects which are both electric dipoles and magnetic dipoles, then an applied electric feld, by aligning these objects, produces not only electric polarization but also a magnetic polarization, and the converse situation occurs for an applied magnetic field. However, with physical molecules, this cannot occur, because of parity symmetry. If the electric dipole moment operator has a non-vanishing matrix element between two quantum states n and n', then the magnetic dipole moment matrix element vanishes, and vice versa.

ric constant of an ideal gas of molecules, thereby showing how our more general formulae reduce to the ones of the conventional theory of electric susceptibilities.[4]

Although our formulae apply to arbitrary media, some care must be used in their interpretation when the medium in question is a superconductor. The assumption of localizability used in the separation of the electric and magnetic contributions may not be a reasonable assumption in superconducting materials. It is certainly not reasonable if the material satisfies the London[5] phenomenological equations. If the splitting in question is nevertheless used for such a material, the results, though still correct, appear in a rather awkward form. Even in a normal material, care must be exercised if the applied fields are space- and time-independent. These points will be discussed in more detail in a forthcoming publication.[6] For normal materials, however, it is sufficient to start with space- and time-dependent fields and go to the limit of zero frequency and zero wave number.

Some additional discussion of the expressions found here is given in section 5.

§ 2. Reduction to local and causal form

Since we are interested in the *linear* part of the response, it is convenient to make a Fourier decomposition of the fields and consider one Fourier component at a time. For the moment, we shall make a Fourier decomposition in space only, not in time. We therefore write our fields and potentials in the form*

$$\boldsymbol{B}(\boldsymbol{x}, t)=\boldsymbol{B}(\boldsymbol{q}, t)(2/V)^{1/2}\cos(\boldsymbol{q}\cdot\boldsymbol{x}), \tag{2·1a}$$

$$\boldsymbol{E}(\boldsymbol{x}, t)=\boldsymbol{E}(\boldsymbol{q}, t)(2/V)^{1/2}\sin(\boldsymbol{q}\cdot\boldsymbol{x}), \tag{2·1b}$$

$$\boldsymbol{A}(\boldsymbol{x}, t)=\boldsymbol{A}(\boldsymbol{q}, t)(2/V)^{1/2}\sin(\boldsymbol{q}\cdot\boldsymbol{x}), \tag{2·1c}$$

$$\phi(\boldsymbol{x}, t)=\phi(\boldsymbol{q}, t)(2/V)^{1/2}\cos(\boldsymbol{q}\cdot\boldsymbol{x}). \tag{2·1d}$$

The Maxwell equations (1·12) and (1·13) become:

$$\boldsymbol{q}\cdot\boldsymbol{B}(\boldsymbol{q}, t)=0, \tag{2·2a}$$

$$c\boldsymbol{q}\times\boldsymbol{E}(\boldsymbol{q}, t)=-\partial\boldsymbol{B}(\boldsymbol{q}, t)/\partial t, \tag{2·2b}$$

and the Fourier decomposition of the induced current $\boldsymbol{J}(\boldsymbol{x}, t)$ is

$$\boldsymbol{J}(\boldsymbol{x}, t)=\boldsymbol{J}(\boldsymbol{q}, t)(2/V)^{1/2}\sin(\boldsymbol{q}\cdot\boldsymbol{x}). \tag{2·3}$$

With this notation, Nakajima's result[3] (i.e., our equation (1·17)) assumes the form:

$$J_\mu(\boldsymbol{q}, t)=c\Big[S_{\mu\nu}(\boldsymbol{q}, 0)-\bar{L}\delta_{\mu\nu}\Big]A_\nu(\boldsymbol{q}, t)+c^2\int_{-\infty}^{t}dt'\ S_{\mu\nu}(\boldsymbol{q}, t-t')E_\nu(\boldsymbol{q}, t') \tag{2·4}$$

where the quantities $S_{\mu\nu}(\boldsymbol{q}, t)$ and $\bar{L}$ are defined as follows: We introduce the

* The alternation of sines and cosines in these expressions is required by the Maxwell equations and by the relations (1·3), (1·4).

operator for the q'th Fourier component of the electric charge density:

$$\rho(\boldsymbol{q})=\sum_{i=1}^{N} e_i(2/V)^{1/2}\cos(\boldsymbol{q}\cdot\boldsymbol{x}_i), \tag{2·5}$$

and the operator for the q'th Fourier component of the current density:*

$$\boldsymbol{j}(\boldsymbol{q})=\sum_{i=1}^{N}(e_i/2m_i)(2/V)^{1/2}[\boldsymbol{p}_i\sin(\boldsymbol{q}\cdot\boldsymbol{x}_i)+\sin(\boldsymbol{q}\cdot\boldsymbol{x}_i)\boldsymbol{p}_i]. \tag{2·6}$$

With this notation, the operator L is given by

$$L=-\frac{[[H_0,\rho(\boldsymbol{q})],\rho(\boldsymbol{q})]}{(\hbar c)^2q^2}=\sum_{i=1}^{N}\frac{2e_i^2}{Vm_ic^2}\sin^2(\boldsymbol{q}\cdot\boldsymbol{x}_i) \tag{2·7}$$

and $\bar{L}$ in equation (2·4) is the statistical average of L over *field-free* distribution U_0:

$$\bar{L}=\mathrm{Tr}(LU_0)=V^{-1}\sum_{i=1}^{N}e_i^2/m_ic^2. \tag{2·8}$$

The last form follows form the translation invariance of the field-free system, which allows us to replace $\sin^2(\boldsymbol{q}\cdot\boldsymbol{x}_i)$ by its average value, 1/2. Finally, the quantity $S_{\mu\nu}(\boldsymbol{q},t)$ in (2·4) is defined by

$$S_{\mu\nu}(\boldsymbol{q},t)=(1/c^2)\mathrm{Tr}\int_0^\beta ds\,U_0j_\nu(\boldsymbol{q})j_\mu\Big(\boldsymbol{q},s-\frac{it}{\hbar}\Big), \tag{2·9}$$

where

$$j_\mu(\boldsymbol{q},s)\equiv\exp(-sH_0)j_\mu(\boldsymbol{q})\exp(+sH_0). \tag{2·10}$$

Although equation (2·4) involves the vector potential explicitly, it is actually gauge invariant, due to the Buckingham identity[7]

$$\bar{L}q_\mu-S_{\mu\nu}(\boldsymbol{q},0)q_\nu=q_\nu\bar{L}-q_\mu S_{\mu\nu}(\boldsymbol{q},0)=0. \tag{2·11}$$

For systems of cubic symmetry, this identity assumes the form

$$\sum_{\mu,\nu}(q_\mu q_\nu/q^2)S_{\mu\nu}(\boldsymbol{q},0)-\bar{L}=0. \tag{2·12}$$

Let us now transform (2·4) so as to satisfy the localizability condition discussed in section 1. The decomposition of the electric field $\boldsymbol{E}$ into two parts becomes

$$\boldsymbol{E}=\boldsymbol{E}_p+\boldsymbol{E}_t=(1/q^2)[(\boldsymbol{q}\cdot\boldsymbol{E})\boldsymbol{q}-\boldsymbol{q}\times(\boldsymbol{q}\times\boldsymbol{E})]. \tag{2·13}$$

If we let the direction of the vector $\boldsymbol{q}$ be direction number 3, for the moment, then $\boldsymbol{E}_p$ is just the component E_3. Furthermore, with cubic symmetry, the tensor $S_{\mu\nu}(\boldsymbol{q},t)$ is diagonal in this system, with $S_{11}=S_{22}$, but not equal to S_{33}, since the q-direction is clearly a preferred direction. Thus the integral in (2·4) becomes

$$I=\int_{-\infty}^{t}dt'\left[c^2S_{11}(\boldsymbol{q},t-t')\boldsymbol{E}_t(\boldsymbol{q},t')+c^2S_{33}(\boldsymbol{q},t-t')\boldsymbol{E}_p(\boldsymbol{q},t')\right]. \tag{2·14}$$

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* This is the point at which we ignore electron spin and the currents due to it

This is precisely the Fourier image of the form (1·15), and *we see that we must identify* $c^2S_{33}(\boldsymbol{q}, t)$ *as the true electric response function* Y:

$$Y(\boldsymbol{q}, t)=\frac{1}{q^2}\mathrm{Tr}\int_0^\beta d\lambda\, U_0(\boldsymbol{q}\cdot\boldsymbol{j}(\boldsymbol{q}))\Big(\boldsymbol{q}\cdot\boldsymbol{j}(\boldsymbol{q}, \lambda-\frac{it}{\hbar})\Big). \tag{2·15}$$

By using the time-dependent charge density operator

$$\rho(\boldsymbol{q}, t)=\exp(itH_0/\hbar)\rho(\boldsymbol{q})\exp(-itH_0/\hbar) \tag{2·16}$$

and the continuity equation

$$\boldsymbol{q}\cdot\boldsymbol{j}(\boldsymbol{q})=(i\hbar)^{-1}[H_0,\ \rho(\boldsymbol{q})]_-, \tag{2·17}$$

it can be shown that (2·15) is equal to

$$Y(\boldsymbol{q}, t)=-(\hbar^2q^2)^{-1}\mathrm{Tr}\Big\{U_0\Big[[H_0,\ \rho(\boldsymbol{q},\ t)],\rho(\boldsymbol{q})\Big]\Big\}. \tag{2·18}$$

The quantity $\bar{L}$, (2,7) and (2·8), is related to the electrical response function $Y(\boldsymbol{q}, t)$ for $t=0$:

$$\bar{L}=Y(\boldsymbol{q},\ 0)/c^2. \tag{2·19}$$

We have now identified the "electric" part of the current, and found an expression for the function $Y(\boldsymbol{x}, t)$ in (1·8), namely the Fourier inverse of (2·18). It remains to reduce the magnetic part of $\boldsymbol{J}$ to the form $c\,\mathrm{curl}\,\boldsymbol{M}$, or, in Fourier notation:

$$\boldsymbol{J}_{mag}(\boldsymbol{q}, t)=-c\boldsymbol{q}\times\boldsymbol{M}(\boldsymbol{q}, t) \tag{2·20a}$$

with

$$\boldsymbol{M}(\boldsymbol{x}, t)=\boldsymbol{M}(\boldsymbol{q}, t)\,(2/V)^{1/2}\cos\,(\boldsymbol{q}\cdot\boldsymbol{x}). \tag{2·20b}$$

The "magnetic" part of (2·14) equals

$$I_{mag}=c^2\int_{-\infty}^{t}dt'\Big[S_{11}(\boldsymbol{q}, t-t')-S_{33}(\boldsymbol{q}, t-t')\Big]\boldsymbol{E}_t(\boldsymbol{q}, t'). \tag{2·21}$$

It is convenient to define a quantity $T(\boldsymbol{q}, t)$ by

$$T(\boldsymbol{q},\ t)=\frac{S_{11}(\boldsymbol{q}, t)-S_{33}(\boldsymbol{q}, t)}{q^2}=\frac{1}{q^2}\sum_{\mu,\nu}\frac{1}{2}\Big(\delta_{\mu\nu}-\frac{3q_\mu q_\nu}{q^2}\Big)S_{\mu\nu}(\boldsymbol{q},\ t). \tag{2·22}$$

Using (2·2b), (2·13), and (2·22), the magnetic part (2·21) becomes

$$I_{mag}=c\int_{-\infty}^{t}dt'\ T(\boldsymbol{q}, t-t')\boldsymbol{q}\times\frac{\partial\boldsymbol{B}(\boldsymbol{q}, t')}{\partial t'}.$$

An integration by parts is required to obtain an expression of the form (1·9); we therefore define the Fourier transform of $K(\boldsymbol{x}, t)$ in (1·9) by:

$$K(\boldsymbol{q}, t)=-dT(\boldsymbol{q}, t)/dt \tag{2·23}$$

and we get for the magnetic part of $\boldsymbol{J}(\boldsymbol{q}, t)$, combining all terms:

$$[\boldsymbol{J}_{mag}(\boldsymbol{q}, t)]_\mu = c[S_{\mu\nu}(\boldsymbol{q}, 0) - \bar{L}\delta_{\mu\nu}]A_\nu(\boldsymbol{q}, t) + cT(\boldsymbol{q}, 0)[\boldsymbol{q}\times\boldsymbol{B}(\boldsymbol{q}, t)]_\mu - \left[c\boldsymbol{q}\times\int_{-\infty}^{t} dt'\, K(\boldsymbol{q}, t-t')\boldsymbol{B}(\boldsymbol{q}, t')\right]_\mu . \tag{2·24}$$

It is now an easy matter to check that the "instantaneous terms" which appear in the first line of (2·24) cancel identically; that is, *the requirement of localizability has led automatically to an expression for* $\boldsymbol{J}_{mag}$ *which is also causal;* $\boldsymbol{J}_{mag}(\boldsymbol{q}, t)$ is given by (2·20) with

$$\boldsymbol{M}(\boldsymbol{q}, t) = \int_{-\infty}^{t} dt'\, K(\boldsymbol{q}, t-t')\boldsymbol{B}(\boldsymbol{q}, t'), \tag{2·25}$$

where $K(\boldsymbol{q}, t)$ is defined by (2·22) and (2·23).

The transformations carried out in this section have now led to expressions in which the "electric" and "magnetic" parts of the current density are clearly separated, and the existence of magnetic time-relaxation is manifest.

Kubo's[1] expression for the *electric* response of the system is simply the integral in (2·4). This is perfectly correct under the assumptions Kubo used, i.e., a purely longitudinal electric field, $\boldsymbol{E} = -\text{grad}\,\phi$, but should not be used for arbitrary electromagnetic fields.

We can recover the *equilibrium* response of the system to a time-independent but space-dependent magnetic field $\boldsymbol{B}(\boldsymbol{x})$ by imagining that this field is switched on very slowly, i.e., we set

$$\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{q})\exp(\alpha t)\,(2/V)^{1/2}\cos\,(\boldsymbol{q}\cdot\boldsymbol{x}) \tag{2·26}$$

and then let α go to zero. In this limit, (2·2b) shows that the electric field goes to zero proportional to α, and we get the equilibrium magnetization

$$\boldsymbol{M}(\boldsymbol{q}) = \bar{K}(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q}) \tag{2·27}$$

where*

$$\bar{K}(\boldsymbol{q}) = \lim_{\alpha\to 0}\int_0^\infty dt K(\boldsymbol{q}, t)\exp(-\alpha t) = T(\boldsymbol{q}, 0) \tag{2·28}$$

or, remembering (2·22),

$$\bar{K}(\boldsymbol{q}) = (1/q^2)\sum_{\mu,\nu}(1/2)\,(\delta_{\mu\nu} - 3q_\mu q_\nu/q^2)S_{\mu\nu}(\boldsymbol{q}, 0). \tag{2·29}$$

Since $S_{\mu\nu}(\boldsymbol{q}, 0)$ is just the $S_{\mu\nu}(\boldsymbol{q})$ of Buckingham[7], equation (2·29) is identical with one of the forms derived by Buckingham. We shall call it the "Buckingham canonical form".

The identity of (2·15) and (2·18) is the time-dependent generalization of the

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* In (2·28) it is necessary to assume that the magnetic "memory" is limited, i.e., $\lim_{t\to\infty} K(\boldsymbol{q}, t) = 0$. As Professor Nakajima has kindly pointed out to us, this assumption may fail for exceptional systems, e.g., the condensed ideal Bose gas.

Buckingham identity (2·12); since such an identity exists, the $S_{33}(\boldsymbol{q}, t)$ in (2·22) can also be expressed in terms of the charge density operator $\rho(\boldsymbol{q}, t)$. Thus the possibility of many apparently different forms for the magnetic response, all of which are equal to each other as a consequence of an identity, exists for the time dependent response function $K(\boldsymbol{q}, t)$ as well as for the equilibrium response function $\bar{K}(\boldsymbol{q})$.

Since electric response, unlike magnetic response, is basically a classical phenomenon, one may expect that there exists a purely classical derivation of the simple formula (2·7) for L, based on the interpretation (2·19) of $\bar{L}$ as an electrical response. This is indeed the case. Let us consider a classical system of particles i with positions $\boldsymbol{x}_i(t)$, velocities $\boldsymbol{v}_i(t)$, masses m_i, and charges e_i. At $t=0$ we introduce a delta-function (in time) electric field of form:

$$\boldsymbol{E}(\boldsymbol{x}, t)=\boldsymbol{E}(\boldsymbol{q})\,(2/V)^{1/2}\sin(\boldsymbol{q}\cdot\boldsymbol{x})\delta(t). \tag{2·30}$$

The effect of this field is to impart an impulse to each particle. The velocities just before and just after the impulse are related by

$$\boldsymbol{v}_i(0+)=\boldsymbol{v}_i(0-)+(e_i/m_i)\boldsymbol{E}(\boldsymbol{q})\,(2/V)^{1/2}\sin(\boldsymbol{q}\cdot\boldsymbol{x}_i). \tag{2·31}$$

The current density Fourier transform $\boldsymbol{j}(\boldsymbol{q})$ is given classically by

$$\boldsymbol{j}(\boldsymbol{q})=\sum_{i=1}^{N} e_i\,\boldsymbol{v}_i(2/V)^{1/2}\sin(\boldsymbol{q}\cdot\boldsymbol{x}_i). \tag{2·32}$$

Inserting (2·31) into (2·32) and taking the statistical average gives

$$\boldsymbol{j}(\boldsymbol{q},0+)=\boldsymbol{j}(\boldsymbol{q},0-)+\overline{\Big(\sum_{i=1}^{N}\frac{2e_i^{\,2}\sin^2(\boldsymbol{q}\cdot\boldsymbol{x}_i)}{Vm_i}\Big)}\boldsymbol{E}(\boldsymbol{q}). \tag{2·33}$$

By definition, the average which appears here is $Y(\boldsymbol{q}, 0)$, and it is indeed related to $\bar{L}$, (2·8), by equation (2·19).

§ 3. Conduction and polarization currents

It is conventional[2] to write the "electric" part of the current as a sum of two parts, the "conduction" and "dielectric polarization" currents:

$$\boldsymbol{J}_{el}(\boldsymbol{x}, t)=\boldsymbol{J}_{con}(\boldsymbol{x}, t)+\frac{\partial\boldsymbol{P}(\boldsymbol{x}, t)}{\partial t}. \tag{3·1}$$

The distinction between these two contributions is usually based upon a specific model of the substance: the conduction current is thought to be the contribution of moving free charges, the polarization current is due to bound charges moving about their equilibrium positions.

Although we cannot use such specific models in a general approach, the separation (3·1) is useful and we therefore wish to find a model-independent way of performing it. For this purpose we shall do, theoretically, just what the experimentalists do to make the separation: we shall suppose that the impressed electric

field $\boldsymbol{E}(\boldsymbol{x}, t)$ has a sine-wave variation in time, and we shall identify $\boldsymbol{J}_{con}(\boldsymbol{x}, t)$ as that component of $\boldsymbol{J}_{el}(\boldsymbol{x}, t)$ which is in phase with the electric field, and we shall identify the polarization current with the out-of-phase component.

Since the Kubo formalism demands that $\boldsymbol{E}(\boldsymbol{x}, t)$ approaches zero as $t \to -\infty$, we introduce a time factor exp (αt) into the field, where α is a small positive constant which will eventually be made to approach zero. Furthermore, we shall make a Fourier analysis in space as well as in time. Thus we write the impressed electric field in the form:

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}(\boldsymbol{q}, \omega)\,(2/V)^{1/2} \sin\,(\boldsymbol{q}\cdot\boldsymbol{x}) \mathrm{Re}\{\exp[(\alpha + i\omega)t]\}. \tag{3·2}$$

Substitution into (1·8) gives, in the limit $\alpha \to 0$,

$$\boldsymbol{J}(\boldsymbol{x}, t) = \boldsymbol{E}(\boldsymbol{q}, \omega)\,(2/V)^{1/2} \sin\,(\boldsymbol{q}\cdot\boldsymbol{x}) \mathrm{Re}[\sum(\boldsymbol{q}, \omega) \exp\,(i\omega t)] \tag{3·3}$$

where

$$\sum(\boldsymbol{q}, \omega) = \lim_{\alpha \to 0} \int_0^\infty dt\; Y(\boldsymbol{q}, t) \exp\,[-(\alpha + i\omega)t]. \tag{3·4}$$

The quantity $\sum(\boldsymbol{q}, \omega)$ is complex:

$$\sum(\boldsymbol{q}, \omega) = \sum_1(\boldsymbol{q}, \omega) + i\sum_2(\boldsymbol{q}, \omega). \tag{3·5}$$

The real part, $\sum_1$, is the electrical conductivity σ at frequency ω and wave number $\boldsymbol{q}$. The dielectric constant $\varepsilon(\boldsymbol{q}, \omega)$ is related to the imaginary part $\sum_2$ through

$$\varepsilon(\boldsymbol{q}, \omega) - 1 = 4\pi\sum_2(\boldsymbol{q}, \omega)/\omega. \tag{3·6}$$

We shall use equation (2·18) for $Y(\boldsymbol{q}, t)$; the only time-dependent part of this is the operator $\rho(\boldsymbol{q}, t)$ given by (2·16). We use the operator identity

$$\frac{d\rho(\boldsymbol{q}, t)}{dt} = \frac{i}{\hbar}[H_0, \rho(\boldsymbol{q}, t)]_- \tag{3·7}$$

to obtain

$$\int_0^\infty dt \exp\,[-(\alpha + i\omega)t]\Big[H_0, \rho(\boldsymbol{q}, t)\Big]_- = (i\hbar)\Big\{\rho(\boldsymbol{q}) - (\alpha + i\omega)P(\boldsymbol{q}, \alpha + i\omega)\Big\}, \tag{3·8}$$

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* As Professor Nagamiya has kindly pointed out to us, this identification agrees with the usual one only in the limit of low frequencies. For example, consider a medium of electric dipoles imbedded in a viscous fluid. The electric current induced in such a medium would normally by classified as a dielectric polarization current; yet it has a non-vanishing in-phase component at non-zero frequencies. This in-phase component vanishes in the zero-frequency limit. At non-zero frequencies, therefore, we get a "complex conductivity" $\sum(q, \omega)$, equation (3·5), rather than the separation (3·1) in its usual form. However, it is just this complex conductivity coefficient which is measured by the usual experimental methods.

where

$$P(\boldsymbol{q}, \alpha+i\omega) \equiv \int_0^\infty dt \, \exp\left[-(\alpha+i\omega)t\right] \rho(\boldsymbol{q}, t). \tag{3·9}$$

Substitution into (3·4) then gives

$$\sum(\boldsymbol{q}, \omega) = \lim_{\alpha \to 0} \left\{ \frac{\alpha+i\omega}{i\hbar q^2} \mathrm{Tr}\Big(U_0[\rho(\boldsymbol{q}), P(\boldsymbol{q}, \alpha+i\omega)]_- \Big) \right\}. \tag{3·10}$$

Let us denote the quantum states of H_0 by $|k\rangle$, with energies E_k. Then it is easily seen that

$$\langle k|P(\boldsymbol{q}, \alpha+i\omega)|k'\rangle = \frac{\langle k|\rho(\boldsymbol{q})|k'\rangle}{\alpha+i\omega+(i/\hbar)(E_{k'}-E_k)} \tag{3·11}$$

and hence

$$\sum(\boldsymbol{q}, \omega) = \lim_{\alpha \to 0} \left\{ \frac{\alpha+i\omega}{q^2} \sum_{k,k'} \exp\left[\beta(F_0-E_k)\right] \, |\langle k|\rho(\boldsymbol{q})|k'\rangle|^2 \right.$$
$$\left. \times \left[\frac{1}{E_{k'}-E_k+\hbar\omega-i\hbar\alpha} + \frac{1}{E_{k'}-E_k-\hbar\omega+i\hbar\alpha} \right] \right\}. \tag{3·12}$$

Separation of this into real and imaginary parts gives, for the conductivity:

$$\sigma(\boldsymbol{q}, \omega) = \lim_{\alpha \to 0} \left\{ \frac{\alpha}{q^2} \sum_{k,k'} \exp[\beta(F_0-E_k)](E_{k'}-E_k) \, |\langle k|\rho(\boldsymbol{q})|k'\rangle|^2 \right.$$
$$\left. \times \left[\frac{1}{(E_{k'}-E_k+\hbar\omega)^2+(\hbar\alpha)^2} + \frac{1}{(E_{k'}-E_k-\hbar\omega)^2+(\hbar\alpha)^2} \right] \right\} \tag{3·13}$$

and for the dielectric constant:

$$\varepsilon(\boldsymbol{q}, \omega) - 1 = \lim_{\alpha \to 0} \left\{ \frac{8\pi}{q^2} \sum_{k,k'} \exp[\beta(F_0-E_k)](E_{k'}-E_k) \, |\langle k|\rho(\boldsymbol{q})|k'\rangle|^2 \right.$$
$$\left. \times \frac{(E_{k'}-E_k)^2-(\hbar\omega)^2}{[(E_{k'}-E_k+\hbar\omega)^2+(\hbar\alpha)^2][(E_{k'}-E_k-\hbar\omega)^2+(\hbar\alpha)^2]} \right\}. \tag{3·14}$$

In general, the process of taking the limit $\alpha \to 0$ in these expressions cannot be interchanged with the sum over states k, k', since some of the energy denominators may then vanish. We note also that there is no contribution to either (3·13) and (3·14) from pairs of states (k, k') with equal energies. The expressions (3·13) and (3·14) have the same generality as the equations of equilibrium statistical mechanics. They suffer of course from the same defect, i.e., that they are hard to evaluate for complex systems.

Although the evaluation (3·12) of the general fourmula (3·10) is in principle adequate for all cases, it is possible to write very much simpler expressions if the system in question is an "independent particle system", for example, an ideal gas of molecules. Formally, an independent particle system has single-particle energy

levels ε_s and a Hamiltonian:

$$H_0=\sum_s \varepsilon_s\, a_s^+\, a_s \tag{3·15}$$

where a_s^+ and a_s are the usual creation and destruction operators for single particle state s. The index "s" is meant to describe the single-particle state completely, i.e., in the case of a molecular ideal gas, s contains both the centre-of-gravity momentum of the molecule and the internal state of excitation of the molecule. The operator $\rho(\boldsymbol{q})$, (2·5), assumes the form:

$$\rho(\boldsymbol{q})=\sum_{s,s'} \rho_{ss'}\, a_s^+\, a_{s'}, \tag{3·16}$$

where $\rho_{ss'}$ is the matrix element of $\rho(\boldsymbol{q})$ for a single particle between single-particle states s and s'. A straightforward calculation also gives the operator identity:

$$P(\boldsymbol{q}, \alpha+i\omega)=\sum_{s,s'} \frac{\rho_{ss'}}{\alpha+i\omega-(i/\hbar)(\varepsilon_s-\varepsilon_{s'})}\, a_s^+\, a_{s'}. \tag{3·17}$$

The evaluation of (3·10) is simplified by transforming to the grand canonical ensemble, i.e., we replace the statistical matrix U_0 by

$$U_0 \to \exp[\beta(\Omega+\mu N-H_0)]=\exp(\beta\Omega)\exp\Big[\sum_s \beta(\mu-\varepsilon_s)\, a_s^+\, a_s\Big] \tag{3·18}$$

where μ is the chemical potential and Ω is the grand canonical potential, $\Omega=-pV$.

Using (3·16), (3·17), and (3·18), the evaluation of the trace in (3·10) is straightforward, using the commutation (or anti-commutation) relations for the operators a, a^+. We get:

$$\begin{aligned}\text{Trace}\,(U_0[\rho, P]_-) &=\sum_{s,s} \bar{n}_s(\rho_{ss'} P_{s's}-P_{ss'}\rho_{s's}) \\ &=\sum_{s,s'} \bar{n}_s|\rho_{ss'}|^2\left[\frac{1}{\alpha+i\omega+(i/\hbar)(\varepsilon_s-\varepsilon_{s'})}-\frac{1}{\alpha+i\omega-(i/\hbar)(\varepsilon_s-\varepsilon_{s'})}\right]\end{aligned} \tag{3·19}$$

where we have made use of the fact that Trace $U_0=1$, and where $\bar{n}_s$ is the average occupation number of state s:

$$\bar{n}_s=\frac{1}{\exp[\beta(\varepsilon_s-\mu)]\mp 1}. \tag{3·20}$$

The upper sign refers to Bose-Einstein statistics, the lower to Fermi-Dirac statistics. Formula (3·19) is correct with either statistics. In practice, in the case of Fermi-Dirac statistics it is sometimes preferable to replace the factor $\bar{n}_s$ in (3·19) by $\bar{n}_s(1-\bar{n}_{s'})$. This is permissible since the product $\bar{n}_s\bar{n}_{s'}$ is symmetric under exchange of s and s', whereas the other factor is antisymmetric.

(3·19) simplifies further in the high-temperature limit of a non-degenerate gas. We use the approximation:

$$\bar{n}_s \cong \exp[\beta(\mu-\varepsilon_s)] \quad \text{(high temperature)} \tag{3·21}$$

and introduce the total number of particles N and the free energy F through:

$$N=\sum \bar{n}_s \cong \exp[\beta(\mu-F)]. \tag{3·22}$$

We use (3·22) to eliminate the chemical potential μ, and finally obtain the following result for $\sum$, equation (3·10):

$$\sum(q,\ \omega)=\lim_{\alpha\to 0}\left\{\frac{(\alpha+i\omega)N}{i\hbar q^2}\sum_{s,s'}\exp[\beta(F-\varepsilon_s)]|\rho_{ss'}|^2\right.$$
$$\left.\times\left[\frac{1}{\alpha+i\omega+(i/\hbar)(\varepsilon_s-\varepsilon_{s'})}-\frac{1}{\alpha+i\omega-(i/\hbar)(\varepsilon_s-\varepsilon_{s'})}\right]\right\}. \tag{3·23}$$

This formula is less general than (3·19) since it restricted to a nondegenerate ideal gas; both (3·19) and (3·23) are less general than (3·12), which latter is not restricted to ideal gases. However, when (3·23) is applicable, it is very much simpler to evaluate than (3·12).

§ 4. The dielectric constant of an ideal gas

It is instructive to see just how the very general expression (3·10) for the complex admittance of an arbitrary medium reduces to the well-known expression which forms the starting point of the theory of electric susceptibilities:[4]

$$\chi=\frac{\varepsilon-1}{4\pi}=\frac{N}{V}\sum_n\exp[\beta(f-\varepsilon_n)]\left\{\frac{|\langle n|p_z|n\rangle|^2}{kT}+2\sum_{n'}{}'\frac{|\langle n|p_z|n'\rangle|^2}{\varepsilon_{n'}-\varepsilon_n}\right\}. \tag{4·1}$$

In this formula, N is the number of molecules, V the volume, ε_n is the energy of molecular state n in the absence of centre-of-gravity motion, f is the "internal free energy" of the molecule:

$$\exp(-\beta f)=\sum_n \exp(-\beta\varepsilon_n) \tag{4·2}$$

and the operator p_z is the z-component of the electric dipole moment of the molecule, referred to its centre-of-gravity:

$$p_z=\sum_{j=1}^{g} e_j(z_j-Z) \tag{4·3}$$

where the sum goes over all charged particles in the molecule. The derivation of the Langevin-Debye formula starting from (4·1) is given in reference 4. Hence, once we have arrived at equation (4·1), our task is finished.

It should be noted that the operator (4·3) has no matrix elements connecting molecular states of the same parity. Thus, in particular, all diagonal matrix elements $\langle n|p_z|n\rangle$ vanish identically. (4·1) can therefore be written in the equivalent form:

$$\chi=\frac{2N}{V}\sum_{n,n'}\exp[\beta(f-\varepsilon_n)]\frac{|\langle n|p_z|n'\rangle|^2}{\varepsilon_{n'}-\varepsilon_n}. \tag{4·4}$$

No prime is needed on the sum to exclude $n=n'$, since these terms are zero as a result of the matrix element vanishing (barring accidental degeneracies between states of opposite parity).

Since we are dealing with an ideal gas, by assumption, we can use equation (3·23). We replace the index s by the two indices $\boldsymbol{K}$, n, where $\boldsymbol{K}$ is the wave number of the centre-of-gravity motion of the molecule, and n is the quantum number defining its internal state. The wave function of state $\boldsymbol{K}$, n is given by

$$\Psi_{\boldsymbol{K},n} = V^{-1/2} \exp(i\boldsymbol{K}\cdot\boldsymbol{R})\phi_n(\boldsymbol{r}_1-\boldsymbol{R},\ \boldsymbol{r}_2-\boldsymbol{R},\cdots,\ \boldsymbol{r}_g-\boldsymbol{R}) \tag{4·5}$$

with the energy eigenvalue

$$\varepsilon_{\boldsymbol{K},n}=\hbar^2K^2/2M+\varepsilon_n. \tag{4·6}$$

The operator $\rho(\boldsymbol{q})$ is given by:

$$\begin{aligned}\rho(\boldsymbol{q}) &= (2/V)^{1/2}\sum_{j=1}^{g} e_j \cos\ (\boldsymbol{q}\cdot\boldsymbol{r}_j) \\ &= (2V)^{-1/2}\Big[\exp(i\boldsymbol{q}\cdot\boldsymbol{R})D(\boldsymbol{q})+\exp(-i\boldsymbol{q}\cdot\boldsymbol{R})D(-\boldsymbol{q})\Big]\end{aligned} \tag{4·7}$$

where

$$D(\boldsymbol{q}) = \sum_{j=1}^{g} e_j \exp[i\boldsymbol{q}\cdot(\boldsymbol{r}_j-\boldsymbol{R})]\,. \tag{4·8}$$

Thus the matrix element $\rho_{ss'}$ in (3·23) becomes:

$$\begin{aligned}\rho_{ss'} &= \langle \boldsymbol{K}, n|\rho(\boldsymbol{q})|\boldsymbol{K}', n'\rangle \\ &= (2V)^{-1/2}\Big[\delta(\boldsymbol{K}-\boldsymbol{K}'-\boldsymbol{q})D_{nn'}(\boldsymbol{q})+\delta(\boldsymbol{K}-\boldsymbol{K}'+\boldsymbol{q})D_{nn'}(-\boldsymbol{q})\Big],\end{aligned} \tag{4·9}$$

where $D_{nn'}(\boldsymbol{q})$ is the matrix element of the operator (4·8) between the internal states ϕ_n and $\phi_{n'}$. Straightforward substitution of (4·6)–(4·9) into (3·23) gives

$$\begin{aligned}\sum(\boldsymbol{q},\omega) = \lim_{\alpha\to 0}\Big\{ &\frac{(\alpha+i\omega)N}{2i\hbar q^2 V}\sum_{\boldsymbol{K},n,n'}\exp[\beta(F-\varepsilon_{\boldsymbol{K},n})]|D_{nn'}(\boldsymbol{q})|^2 \\ &\times\Big[\frac{1}{\alpha+i\omega+(i/\hbar)(\varepsilon_{\boldsymbol{K}n}-\varepsilon_{\boldsymbol{K}-q,n'})}-\frac{1}{\alpha+i\omega-(i/\hbar)(\varepsilon_{\boldsymbol{K}n}-\varepsilon_{\boldsymbol{K}-q,n'})}\Big] \\ &+(\boldsymbol{q}\to-\boldsymbol{q})\Big\}.\end{aligned} \tag{4·10}$$

Although the sum over $\boldsymbol{K}$ can be evaluated as it stands, a great simplification is achieved when we replace the energy differences $\varepsilon_{\boldsymbol{K},n}-\varepsilon_{\boldsymbol{K}-q,n'}$ by $\varepsilon_n-\varepsilon_{n'}$. This replacement is certainly permissible in the limit $q\to 0$, for example, for a parallel plate condenser. More physically, the characteristic length which must be compared to q^{-1} is the de Broglie wave length of a molecule moving with kinetic energy kT, i.e., the quantity λ:

$$\lambda=(2\pi\hbar^2/MkT)^{1/2}. \tag{4·11}$$

This length is considerably less than 10^{-8} cm for all molecular gases, so that the non-dimensional quantity λ_q is always much less than unity. We then perform the sum over $\boldsymbol{K}$ trivially:

$$\sum_{K}\exp[\beta(F-\varepsilon_{K,n})]=\exp[\beta(f-\varepsilon_n)]. \tag{4·12}$$

Furthermore, in the limit of small q(i.e., q^{-1} much larger than internal dimensions of the molecule), we may expand the operator $D(\boldsymbol{q})$, (4·8), in a power series in q. The leading term vanishes if the molecule is electrically neutral. Letting the direction of the vector $\boldsymbol{q}$ be the z-direction, we get:

$$\lim_{q\to 0}\ (|D_{nn'}(\boldsymbol{q})|^2/q^2)=|\langle n|p_z|n'\rangle|^2. \tag{4·13}$$

Substitution of (4·12) and (4·13) into (4·10) and going to the limit $\alpha\to 0$ gives*

$$\lim_{q\to 0}\sum(\boldsymbol{q},\omega)=\frac{i\omega N}{V}\sum_{n,n'}\exp[\beta(f-\varepsilon_n)]\ |\langle n|p_z|n'\rangle|^2$$

$$\times\Big[(\hbar\omega-\varepsilon_n+\varepsilon_{n'})^{-1}-(\hbar\omega+\varepsilon_n-\varepsilon_{n'})^{-1}\Big]. \tag{4·14}$$

If we now go to zero frequency as well, we obtain:

$$\lim_{\omega\to 0}\lim_{q\to 0}\sum(q,\omega)/i\omega=\chi \tag{4·15}$$

where χ is given by equation (4·4). This concludes the demonstration that the present formalism reduces to the conventional one in the case of an ideal gas of molecules. However, some difficult points in this derivation need further discussion, which will be given in the following section.

§ 5. Discussion

Although we were able to obtain the conventional expression for the dielectric susceptibility in section 4, it should be noted that this was in a sense accidental. We obtained equation (4·4), not equation (4·1), and the two are equal only because the diagonal matrix elements of the electric dipole moment vanish. If we had used the corresponding simple model to evaluate the spin paramagnetism of a gas of free electrons, this spin paramagnetism would have turned out to be zero.

This is not a fundamental difficulty, but it illustrates that the Kubo formalism must be interpreted with some care. The reason for the discrepancy is simply that our rudimentary model does not contain any mechanism for a return of the system to thermal equilibrium, once this equilibrium has been disturbed by some external influence. The "dust particle" is missing.

In the usual equilibrium computation of the average dipole moment, as given

* The only effect of the limit $\alpha\to 0$ is to exclude terms with $\varepsilon_n=\varepsilon_{n'}$. These terms vanish anyway, however, since the matrix element vanishes between states of equal parity.

in reference 4, chapter VII, the electric field strength appears in two places: (1) In the operator p_E, through the possibility of induced moments, and (2) in the Boltzmann factors $\exp(-\beta W)$, through the change of the energy as a result of switching on the field. The Kubo formalism applied to our simple model gives (1) but not (2). That is, the system fails to rearrange its probabilities of being in certain quantum states in accordance with the altered energies.

Thus, the result obtained with this simple model is the "adiabatic" result, in which occupation numbers are frozen to their initial, field-free equilibrium values. This is the reason for the vanishing spin paramagnetism. The equilibrium orbital diamagnetism was obtained correctly (see equation (2·29)) because the equilibrium Boltzmann distribution of velocities is restored automatically when the magnetic field is switched off, even in the absence of collision mechanisms.

The Kubo formalism is more general than ordinary equilibrium statistical mechanics, in that it allows the computation of relaxation times as well as the computation of equilibrium properties. However, one must pay for this added generality by having to use more complicated models of a physical substance. The "dust particle" which can be omitted in equilibrium computations must be inserted explicitly in the Kubo formalism. Applied to spin paramagnetism, for instance, one must include in the model a mechanism for "spin-lattice" relaxation, otherwise the spin-lattice relaxation time becomes infinite, and no spin paramagnetism is obtained.

Furthermore, even if internal relaxation mechanisms are included in the model, the Kubo formalism does not yield results equivalent to thermal equilibrium at constant temperature, but rather results corresponding to a thermally insulated system. This should not make any difference for the usual linear response, since Joule heating is an effect quadratic in the fields. But it would be preferable to have some way of allowing the influence of the "constant temperature bath" within the Kubo formalism, so as to be able to treat more complicated effects (Hall effect, Faraday effect, etc.) under isothermal rather than adiabatic conditions.

Since it is necessary to include relaxation mechanisms within the model, it becomes highly tempting to employ a perturbation expansion of $\sum(\boldsymbol{q}, \omega)$ around the Hamiltonian of the over-simplified model. That is, we write

$$H_0 = H_{00} + gH'_0 \tag{5·1}$$

where H_{00} is the Hamiltonian for a system incapable of thermal relaxation (e.g., (3·5)), and gH_0' contains the "dust particles".

Unfortunately, such an expansion is usually badly behaved. The perturbation expansion of the operator $\rho(\boldsymbol{q}, t)$, (2·16), in powers of g has an n'th term proportional to t^n, and thus becomes useless for very large t. In the limit $\alpha \to 0$, large values of t make significant contributions to the operator $P(\boldsymbol{q}, \alpha + i\omega)$, equation (3·9). However, a radiation damping kind of expansion[8] is possible and meaningful in this problem.

The general relations of section 3 have by no means been exploited fully as

yet. Of course, the Kramers-Kronig dispersion relations follow immediately from (3·12), but (3·12) contains much more detailed information about the system. For example, it should now be possible to make a "virial" type expansion of the dielectric constant of a real gas, in full analogy to the virial expansion of the specific heat.

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Two-Body Correlation of Interacting Fermions

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The electron-pair model, proposed for superconductivity by Bardeen, Cooper and Schrieffer, is discussed in the configuration space. The wave function for the system of electron-pairs is taken as follows,

$$\Psi_0=\sum_{\mathrm{P}}(-1)^p\mathrm{P}\{\chi(1,2)\chi(3,4)\cdots\cdots\chi(2\mathrm{N}-1,2\mathrm{N})\},$$

where $\chi(1,2)$ is the wave function for the singlet state of a two electron system. By a method having a remarkable similarity to Mayer's theory of an imperfect gas, the expectation value of the system can be obtained rigorously. The expression for energy coincides with that which is given by Bogoliubov in his theory of superconductivity. By a variational method, Bogoliubov's integral equation is obtained.

§ 1. Introduction

In the works of Bardeen, Cooper and Schrieffer[1] and of Bogoliubov[2] a systematic theory of superconductivity has been erected. The BCS theory is based on the electron-pair model for a non-ideal Fermi gas and is formulated in terms of second quantization. Here we shall discuss the electron-pair model in the configuration space.*

If all of 2N electrons correlate in pairs, the ground state wave function for the system will be taken as

$$\Psi_0=\sum_{\mathrm{P}}(-1)^p\mathrm{P}\{\chi(1,2)\chi(3,4)\cdots\cdots\chi(2\mathrm{N}-1,2\mathrm{N})\}, \tag{1}$$

where P is a permutation operator working on the indices of 2N coordinates (space coordinates $\boldsymbol{r}_1\cdots\boldsymbol{r}_i\cdots\boldsymbol{r}_{2\mathrm{N}}$ and spin coordinates $\mathrm{s}_1\cdots\mathrm{s}_i\cdots\mathrm{s}_{2\mathrm{N}}$), p its parity and $\chi(1,2)$, the wave function for the singlet state of a two electron system, is given by

$$\chi(1,2)=\psi(r_{12})\frac{\alpha(1)\beta(2)-\beta(1)\alpha(2)}{\sqrt{2}}, \tag{2}$$

in which $\psi(r_{12})$ is a spatial wave function giving correlation between two electrons, $\boldsymbol{r}_{12}=|\boldsymbol{r}_1-\boldsymbol{r}_2|$, and α and β the two spin functions.

We assume that the system has the Hamiltonian of the following form,

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* A part of this work was reported at the annual meeting of the Physical Society of Japan, Oct., 1957. Also this discussion was attempted by Professor S. Nakajima in the symposium on superconductivity at the annual meeting of the Physical Society of Japan, Oct., 1958.

$$H=\sum_i{}' \frac{\boldsymbol{p}_i^2}{2\mu}+\frac{1}{2}\sum_{i\neq j}\phi_{ij}, \tag{3}$$

$$\phi_{ij}=-\frac{4F\zeta_0}{3N_A}\sum_{k<k_0}{}' e^{i(\boldsymbol{k}\cdot\boldsymbol{r}_{ij})}\cdot\left(\frac{sk}{sk-\boldsymbol{k}\cdot\boldsymbol{p}_i/\mu-\hbar k^2/2\mu}+\frac{sk}{sk-\boldsymbol{k}\cdot\boldsymbol{p}_j/\mu+\hbar k^2/2\mu}\right)+v(r_{ij}), \tag{4}$$

where μ is the electronic mass, N_A the number of atoms in volume Ω, F the Fröhlich coupling constant for the electron-electron interaction via phonon exchange, ζ_0 the Fermi energy, s the sound velocity, k_0 the Debye maximum wave number, and the prime of $\sum'$ denotes the omission of the term $\boldsymbol{k}=0$ from the summation over $\boldsymbol{k}$. The first term in eq. (4) is the effective potential* proposed by Fröhlich and by Bardeen[3] in their earlier theories of superconductivity. In the formalism of second quantization, the FB-potential contains the diagonal term investigated by Fröhlich and the nondiagonal term playing an important role in the BCS theory. The second term in eq. (4) is the screened Coulomb potential.

We shall calculate the expectation value of Hamiltonian (3), making use of the wave function (1), and show that the expression for energy is equivalent to that of Bogoliubov. In this calculation, we adopt a method having a remarkable similarity to the methods used by Mayer[4] in the theory of an imperfect gas and those by Kahn and Uhlenbeck[5] in the quantum statistical mechanics. In § 3, it is shown that the Bogoliubov integral equation is derived by a variational method, and that the wave function (1) is equivalent to BCS' or Bogoliubov's wave function.

§ 2. Calculation of expectation value of energy

The expectation value of the Hamiltonian (3) is given by

$$E_0=(\Psi_0,\ H\Psi_0)/(\Psi_0,\ \Psi_0). \tag{5}$$

Using the wave functions given in eqs. (1) and (2), we can write for $(\Psi_0,\ \Psi_0)/(2N)!$

$$\frac{1}{(2N)!}(\Psi_0,\ \Psi_0)=2^N(N!)^2\sum_{(\Sigma l m_l=N)}\prod_l\frac{(\Omega b_l)^{m_l}}{m_l!}, \tag{6}$$

and for $(\Psi_0,\ H\Psi_0)/(2N)!$

$$\frac{1}{(2N)!}(\Psi_0,\ H\Psi_0)=2^N(N!)^2\sum_{(\Sigma l m_l=N)}\varepsilon(\cdots m_l\cdots)\prod_l\frac{(\Omega b_l)^{m_l}}{m_l!}, \tag{7}$$

where

$$b_l=\frac{(-1)^{l-1}}{\Omega l}\int\psi(r_{12})\psi(r_{23})\cdots\cdots\psi(r_{2l,1})\,d\boldsymbol{r}^{2l}, \tag{8}$$

and $\varepsilon(\cdots m_l\cdots)$ has the following form,

* Hereafter this potential will be called the FB-potential, ϕ_{ij}^{FB}.

$$\varepsilon(\cdots m_l\cdots)=\sum_l m_l(t_l+v_l)+\sum_{l\neq l'} m_l m_{l'} v_{l,l'}+\sum_l \frac{m_l(m_l-1)}{2}v_{ll}, \tag{9}$$

in which

$$t_l=\frac{\int \psi(r_{23})\psi(r_{45})\cdots\psi(r_{2l,1})\sum_i (p_i^2/2\mu)\psi(r_{12})\psi(r_{34})\cdots\cdots\psi(r_{2l-1,2l})\,d\boldsymbol{r}^{2l}}{\int \psi(r_{12})\psi(r_{23})\cdots\cdots\psi(r_{2l,1})\,d\boldsymbol{r}^{2l}}, \tag{10}$$

$$v_l=\frac{\int \psi(r_{23})\psi(r_{45})\cdots\cdots\psi(r_{2l,1})\,\frac{1}{2}\sum_{i\neq j}\phi_{ij}\psi(r_{12})\psi(r_{34})\cdots\cdots\psi(r_{2l-1,2l})\,d\boldsymbol{r}^{2l}}{\int \psi(r_{12})\psi(r_{23})\cdots\cdots\psi(r_{2l,1})\,d\boldsymbol{r}^{2l}}, \tag{11}$$

and

$$\begin{aligned} v_{l,l'}=&\iint \psi(r_{23})\psi(r_{45})\cdots\cdots\psi(r_{2l,1})\psi(r_{2'3'})\psi r_{4'5'})\cdots\psi(r_{2l',1'})\,\tfrac{1}{2}\sum_{i,j'}\phi_{ij'}\\ &\times\psi(r_{12})\psi(r_{34})\cdots\cdots\psi(r_{2l-1,2l})\psi(r_{1'2'})\psi(r_{3'4'})\cdots\cdots\psi(r_{2l'-1',2l'})\,d\boldsymbol{r}^{2l}\,d\boldsymbol{r}^{2l'}/\\ &\iint \psi(r_{12})\psi(r_{23})\cdots\cdots\psi(r_{2l,1})\psi(r_{1'2'})\psi(r_{2'3'})\cdots\cdots\psi(r_{2l',1'})\,d\boldsymbol{r}^{2l}\,d\boldsymbol{r}^{2l'}. \end{aligned} \tag{12}$$

The derivation of eqs. (6) and (7) is given in Appendix A.

From eqs. (5), (6) and (7), we get

$$E_0=\frac{\sum_{(\Sigma\, l m_l=N)}\varepsilon(\cdots m_l\cdots)\prod_l(\Omega b_l)^{m_l}/m_l!}{\sum_{(\Sigma\, l m_l=N)}\prod_l(\Omega b_l)^{m_l}/m_l!}. \tag{13}$$

Eq. (13) shows a remarkable analogy to the expression for energy given by Mayer in the theory of an imperfect gas and by Kahn and Uhlenbeck in the quantum statistical mechanics. This analogy to the statistical mechanics will be useful in the following calculation. b_l corresponds to the cluster integral. t_l and v_l are the kinetic and potential energies of a cluster l (hereafter called chain-cluster), and $v_{ll'}$ is the interaction energy between the two chain-cluster, l and l'.

The screened Coulomb potential v_{ij} can be expressed in the form of the Fourier series:

$$v(r_{ij})=\sum_{k>k_c}\nu(\boldsymbol{k})\,\exp i(\boldsymbol{k}\cdot\boldsymbol{r}_{ij}) \tag{14}$$

where

$$\nu(\boldsymbol{k})=4\pi e^2/\Omega k^2. \tag{15}$$

Then it is easy to prove that the interaction term $v_{ll'}$ vanishes. This proof is given in Appendix B. Thus eq. (9) is written as

$$\varepsilon(\cdots m_l\cdots)=\sum_l m_l(t_l+v_l). \tag{16}$$

Substituting eq. (16) in eq. (13), we obtain

$$E_0=\sum_l \langle m_l\rangle (t_l+v_l), \tag{17}$$

where $\langle m_l\rangle$ is given by

$$\langle m_l\rangle = b_l \frac{\partial}{\partial b_l}(\log Q_N), \tag{18}$$

in which

$$Q_N=\sum_{(\Sigma l m_l=N)} \prod_l (\Omega b_l)^{m_l}/m_l!. \tag{19}$$

The above expression for Q_N corresponds to the partition function in the statistical mechanics. As shown by Born and Fuchs,[6] for large N

$$\langle m_l\rangle = \Omega b_l z^l, \tag{20}$$

where z is a parameter corresponding to the fugacity in the thermodynamics. The parameter z is determined by

$$\Omega \sum_l l b_l z^l = N. \tag{21}$$

In the following calculation, it is convenient to use Fourier transform for the spatial wave function $\psi(r)$. The Fourier transforms are defined by a series or integral as follows,

$$\psi(r)=\frac{1}{\Omega}\sum_{\mathbf{K}} f(\mathbf{K})\, e^{i(\mathbf{K}\cdot r)}=\frac{1}{(2\pi)^3}\int f(\mathbf{K})\, e^{i(\mathbf{K}\cdot r)}\, d\mathbf{K}. \tag{22}$$

Substituting eq. (22) in eq. (8), we obtain

$$b_l=\frac{(-1)^{l-1}}{\Omega l}\sum_{\mathbf{K}}\{f(\mathbf{K})\}^{2l}. \tag{23}$$

Making use of eq. (23), we can then rewrite eq. (21) as follows,

$$\sum_{\mathbf{K}} \frac{1}{z^{-1}\{f(\mathbf{K})\}^{-2}+1}=N. \tag{24}$$

The calculation of the kinetic energy is carried out in the following manner. Substituting eq. (23) in eq. (10), we obtain as the kinetic energy of a chain-cluster

$$t_l=\frac{(-1)^{l-1}}{\Omega b_l}\sum_{\mathbf{K}}\frac{(\hbar\mathbf{K})^2}{\mu}\cdot\{f(\mathbf{K})\}^{2l}. \tag{25}$$

The total kinetic energy is given by

$$T=\sum_l \langle m_l\rangle t_l. \tag{26}$$

From eqs. (20) and (25), eq. (26) can be rewritten as

$$T=\frac{\hbar^2}{\mu}\sum_{\mathbf{K}}\frac{K^2}{z^{-1}\{f(\mathbf{K})\}^{-2}+1}. \tag{27}$$

The potential energy is calculated in the same manner. As shown in Appendix C, the potential energy of a chain-cluster can be expressed as

$$v_l=2\frac{(-1)^{l-1}}{\Omega b_l}\sum_{\mathbf{K},\mathbf{K}'}\frac{f(\mathbf{K}')}{f(\mathbf{K})-f(\mathbf{K}')}\{f(\mathbf{K})\}^{2l}\mathrm{V}_{\mathbf{K},\mathbf{K}'}, \tag{28}$$

where

$$\mathrm{V}_{\mathbf{K},\mathbf{K}'}=\frac{16\mathrm{F}\zeta_0}{3\mathrm{N}_A}\cdot\frac{(\sigma|\mathbf{K}-\mathbf{K}'|)^2}{(\mathbf{K}^2-\mathbf{K}'^2)^2-(\sigma|\mathbf{K}-\mathbf{K}'|)^2}+\nu(\mathbf{K}-\mathbf{K}'), \tag{29}$$

in which $\sigma=2\mu s/\hbar$. The total potential energy is given by

$$\Phi=\sum_l\langle m_l\rangle v_l. \tag{30}$$

Substituting eqs. (20) and (28) in eq. (30), we get

$$\Phi=2\sum_{\mathbf{K},\mathbf{K}'}\frac{1}{z^{-1}\{f(\mathbf{K})\}^{-2}+1}\cdot\frac{f(\mathbf{K}')}{f(\mathbf{K})-f(\mathbf{K}')}\cdot\mathrm{V}_{\mathbf{K},\mathbf{K}'}. \tag{31}$$

Thus, from eqs. (17), (27) and (31), we can write for the total energy E_0

$$\mathrm{E}_0=2\sum_{\mathbf{K}}\frac{\varepsilon_{\mathbf{K}}}{z^{-1}\{f(\mathbf{K})\}^{-2}+1}+2\sum_{\mathbf{K},\mathbf{K}'}\frac{1}{z^{-1}\{f(\mathbf{K})\}^{-2}+1}\cdot\frac{f(\mathbf{K}')}{f(\mathbf{K})-f(\mathbf{K}')}\cdot\mathrm{V}_{\mathbf{K},\mathbf{K}'}, \tag{32}$$

where

$$\varepsilon_{\mathbf{K}}=(\hbar\mathbf{K})^2/2\mu.$$

Eq. (32) is the expectation value of energy of the electron-pair system given by the wave function (1). From eqs. (24) and (32), it is easy to see that $1/[z^{-1}\{f(\mathbf{K})\}^{-2}+1]$ corresponds to the Fermi distribution function in the quantum-statistical mechanics.

§ 3. Bogoliubov's integral equation

We want to show that eq. (32) is equivalent to Bogoliubov's expression for energy. If we put

$$h_{\mathbf{K}}=\frac{1}{z^{-1}\{f(\mathbf{K})\}^{-2}+1}, \tag{33}$$

we can rewrite eq. (32) as follows,

$$\mathrm{E}_0=2\sum_{\mathbf{K}}\varepsilon_{\mathbf{K}}h_{\mathbf{K}}+\sum_{\mathbf{K},\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'}[h_{\mathbf{K}}(1-h_{\mathbf{K}})h_{\mathbf{K}'}(1-h_{\mathbf{K}'})]^{1/2}-\sum_{\mathbf{K},\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'}h_{\mathbf{K}}h_{\mathbf{K}'}. \tag{34}$$

Similarly, from eq. (24), we get

$$\sum_{\mathbf{K}}h_{\mathbf{K}}=\mathrm{N}. \tag{35}$$

Eq. (35) is the subsidiary condition of eq. (34). The second term in eq. (34) is the BCS interaction term originating from the nondiagonal part of the F B-potential.

In order to derive Bogoliubov's expression from eq. (34), we introduce $u_{\mathbf{K}}$ and $v_{\mathbf{K}}$ by the following relation,

$$u_{\mathbf{K}}{}^2=1-h_{\mathbf{K}},$$

and

$$v_{\mathbf{K}}{}^2=h_{\mathbf{K}}. \tag{36}$$

From eqs. (34), (35) and (36), we obtain

$$\mathrm{E}_0-2\lambda\mathrm{N}=2\sum_{\mathbf{K}}(\varepsilon_{\mathbf{K}}-\lambda-\tfrac{1}{2}\sum_{\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'})h_{\mathbf{K}}+\sum_{\mathbf{K},\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'}\{h_{\mathbf{K}}(1-h_{\mathbf{K}'})$$
$$+[h_{\mathbf{K}}(1-h_{\mathbf{K}})h_{\mathbf{K}'}(1-h_{\mathbf{K}'})]^{1/2}\}, \tag{37}$$
$$=2\sum_{\mathbf{K}}(\mathrm{E}_{\mathbf{K}}-\lambda)v_{\mathbf{K}}{}^2+\sum_{\mathbf{K},\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'}[u_{\mathbf{K}}{}^2v_{\mathbf{K}'}{}^2+u_{\mathbf{K}}v_{\mathbf{K}}u_{\mathbf{K}'}v_{\mathbf{K}'}], \tag{38}$$

where

$$\mathrm{E}_{\mathrm{K}}=\varepsilon_{\mathrm{K}}-\tfrac{1}{2}\sum_{\mathrm{K}'}\mathrm{V}_{\mathrm{K},\mathrm{K}'}. \tag{39}$$

Eq. (38) is equivalent to Bogoliubov's expression.

It is easy to derive the Bogoliubov integral equation from eq. (37). The best form of the distribution function $h_{\mathbf{K}}$ is determined by solving the integral equation, which is derived from $\partial(\mathrm{E}_0-2\lambda\mathrm{N})/\partial h_{\mathbf{K}}=0$. From eq. (37), this integral equation can be expressed as follows,

$$2(\varepsilon_{\mathbf{K}}-\lambda-\sum_{\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'}h_{\mathbf{K}'})+\frac{1-2h_{\mathbf{K}}}{2[h_{\mathbf{K}}(1-h_{\mathbf{K}})]^{1/2}}\sum_{\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'}[h_{\mathbf{K}'}(1-h_{\mathbf{K}'})]^{1/2}=0. \tag{40}$$

Let us define $\tilde{\xi}_{\mathbf{K}}$ by the following form,

$$\tilde{\xi}_{\mathbf{K}}=\varepsilon_{\mathbf{K}}-\lambda-\sum_{\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'}h_{\mathbf{K}'}. \tag{41}$$

If we put

$$v_{\mathbf{K}}{}^2=\tfrac{1}{2}[1-\tilde{\xi}_{\mathbf{K}}/\sqrt{\mathrm{C}_{\mathbf{K}}{}^2+\tilde{\xi}_{\mathbf{K}}{}^2}],$$

eq. (40) can be rewritten from eq. (36) as

$$\mathrm{C}_{\mathbf{K}}=-\tfrac{1}{2}\sum_{\mathbf{K}'}\mathrm{V}_{\mathbf{K},\mathbf{K}'}(\mathrm{C}_{\mathbf{K}'}/\sqrt{\mathrm{C}_{\mathbf{K}'}{}^2+\tilde{\xi}_{\mathbf{K}'}{}^2}). \tag{42}$$

Eq. (42) is Bogoliubov's integral equation.

§ 4. Discussion

In the previous sections we have discussed the electron-pair model in the configuration space, and found that the wave function (1) is equivalent to the BCS wave function and that the Bogoliubov integral equation is equivalent to the condition of minimizing the expectation value of the Hamiltonian. K. Yosida[7] discussed the relation between the BCS theory and the Bogoliubov theory, and showed the latter result, using the method of the canonical transforms. Our calculation in the

configuration space is more complicated than that in the scheme of second quantization. It is, however, interesting that our method of calculation has a remarkable similarity to Mayer's theory of an imperfect gas. Recently, in connection with the many-body problems, the cluster expansion methods are prevailing in the nuclear physics. Our method used here is an example of the cluster expansion methods in the quantum-mechanics. Furthermore, it will be interesting to discuss the pair model in the system with potential of hard core.[8] The extension of our method of calculation to these system will be given in the subsequent papers.

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Appendix A. Derivation of eqs. (6) and (7)

In the first place, let us begin with the proof of eq. (6). From wave function (1), we have

$$\frac{1}{(2N)!}(\Psi_0,\ \Psi_0)=\sum_P(-1)^P\int P\{\chi(1,2)\chi(3,4)\cdots\chi(2N-1,2N)\}$$
$$\cdot\chi(1,2)\chi(3,4)\cdots\chi(2N-1,2N)\,d\tau^{2N}, \qquad (A\cdot1)$$

where $d\tau^{2N}=d\boldsymbol{r}_1 ds_1\cdots d\boldsymbol{r}_i ds_i\cdots d\boldsymbol{r}_{2N} ds_{2N}$. As seen easily, eq. (A·1) can be divided into the sum of the products of the so-called chain cluster integrals containing the even number of electrons. One of the products which represents m_1 clusters of two electrons, m_2 clusters of four electrons, $\cdots m_l$ clusters of $2l$ electrons, $\cdots$, has the form

$$I(m_1\cdots m_l\cdots)=(I_1)^{m_1}\cdot(I_2)^{m_2}\cdot\cdots\cdot(I_l)^{m_l}\cdot\cdots, \qquad (A\cdot2)$$

where

$$I_l=\int\chi(1,2)\chi(2,3)\chi(3,4)\cdots\chi(2l-1,2l)\chi(2l,1)\,d\tau^{2l}. \qquad (A\cdot3)$$

Here we need to know how many times this products $I(m_1\cdots m_l\cdots)$ occur in the summation. Of course, this is equal to the numbers of ways in which 2N electrons are divided into groups or clusters so that there are m_1 chain clusters of two electrons, m_2 chain clusters of four electrons, $\cdots m_l$ chain clusters of $2l$ electrons,$\cdots$. The coefficient of $I(m_1\cdots m_l\cdots)$ in the summation can be obtained as follows.

i) Among (2N)! permutations in eq. (A·1), exchanges inside pairs and between pairs lead to the same products. Then we have a factor

$$2^N\cdot N!. \qquad (A\cdot4)$$

ii) The number of ways in which N objects are distributed into m_l unnumbered piles of objects each is

$$\mathrm{N}!/\prod_l (l!)^{m_l} m_l!. \tag{A·5}$$

iii) In eq. (A·1), the chain-cluster integral I_l comes from the following expression,

$$(-1)^{p_l}\int \mathrm{P}_l\{\chi(1,2)\chi(3,4)\cdots\chi(2l-1,2l)\}\chi(1,2)\chi(3,4)\cdots\chi(2l-1,2l)\,d\tau^{2l}, \tag{A·6}$$

where P_l is the permutation operator leading eq. (A·6) to the chain cluster integral I_l, and p_l its parity. As seen easily, the number of those permutation is given by

$$2^{l-1}(l-1)!,$$

excluding the permutations considered in i). Thus, the total number of those permutations in the set $\{m_l\}$ is

$$\prod_l \{2^{l-1}(l-1)!\}^{m_l}. \tag{A·7}$$

Also the sign, originating from the rearrangement of eq. (A·6) into the chain cluster integral I_l, is negative. Then we have a factor

$$\prod_l (-1)^{m_l}. \tag{A·8}$$

Thus the number of ways in which 2N electrons are divided into a set of numbers $\{m_l\}$ is

$$\frac{2^{\mathrm{N}}(\mathrm{N}!)^2}{\prod_l (l!)^{m_l} m_l!}\prod_l \{2^{l-1}(l-1)!\}^{m_l}. \tag{A·9}$$

Let us prove that the sum of eq. (A·9) under consideration of the condition $\sum l m_l = \mathrm{N}$ is consistent with the number of the permutation of 2N electrons. It is adequate to prove that

$$2^{2\mathrm{N}}(\mathrm{N}!)^2 \sum_{(\sum l m_l = \mathrm{N})} \prod_l \frac{1}{(2l)^{m_l} m_l!} = (2\mathrm{N})!. \tag{A·10}$$

For the convenience, let us introduce the following generating function:

$$\mathrm{G}(x) = \sum_{m_l} \prod_l \frac{x^{2l m_l}}{(2l)^{m_l} m_l!}. \tag{A·11}$$

Let us calculate the coefficient of $x^{2\mathrm{N}}$ of eq. (A·11). As seen easily, $\mathrm{G}(x)$ can be rewritten as follows:

$$\mathrm{G}(x) = \prod_l \exp(x^{2l}/2l) = \exp\Big(\sum_l x^{2l}/2l\Big) = \exp\log\Big(\frac{1}{\sqrt{1-x^2}}\Big) = \frac{1}{\sqrt{1-x^2}}. \tag{A·12}$$

The coefficient of x^{2N} of eq. (A·12) is equal to $1\cdot3\cdot5\cdots\cdots(2N-1)/2^N N!$. Thus we obtain

$$2^{2N}(N!)^2 \sum_{(\Sigma l m_l=N)} \prod_l \frac{1}{(2l)^{m_l} m_l!} = (2N)!.$$

From eqs. (A·2), (A·8) and (A·9), eq. (A·1) can be written as follows:

$$\frac{1}{(2N)!}(\Psi_0, \Psi_0) = 2^N (N!)^2 \sum_{(\Sigma l m_l=N)} \frac{1}{\prod_l (l!)^{m_l} m_l!} \cdot \prod_l \{-2^{l-1}(l-1)!\, I_l\}^{m_l}. \quad (A\cdot13)$$

Let us carry out the integration over spin variables s_i in (A·3). It follows easily that

$$\int \frac{\alpha(1)\beta(2)-\alpha(2)\beta(1)}{\sqrt{2}} \cdot \frac{\alpha(2)\beta(3)-\alpha(3)\beta(2)}{\sqrt{2}} \cdots\cdots$$
$$\cdot \frac{\alpha(2l)\beta(1)-\alpha(1)\beta(2l)}{\sqrt{2}} ds^{2l} = \frac{(-1)^l}{2^{l-1}}. \quad (A\cdot14)$$

Substituting eq. (A·14) in eq. (A·3), we get

$$I_l = \frac{(-1)^l}{2^{l-1}} \int \psi(r_{12})\psi(r_{23}) \cdots\cdots \psi(r_{2l,1})\, d\boldsymbol{r}^{2l}. \quad (A\cdot15)$$

By making use of eq. (A·15), eq. (A·13) can be written as follows:

$$\frac{1}{(2N)!}(\Psi_0, \Psi_0) = 2^N (N!)^2 \sum_{(\Sigma l m_l=N)} \frac{1}{\prod_l (l)^{m_l} m_l!}$$
$$\cdot \prod_l (-1)^{l-1} \int \psi(r_{12})\psi(r_{23}) \cdots \psi(r_{2l,1})\, d\boldsymbol{r}^{2l})^{m_l}. \quad (A\cdot16)$$

Using b_l given by eq. (8), we obtain

$$\frac{1}{(2N)!}(\Psi_0, \Psi_0) = 2^N (N!)^2 \sum_{(\Sigma l m_l=N)} \prod_l \frac{(\Omega b_l)^{m_l}}{m_l!}.$$

Thus eq. (6) has been proved.

Eq. (7) is proved in the same manner. From wave function (1), we have

$$\frac{1}{(2N)!}(\Psi_0, H\Psi_0) = \sum_P (-1)^p \int P\{\chi(1,2)\chi(3,4)\cdots\chi(2N-1, 2N)\} H$$
$$\times \chi(1,2)\chi(3,4)\cdots\chi(2N-1, 2N)\, d\tau^{2N}. \quad (A\cdot17)$$

Of course, as in the preceding investigation, eq. (A·17) can be divided into the sum of the products of the integrals. In the first place, let us consider the part of the kinetic energy. The kinetic energy operator T can be divided in such a way as to operate on each clusters. Then the integral $I_l(\cdots m_l\cdots)$ corresponding to eq. (A·2) is taken as

$$I_l(\cdots m_l\cdots) = m_l K_l (I_1)^{m_1} (I_2)^{m_2} \cdots (I_{l-1})^{m_{l-1}} (I_l)^{m_l-1} (I_{l+1})^{m_l+1} \cdots, \quad (A\cdot18)$$

where

$$K_l=\int\chi(2,3)\chi(4,5)\cdots\chi(2l,1)\sum_i\frac{\boldsymbol{p}_i^2}{2\mu}\chi(1,2)\chi(3,4)\cdots\chi(2l-1,2l)\,d\tau^{2l}. \quad (A\cdot19)$$

Eq. (A·18) can be rewritten as follows:

$$I_l(\cdots m_l\cdots)=m_l\frac{K_l}{I_l}(I_1)^{m_1}(I_2)^{m_2}\cdots(I_l)^{m_l}\cdots. \quad (A\cdot20)$$

As seen easily from eq. (11), we have

$$K_l/I_l=t_l. \quad (A\cdot21)$$

Thus we have

$$I_l(\cdots m_l\cdots)=m_l t_l I(\cdots m_l\cdots). \quad (A\cdot22)$$

Applying the same method as in the proof of eq. (6) to eq. (A·22), we get

$$\frac{1}{(2N)!}(\Psi_0,T\Psi_0)=2^N(N!)^2\sum_{(\Sigma_l m_l=N)}\sum_l m_l t_l\prod_l\frac{(\Omega b_l)^{m_l}}{m_l!}. \quad (A\cdot23)$$

Secondly, let us consider the part of the potential energy. The potential operator Φ can be divided into the two groups: (1) the group operating on each clusters and (2) the group operating between two clusters. Since the first case is the same as the case of the kinetic operator, we have a term

$$2^N(N!)^2\sum_{(\Sigma_l m_l=N)}\sum_l m_l v_l\prod_l\frac{(\Omega b_l)^{m_l}}{m_l!}. \quad (A\cdot24)$$

Let us consider the second case. In this case, the integral $I_p(\cdots m_l\cdots)$ corresponding to eq. (A·2) or eq. (A·18) is taken as

$$\begin{aligned}I_p(\cdots m_l\cdots)=&m_l m_{l'}J_{ll'}(I_1)^{m_1}(I_2)^{m_2}\cdots(I_{l-1})^{m_{l-1}}(I_l)^{m_{l-1}}\\&\times(I_{l+1})^{m_{l+1}}\cdots(I_{l'-1})^{m_{l'-1}}(I_{l'})^{m_{l'-1}}(I_{l'+1})^{m_{l'+1}}\cdots,\end{aligned} \quad (A\cdot25)$$

where

$$\begin{aligned}J_{ll'}=&\int\!\!\int\chi(2,3)\chi(4,5)\cdots\chi(2l,1)\chi(2',3')\chi(4',5')\cdots\chi(2l',1')\sum_{i\in l,j\in l'}\phi_{ij}\\&\times\chi(1,2)\chi(3,4)\cdots\chi(2l-1,2l)\chi(1',2')\chi(3',4')\cdots\chi(2l'-1',2l')\,d\tau^{2l}\,d\tau^{2l'}.\end{aligned} \quad (A\cdot26)$$

When $l=l'$, the coefficient $m_l m_{l'}$ in eq. (A·25) should be replaced by $m_l(m_l-1)/2$. As seen easily, eq. (A·25) can be rewritten as

$$I_p(\cdots m_l\cdots)=m_l m_{l'}\frac{J_{ll'}}{I_l I_{l'}}I(\cdots m_l\cdots). \quad (A\cdot27)$$

Since the factor $J_{ll'}/I_l\cdot I_{l'}$ is identical with $v_{ll'}$ given by eq. (12), we have

$$I_p(\cdots m_l\cdots)=m_l m_{l'}v_{ll'}I(\cdots m_l\cdots). \quad (A\cdot28)$$

From the above equation, therefore, we have a term

$$2^{N}(N!)^{2}\sum_{(\Sigma\, l m_l=N)}\sum_{l,l'} m_l m_{l'} v_{ll'}\prod_{l}\frac{(\Omega b_l)^{m_l}}{m_l!}. \qquad (A\cdot 29)$$

Making use of (A·24) and (A·29), we get

$$\frac{1}{(2N)!}(\Psi_0, \Phi\Psi_0)=2^{N}(N!)^{2}\sum_{(\Sigma\, l m_l=N)}\Big(\sum_{l} m_l v_l+\sum_{l\neq l'} m_l m_{l'} v_{ll'} + \sum_{l}\frac{m_l(m_l-1)}{2}v_{l,l}\Big)\prod_{l}\frac{(\Omega b_l)^{m_l}}{m_l!}. \qquad (A\cdot 30)$$

Substituting eqs. (A·23) and (A·30) in eq. (A·17), we can then write

$$\frac{1}{(2N)!}(\Psi_0, H\Psi_0)=2^{N}(N!)^{2}\sum_{(\Sigma\, l m_l=N)}\Big(\sum_{l} m_l t_l+\sum_{l} m_l v_l + \sum_{l\neq l'} m_l m_{l'} v_{ll'}+\sum_{l}\frac{m_l(m_l-1)}{2}v_{l,l}\Big)\prod_{l}\frac{(\Omega b_l)^{m_l}}{m_l!}.$$

Thus eq. (7) has been proved.

Appendix B

It will be sufficient to prove that

$$\iint\psi(r_{23})\psi(r_{45})\cdots\psi(r_{2l,1})\psi(r_{2l'3'})\psi(r_{4l'5'})\cdots\psi(r_{2l',1'})\phi_{1,1'} \times\psi(r_{12})\psi(r_{34})\cdots\psi(r_{2l-1,2l})\psi(r_{1'2'})\psi(r_{3l'4'})\cdots\psi(r_{2l'-1',2l'})\,d\boldsymbol{r}^{2l}\,d\boldsymbol{r}^{2l'}=0. \qquad (B\cdot 1)$$

Substituting eqs. (4) and (22) in eq. (B·1), we find that it is sufficient to prove the following relation:

$$\iint e^{i\mathbf{K}_1(\boldsymbol{r}_1-\boldsymbol{r}_2)}\,e^{i\mathbf{K}_2(\boldsymbol{r}_2-\boldsymbol{r}_3)}\cdots e^{i\mathbf{K}_{2l}(\boldsymbol{r}_{2l}-\boldsymbol{r}_1)}\,e^{i\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_{1'})} \times e^{i\mathbf{K}_{1'}(\boldsymbol{r}_{1'}-\boldsymbol{r}_{2'})}\,e^{i\mathbf{K}_{2'}(\boldsymbol{r}_{2'}-\boldsymbol{r}_{3'})}\cdots e^{i\mathbf{K}_{2l'}(\boldsymbol{r}_{2l'}-\boldsymbol{r}_{1'})}\,d\boldsymbol{r}^{2l}\,d\boldsymbol{r}^{2l'}=0. \qquad (B\cdot 2)$$

We can find easily that the left-hand side of eq. (B·2) vanishes, because of the omission of the term $\boldsymbol{k}=0$ from the summation over $\boldsymbol{k}$ in eqs. (4) and (14). Thus it follows that $v_{ll'}=0$.

Appendix C

First of all, we shall calculate the numerator of eq. (11), in order to derive eq. (28). Let us denote this numerator by U and divide U into the part coming from the FB-potential, U_1 and the one from the screened Coulomb potential, U_2:

$$U=U_1+U_2. \qquad (C\cdot 1)$$

U_1 can be written as follows

$$U_1=l(J_1+J_2), \tag{C·2}$$

where

$$J_1=\sum_{i=1}^{l}\int\psi(r_{23})\psi(r_{45})\cdots\psi(r_{2l,1})\phi_{1,2i}^{F.B.}\psi(r_{12})\psi(r_{34})\cdots\psi(r_{2l-1,2l})\,d\boldsymbol{r}^{2l}, \tag{C·3}$$

and

$$J_2=\sum_{i=2}^{l}\int\psi(r_{23})\psi(r_{4,5})\cdots\psi(r_{2l,1})\phi_{1,2i-1}^{F.B.}\psi(r_{12})\psi(r_{34})\cdots\psi(r_{2l-1,2l})\,d\boldsymbol{r}^{2l}. \tag{C·4}$$

Substituting the first term of eqs. (4) and (22) in eqs. (C·3) and (C·4), we can then write for eq. (C·3)

$$J_1=-\frac{8F\zeta_0}{3N_A}\sum_{i=1}^{l}\sum_{\mathbf{K}}\sum_{k}{}'\{f(\mathbf{K})\}^{2i-1}\cdot\{f(\mathbf{K}+\boldsymbol{k})\}^{2l-2i+1}\cdot\left(\frac{\sigma k}{\sigma k-2\boldsymbol{k}\mathbf{K}-k^2}+\frac{\sigma k}{\sigma k+2\boldsymbol{k}\mathbf{K}+k^2}\right) \tag{C·5}$$

and for eq. (C·4)

$$J_2=-\frac{8F\zeta_0}{3N_A}\sum_{i=2}^{l}\sum_{\mathbf{K}}\sum_{k}{}'\{f(\mathbf{K})\}^{2i-2}\cdot\{f(\mathbf{K}+\boldsymbol{k})\}^{2l-2i+2}\cdot\left(\frac{2\sigma k}{\sigma k-2\boldsymbol{k}\mathbf{K}+k^2}\right). \tag{C·6}$$

Let us carry out the summation over i in eqs. (C·5) and (C·6). Then we obtain for eq. (C·5)

$$\begin{aligned}J_1=&-\frac{8F\zeta_0}{3N_A}\sum_{\mathbf{K}}\sum_{k}{}'\frac{\{f(\mathbf{K})/f(\mathbf{K}+\boldsymbol{k})\}}{1-\{f(\mathbf{K})/f(\mathbf{K}+\boldsymbol{k})\}^2}\cdot\{f(\mathbf{K}+\boldsymbol{k})\}^{2l}\\&\cdot\left(\frac{\sigma k}{\sigma k-2\boldsymbol{k}\mathbf{K}-k^2}+\frac{\sigma k}{\sigma k+2\boldsymbol{k}\mathbf{K}+k^2}\right)\\&+\frac{8F\zeta_0}{3N_A}\sum_{\mathbf{K}}\sum_{k}{}'\frac{\{f(\mathbf{K})/f(\mathbf{K}+\boldsymbol{k})\}}{1-\{f(\mathbf{K})/f(\mathbf{K}+\boldsymbol{k})\}^2}\cdot\{f(\mathbf{K})\}^{2l}\\&\cdot\left(\frac{\sigma k}{\sigma k-2\boldsymbol{k}\mathbf{K}-k^2}+\frac{\sigma k}{\sigma k+2\boldsymbol{k}\mathbf{K}+k^2}\right),\end{aligned} \tag{C·7}$$

and for eq. (C·6)

$$\begin{aligned}J_2=&-\frac{8F\zeta_0}{3N_A}\sum_{\mathbf{K}}\sum_{k}{}'\frac{\{f(\mathbf{K})/f(\mathbf{K}+\boldsymbol{k})\}^2}{1-\{f(\mathbf{K})/f(\mathbf{K}+\boldsymbol{k})\}^2}\cdot\{f(\mathbf{K}+\boldsymbol{k})\}^{2l}\cdot\frac{2\sigma k}{\sigma k+2\boldsymbol{k}\mathbf{K}+k^2}\\&+\frac{8F\zeta_0}{3N_A}\sum_{\mathbf{K}}\sum_{k}{}'\frac{1}{1-\{f(\mathbf{K})/f(\mathbf{K}+\boldsymbol{k})\}^2}\cdot\{f(\mathrm{K})\}^{2l}\cdot\frac{2\sigma k}{\sigma k-2\boldsymbol{k}\mathbf{K}-k^2}.\end{aligned} \tag{C·8}$$

Introducing new coordinates $\mathbf{K}'$ and $\boldsymbol{k}'$ by $\mathbf{K}'=\mathbf{K}+\boldsymbol{k}$ and $\boldsymbol{k}'=-\boldsymbol{k}$ in the first terms of eqs. (C·7) and (C·8), we can then write for (C·7)

$$\begin{aligned}J_1=&-\frac{16F\zeta_0}{3N_A}\sum_{\mathbf{K}}\sum_{k}{}'\frac{\{f(\mathbf{K}+\boldsymbol{k})/f(\mathbf{K})\}}{1-f\{(\mathbf{K}+\boldsymbol{k})/f(\mathbf{K})\}^2}\cdot\{f(\mathbf{K})\}^{2l}\\&\cdot\left(\frac{\sigma k}{\sigma k-2\boldsymbol{k}\mathbf{K}-k^2}+\frac{\sigma k}{\sigma k+2\boldsymbol{k}\mathbf{K}+k^2}\right),\end{aligned} \tag{C·9}$$

and for eq. (C·8)

$$J_2=-\frac{16F\zeta_0}{3N_A}\sum_{\mathbf{K}}\sum_{k}'\frac{\{f(\mathbf{K}+\boldsymbol{k})/f(\mathbf{K})\}}{1-\{(\mathbf{K}+\boldsymbol{k})/f(\mathbf{K})\}^2}\cdot\{f(\mathbf{K})\}^{2l}$$

$$\cdot\left(\frac{\sigma k}{\sigma k-2\boldsymbol{k}\mathbf{K}-k^2}+\frac{\sigma k}{\sigma k+2\boldsymbol{k}\mathbf{R}+k^2}\right). \qquad (C\cdot10)$$

Substituting eqs. (C·9) and (C·10) in eq. (C·2), we get

$$U_1=-l\frac{32F\zeta_0}{3N_A}\sum_{\mathbf{K}}\sum_{k}'\frac{f(\mathbf{K}+\boldsymbol{k})}{f(\mathbf{K})-f(\mathbf{K}+\boldsymbol{k})}\cdot\{f(\mathbf{K})\}^{2l}\cdot\frac{(\sigma k)^2}{(\sigma k)^2-(k^2+2\boldsymbol{k}\mathbf{K})^2}. \qquad (C\cdot11)$$

The calculation of U_2 can be carried out in the same manner. U_2 can be written as

$$U_2=lJ_c, \qquad (C\cdot12)$$

where

$$J_c=\int\psi(r_{23})\psi(r_{45})\cdots\psi(r_{2l,1})\sum_{i=2}^{2l}v_{1i}\psi(r_{12})\psi(r_{34})\cdots\psi(r_{2l-1,2l})\,d\boldsymbol{r}^{2l}. \qquad (C\cdot13)$$

Substituting eqs. (14) and (22) in eq. (C·13), we get

$$J_c=\sum_{i=1}^{2l-1}\sum_{\mathbf{K}}\sum_{k}'\{f(\mathbf{K})\}^{i}\cdot\{f(\mathbf{K}+\boldsymbol{k})\}^{2l-i}\cdot\nu(\boldsymbol{k}), \qquad (C\cdot14)$$

$$=2\sum_{\mathbf{K}}\sum_{k}'\frac{f(\mathbf{K}+\boldsymbol{k})}{f(\mathbf{K})-f(\mathbf{K}+\boldsymbol{k})}\cdot\{f(\mathbf{K})\}^{2l}\nu(\boldsymbol{k}). \qquad (C\cdot15)$$

From eqs. (C·15) and (C·12), we have

$$U_2=2l\sum_{\mathbf{K}}\sum_{k}'\frac{f(\mathbf{K}+\boldsymbol{k})}{f(\mathbf{K})-f(\mathbf{K}+\boldsymbol{k})}\cdot\{f(\mathbf{K})\}^{2l}\cdot\nu(\boldsymbol{k}). \qquad (C\cdot16)$$

Substituting eqs. (C·16) and (C·11) in eq. (C·1), we obtain

$$U=2l\sum_{\mathbf{K}}\sum_{k}'\frac{f(\mathbf{K}+\boldsymbol{k})}{f(\mathbf{K})-f(\mathbf{K}+\boldsymbol{k})}\cdot\{f(\mathbf{K})\}^{2l}\cdot\left(\frac{16F\zeta_0}{3N_A}\cdot\frac{(\sigma k)^2}{(k^2+2\boldsymbol{k}\mathbf{K})^2-(\sigma k)^2}+\nu(\boldsymbol{k})\right), \qquad (C\cdot17)$$

$$=2l\sum_{\mathbf{K}}\sum_{\mathbf{K}'}\frac{f(\mathbf{K}')}{f(\mathbf{K})-f(\mathbf{K}')}\cdot\{f(\mathbf{K})\}^{2l}\cdot V_{\mathbf{K},\mathbf{K}'}. \qquad (C\cdot18)$$

Substituting eqs. (8), (C·18) in eq. (10), we get

$$v_l=2\frac{(-1)^{l-1}}{\Omega b_l}\sum_{\mathbf{K},\mathbf{K}'}\frac{f(\mathbf{K}')}{f(\mathbf{K})-f(\mathbf{K}')}\cdot\{f(\mathbf{K})\}^{2l}\cdot V_{\mathbf{K},\mathbf{K}'}. \qquad (C\cdot19)$$

Thus eq. (28) has been derived.

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Quantum Theory in Pseudo-Hilbert Space. II

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In the previous paper we could remove the difficulty of the Lorentz non-invariance of Gupta's old theory in quantum electrodynamics in the case of the free field. The same problem is investigated in the case where the interaction is present.

In the works on quantum electrodynamics by Källén and others[1] it is assumed that the norm of the Heisenberg state Ψ is given by

$$\Psi^* \eta \Psi = (-1)^{n_0} \tag{1}$$

and the Green function of a photon with interaction is defined in the form of

$$\Psi_0^* \eta A_\mu(x) A_\nu(y) \Psi_0, \tag{2}$$

where n_0 is the number of "physical" scalar photons, η is the metric operator defined by Gupta only in the case of the free field, Ψ_0 and $A_\mu(x)$ are the true vacuum state and the Heisenberg operator, respectively. In Gupta's theory η has a simple expression $(-1)^{N_0}$, where N_0 is the number operator of the bare scalar photons, but not of the physical ones. Therefore, there may arise the question if the above definition of the norm and the Green function are adequate.

Recently S. Sunakawa answered to this question in favour of the above method[2]. However, his investigation proved to be invalid, for Gupta's old method[3] depends on a special Lorentz frame as shown by Sunakawa himself[2]. Now we want to reformulate his investigation in the light of the method developed in our preceding paper[4].

We start with the interaction picture, in which the electromagnetic four-potential $A_\mu(x)$, as well as its Hermitian conjugate $A_\mu(x)^*$, satisfies the homogenious free field equation and the free commutation relation. Then we can follow the quite same procedure as that for the free field, namely, we adopt as the metric operator the one which connects $A_\mu(x)$ and $A_\mu(x)^*$: $A_\mu(x) = \eta^{-1} A_\mu(x)^* \eta$.(†) Accordingly, $A_\mu(x)$'s are self-adjoint mixed operators of the first kind, $A_\cdot^{\cdot} = A^\dagger{}_\cdot^{\cdot} = \eta_{\cdot\cdot} A^*{}_{\cdot}^{\cdot} \eta^{\cdot\cdot}$ or briefly $A = A^\dagger = \eta^{-1} A^* \eta$. (In the following we omit the dot symbols. All

† As the electromagnetic field is the neutral one, A and A* are not mutually independent. This note should be added to the preceding paper in showing the Hermitian character of η. The authors are indebted to the members of Research Institute for Fundamental Physics on this point.

operators except for η and η^{-1} are mixed operators of the first kind.) And any transformation in the present theory should be "pseudo-unitary" in the sense

$$U^{-1}=\eta^{-1}U^{*}\eta, \tag{3}$$

where η is the one introduced in the interaction picture. The same is true for the transformations between the various pictures in the field theory. This can be seen in the following way. The state vactors in the interaction, Schrödinger and Heisenberg pictures are mutually related as

$$\Psi_s(t)=\exp(-iH_0t)\Psi_I(t),\ \Psi=\exp(iHt)\Psi_s(t) \text{ and } \Psi=\exp(iHt)\cdot\exp(-iH_0t)\Psi_I(t), \tag{4}$$

where H_0 is the free Hamiltonian in the interaction picture as well as in the Schrödinger one, and H is the total Hamiltonian in the Schrödinger and Heisenberg pictures defined by

$$H=\exp(-iH_0t)\ H(t)\exp(iH_0t)$$

with $H(t)$ the total Hamiltonian in the interaction picture. Then we easily see that H_0 and $H(t)$ are self-adjoint and the transformation functions introduced above are all pseudo-unitary. The self-adjointness of the four-potential is also guaranteed in the Schrödinger and Heisenberg pictures:

$$\begin{aligned} A_\mu^s(x)&\equiv\exp(-iH_0t)A_\mu(x)\exp(iH_0t)=\eta^{-1}A_\mu^s(x)^*\eta,\\ \boldsymbol{A}_\mu(x)&\equiv\exp(iHt)\exp(-iH_0t)A_\mu(x)\exp(iH_0t)\exp(-iHt)=\eta^{-1}\boldsymbol{A}_\mu(x)^*\eta. \end{aligned} \tag{5}$$

Thus we can use the same metric operator η without regard to the picture we adopt.

If Ψ and Φ are the eigenstates of H and H_0 with a same eigenvalue E, then the adiabatic theorem gives

$$\Psi=U(0,\ -\infty)\Phi, \tag{6}$$

with

$$U(0,-\infty)=\lim_{t\to-\infty}U(0,\ t),\quad U(0,\ t)=\exp\ (iHt)\ \exp\ (-iH_0t), \tag{7}$$

where $U(0,\ -\infty)$ is pseudo-unitary. By means of this $U(0,\ -\infty)$ we define the "in" field $\boldsymbol{A}_\mu^{in}(x)$ as follows:

$$\boldsymbol{A}_\mu^{in}(x)=U(0,\ -\infty)A_\mu(x)U(0,\ -\infty)^{-1}. \tag{8}$$

$\boldsymbol{A}_\mu^{in}(x)$ is, of course, self-adjoint. The Fourier expansion of $\boldsymbol{A}_\mu^{in}$ is then written as

$$\boldsymbol{A}_\mu^{in}(x)=\sum_k\frac{1}{\sqrt{2Vk_0}}[\boldsymbol{a}_\mu^{in}(\boldsymbol{k})\exp(ik\cdot x)+\boldsymbol{a}_\mu^{in\dagger}(\boldsymbol{k})\exp(-ik\cdot x)], \tag{9}$$

where the coefficients have the following properties,

$$\left.\begin{aligned} &\boldsymbol{a}_\mu^{in}=U(0,-\infty)\, a_\mu U(0,-\infty),^{-1} \\ &\boldsymbol{a}_\mu^{in\dagger}=U(0,-\infty)\, a_\mu^\dagger U(0,-\infty),^{-1} \\ &\text{and} \\ &[\boldsymbol{a}_\mu^{in}, \boldsymbol{a}_\nu^{in\dagger}]=U(0,-\infty)[a_\mu, a_\nu^\dagger]U(0,-\infty)^{-1}=g_{\mu\nu}. \end{aligned}\right\} \quad (10)$$

$\boldsymbol{a}_\mu^{in}$ and $\boldsymbol{a}_\mu^{in\dagger}$ (including the case of $\mu=0$) can be interpreted as the annihilation and the creation operators of the physical photons in contrast to the case of the interaction picture where the bare photons are concerned, if we adopt the number operators of the "in" field as

$$\boldsymbol{N}_\mu^{in}=g_{\mu\mu}\boldsymbol{a}_\mu^{in\dagger}\boldsymbol{a}_\mu^{in}, \text{ (no summation over } \mu), \quad (11)$$

where $\boldsymbol{N}_\mu^{in}$'s are apparently self-adjoint. Assuming the existence of the "covariant true vacuum state" $\boldsymbol{\Psi}_0$ with the properties

$$\boldsymbol{a}_\mu^{in}\,\boldsymbol{\Psi}_0=0 \text{ and } \boldsymbol{\Psi}_0{}^*\,\eta\,\boldsymbol{\Psi}_0=1, \quad (12)$$

we can construct, in quite a similar way to the case of the free field, a set of states $\boldsymbol{\Psi}(n_\mu)$'s with n physical photons from $\boldsymbol{\Psi}_0$. These states are orthonormal in the sense,

$$\boldsymbol{\Psi}^*(n_\mu)\eta\boldsymbol{\Psi}(m_\mu)=(g_{\mu\mu})^{n_\mu}\ \delta_{n_\nu m_\mu}, \quad (13)$$

(no summation over μ).

This is just the generalized form of (1) as we expected. Therefore one can see that the definitions of the norm (1) and of the propagator (2) for the interacting fields are quite satisfactory from the view-point of the Lorentz invariance in spite of the fact that $\eta\boldsymbol{\Psi}\neq(-1)^{n_0}\boldsymbol{\Psi}$.

In Gupta's old formalism the set of the eigenvectors $\Phi(n_0)$ is orthonormal in the ordinary sense as well as in the sense of the indefinite metric. Therefore the expression $\Phi(n_0)^*\ \eta\Phi(n_0)=(-1)^{n_0}$ compels us to adopt the representation of η, i.e.

$$\eta\Phi(n_0)=(-1)^{n_0}\Phi\ (n_0). \quad (14)$$

In the present formalism the set $\{\boldsymbol{\Psi}(n_\mu)\}$ is orthonormal only in the latter sense, and we need only (13) in the practical calculations, while (14) neither holds in general nor is necessary. In order to get the expression similar to (14) in the Heisenberg picture, Sunakawa introduced $\eta^{in}=U(0,-\infty)\ \eta U(0,-\infty)^{-1}$ together with the "modified" Hermitian conjugation and "modified" adjoint. The necessity of introducing the modified Hermitian conjugation is apparent, because the Hermitian conjugation has no invariant meaning in the theory with the indefinite metric operator. On the other hand, the adjoint conjugation in our theory has certainly an invariant meaning. The modified adjoint of Sunakawa is essentially the same thing as the "unmodified" one, as can be easily proved.

In conclusion the authors wish to express their sincere thanks to Prof. R. Utiyama for his kind guidance and encouragement.

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Collective Excitations in Superconductors

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The collective excitations in superconductors are investigated on the basis of the Bogoliubov method. The result obtained by Anderson is confirmed that the plasma frequency in superconductors is completely the same as that in the normal state. It is also shown that the B. C. S. Meissner effect is really gauge invariant.

§ 1. Introduction

In the Bardeen-Cooper-Schrieffer theory of superconductivity,[1] they started with the following reduced Hamiltonian:

$$H_{\text{Red}}=-1/V\sum_{(k,k')}J(\boldsymbol{k},\ \boldsymbol{k}')\,a^{+}_{-k'\downarrow}a_{-k\downarrow}a^{+}_{k'\uparrow}a_{k\uparrow}\,, \tag{1·1}$$

which is only a part of the complete Fröhlich Hamiltonian,

$$H_{F}=-1/2V\sum_{(s_1,s_2)}\sum_{\substack{k_1,k_2,k_1',k_2'\\ k_1+k_2=k_1'+k_2'\\ k_1\neq k_1'}}J(\boldsymbol{k},\ \boldsymbol{k}')\,a^{+}_{k_2's_2}a_{k_2s_2}a^{+}_{k_1's_1}a_{k_1s_1}\,, \tag{1·2}$$

where $J(\boldsymbol{k},\ \boldsymbol{k}')$ is the coupling constant of the Fröhlich interaction, and s_1 and s_2 represent two spin states of electrons. The fact that the part of the Hamiltonian, $H_F-H_{\text{Red.}}$, which has been neglected in the B. C. S. calculation has no essential effects on the superconducting ground state and individual excitations, has already been shown by B. C. S. themselves and also in the Bogoliubov calculation.[2] However, this part of Hamiltonian plays the important role for the collective excitation and also for the gauge invariance. Anderson[3] and Bogoliubov, Tolmahcev and Shirkov[4] have shown that this part gives rise to a longitudinal collective excitation whose energy lies in the energy gap in the neutral case.

Anderson[5] has also shown that the Coulomb interaction changes this collective excitation to the plasma excitation and that its energy is high and just equal to the plasma excitation energy in the normal state. On the other hand, B. T. S. have obtained a preliminary conclusion that, in the presence of Coulomb repulsion, the longitudinal collective mode still survives in the energy gap and that the plasma frequency is highly modified at the long wave length side and it is quite different from that in the normal state.

One of the purposes of this paper is to reinvestigate the collective excitations in superconductors with the use of the B. T. S. formalism and to check the B. T.

S. calculation. The result we have obtained is that Anderson's result[5] is completely correct; thus, in the presence of Coulomb interaction, the plasma frequency becomes exactly equal to that of the normal state and there is no collective excitation in the energy gap. Furthermore, we calculate the wave function of the plasma excitation and the paramagnetic current due to the longitudinal component of the vector potential and show that this paramagnetic current completely cancels the diamagnetic current for this component. Finally, remarks on the spin-wave like excitation are also given.

§ 2. Secular equation for collective excitations

We shall start with the complete Hamiltonian consisting of the Fröhlich Hamiltonian and the Coulomb interaction,

$$H=1/2V\sum_{(s_1,s_2)}\sum_{\substack{k_1,k_2,k_1',k_2'\\ k_1+k_2=k_1'+k_2'\\ k_1\neq k_1'}}\{-J(\boldsymbol{k}_1,\boldsymbol{k}_1')+v(|\boldsymbol{k}_1-\boldsymbol{k}_1'|)\}a^{-}_{k_2's_2}a_{k_2s_2}a^{+}_{k_1's_1}a_{k_1s_1}, \qquad (2\cdot1)$$

where $v(|\boldsymbol{k}-\boldsymbol{k}'|)$ represents the Fourier component of the Coulomb interaction, namely

$$v(|\boldsymbol{k}-\boldsymbol{k}'|)=4\pi e^2/|\boldsymbol{k}-\boldsymbol{k}'|^2,$$

and V is the total volume of this system. The ground state of the superconducting state is represented by the vacuum state with respect to the Bogoliubov Fermions α_{k1} and α_{k0}. α_{k1} and α_{k0} are derived by the Bogoliubov transformation,

$$\begin{aligned}\alpha_{k0}&=u_k a_{k\uparrow}-v_k a^{+}_{-k\downarrow},\\ \alpha_{k1}&=u_k a_{-k\downarrow}+v_k a^{+}_{k\uparrow},\end{aligned} \qquad (2\cdot2)$$

where u_k and v_k are the following real function of the kinetic energy ε_k of an electron with wave number $\boldsymbol{k}$, taking the Fermi energy as the origin:

$$\begin{aligned}u_k^2&=\tfrac{1}{2}\{1+\varepsilon_k/\sqrt{\varepsilon_k^2+I^2}\},\\ v_k^2&=\tfrac{1}{2}\{1-\varepsilon_k/\sqrt{\varepsilon_k^2+I^2}\}.\end{aligned} \qquad (2\cdot3)$$

I is the quantity corresponding to half the energy gap and is determined by the following B. C. S. condition:

$$1/2V\sum_{k'}J(\boldsymbol{k},\boldsymbol{k}')\,1/\sqrt{\varepsilon_k^2+I^2}=1. \qquad (2\cdot4)$$

Here, $J(\boldsymbol{k},\boldsymbol{k}')$ is regarded as the difference of the coupling constant $J(\boldsymbol{k},\boldsymbol{k}')$ of (2·1) and the Coulomb interaction $v(\boldsymbol{k}-\boldsymbol{k}')$. u_k^2 and v_k^2 correspond to $1-h_k$ and h_k of B. C. S.

Now we shall introduce the following Bose operators $\beta_p^{+}(\boldsymbol{k})$ and $\beta_p(\boldsymbol{k})$ for a pair excitation from the ground state:

$$\beta_p^{+}(\boldsymbol{k})=\alpha^{+}_{k+p0}\alpha^{+}_{k1},\quad \beta_p(\boldsymbol{k})=\alpha_{k1}\alpha_{k+p0}. \qquad (2\cdot5)$$

By the random phase approximation, these operators satisfy the Bose commutation relation

$$[\beta_{p'}(\boldsymbol{k}'),\ \beta_p^+(\boldsymbol{k})]=\delta_{pp'}\delta_{kk'}. \tag{2·6}$$

We shall proceed in the same way as that developed in section 7 of the B. T. S. paper.[4] After Gell-Mann, Brueckner, Sawada and Brout[6] we approximate the Hamiltonian (2·1) with those parts of (2·1) which have the following quadratic form in $\beta_p(\boldsymbol{k})$ and $\beta_p^+(\boldsymbol{k}')$:

$$\begin{aligned}H'=&\sum_{k,p}E_{kp}\beta_p^+(\boldsymbol{k})\beta_p(\boldsymbol{k})+\sum_{\substack{k,k',p\\ k\neq k'}}[A_p(\boldsymbol{k},\boldsymbol{k}')\beta_p^+(\boldsymbol{k})\beta_p(\boldsymbol{k})\\ &+\tfrac{1}{2}B_p(\boldsymbol{k},\boldsymbol{k}')\{\beta_p^+(\boldsymbol{k})\beta_{-p}^+(\boldsymbol{k}')+\beta_{-p}(\boldsymbol{k}')\beta_p(\boldsymbol{k})\}],\end{aligned} \tag{2·7}$$

where E_{kp} is the individual excitation energy for $\beta_p^+(\boldsymbol{k})$, and it is given by

$$E_{kp}=\sqrt{\varepsilon_k^{\,2}+I^2}+\sqrt{\varepsilon_{k+p}^2+I^2}. \tag{2·8}$$

$A_p(\boldsymbol{k},\boldsymbol{k}')$ and $B_p(\boldsymbol{k},\boldsymbol{k}')$ are obtained by picking up the terms in (2·7) from the total Hamiltonian (2·1) as follows:

$$\begin{aligned}A_p(\boldsymbol{k},\boldsymbol{k}')=&1/V\{-\tfrac{1}{2}J(\boldsymbol{k}',\boldsymbol{k})-\tfrac{1}{2}J(\boldsymbol{k}'+\boldsymbol{p},\boldsymbol{k}+\boldsymbol{p})\\ &+v(|\boldsymbol{k}'-\boldsymbol{k}|)\}L(\boldsymbol{k},\boldsymbol{k}')L(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k}'+\boldsymbol{p})\\ &+1/V\{-\tfrac{1}{2}J(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{p})-\tfrac{1}{2}J(\boldsymbol{k}',\boldsymbol{k}'+\boldsymbol{p})\\ &+v(p)\}M(\boldsymbol{k}',\boldsymbol{k}'+\boldsymbol{p})M(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{p}),\end{aligned} \tag{2·9a}$$

$$\begin{aligned}B_p(\boldsymbol{k},\boldsymbol{k}')=&1/V\{-\tfrac{1}{2}J(\boldsymbol{k}',\boldsymbol{k}'-\boldsymbol{p})-\tfrac{1}{2}J(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{p})\\ &+v(p)\}M(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{p})M(\boldsymbol{k}',\boldsymbol{k}'-\boldsymbol{p})\\ &-1/V\{-\tfrac{1}{2}J(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{p})-\tfrac{1}{2}J(\boldsymbol{k},\boldsymbol{k}'-\boldsymbol{p})\\ &+v(|\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{p}|)\}M(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{p})M(\boldsymbol{k},\boldsymbol{k}'-\boldsymbol{p}),\end{aligned} \tag{2·9b}$$

where

$$\begin{aligned}L(\boldsymbol{k},\boldsymbol{k}')&=u_{k'}u_k-v_{k'}v_k,\\ M(\boldsymbol{k},\boldsymbol{k}')&=u_{k'}v_k+v_{k'}u_k.\end{aligned} \tag{2·10}$$

Here, we make the same approximation as B. T. S., that is, we neglect J in the second term of (2·9a) and in the first term of (2·9b) compared with the Coulomb part $v(p)$ which is infinitely large for small values of p, and we use the same notation J for the difference of J and v in the first term of (2·9a) and in the second term of (2·9b). Then, (2·9a) and (2·9b) are simplified as

$$\begin{aligned}A_p(\boldsymbol{k},\boldsymbol{k}')=&-1/2V\{J(\boldsymbol{k},\boldsymbol{k}')+J(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k}'+\boldsymbol{p})\}L(\boldsymbol{k},\boldsymbol{k}')L(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k}'+\boldsymbol{p})\\ &+(1/V)v(p)M(\boldsymbol{k}',\boldsymbol{k}'+\boldsymbol{p})M(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{p}),\end{aligned} \tag{2·11a}$$

$$\begin{aligned}B_p(\boldsymbol{k},\boldsymbol{k}')=&1/2V\{J(\boldsymbol{k},\boldsymbol{k}'-\boldsymbol{p})+J(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k}')\}M(\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{p})M(\boldsymbol{k},\boldsymbol{k}'-\boldsymbol{p})\\ &+(1/V)v(p)M(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{p})M(\boldsymbol{k}',\boldsymbol{k}'-\boldsymbol{p}).\end{aligned} \tag{2·11b}$$

Besides the collective excitations given in terms of the operators of (2·5), there might be spin wave like excitations described in terms of the linear combination of $\alpha^{+}_{-k-p1}\alpha^{+}_{k1}$ and $\alpha^{+}_{k+p0}\alpha^{+}_{-k0}$. The discussion on this type of excitation will be given in the last section.

In order to diagonalize the Hamiltonian (2·7), we introduce the eigenoperator

$$U^{+}_{p}=\sum_{k}\{\varphi_{p}(\boldsymbol{k})\beta^{+}_{p}(\boldsymbol{k})-\chi_{p}(\boldsymbol{k})\beta_{-p}(\boldsymbol{k})\}. \tag{2·12}$$

$\varphi_{p}(\boldsymbol{k})$ and $\chi_{p}(\boldsymbol{k})$ are determined by the following simultaneous equations:

$$\begin{aligned}
&-(E_{kp}-E)\varphi_{p}(\boldsymbol{k})\\
&\quad=\sum_{k'}\{A_{p}(\boldsymbol{k},\boldsymbol{k}')\varphi_{p}(\boldsymbol{k}')+\tfrac{1}{2}(B_{p}(\boldsymbol{k},\boldsymbol{k}')+B_{-p}(\boldsymbol{k}',\boldsymbol{k}))\chi_{p}(\boldsymbol{k}')\}, \qquad (2·13)\\
&-(E_{k,-p}+E)\chi_{p}(\boldsymbol{k})\\
&\quad=\sum_{k'}\{A_{-p}(\boldsymbol{k}',\boldsymbol{k})\chi_{p}(\boldsymbol{k}')+\tfrac{1}{2}(B_{p}(\boldsymbol{k}',\boldsymbol{k})+B_{-p}(\boldsymbol{k},\boldsymbol{k}'))\varphi_{p}(\boldsymbol{k}')\},
\end{aligned}$$

and the normalization condition,

$$[U_{p},\ U^{+}_{p}]=1=\sum_{k}\{|\varphi_{p}(\boldsymbol{k})|^{2}-|\chi_{p}(\boldsymbol{k})|^{2}\}. \tag{2·14}$$

If we approximate $\frac{1}{2}\{J(\boldsymbol{k},\boldsymbol{k}')+J(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k}'+\boldsymbol{p})\}$ by $J(\boldsymbol{k}+\frac{1}{2}p,\boldsymbol{k}'+\frac{1}{2}\boldsymbol{p})$, insert (2·11) into (2·13) and make the usual B. C. S. approximation that $J(\boldsymbol{k},\boldsymbol{k}')$ is constant in the limited region near the Fermi surface, where $|\varepsilon_{k}|$ and $|\varepsilon_{k'}|$ are less than $\hbar\omega_{D}$, and zero outside of it, we obtain

$$\begin{aligned}
(E-E_{kp})\varphi_{p}=&-J/V\sum\nolimits'_{k'}L(\boldsymbol{k},\boldsymbol{k}')L(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k}'+\boldsymbol{p})\varphi_{p'}\\
&+v(p)/V\sum\nolimits_{k'}M(\boldsymbol{k}',\boldsymbol{k}'+\boldsymbol{p})M(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{p})\varphi_{p'}\\
&+J/V\sum\nolimits'_{k'}M(\boldsymbol{k}'+\boldsymbol{p},\boldsymbol{k}+\boldsymbol{p})M(\boldsymbol{k},\boldsymbol{k}')\chi_{p'}\\
&+v(p)/V\sum\nolimits_{k'}M(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{p})M(\boldsymbol{k}'+\boldsymbol{p},\boldsymbol{k}')\chi_{p'}, \qquad (2·15a)\\
-(E+E_{kp})\chi_{p}=&-J/V\sum\nolimits'_{k'}L(\boldsymbol{k}'+\boldsymbol{p},\boldsymbol{k}+\boldsymbol{p})L(\boldsymbol{k}',\boldsymbol{k})\chi_{p'}\\
&+v(p)/V\sum\nolimits_{k'}M(\boldsymbol{k}',\boldsymbol{k}'+\boldsymbol{p})M(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k})\chi_{p'}\\
&+J/V\sum\nolimits'_{k'}M(\boldsymbol{k}',\boldsymbol{k})M(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k}'+\boldsymbol{p})\varphi_{p'}\\
&+v(p)/V\sum\nolimits_{k'}M(\boldsymbol{k}+\boldsymbol{p},\boldsymbol{k})M(\boldsymbol{k}',\boldsymbol{k}'+\boldsymbol{p})\varphi_{p'}, \qquad (2·15b)
\end{aligned}$$

where a prime attached to $\sum$ means the sum over the limited region mentioned above, where $|\varepsilon_{k+(1/2)p}|$ is less than $\hbar\omega_{D}$, and the following abbreviation has been used:

$$\chi_{p}(\boldsymbol{k}+\boldsymbol{p})=\chi_{p},\quad \varphi_{p}(\boldsymbol{k})=\varphi_{p}.$$

Adding and subtracting these two equations, we obtain the following two expressions:

$$\begin{aligned}
&E(\varphi_{p}-\chi_{p})-E_{kp}(\varphi_{p}+\chi_{p})\\
&\quad=-J(u_{k}u_{k+p}-v_{k}v_{k+p})x+J(u_{k}v_{k+p}+v_{k}u_{k+p})y\\
&\quad+2v(p)\,(u_{k}v_{k+p}+v_{k}u_{k+p})\zeta, \qquad (2·16a)
\end{aligned}$$

$$E(\varphi_p+\chi_p)-E_{kp}(\varphi_p-\chi_p)$$
$$=-J(u_k u_{k+p}+v_k v_{k+p})\xi+J(u_k v_{k+p}-v_k u_{k+p})\eta, \qquad (2\cdot16\text{b})$$

where L and M have been explicitly written down in terms of u_k and v_k, and the newly introduced quantities x, y, ξ, η, and ζ are defined by the following expressions:

$$\begin{aligned}
x&=1/V\sum_k'(u_k u_{k+p}-v_k v_{k+p})(\varphi_p+\chi_p),\\
y&=1/V\sum_k'(u_k v_{k+p}+v_k u_{k+p})(\varphi_p+\chi_p),\\
\xi&=1/V\sum_k'(u_k u_{k+p}+v_k v_{k+p})(\varphi_p-\chi_p),\\
\eta&=1/V\sum_k'(u_k v_{k+p}-v_k u_{k+p})(\varphi_p-\chi_p),\\
\zeta&=1/V\sum_k(u_k v_{k+p}+v_k u_{k+p})(\varphi_p+\chi_p).
\end{aligned} \qquad (2\cdot17)$$

If we solve $\varphi_p-\chi_p$ and $\varphi_p+\chi_p$ from (2·16a, b) and insert them into the right-hand side of (2·17), we obtain the following five linear homogeneous equations for the five unknown quantities x, y, ξ, η, and ζ:

$$\begin{aligned}
x&=-Ja'(E)x+Jb'(E)\eta,\\
\eta&=-Jb'(E)x+Jc'(E)\eta,
\end{aligned} \qquad (2\cdot18)$$

and

$$\begin{aligned}
y&=Jf'(E)y+2v(p)f'(E)\zeta-Jg'(E)\xi,\\
\xi&=Jg'(E)y+2v(p)g'(E)\zeta-Jh'(E)\xi,\\
\zeta&=Jf'(E)y+2v(p)f(E)\zeta-Jg'(E)\xi,
\end{aligned} \qquad (2\cdot19)$$

where $a'(E)$, $b'(E)$, $c'(E)$, $f'(E)$, $g'(E)$ and $h'(E)$ are defined by

$$\begin{aligned}
a'(E)&=\frac{1}{V}\sum'\frac{E_{kp}}{E^2-E_{kp}^2}(u_k u_{k+p}-v_k v_{k+p})^2,\\
b'(E)&=\frac{1}{V}\sum'\frac{E}{E^2-E_{kp}^2}(u_{k+p}v_{k+p}-u_k v_k),\\
c'(E)&=\frac{1}{V}\sum'\frac{E_{kp}}{E^2-E_{kp}^2}(u_k v_{k+p}-v_k u_{k+p})^2,\\
f'(E)&=\frac{1}{V}\sum'\frac{E_{kp}}{E^2-E_{kp}^2}(u_k v_{k+p}+v_k u_{k+p})^2,\\
g'(E)&=\frac{1}{V}\sum'\frac{E}{E^2-E_{kp}^2}(u_k v_k+u_{k+p}v_{k+p}),\\
h'(E)&=\frac{1}{V}\sum'\frac{E_{kp}}{E^2-E_{kp}^2}(u_k u_{k+p}+v_k v_{k+p})^2,
\end{aligned} \qquad (2\cdot20)$$

and in f without prime the summation with respect to $\boldsymbol{k}$ is taken over the whole

region. In obtaining (2·18) and (2·19), we have assumed that E_{kp}, $u_k u_{k+p}+v_k v_{k+p}$ and $u_k v_{k+p}+v_k u_{k+p}$ are even functions of ε_k, while $u_k v_{k+p}-v_k u_{k+p}$ and $u_k u_{k+p}-v_k v_{k+p}$ are odd. This assumption rigorously holds at $\boldsymbol{p}=0$. From (2·18) we obtain a two-dimensional secular equation for the excitation energy and a three-dimensional secular equation from (2·19) as follows:

$$\begin{vmatrix} 1+Ja'(E) & -Jb'(E) \\ Jb'(E) & 1-Jc'(E) \end{vmatrix}=0, \tag{2·21}$$

$$\begin{vmatrix} 1-Jf'(E) & -2v(p)f'(E) & Jg'(E) \\ -Jg'(E) & -2v(p)g'(E) & 1+Jh'(E) \\ -Jf'(E) & 1-2v(p)f(E) & Jg'(E) \end{vmatrix}=0. \tag{2·22}$$

§ 3. Solution of secular equations

(2·18) does not include $v(p)$, namely the Coulomb interaction does not affect this mode. For simplicity, let us consider the limiting case of $p=0$. In this limit $b'(E)$ and $c'(E)$ tend to zero so that $\gamma=0$ and the eigenvalue of the energy is determined by

$$1+Ja'(E)_{p=0}=1+\frac{4J}{V}\sum{}'\frac{\varepsilon_k^2}{(E^2-E_{k0}^2)E_{k0}}=0. \tag{3·1}$$

This equation has a solution of $E=2I$ and there is no other solution in the energy gap.

The physically interesting mode arises from (2·19). This mode is highly influenced by the Coulomb interaction. Without Coulomb interaction (2·22) is reduced to the following two-dimensional determinant:

$$\begin{vmatrix} 1-Jf'(E), & Jf'(E) \\ -Jf'(E), & 1+Jh'(E) \end{vmatrix}=0. \tag{3·2}$$

It is easily shown that this equation has a solution of $E=0$. This mode keeps the total number of electrons constant as was pointed out by Anderson.[5] For small p the dispersion relation is obtained by a straightforward calculation. Here, omitting the detailed process of calculation, we write down only the result:

$$E=\frac{1}{\sqrt{3}}v_F p\hbar(1+n(E_F)J)^{1/2} \tag{3·3}$$

where v_F is the velocity of electrons at the Fermi surface and $n(E_F)$ is the state density per unit volume at the Fermi surface. A similar result has been obtained by Anderson and B. T. S.

For the charged case, we have to solve the secular equation (2·22). This is written as

$$(1+Jh')-(2v(p)+J)\{f'(1+Jh')-Jg'^2\}$$
$$-2v(p)(f-f')\{(1+Jh')-[f'(1+Jh')-Jg'^2]\}=0 \qquad (3\cdot4)$$

Taking the limit of $p\to0$, we obtain the following relations:

$$f'(E)_{p=0}=\frac{4I^2}{V}\sum{}'\frac{1}{(E^2-E_{k0}^2)E_{k0}},$$
$$g'(E)_{p=0}=\frac{2IE}{V}\sum{}'\frac{1}{(E^2-E_{k0}^2)E_{k0}}, \qquad (3\cdot5)$$
$$h'(E)_{p=0}=\frac{1}{V}\sum{}'\frac{E_{k0}}{E^2-E_{k0}^2}.$$

If we use the relation (2·4), we obtain the relation

$$[1+Jh'(E)]_{p=0}=\frac{J}{V}\sum{}'\left\{\frac{1}{E_{k0}}+\frac{E_{k0}}{E^2-E_{k0}^2}\right\}=E^2\frac{J}{V}\sum{}'\frac{1}{(E^2-E_{k0}^2)E_{k0}}. \qquad (3\cdot6)$$

Using (3·5) and (3·6), we obtain

$$[f'(1+Jh')-Jg'^2]_{p=0}=0. \qquad (3\cdot7)$$

Thus, in the limit of $p=0$, (3·4) becomes

$$(1+Jh_0')-2v(p)[f'(1+Jh')-Jg'^2]-2v(p)(f-f')(1+Jh_0')=0, \qquad (3\cdot4')$$

where the suffix 0 means the value at $p=0$. From (3·5) we see that for $p=0$ f becomes equal to f' because I vanishes outside of the limited region of $|\varepsilon_k|<\hbar\omega_D$ around the Fermi surface. Therefore, $f-f'$ can be expanded as

$$f-f'=cp^2+\cdots,$$

because both f and f' are even functions of p. For $E=0$, $1+Jh_0'$ vanishes, so that it is confirmed that $E=0$ is not a solution of (3·4′) for $p=0$ in contrast with the neutral case. The last term in (3·4) would be expected to become finite in the limit of $p=0$. However, this finite value will be vanishingly small compared with the first and second terms because of rapid convergence of the summation in c. Therefore, we ignore its contribution.

If we neglect the difference between primed and unprimed quantities, y becomes equal to ζ, and (2·22) is reduced to a two-dimensional determinant. It will be seen later that this determinant gives the energy eigenvalue which is identical to that obtained by Anderson. Now we expand f', g' and h' with respect to p^2 and denote the parts proportional to p^2 by $\delta f'$, $\delta g'$ and $\delta h'$. Retaining non-vanishing terms in (3·4) we obtain the equation determining the energy eigenvalue for $p=0$ as

$$(1+Jh_0')-2v(p)\{(1+Jh_0')\delta f'-2Jg_0'\delta g'+Jf_0'\delta h'\}=0, \qquad (3\cdot8)$$

where the suffix 0 means the value for $p=0$. It is seen from (3·5) and (3·6)

that $1+Jh_0'$, Jf_0' and Jg_0' are proportional to each other. Therefore, removing a common factor of

$$\frac{J}{V}\sum_k'\frac{1}{(E^2-E_{k0}^2)E_{k0}}\neq 0$$

(3·8) becomes

$$E^2-2v(p)\{E^2\delta f'-4IE\delta g'+4I^2\delta h'\}=0. \tag{3·9}$$

For calculating $\delta f'$, $\delta g'$ and $\delta h'$, we use the following expansions:

$$E_{kp}=E_{k0}\left\{1+\frac{2\varepsilon_k}{E_{k0}^2}\delta\varepsilon_k+4\frac{I^2}{E_{k0}^4}(\delta\varepsilon_k)^2\right\}, \tag{3·10}$$

$$(u_kv_{k+p}+v_ku_{k+p})^2=4\frac{I^2}{E_{k0}^2}\left\{1-4\frac{\varepsilon_k}{E_{k0}^2}\delta\varepsilon_k+\frac{4}{E_{k0}^4}(5\varepsilon_k^2-I^2)(\delta\varepsilon_k)^2\right\}, \tag{3·11}$$

$$u_kv_k+u_{k+p}v_{k+p}=\frac{2I}{E_{k0}}\left\{1-2\frac{\varepsilon_k}{E_{k0}^2}\delta\varepsilon_k+\frac{4}{E_{k0}^4}(2\varepsilon_k^2-I^2)(\delta\varepsilon_k)^2\right\}, \tag{3·12}$$

$$(u_ku_{k+p}+v_kv_{k+p})^2=1-4\frac{I^2}{E_{k0}^4}(\delta\varepsilon_k)^2, \tag{3·13}$$

where

$$\delta\varepsilon_k=\frac{\hbar^2}{m}\{(k\cdot p)+\tfrac{1}{2}p^2\}. \tag{3·14}$$

With the use of these expansions, we obtain the expressions for $\delta f'$, $\delta g'$ and $\delta h'$,

$$\delta f'=\frac{16I^2}{V}\sum_k'\frac{1}{E_{k0}(E^2-E_{k0}^2)}\left\{\frac{2I^2-\varepsilon_k^2}{E_{k0}^2(E^2-E_{k0}^2)}+\frac{4\varepsilon_k^2}{(E^2-E_{k0}^2)^2}+\frac{3\varepsilon_k^2}{E_{k0}^4}\right\}(\delta\varepsilon_k)^2, \tag{3·15}$$

$$\delta g'=\frac{8IE}{V}\sum_k'\frac{1}{(E^2-E_{k0}^2)E_{k0}}\left\{\frac{2I^2-\varepsilon_k^2}{(E^2-E_{k0}^2)E_{k0}^2}+\frac{4\varepsilon_k^2}{(E^2-E_{k0}^2)^2}+\frac{2\varepsilon_k^2-I^2}{E_{k0}^4}\right\}(\delta\varepsilon_k)^2, \tag{3·16}$$

$$\delta h'=\frac{4}{V}\sum_k'\frac{E_{k0}}{E^2-E_{k0}^2}\left\{\frac{2I^2+3\varepsilon_k^2}{(E^2-E_{k0}^2)E_{k0}^2}+\frac{4\varepsilon_k^2}{(E^2-E_{k0}^2)^2}\right\}(\delta\varepsilon_k)^2. \tag{3·17}$$

Inserting (3·15), (3·16) and (3·17) into (3·9), we obtain

$$E^2-2v(p)16I^2\frac{1}{V}\sum_k'\frac{2I^2-\varepsilon_k^2}{E_{k0}^5}(\delta\varepsilon_k)^2=0. \tag{3·18}$$

The average value of $\delta\varepsilon_k$ with respect to the direction of the wave vector k is expressed as

$$(\delta\varepsilon_k)_{av}^2=\left(\frac{\hbar^2}{m}\right)^2\frac{1}{3}p^2k^2=\frac{1}{3}\left(\frac{\hbar^2}{m}\right)^2p^2k_F^2+\frac{2}{3}\frac{\hbar^2}{m}p^2\varepsilon_k, \tag{3·19}$$

where k_F is the wave vector at the Fermi surface. The summation over $\boldsymbol{k}$ can be replaced by the integration

$$\sum_k = \frac{V}{2\pi^2} k_F \left(\frac{\hbar^2}{m}\right)^{-1} \int_{-\infty}^{\infty} d\varepsilon. \tag{3·20}$$

Using (3·19), (3·20) and the relation

$$N/V = n = (1/3\pi^2) \cdot k_F^3, \tag{3·21}$$

we obtain, as the plasma excitation energy,

$$E_{pl}^2 = 4\pi e^2 n\hbar^2/m = \omega_p^2 \hbar^2, \tag{3·22}$$

where ω_p is the plasma frequency of the normal state. For large p the plasma energy should be equal to the normal value because of a small energy gap. Therefore, we arrive at the conclusion that the plasma frequency in the superconducting state is expected to be the same as its normal value. Thus, we confirm Anderson's result completely.

§ 4. Wave function of plasma excitation

The eigenfunction of a plasma excitation is given by

$$\psi_{pl} = U^+ \psi_0 = \sum_k \{\varphi_p(\boldsymbol{k}) \beta_p^+(\boldsymbol{k}) - \chi_p(\boldsymbol{k}) \beta_{-p}(\boldsymbol{k})\} \psi_0, \tag{4·1}$$

where ψ_0 is the wave function of the ground state. $\varphi_p(\boldsymbol{k})$ and $\chi_p(\boldsymbol{k})$ are determined by (2·16a, b). Noting that for a plasma excitation $\chi = \eta = 0$, we obtain

$$\varphi_p(\boldsymbol{k}) - \chi_p(\boldsymbol{k}+\boldsymbol{p}) = J \frac{E_{pl}(u_k v_{k+p} + v_k u_{k+p})}{E_{pl}^2 - E_{kp}^2} y$$

$$+ 2v(p) \frac{E_{pl}(u_k v_{k+p} + v_k u_{k+p})}{E_{pl}^2 - E_{kp}^2} \zeta - J \frac{E_{kp}(u_k u_{k+p} + v_k v_{k+p})}{E_{pl}^2 - E_{kp}^2} \xi, \tag{4·2}$$

$$\varphi_p(\boldsymbol{k}) + \chi_p(\boldsymbol{k}+\boldsymbol{p}) = J \frac{E_{kp}(u_k v_{k+p} + v_k u_{k+p})}{E_{pl}^2 - E_{kp}^2} y$$

$$+ 2v(p) \frac{E_{kp}(u_k v_{k+p} + v_k u_{k+p})}{E_{pl}^2 - E_{kp}^2} \zeta - J \frac{E_{pl}(u_k u_{k+p} + v_k v_{k+p})}{E_{pl}^2 - E_{kp}^2} \xi. \tag{4·3}$$

Moreover, we obtain the following relations from (2·19):

$$y = \zeta, \quad \xi = \frac{(J + 2v(p)) g'(E_{pl})}{1 + Jh'(E_{pl})} \zeta. \tag{4·4}$$

Inserting (4·2), (4·3) and (4·4) into the normalization condition (2·14), we obtain the relation determining ζ as follows:

$$1=\sum_k\{\varphi_p(\boldsymbol{k})+\chi_p(\boldsymbol{k}+\boldsymbol{p})\}\{\varphi_p(\boldsymbol{k})-\chi_p(\boldsymbol{k}+\boldsymbol{p})\}$$
$$=\zeta^2(J+2v(p))^2\Big\{\sum_k\frac{E_{pl}E_{kp}(u_kv_{k+p}+v_ku_{k+p})^2}{(E_{pl}^2-E_{kp}^2)^2}$$
$$-\frac{Jg'(E_{pl})}{1+Jh'(E_{pl})}\sum_k\frac{(E_{pl}^2+E_{kp}^2)(u_kv_k+u_{k+p}v_{k+p})}{(E_{pl}^2-E_{kp}^2)^2}$$
$$+\frac{J^2g'^2(E_{pl})}{\{1+Jh'(E_{pl})\}^2}\sum_k\frac{E_{pl}E_{kp}(u_ku_{k+p}+v_kv_{k+p})^2}{(E_{pl}^2-E_{kp}^2)^2}\Big\}. \tag{4·5}$$

This can be written as

$$\zeta^{-2}=-\frac{V}{2}(J+2v(p))^2\Big\{\frac{\partial f'(E_{pl})}{\partial E_{pl}}-\frac{2Jg'(E_{pl})}{1+Jh'(E_{pl})}\frac{\partial g'(E_{pl})}{\partial E_{pl}}$$
$$+\frac{J^2g'^2(E_{pl})}{\{1+Jh'(E_{pl})\}^2}\frac{\partial h'(E_{pl})}{\partial E_{pl}}\Big\}. \tag{4·6}$$

On the other hand, we obtain the following equation by differentiating the secular equation (3·4) by e^2:

$$\frac{1}{e^2(\partial E_{pl}/\partial e^2)}=-\frac{\{J+2v(p)\}^2}{2v(p)}\Big\{\frac{\partial f'}{\partial E_{pl}}-\frac{2Jg'}{1+Jh'}\frac{\partial g'}{\partial E_{pl}}+\frac{J^2g'^2}{(1+Jh')^2}\frac{\partial h'}{\partial E_{pl}}\Big\}. \tag{4·7}$$

Using (4·7) in (4·6), we can simplify the expression for ζ^2 as

$$\zeta^2=\frac{1}{Vv(p)}e^2\frac{\partial E_{pl}}{\partial e^2}. \tag{4·8}$$

(4·2) can also be written as

$$\varphi_p(\boldsymbol{k})-\chi_p(\boldsymbol{k}+\boldsymbol{p})=\zeta(J+2v(p))\Big\{\frac{E_{pl}(u_kv_{k+p}+v_ku_{k+p})}{E_{pl}^2-E_{kp}^2}$$
$$-\frac{Jg'(E_{pl})}{1+Jh'(E_{pl})}\frac{E_{kp}(u_ku_{k+p}+v_kv_{k+p})}{E_{pl}^2-E_{kp}^2}\Big\}. \tag{4·9}$$

§ 5. Gauge invariance

In this section we shall consider the effect of a magnetic field. Let us decompose the vector potential into Fourier components, the p-component of which is written as

$$\boldsymbol{A}e^{ipr}+\boldsymbol{A}^+e^{-ipr}. \tag{5·1}$$

The component of the current operator becomes

$$\boldsymbol{j}=(\boldsymbol{j}_p+\boldsymbol{j}_A)e^{ipr}+(\boldsymbol{j}_p^+\boldsymbol{j}_A^+)e^{-ipr}, \tag{5·2}$$

where $\boldsymbol{j}_p$ and $\boldsymbol{j}_A$ are expressed in terms of a_{ks}^+ and a_{ks} as

$$\boldsymbol{j}_p=\boldsymbol{j}_{-p}^{+}=-\frac{e\hbar}{2mV}\sum_{ks} a_{ks}^{+} a_{k+ps}(2\boldsymbol{k}+\boldsymbol{p}), \tag{5·3}$$

$$\boldsymbol{j}_A=-\frac{e^2}{mc}\frac{1}{V}\boldsymbol{A}\sum_{ks} a_{ks}^{+} a_{ks}-\frac{e^2}{mc}\boldsymbol{A}^{+}\sum_{ks} a_{k-ps}^{+} a_{k+ps}. \tag{5·4}$$

The expectation values of $\boldsymbol{j}_p$ and $\boldsymbol{j}_A$ in the ground state are given, to the lowest order in $\boldsymbol{A}$, by

$$\langle \boldsymbol{j}_p\rangle_{av}=\sum_i \frac{1}{E_0-E_i}\{(0|H_A|i)\,(i|\boldsymbol{j}_p|0)+(0|\boldsymbol{j}_p|i)\,(i|H_A|0)\}, \tag{5·5}$$

$$\langle \boldsymbol{j}_A\rangle_{av}=-\frac{e^2}{mc}\frac{N}{V}\boldsymbol{A}, \tag{5·6}$$

where i and 0 denote the excited states and ground state and E_0 and E_i are the unperturbed energies of the ground state and excited states. H_A is the perturbing Hamiltonian linear in vector potential and is expressed as

$$H_A=-V/c(\boldsymbol{j}_p\cdot\boldsymbol{A}^{+}+\boldsymbol{j}_{-p}\cdot\boldsymbol{A}). \tag{5·7}$$

If we insert (5·7) into (5·5) we obtain the following expression:

$$\langle \boldsymbol{j}_p\rangle_{av}=-\frac{2V}{c}\sum_i \frac{1}{E_0-E_i}(0|\boldsymbol{j}_{-p}\cdot\boldsymbol{A}|i)\,(i|\boldsymbol{j}_p|0). \tag{5·8}$$

$\langle \boldsymbol{j}_A\rangle_{av}$ is the diamagnetic or London current and $\langle \boldsymbol{j}_p\rangle_{av}$ is the paramagnetic current.

According to the B. C. S. theory, if we take individual excitations as excited states i, the paramagnetic current $\langle \boldsymbol{j}_p\rangle_{av}$ vanishes in the limit of $p=0$ on account of the existence of an energy gap and only the diamagnetic current remains. In this calculation, the gauge of the vector potential is chosen as divergenceless, namely $(\boldsymbol{A}\cdot\boldsymbol{p})=0$. The gauge invariance means that for the longitudinal component of the vector potential $(\boldsymbol{A}|\boldsymbol{p})$ the paramagnetic current $\langle \boldsymbol{j}_p\rangle_{av}$ cancels the diamagnetic current $\langle \boldsymbol{j}_A\rangle_{av}$.

Anderson[3] pointed out that if the potential part of the Hamiltonian is commutable with the operator exciting the plasma mode, namely

$$[H_{\text{potential}},\sum_{ks} a_{k+ps}^{+} a_{ks}]=0, \tag{5·9}$$

the gauge invariance is satisfied, and moreover, that the paramagnetic current arising from the plasma excitation cancels the diamagnetic current for a longitudinal component of the vector potential. Here, we shall show that this is actually true by using the wave function derived previously and evaluating the paramagnetic current due to this excited state.

The paramagnetic current operator $\boldsymbol{j}_p$ is expressed in terms of the Bogoliubov Fermion operators as

$$\begin{aligned}\boldsymbol{j}_p=-e\hbar/2mV\sum_k (2\boldsymbol{k}+\boldsymbol{p})\{&(u_k u_{k+p}-v_k v_{k+p})(\alpha_{k0}^{+}\alpha_{k+p0}-\alpha_{k+p1}^{+}\alpha_{k1})\\&+(u_k v_{k+p}-v_k u_{k+p})(\alpha_{k0}^{+}\alpha_{k+p1}^{+}+\alpha_{k+p0}\alpha_{k1})\},\end{aligned} \tag{5·10}$$

and the operator of the second term of this expression is written as

$$\alpha^{+}_{k0}\alpha^{+}_{k+p1}+\alpha_{k+p0}\alpha_{k1}=\beta^{+}_{-p}(\boldsymbol{k}+\boldsymbol{p})-\beta_{p}(\boldsymbol{k}). \tag{5·11}$$

Since ψ_0 is the state with the lowest energy, the matrix element of the current operator between the ground state and the state of the plasma excitation can be written as

$$\begin{aligned}(\psi_{pl}|\boldsymbol{j}_p|\psi_0)&=(U^{+}_{p'}\psi_0,\ \boldsymbol{j}_p\psi_0)=(\psi_0[U_{p'},\ \boldsymbol{j}_p]\psi_0),\\ (\psi_0|\boldsymbol{j}_{-p}|\psi_{pl})&=(\psi_0\boldsymbol{j}_{-p}U^{+}_{p'}\psi_0)=(\psi_0[\boldsymbol{j}_{-p}U^{+}_{p'}]\psi_0).\end{aligned} \tag{5·12}$$

Using (2·12) and (2·6), the following commutation relations are obtained:

$$\begin{aligned}[U_{p'},\ \beta^{+}_{-p}(\boldsymbol{k}+\boldsymbol{p})-\beta_p(\boldsymbol{k})]&=\{\varphi^{+}_{-p}(\boldsymbol{k}+\boldsymbol{p})-\chi^{+}_{-p}(\boldsymbol{k})\}\delta_{p,-p'},\\ [\beta^{+}_{p}(\boldsymbol{k}-\boldsymbol{p})-\beta_{-p}(\boldsymbol{k}),\ U^{+}_{p'}]&=\{-\varphi_{-p}(\boldsymbol{k})+\chi_{-p}(\boldsymbol{k}-\boldsymbol{p})\delta_{p',-p}.\end{aligned} \tag{5·13}$$

Inserting (5·10) into (5·12) and applying (5·13) to it, we obtain

$$\begin{aligned}(\psi_{pl}(-p)|\boldsymbol{j}_p|\psi_0)&=(\psi_0|[U_{-p},\ \boldsymbol{j}_p]|\psi_0)\\ &=-e\hbar/2mV\textstyle\sum_k(2\boldsymbol{k}+\boldsymbol{p})(u_kv_{k+p}-v_ku_{k+p})(\varphi_{-p}(\boldsymbol{k}+\boldsymbol{p})-\chi_{-p}(\boldsymbol{k})),\\ (\psi_0|\boldsymbol{j}_{-p}|\psi_{pl}(-p))&=(\psi_0|[\boldsymbol{j}_{-p},\ U^{+}_{-p}]|\psi_0)\\ &=e\hbar/2mV\textstyle\sum_k(2\boldsymbol{k}-\boldsymbol{p})(u_kv_{k-p}-v_ku_{k-p})(-\varphi_{-p}(\boldsymbol{k})+\chi_{-p}(\boldsymbol{k}-\boldsymbol{p})).\end{aligned} \tag{5·14}$$

Using (5·14) in (5·8), we obtain the expression for the paramagnetic current as

$$\begin{aligned}\langle\boldsymbol{j}_p\rangle_{av}=\frac{2V}{c}\frac{1}{E_{pl}(p)}\Big(\frac{e\hbar}{2mV}\Big)^2&\textstyle\sum_k(2\boldsymbol{k}+\boldsymbol{p})\cdot\boldsymbol{A}(u_kv_{k+p}-v_ku_{k+p})(\varphi_p(\boldsymbol{k})-\chi_p(\boldsymbol{k}+\boldsymbol{p}))\\ &\times\textstyle\sum_k(2\boldsymbol{k}+\boldsymbol{p})(u_kv_{k+p}-v_ku_{k+p})(\varphi_p(\boldsymbol{k})-\chi_p(\boldsymbol{k}+\boldsymbol{p})),\end{aligned} \tag{5·15}$$

where in the right-hand side p is replaced by $-p$.

In order to verify the gauge invariance, we assume that the Fourier component of the vector potential $\boldsymbol{A}$ is parallel to the vector $\boldsymbol{p}$, the direction of which is taken as the z-axis. Since we are interested in the limiting case of $p=0$, we evaluate $\langle\boldsymbol{j}_p\rangle_{av}$ in this limit. Picking up the terms of lowest order in p, we obtain

$$(2k+p)\cdot A=2k_zA, \tag{5·16}$$

and

$$u_kv_{k+p}-v_ku_{k+p}=\frac{2I}{E^2_{k0}}\delta\varepsilon_k=\frac{2I}{E^2_{k0}}\frac{\hbar^2}{m}k_zp. \tag{5·17}$$

From (4·9), we also obtain

$$\varphi_p(\boldsymbol{k})-\chi_p(\boldsymbol{k}+\boldsymbol{p})=2\zeta v(p)\Big\{\frac{2IE_{pl}}{E_{k0}(E^2_{pl}-E^2_{k0})}-\frac{Jg_0'(E_{pl})}{\{1+Jh_0'(E_{pl})\}}\frac{E_{k0}}{(E^2_{pl}-E^2_{k0})}\Big\}.$$

On account of (3·5) and (3·6), this becomes

$$\varphi_p(\boldsymbol{k})-\chi_p(\boldsymbol{k}+\boldsymbol{p})=4\zeta v(p)\,(I/E_{pl}E_{k0}). \tag{5·18}$$

Inserting (5·16), (5·17), (5·18) into (5·15) and using (4·8), we obtain

$$\langle \boldsymbol{j}_p \rangle_{av} = \frac{2V}{c} \frac{1}{E_{pl}} \left(\frac{e\hbar}{2mV} \right)^2 \cdot 4A(2I)^2 \left(\frac{\hbar^2}{m} \right)^2 p^2$$
$$\times \frac{e^2(\partial E_{pl}/\partial e^2)}{Vv(p)} (2v(p))^2 \frac{(2I)^2}{E_{pl}^2} \left(\sum_k \frac{1}{E_{k0}^3} k_z^2 \right)^2. \qquad (5\cdot19)$$

Taking into consideration the following relations:

$$\sum_k \frac{1}{E_{k0}^3} = \frac{V}{\pi^2} k_F \left(\frac{\hbar^2}{m} \right)^{-1} \int_0^\infty \frac{d\varepsilon}{E_{k0}^3} = \frac{V}{8\pi^2} k_F \left(\frac{\hbar^2}{m} \right)^{-1} \frac{1}{I^2},$$

$$e^2 \frac{\partial E_{pl}}{\partial e^2} = \frac{1}{2} E_{pl},$$

and (3·21), we arrive at the final value for $\langle \boldsymbol{j}_p \rangle_{av}$

$$\langle \boldsymbol{j}_p \rangle_{av} = (e^2/mc) n \boldsymbol{A}. \qquad (5\cdot20)$$

Thus, we realize that the plasma excitation guarantees the gauge invariance as Anderson[8] pointed out.

The paramagnetic current for a longitudinal component arising from the individual excitation begins with the term linear in p^2. This is expected to vanish combined with the p^2-term of the paramagnetic current arising from the plasmon state.

§ 6. Spin wave-like excitation

In this section we shall deal with collective excitations of spin reversal. This type of excitation is given by applying

$$a_{k+p\downarrow}^+ a_{k\uparrow} \quad \text{and} \quad a_{k+p\uparrow}^+ a_{k\downarrow}$$

to the ground state. Referring to the Bogoliubov Fermion operators, these correspond to the following two types of operators:

$$\sigma_p^+(\boldsymbol{k}) = \alpha_{-k-p1}^+ \alpha_{k1}^+,$$
$$\tau_p^+(\boldsymbol{k}) = \alpha_{k+p0}^+ \alpha_{-k0}^+. \qquad (6\cdot1)$$

The Hamiltonian for these two kinds of excitations can be expressed, in the Brueckner-Sawada approximation, as

$$H = \sum_{k,p} E_{kp} (\sigma_p^+(k)\, \sigma_p(k) + \tau_p^+(k)\, \tau_p(k))$$
$$+ \sum_{k\,k',p} C_p(\boldsymbol{k}, \boldsymbol{k}') \{\sigma_p^+(\boldsymbol{k})\, \sigma_p(\boldsymbol{k}') + \tau_p^+(\boldsymbol{k})\, \tau_p(\boldsymbol{k}')\}$$
$$+ \sum_{k,k'\,p} D_p(\boldsymbol{k}, \boldsymbol{k}') \{\sigma_p^+(\boldsymbol{k})\, \tau_{-p}^+(\boldsymbol{k}') + \tau_{-p}(\boldsymbol{k}')\, \sigma_p(\boldsymbol{k})\}. \qquad (6\cdot2)$$

$C_p(\boldsymbol{k}, \boldsymbol{k}')$ and $D_p(\boldsymbol{k}, \boldsymbol{k}')$ can be evaluated in a similar way to the case of obtaining $A_p(\boldsymbol{k}, \boldsymbol{k}')$ and $B_p(\boldsymbol{k}, \boldsymbol{k}')$ in (2·7) as follows:

$$C_p(\boldsymbol{k}, \boldsymbol{k}') = \{-J(\boldsymbol{k}, \boldsymbol{k}') + v(|\boldsymbol{k}-\boldsymbol{k}'|)\} L(\boldsymbol{k}, \boldsymbol{k}') L(\boldsymbol{k}+\boldsymbol{p}, \boldsymbol{k}'+\boldsymbol{p}) - \{-J(\boldsymbol{k}, -\boldsymbol{k}') + v(|\boldsymbol{k}+\boldsymbol{k}'|)\} L(\boldsymbol{k}, \boldsymbol{k}'+\boldsymbol{p}) L(\boldsymbol{k}', \boldsymbol{k}+\boldsymbol{p}), \quad (6\cdot3)$$

$$D_p(\boldsymbol{k}, \boldsymbol{k}') = \{-J(\boldsymbol{k}, \boldsymbol{k}') + v(|\boldsymbol{k}-\boldsymbol{k}'|)\} M(\boldsymbol{k}, \boldsymbol{k}'-\boldsymbol{p}) M(\boldsymbol{k}', \boldsymbol{k}+\boldsymbol{p}) - \{-J(\boldsymbol{k}, -\boldsymbol{k}') + v(|\boldsymbol{k}+\boldsymbol{k}'|)\} M(\boldsymbol{k}, \boldsymbol{k}') M(\boldsymbol{k}+\boldsymbol{p}, \boldsymbol{k}'+\boldsymbol{p}), \quad (6\cdot4)$$

where p has been put as zero in J and v.

Throughout this paper we have used the same assumption as B. C. S. that $-J(\boldsymbol{k}, \boldsymbol{k}') + v(|\boldsymbol{k}-\boldsymbol{k}'|)$ is constant in the vicinity of the Fermi surface and zero outside this region. In this approximation, it is found from (6·3) and (6·4) that both $C_p(\boldsymbol{k}, \boldsymbol{k}')$ and $D_p(\boldsymbol{k}, \boldsymbol{k}')$ vanish in the limit of $p=0$. Therefore, we obtain the conclusion that there is no collective excitation with reversed spin in the superconducting state and confirm the result previously[7] obtained that the paramagnetic susceptibility in superconductors vanishes at the absolute zero of temperature, as far as this approximation is valid. This result is inconsistent with Reif's experiment.[8]

§ 7. Conclusion

It has been shown that the plasma frequency in the superconducting state is the same as that in the normal state, and that the paramagnetic current from the plasmon state cancels exactly the diamagnetic current for the longitudinal component of the vector potential in the limit of $p=0$. Thus, we confirm Anderson's result and arrive at the conclusion that the B. C. S. theory of Meissner effect is really gauge invariant.

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Gauge Invariance in the Theory of Superconductivity

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By using identities first pointed out by Buckingham, it is possible to derive explicit and manifestly gauge invariant expressions for the linear magnetic response of a statistical system. If the approximation employed for the statistical operator fails to satisfy these identities, the lack of gauge invariance shows itself through the fact that different explicit expressions, which ought to give identical results, now yield a whole range of results depending on an arbitrary function $\alpha(q)$. However, it is possible to select the best function $\alpha(q)$ by a variational procedure. The form selected in this way is usually different from the prescription of calculating everything in London gauge, without making use of identities.

§ 1. Introduction

Recently there has been considerable discussion[1,2] concerning the question of gauge invariance in the theory of superconductivity. An alternative way of looking at the problem is presented in this paper. This way amounts to a rephrasing of the question, and leads to a unique method of evaluating the magnetic response of any system, normal or superconducting. The method is applicable even if the approximation used for the statistical matrix of the system is not gauge-invariant, and hence in particular fails to satisfy the Buckingham sum rule.[3] From the practical point of view, this is helpful because full gauge invariance, or even the weaker condition imposed by the Buckingham sum rule,**** is a very stringent condition on the permissible approximations. Some very useful approximations, in particular the Bogoliubov approximation,[4] without "collective excitations", violate this condition. The evaluation of the Meissner-Ochsenfeld effect in the Bogoliubov theory, making use of the present method, has been performed by May and

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**** The Buckingham sum rule ensures gauge invariance for the *linear* response of the system to an applied magnetic field. An approximation may satisfy the Buckingham sum rule, and yet fail to be fully gauge-invariant; the lack of gauge invariance would then show up in terms of higher order in the applied field.

Schafroth.[5]

The usual way of calculating the response of a physical system to an applied magnetic field, i.e., the magnetization produced by this field, is as follows: One makes some approximate treatment of the field-free system; then one introduces a vector potential $\boldsymbol{A}$ into the approximate Hamiltonian by replacing $\boldsymbol{p}$ by $\boldsymbol{p}-(e/c)\boldsymbol{A}$; thereafter a series of steps is required to eliminate this vector potential again in favour of physical quantities (the magnetic field $\boldsymbol{B}$); and all too frequently this elimination proves to be impossible: the free energy can not be expressed in terms of curl $\boldsymbol{A}$ only.

In such a case the approximation is said to be not gauge-invariant, and such approximations have had to be discarded, as being of insufficient accuracy to allow any conclusions to be drawn.

We would like to suggest an alternative way of looking at the problem. *If* the exact Hamiltonian of the system is known, both for the field-free system and for the system in the presence of an arbitrary magnetic field, then it is possible to eliminate the vector potential $\boldsymbol{A}$ in favour of the magnetic field $\boldsymbol{B}$ once and for all at the beginning, making use of identities which must hold.[3] For the *linear* response of the system one then gets an expression of the type:

$$\boldsymbol{M}(\boldsymbol{q})=K(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q}), \tag{1·1}$$

where $\boldsymbol{M}(\boldsymbol{q})$ is the $\boldsymbol{q}$'th Fourier component (in space) of the magnetization density, $\boldsymbol{B}(\boldsymbol{q})$ is the $\boldsymbol{q}$'th Fourier component of the impressed magnetic field, and $K(\boldsymbol{q})$ is given by an explicit formula which involves expectation values of well-defined operators over the statistical matrix U of the *field-free* system. Thus (1·1) is "manifestly" gauge invariant.

Of course, the problem which showed up previously as a problem of gauge invariance has not simply disappeared; it has merely changed its form. It turns out that not only is there an explicit formula for $K(\boldsymbol{q})$ in (1·1), but there are infinitely many formulas, of type:

$$K(\boldsymbol{q})=K_B(\boldsymbol{q})+\frac{\alpha(\boldsymbol{q})G(\boldsymbol{q})}{q^2}, \tag{1·2}$$

where $K_B(\boldsymbol{q})$ and $G(\boldsymbol{q})$ are given by explicit formulas, and $\alpha(\boldsymbol{q})$ is a completely arbitrary function of $\boldsymbol{q}$. The condition that all the forms (1·2) give identical results is the "Buckingham sum rule",

$$G(\boldsymbol{q})=0. \tag{1·3}$$

As long as no approximations are made, there is no difficulty. However, let us now replace the exact statistical matrix U of the system by some approximate statistical matrix $\hat{U}$. We then get a set of approximations to the magnetic response function $K(\boldsymbol{q})$, namely

$$\hat{K}[\boldsymbol{q}, \alpha(\boldsymbol{q})]=\hat{K}_B(\boldsymbol{q})+\frac{\alpha(\boldsymbol{q})\hat{G}(\boldsymbol{q})}{q^2}, \tag{1·4}$$

where the $\wedge$ on top of any quantity indicates that it is computed from the explicit formula, but using $\hat{U}$ instead of the exact U. Unlike the exact $G(\boldsymbol{q})$, (1·3), the approximate $\hat{G}(\boldsymbol{q})$ may, and often does, differ from zero. We then say that the approximation in question violates the Buckingham identity.* When this happens, instead of getting a unique, well-defined, and manifestly gauge-invariant result, we have an infinity of different results for $K(\boldsymbol{q})$, one for each choice of $\alpha(\boldsymbol{q})$ in (1·4).

However, *there must be some best function* $\alpha(\boldsymbol{q})$ *which makes* (1·4) *closest to the correct kernel* $K(\boldsymbol{q})$. *We shall determine this best* $\alpha(\boldsymbol{q})$ *by a variational condition.* The exact statistical matrix of the field-free system is

$$U=\exp[\beta(F-H)]. \tag{1·5}$$

We shall assume that the approximate statistical matrix $\hat{U}$ can be written also as the exponential of an operator (this is always possible in principle if $\hat{U}$ is positive definite):

$$\hat{U}=\exp[W(\beta)]. \tag{1·6}$$

Let us consider the difference between the exponents as a small quantity:**

$$W(\beta)-\beta(F-H)\equiv\varepsilon H' \tag{1·7}$$

and let us expand the approximate kernel $\hat{K}[\boldsymbol{q}.\,\alpha(\boldsymbol{q})]$, (1·4), in powers of the the parameter ε, around the exact result. This gives, in principle:

$$\hat{K}(\boldsymbol{q},\,\alpha)=K(\boldsymbol{q})+\varepsilon K_1(\boldsymbol{q},\,\alpha)+\varepsilon^2K_2(\boldsymbol{q},\,\alpha)+\cdots, \tag{1·8}$$

where the leading term is the exact result, and hence independent of the choice of $\alpha(\boldsymbol{q})$. Let n be the *lowest* order of ε for which we get a non-vanishing variation $K_n(\boldsymbol{q},\,\alpha)$. We now demand that this variation vanish identically, thereby getting a condition to determine the function $\alpha(\boldsymbol{q})$:

$$K_n(\boldsymbol{q},\,\alpha)=K_{nB}(\boldsymbol{q})+\alpha(\boldsymbol{q})G_n(\boldsymbol{q})/\boldsymbol{q}^2=0. \tag{1·9}$$

It should be noted that different values of $\boldsymbol{q}$ mean the imposition of different external magnetic fields on the system, thus condition (1·9) is a separate condition for each value of $\boldsymbol{q}$, and defines a function $\alpha(\boldsymbol{q})$ of $\boldsymbol{q}$.***

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* Every approximation which violates the Buckingham identity violates gauge invariance; but an approximation which satisfies the Buckingham identity may yet violate full gauge invariance (see earlier footnote).

** Sometimes the nature of the approximation used demands that different orders of smallness be assigned to separate parts of this difference; e.g. the right side of (1·7) may have to be written as $\varepsilon H'+\varepsilon^2H''$. This is true of the Bogoliubov scheme. The modifications introduced by this change are of course obvious: we consider the lowest non-zero variation (in fact of order ε^2) and demand that it vanishes for all $\boldsymbol{q}$.

*** It is typical of variational methods of this kind that linearity has to be postulated a posteriori, and does not follow from the method itself. That is, suppose we impose an external field which is a superposition of two different Fourier components $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. We then *postulate* that the magnetic response of the system to this field shall be calculated by a linear superposition, using values $\alpha(\boldsymbol{q}_1)$ and $\alpha(\boldsymbol{q}_2)$ determined from (1·9), i.e., on the basis of each Fourier component being the only one present. To the extent that the Fourier decomposition used depends on the shape of the specimen (e.g., it becomes a Fourier-Bessel decomposition for a cylindrical specimen) the values $\alpha(\boldsymbol{q})$ are also shape-dependent. This is of no practical importance, however, and we shall always assume we are working with a periodicity cube, and ordinary Fourier series.

In practice, this condition must be rephrased slightly, because we cannot expand around the exact results as in (1·8), but must expand around the quantity we can calculate, i.e., around $\hat{K}(\boldsymbol{q}, \alpha)$. Thus:

$$K(\boldsymbol{q}) = \hat{K}(\boldsymbol{q}, \alpha) + \varepsilon \hat{K}_1(\boldsymbol{q}, \alpha) + \varepsilon^2 \hat{K}_2(\boldsymbol{q}, \alpha) + \cdots, \tag{1·10}$$

where the leading term is just (1·4), and the correction terms come from an expansion of

$$U = \exp[W(\beta) - \varepsilon H'] \tag{1·11}$$

in powers of ε. The variational condition determining $\alpha(\boldsymbol{q})$ is:

$$\hat{K}_n(\boldsymbol{q}, \alpha) = \hat{K}_{nB}(\boldsymbol{q}) + \alpha(\boldsymbol{q}) \hat{G}_n(\boldsymbol{q}) \, q^2 = 0, \tag{1·12}$$

where n is the lowest order of ε in which $\hat{K}_{nB}$ and $\hat{G}_n$ are not both separately zero. To the extent that ε is really a small quantity, i.e., that $\hat{U}$ is a good approximation to the exact U, conditions (1·9) and (1·12) are equivalent.

Once $\alpha(\boldsymbol{q})$ has been determined from (1·12), the best result for the magnetic response function $K(\boldsymbol{q})$ which can be deduced from the approximation $\hat{U}$ is obtained by substituting this $\alpha(\boldsymbol{q})$ into (1·4), i.e.,

$$\hat{K}(\boldsymbol{q}) = \hat{K}_B(\boldsymbol{q}) - \frac{\hat{G}(\boldsymbol{q}) \hat{K}_{nB}(\boldsymbol{q})}{\hat{G}_n(\boldsymbol{q})} . \tag{1·13}$$

As usual with variation principles, the method does not allow one to improve the approximation systematically. If a better result than (1·13) is desired, a better approximation $\hat{U}$ is necessary, and this better approximation must be obtained in some other way. But the variation principle allows us to find the best possible explicitly gauge-invariant magnetic responce function for any given approximation $\hat{U}$.

The form (1·13) is usually *not* equivalent to the prescription[6] of calculating everything in the London gauge ($\mathrm{div}\boldsymbol{A}=0$) without making use of identities such as (1·3). This prescription in fact means the choice:

$$\alpha(\boldsymbol{q}) = 1 \quad (\text{all } \boldsymbol{q}) \tag{1·14}$$

and it is only very indeed that this choice follows also from (1·12).

The fact that (1·13) is variationally correct means that the function $\hat{K}(\boldsymbol{q})$ is considerably better than the approximation $\hat{U}$ on which it is based. For example, in the calculation of reference (5) it is seen that even the straight-forward Bogoliubov theory in lowest order approximation gives a $K(\boldsymbol{q})$ which contains enough of the final (infinite order of perturbation theory) result to be nearly indistinguishable from it experimentally, even though there are significant differences in principle.

§ 2. Elimination of the vector potential*

We consider a system which, in the absence of magnetic fields, has the

* The derivation given here differs only in details from the one of Buckingham,[3] as extended to systems at finite temperature by Schafroth (unpublished).

Hamiltonian

$$H=\sum_{i=1}^{N}\frac{(\boldsymbol{p}_i)^2}{2m_i}+V(\boldsymbol{x}_1,\ \boldsymbol{x}_2,\cdots,\ \boldsymbol{x}_n)\,, \tag{2·1}$$

and we assume that particle i interacts with the electromagnetic field only through its charge e_i. This assumption means that we ignore spin magnetic moments; the spin terms do not change anything essential, and are omitted only for the sake of simplicity. We note that the Ansatz (2·1) includes a very accurate model of a solid body, namely a system of N_1 nuclei and N_2 electrons ($N_1+N_2=N$), interacting with each other through Coulomb forces. The electron-phonon interaction is contained in this model.

We now include one Fourier component of the applied vector potential of the form:

$$\boldsymbol{A}(\boldsymbol{x})=\boldsymbol{A}(\boldsymbol{q})\,(2/V)^{1/2}\sin(\boldsymbol{q}\cdot\boldsymbol{x})\,. \tag{2·2}$$

We can restrict ourselves to one Fourier component at a time for purposes of the *linear* part of the response of the system to the applied field. The magnetic field derived from (2·2) is:

$$\boldsymbol{B}(\boldsymbol{x})=\boldsymbol{B}(\boldsymbol{q})\,(2/V)^{1/2}\cos(\boldsymbol{q}\cdot\boldsymbol{x})\,, \tag{2·3a}$$

$$\boldsymbol{B}(\boldsymbol{q})=\boldsymbol{q}\times\boldsymbol{A}(\boldsymbol{q})\,. \tag{2·3b}$$

When we introduce (2·2) into the Hamiltonian (2·1) by the usual prescription of replacing $\boldsymbol{p}_i$ by $\boldsymbol{p}_i-(e_i/c)\boldsymbol{A}(\boldsymbol{x}_i)$, we obtain the new Hamiltonian:

$$H(A)=H+H_A+H_{AA}\,, \tag{2·4}$$

where

$$H_A=-(1/c)\boldsymbol{j}(\boldsymbol{q})\cdot\boldsymbol{A}(\boldsymbol{q})\,, \tag{2·5}$$

$$H_{AA}=\tfrac{1}{2}L(\boldsymbol{q})\,|\boldsymbol{A}(\boldsymbol{q})|^2\,. \tag{2·6}$$

$\boldsymbol{j}(\boldsymbol{q})$ is the $\boldsymbol{q}$'th Fourier component of the current density operator

$$\boldsymbol{j}(\boldsymbol{q})=\left(\frac{2}{V}\right)^{1/2}\sum_{i=1}^{N}\frac{e_i}{2m_i}[\boldsymbol{p}_i\sin(\boldsymbol{q}\cdot\boldsymbol{x}_i)+\sin(\boldsymbol{q}\cdot\boldsymbol{x}_i)\boldsymbol{p}_i]\,, \tag{2·7}$$

and $L(\boldsymbol{q})$ is the operator

$$L(q)=\sum_{i=1}^{N}\frac{2e_i^2}{Vm_ic^2}\sin^2(\boldsymbol{q}\cdot\boldsymbol{x}_i)\,. \tag{2·8}$$

We use the Goldberger-Adams method[7] to expand the normalized density matrix.[8]

$$\begin{aligned}U(A)&=\exp\{\beta[\boldsymbol{F}(A)-H(A)]\}\\&=\exp\{\beta[(\boldsymbol{F}-H)+(\boldsymbol{F}_A-H_A)+(\boldsymbol{F}_{AA}-H_{AA})+\cdots]\}\,.\end{aligned} \tag{2·9}$$

We find that the linear correction to the free energy, $\boldsymbol{F}_A$, vanishes (the Bloch theorem), and the quadratic correction takes the form:

$$F_{AA}=-\tfrac{1}{2}J_{\mu\nu}(\boldsymbol{q})\,A_\mu A_\nu, \tag{2·10}$$

where

$$J_{\mu\nu}=S'_{\mu\nu}-\bar{L}\delta_{\mu\nu}\,. \tag{2·11}$$

Here $\bar{L}$ is the expectation value of the operator L over the *field-free* system,

$$\bar{L}=\mathrm{Tr}\,(LU)=\mathrm{Tr}\{L(\boldsymbol{q})\exp[\beta(F-H)]\} \tag{2·12}$$

and $S'_{\mu\nu}$ is defined in terms of the density matrix U of the *field-free* system and the operator $\boldsymbol{j}(\boldsymbol{q})$, (2·7), by

$$S'_{\mu\nu}(\boldsymbol{q})=\frac{2\beta}{c^2}\int_0^1 ds\int_0^s ds'\ \mathrm{Tr}[U^{1-s+s'}\,j_\mu(\boldsymbol{q})\,U^{s-s'}\,j_\nu(\boldsymbol{q})]. \tag{2·13}$$

Since $A_\mu A_\nu=A_\nu A_\mu$, the quadratic form (2·10) involves only the symmetric part of $S'_{\mu\nu}$, i.e., we can write

$$J_{\mu\nu}=S_{\mu\nu}\quad\bar{L}\delta_{\mu\nu}, \tag{2·14}$$

where

$$S_{\mu\nu}=\tfrac{1}{2}(S'_{\mu\nu}+S'_{\nu\mu})=\frac{\beta}{c^2}\int_0^1 dt\,\mathrm{Tr}[U^{1-t}\,j_\mu(\boldsymbol{q})\,U^t\,j_\nu(\boldsymbol{q})]. \tag{2·15}$$

Although the free energy F_{AA}, (2·10), appears expressed in terms of the vector potential, actually it can depend only on the magnetic field $\boldsymbol{B}(\boldsymbol{q})$; according to (2·3b), the component of $\boldsymbol{A}(\boldsymbol{q})$ parallel to the vector $\boldsymbol{q}$ is quite arbitrary, and must not enter into F_{AA}. This gives the gauge condition:

$$J_{\mu\nu}q_\nu=q_\mu J_{\mu\nu}=0 \tag{2·16}$$

or (taking note of the symmetry of all expressions, and using (2·14)):

$$S_{\mu\nu}q_\nu-\bar{L}q_\mu=0. \tag{2·17}$$

Although (2·17) must hold in one sensible theory, its direct verification starting from the explicit expressions (2·8), (2·12), and (2·15) will now be given; the $\boldsymbol{q}$'th component of the charge density operator

$$\rho(\boldsymbol{q})=\sum_{i=1}^{N} e_i(2/V)^{1/2}\cos(\boldsymbol{q}\cdot\boldsymbol{x}_i) \tag{2·18}$$

is related to the current density operator $\boldsymbol{j}(\boldsymbol{q})$ by the continuity equation

$$\boldsymbol{q}\cdot\boldsymbol{j}(\boldsymbol{q})=(i\hbar)^{-1}[H,\,\rho(\boldsymbol{q})]_-\,. \tag{2·19}$$

Using the operator identity

$$\frac{d}{dt}(e^{-tQ}\rho e^{tQ})=-e^{-tQ}[Q,\,\rho]_-\,e^{tQ} \tag{2·20}$$

to perform the integration over t involved in the first term of (2·17), and using the cyclic property of the trace, we get

$$S_{\mu\nu}q_\nu=(i/\hbar c^2)\,\mathrm{Tr}\{U[\rho(\boldsymbol{q}),\,j_\mu(\boldsymbol{q})]_-\}. \tag{2·21}$$

It is obvious from (2·7) and (2·18) that the components of $\boldsymbol{j}(\boldsymbol{q})$ perpendicular to $\boldsymbol{q}$ commute with $\rho(\boldsymbol{q})$; for the component parallel to $\boldsymbol{q}$ direct calculation yields

$$[\rho(\boldsymbol{q}),\,j_u(\boldsymbol{q})]_-=\frac{iq_u}{\hbar q^2}[[H,\,\rho(\boldsymbol{q})],\,\rho(\boldsymbol{q})]=-iq_u\hbar c^2L(\boldsymbol{q})\,. \tag{2·22}$$

Combination of (2·21) and (2·22) establishes (2·17).

For the sake of simplicity, assume cubic symmetry of the crystal, and let the direction of the vector $\boldsymbol{q}$ be direction number 3. Then we have

$$S_{11}=S_{22}\neq S_{33}\qquad S_{\mu\nu}=0\quad\text{for}\quad\mu\neq\nu \tag{2·23}$$

and the condition (2·17) becomes the Buckingham sum rule

$$S_{33}-\bar{L}=0 \tag{2·24}$$

or, for arbitrary orientation of the coordinate system,

$$G(\boldsymbol{q})\equiv\sum_{\mu,\nu}\frac{q_\mu q_\nu}{q^2}S_{\mu\nu}(\boldsymbol{q})-\bar{L}=0\,. \tag{2·24′}$$

The magnetic free energy becomes, in the special coordinate system,

$$F_{AA}=-\tfrac{1}{2}J_{11}(A_1^2+A_2^2)=-\frac{1}{2q^2}J_{11}(B_1^2+B_2^2)\,. \tag{2·25}$$

The last form is manifestly gauge-invariant. The magnetization of the specimen in thermal equilibrium is given by

$$\boldsymbol{M}(\boldsymbol{x})=\boldsymbol{M}(\boldsymbol{q})\,(2/V)^{1/2}\cos(\boldsymbol{q}\cdot\boldsymbol{x}) \tag{2·26}$$

where

$$M_\mu(\boldsymbol{q})=-\frac{\partial F_{AA}}{\partial B_\mu}=K(\boldsymbol{q})B_\mu(\boldsymbol{q}) \tag{2·27}$$

with

$$K(\boldsymbol{q})=\frac{J_{11}}{q^2}=\frac{S_{11}-\bar{L}}{q^2}\,. \tag{2·28}$$

The form (2·28) is the simplest gauge-invariant expression for the magnetic response of the (cubically symmetric) system ; this form corresponds exactly to the prescription of Bardeen :[6] evaluate everything in London gauge ($A_3=0$), without making use of identities such as (2·24).

On the other hand, we may use (2·24) to eliminate $\bar{L}$ altogether; this yields the "Buckingham canonical form" for $K(\boldsymbol{q})$:

$$K_B(\boldsymbol{q}) = \frac{S_{11} - S_{33}}{q^2}. \tag{2·29}$$

For arbitrary orientation of the coordinate system (but retaining cubic symmetry) the canonical form becomes

$$K_B(\boldsymbol{q}) = \frac{1}{q^2} \sum_{\mu,\nu} \tfrac{1}{2}\left(\delta_{\mu\nu} - \frac{3q_\mu q_\nu}{q^2}\right) S_{\mu\nu}(\boldsymbol{q}). \tag{2·29′}$$

This form is unique in the sense that it is the only form which does not involve $\bar{L}$. The most general form for $K(\boldsymbol{q})$ is given by (1·2), with K_B defined by (2·29′) and $G(\boldsymbol{q})$ by (2·24′). The London gauge form (2·28) is the special case $\alpha = 1$ of (1·2).

§ 3. The "absence of complete cancellation".

In the phenomenological equations of London for superconductivity[9] the magnetic field expulsion (Meissner-Ochsenfeld effect) is described by the kernel

$$K(\boldsymbol{q}) = -\frac{1}{4\pi\lambda^2 q^2} \qquad \text{(London equation)} \tag{3·1}$$

where λ is the London penetration depth. On the other hand, a system with normal magnetic behaviour (diamagnetic or paramagnetic) is described by a kernel $K(\boldsymbol{q})$ nearly independent of $\boldsymbol{q}$ for small $\boldsymbol{q}$, with

$$\lim_{q\to 0} K(\boldsymbol{q}) = \frac{\chi}{1 + 4\pi\chi} \qquad \text{(Normal system)} \tag{3·2}$$

where χ is the isothermal magnetic susceptibility of the system.

It has often been noticed that the London gauge form (2·28) for $K(\boldsymbol{q})$ would lead naturally to a kernel proportional to $1/q^2$, i.e., to the London form (3·1), provided only there is "absence of complete cancellation" between the leading term of S_{11} in an expansion in powers of q^2, and $\bar{L}$. That is, if we assume $S_{\mu\nu}(\boldsymbol{q})$ is regular at $\boldsymbol{q} = 0$ and expand:

$$S_{\mu\nu}(\boldsymbol{q}) = S^{(0)}_{\mu\nu} + S^{(2)}_{\mu\nu\alpha\beta} q_\alpha q_\beta + \cdots, \tag{3·3}$$

we would obtain the London kernel provided only that (in the coordinate system in which $\boldsymbol{q}$ is along the 3-axis)

$$S^{(0)}_{11} \neq \bar{L} \qquad \text{(Absence of complete cancellation)}. \tag{3·4}$$

There are many arguments in the literature purporting to show that "stiff wave functions" can lead to precisely this kind of behaviour.

We now prove the theorem:

Theorem: A material of cubic symmetry in which $S_{\mu\nu}(\boldsymbol{q})$ is an analytic function of the three components q_μ of $\boldsymbol{q}$ at $\boldsymbol{q}=0$ is a normal material, magnetically.

Proof: The cubic symmetry simplifies the general expansion (3·3) to the form:

$$S_{\mu\nu}(\boldsymbol{q})=S_0\delta_{\mu\nu}+S_2q^2\delta_{\mu\nu}+S_2'q_\mu q_\nu+\cdots, \tag{3·5}$$

where S_0, S_2, and S_2' are constants. We substitute (3·5) into the Buckingham canonical form (2·29′); the terms proportional to $\delta_{\mu\nu}$ drop out, and we obtain:

$$K(\boldsymbol{q})=-S_2'+\text{order}(\boldsymbol{q}^2) \tag{3·6}$$

which proves the theorem.

This theorem is a more formal way of stating Buckingham's argument[3] against "absence of complete cancellation". *The state of affairs envisaged in that approach to the theory of superconductivity simply cannot happen; as long as* $S_{\mu\nu}(\boldsymbol{q})$ *is analytic at* $\boldsymbol{q}=0$, *the "Cancellation" is always complete.* Conversely, in a valid theory of superconductivity, $S_{\mu\nu}(\boldsymbol{q})$ must not be analytic at $\boldsymbol{q}=0$.

§ 4. The use of approximate statistical operators

At this stage of the discussion gauge invariance as such is not involved; *all* the forms (1·2) for $K(\boldsymbol{q})$ give manifestly gauge invariant formulas (1·1). The problem has been transformed into the following one: How are we to choose sensibly between an infinity of forms (1·2) for $K(\boldsymbol{q})$, all of which are equivalent to each other as long as the statistical operator for the field-free system,

$$U=\exp[\beta(F-H)] \tag{4·1}$$

is known exactly, but which become inequivalent when U is replaced by some approximate form $\hat{U}$?

Given some approximation $\hat{U}$ to the statistical operator, the corresponding approximation for $\bar{L}$ is:

$$\hat{L}=\text{Tr}(L\hat{U}). \tag{4·2}$$

This is however independent of what approximation is used for $\hat{U}$, and is always equal to the exact value

$$\hat{L}=\bar{L}=\frac{1}{V}\sum_{i=1}^{N}\frac{e_i^2}{m_ic^2}. \tag{4·3}$$

This is true because the expectation value (4·2) is taken over the *field-free* system, so that $\sin^2(\boldsymbol{q}\cdot\boldsymbol{x}_i)$ in (2·8) always gives rise to the average value $\frac{1}{2}$.

The corresponding approximation for $S_{\mu\nu}(\boldsymbol{q})$ is obtained from the exact (2·15) by replacing U by $\hat{U}$, i.e.,

$$\hat{S}_{\mu\nu}(\boldsymbol{q})=\frac{\beta}{c^2}\int_0^1 dt\,\text{Tr}[(\hat{U})^{1-t}j_\mu(\boldsymbol{q})(\hat{U})^t j_\nu(\boldsymbol{q})]. \tag{4·4}$$

Note that the operator $j_\mu(\boldsymbol{q})$ here is exact, i.e., it is based on the exact Hamiltonian (2·1).

Although (4·4) defines a definite approximation for $S_{\mu\nu}$, given *any* approximation $\hat{U}$ to the density matrix, in practice the evaluation of (4·4) is extremely difficult unless $\hat{U}$ can be expressed conveniently as the exponential of some operator, i.e., the form (1·6). However, it is *not* necessary that the operator $W(\beta)$ in the exponent have the simple temperature dependence

$$W(\beta)=\beta[F(\beta)\mathbf{1}-H_{\text{app}}] \quad \text{"Hamiltonian approximation"} \tag{4·5}$$

where $\mathbf{1}$ is the unit operator, $F(\beta)$ is some c-number function of β, and the approximate Hamiltonian H_{app} is independent of temperature. On the contrary, $W(\beta)$ in (1·6) may have a much more complicated temperature dependence without leading to any trouble in the evaluation of (4·4).

An example of a non-Hamiltonian approximation is the Husimi[10] extension of the Hartree-Fock method to statistical mechanics. Another example is the Bogoliubov approximation[4] in the theory of superconductivity, in which the coefficients u_k and v_k are in general complicated functions of the temperature. One the other hand, it has not so far proved possible to write the quasi-chemical equilibrium approximation[11,12,13,14] in the form (1·6).

Given approximations for $S_{\mu\nu}$ and for $\bar{L}$, we define the approximate quantities $\hat{K}_B$ and $\hat{G}$ by

$$\hat{K}_B(\boldsymbol{q})\equiv\frac{1}{q^2}\sum_{\mu,\nu}\tfrac{1}{2}\left(\delta_{\mu\nu}-\frac{3q_\mu q_\nu}{q^2}\right)\hat{S}_{\mu\nu}(\boldsymbol{q}), \tag{4·6}$$

$$\hat{G}(\boldsymbol{q})\equiv\sum_{\mu,\nu}\frac{q_\mu q_\nu}{q^2}\hat{S}_{\mu\nu}(\boldsymbol{q})-\hat{L}\,. \tag{4·7}$$

Finally, the "approximation α" to the kernel $K(\boldsymbol{q})$ is denoted by $\hat{K}[\boldsymbol{q},\alpha(\boldsymbol{q})]$ and is defined by equation (1·4).

We have already discussed, in the introduction, the variational condition which we shall impose on $\hat{K}[\boldsymbol{q},\alpha(\boldsymbol{q})]$ so as to select the best function $\alpha(\boldsymbol{q})$. The condition is (1·12), and it merely remains to write down an explicit formula for the quantity $\hat{K}_n(\boldsymbol{q},\alpha)$ which appears in that equation. We shall assume, for simplicity, that there is a significant variation already for $n=1$, i.e., in the first order in ε. The extension to higher n is obvious.*

Let us expand $S_{\mu\nu}(\boldsymbol{q})$ in powers of ε around $\hat{S}_{\mu\nu}(\boldsymbol{q})$, (4·4), where ε appears through the density matrix (1·11). We write

$$S_{\mu\nu}(\boldsymbol{q})=\hat{S}_{\mu\nu}(\boldsymbol{q})+\varepsilon T_{\mu\nu}(\boldsymbol{q})+\text{order }\varepsilon^2. \tag{4·8}$$

and get by direct calculation:

* Actually, the expansion has to be carried to order ε^2 in the Bogoliubov theory, to obtain a non-vanishing term. See an earlier footnote.

$$T_{\mu\nu}(\boldsymbol{q}) = -\frac{\beta}{c^2}\int_0^1 dt \int_0^1 ds\, t\,\mathrm{Tr}\{(\hat{U})^{(1-s)t} H' (\hat{U})^{st}[\, j_\mu(\boldsymbol{q})\,(\hat{U})^{1-t} j_\nu(\boldsymbol{q}) + (\mu \rightleftarrows \nu)\,]\}\,. \tag{4·9}$$

In terms of the eigenstates $|k\rangle$ of the approximate density matrix $\hat{U}$, and hence also of its logarithm, W, we get*

$$T_{\mu\nu}(\boldsymbol{q}) = \frac{\beta}{c^2}\sum_{k,k',k''} \frac{\langle k|H'|k'\rangle}{W_k - W_{k'}}\left[\frac{e^{W_{k'}} - e^{W_{k''}}}{W_{k'} - W_{k''}} - \frac{e^{W_k} - e^{W_{k''}}}{W_k - W_{k''}}\right]$$
$$\cdot[\langle k'|j_\mu(\boldsymbol{q})|k''\rangle\langle k''|j_\nu(\boldsymbol{q})|k\rangle + (\mu \rightleftarrows \nu)]\,. \tag{4·10}$$

If we let the $\boldsymbol{q}$-direction be direction number 3, for simplicity, the condition (1·12) becomes,

$$\frac{T_{11}(\boldsymbol{q}) - T_{33}(\boldsymbol{q}) + \alpha(\boldsymbol{q})\,T_{33}(\boldsymbol{q})}{q^2} = 0\,. \tag{4·11}$$

There is of course no correction in this order arising from the expansion of $\hat{L}$, since $\hat{L}$ is always exactly correct.

§ 5. An example; The ideal Bose gas

As a simple example of the use of these procedures, let us choose the ideal Bose gas in the limit of zero temperature, which is known to be a superconductor obeying the London equation.[15] In this limit, the exact results are:

$$S_{\mu\nu}(\boldsymbol{q}) = \bar{L}\frac{q_\mu q_\nu}{q^2} \tag{5·1}$$

and

$$K(\boldsymbol{q}) = -\frac{\bar{L}}{q^2}\,, \tag{5·2}$$

where $\bar{L}$ is given by (4·3) as:

$$\bar{L} = \frac{ne^2}{mc^2} = \frac{Ne^2}{Vmc^2}\,. \tag{5·3}$$

We note that (5·1) satisfies the Buckingham sum rule, and also that (5·1) is a non-analytic fuction of the three components q_μ of the vector $\boldsymbol{q}$ at $\boldsymbol{q}=0$.

Let us now "approximate" the true density matrix U of the system by an expression of the same form, but with the mass in the Hamiltonian H changed artificially to some neighbouring value m^*. This altered mass appears in $\hat{U}$ but not in the operator $j_\mu(\boldsymbol{q})$, equation (2·7). The resulting approximation $S_{\mu\nu}(\boldsymbol{q})$ is

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* The states k here are states of the entire system. If W has an "independent-particle form", $W = \sum_s w_s a_s^+ a_s$, it is preferable to evaluate (4·9) differently, in terms of sums over independent particle states s, s', s''.

$$\hat{S}_{\mu\nu}(\boldsymbol{q}) = (m^*/m)\, S_{\mu\nu}(\boldsymbol{q}). \tag{5·4}$$

The Buckingham form $\hat{G}(\boldsymbol{q})$ is no longer zero, but rather equals:

$$\hat{G}(\boldsymbol{q}) = \left(\frac{m^*}{m} - 1\right)\bar{L} \equiv \varepsilon \bar{L}, \tag{5·5}$$

and the approximate kernel $\hat{K}[\boldsymbol{q}, \alpha(\boldsymbol{q})]$, equation (1·4), is:

$$\hat{K}[\boldsymbol{q}, \alpha(\boldsymbol{q})] = -[1 + \varepsilon - \varepsilon\alpha(\boldsymbol{q})]\,\frac{\bar{L}}{q^2}, \tag{5·6}$$

where the smallness parameter ε stands for the fractional change in mass, as in (5·5).

If we now differentiate (5·6) with respect to ε and set the variation equal to zero, we get directly:

$$\alpha(\boldsymbol{q}) - 1 = 0 \tag{5·7}$$

and hence the variational result is, from (5·6) and (4·7),

$$\hat{K}(\boldsymbol{q}) = -\frac{\bar{L}}{q^2}. \tag{5·8}$$

In this case, therefore, the variational result (1·13) agrees identically with the exact result (5·2). This is a consequence of the very simple nature of the variation involved here.

We note that the choice (5·7) of $\alpha(\boldsymbol{q})$ in this case does coincide with the Bardeen[6,16] prescription of using the London gauge. This was to be expected: Reference (16) shows that, *if* the wave functions are "stiff" in any gauge whatsoever, then the London gauge is this preferred gauge, and a variational calculation must lead back to the London gauge prescription. The wave functions are indeed "stiff" in the case of the Bose gas.[15]

However, it is *not* permissible to use this argument the other way around, i.e., to calculate in the London gauge without further ado, for an unknown system. For this procedure amounts essentially to *assuming* stiff wave functions, i.e., a property equivalent to the property we are supposed to prove, (the Meissner effect). The procedure given in this paper does not make such an assumption, and does not in general lead to the London gauge choice $\alpha(\boldsymbol{q}) = 1$.

We are grateful to Drs. S. T. Butler and M. R. Schafroth for many enlightening and useful discussions regarding this subject. One of us (T.M.) also would like to thank Professor H. Messel and the Nuclear Research Foundation within the University of Sydney for making possible his stay in Sydney, and another of us (R.M.M.) would like to thank General Motors Holden for the grant of a Research Fellowship.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in dublicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to couni in enough space for formulas, figures or tables.

On the Time-Variation in Thermonuclear-Fusion Systems

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February 14, 1959

Recent advances in Astrophysics teach us the relations between thermonuclear reactions and the stellar evolution. To study the time-variation in fusion reactors by analogy of these results seems to be worth while. For this purpose, besides the two variables so far used, the plasma temperature T and the number density of ions N, we introduce time t into the set of fundamental variables, which command the basic equations to be common to any thermonuclear fusion system. Since the object of any fusion reactor is to get the energy gain, the basic equations must be those of energy balancing, which should include all conditions necessary for selfsustaining thermonuclear reactors. Therefore we define two kinds of time-dependent energy balancing equations as follows. One of them is " energy balancing in the narrow-sence ", or " power balancing " in which at each instance the energy generation rate overcomes the sum of all kinds of energy loss, i. e. ion-electron and electron-electron bremsstrahlung, the loss in heating and cooling processes of plasma, etc. The other is " energy balancing in the wide sense " in which the generated energy integrated through the whole duration of operation overcomes the integrated energy losses. Only one compound parameter $\tau = Nt$, called reduced time, can take the places of two parameters N and t because of the two-body interactions, when long-range and external forces are negligible. Here N plays a role to renew the unit of time scale from the point of view of the irreversible changes caused by the mutual interaction of N particles in the system. Therefore the solutions of the basic equations give the following results dependent on τ.

i) When the initial fuel consists mostly of deuterons, the energy generation rate increases suddenly with τ owing to the increase of secondarily produced nuclei, T^3 and He^3, which have larger reaction cross sections than D^2. Then the number of the secondary T^3 and He^3 become in the transient equilibrium with the primary D^2, and so the energy generation rate decreases again as the

fuel decreases. But in the case of the mixed fuel with more T^3 than $\sim 10\%$, there is no increase of the reaction rate, since the original tritons have the reaction rate large enough to neglect the contribution of the secondary ones.

ii) The radiation loss caused by ion-electron scattering increases gradually due to the increase of multiplly charged secondary nuclei, He^3 and He^4.

iii) From the energy balancing in the narrow sense, we get two kinds of critical values for reduced time, τ_{min} and τ_{max}. After τ_{min} the energy generation rate becomes larger than the radiation loss, because of the sudden increase of the combustible secondary nuclei, when the initial energy generation rate is lower than the radiation loss. τ_{min} can not exist in the case of the mixed fuel with the large number of T^3. After τ_{max} the radiation loss overcomes the energy generation rate because of the decrease of the fuel and the increase of ion-electron bremsstrahlung caused by secondarily produced He^3 and He^4. τ_{max} gives a practical criterion of time at which the old fuel must be replaced by a new one.

iv) Similarly, from the energy balancing in the wide sense, two kinds of criteria for reduced time, $\bar{\tau}_{min}$ and $\bar{\tau}_{max}$ are found. $\bar{\tau}_{min}$ is the same with Lawson's criterion,[1] which is the minimum duration necessary to regain the energy lost in the heating and the cooling processes. So the available energy gain can be stored only in the interval $\bar{\tau}_{min} < \tau < \tau_{max}$. The available energy gain should again be entirely consumed in the interval of $\tau_{max} < \tau < \bar{\tau}_{max}$, supplying the increasing radiation loss,

Table 1

T		$\bar{\tau}_{min}$	τ_{max}	$\bar{\tau}_{max}$
°K	kev	sec cm^{-3}	sec cm^{-3}	sec cm^{-3}
5×10^7	4.31		1.7×10^{15}	
8×10^7	6.90	1.0×10^{15}	1.57×10^{16}	4.2×10^{16}
1.16×10^8	10.0	8.7×10^{14}	1.27×10^{16}	3.4×10^{16}
3×10^8	25.9	4.8×10^{14}	5.90×10^{15}	1.8×10^{16}
5×10^8	43.1	6.6×10^{14}	4.20×10^{15}	9.6×10^{15}
8×10^8	69.0	1.1×10^{15}	3.04×10^{15}	4.7×10^{15}
1.16×10^9	100.0		2.24×10^{15}	
2.32×10^9	200.0		8.7×10^{14}	

Here the efficiencies of energy carried by various particles except neutrons are assumed to be 1/3 in a combined system of a reactor and subordinate apparatus. And the initial fuel includes 50% T^3.

v) Secondarily produced nuclei act double roles which affect energy balancing considerations in the opposite sense, i. e. the increase of energy generation rate and the increase of radiation loss. Initial ratio of the number of D^2 to the number of T^3 determine which of these two roles overcomes the other. In the case of pure deuterium fuel, the positive role of the secondary nuclei overcomes the negative one, and so the total available gain of energy should increase if the magnetic field is strong enough to trap the secondary nuclei having a higher energy than primary ones due to the reactions. In the case of mixed fuel with 50% T^3, in contrast with the case of pure D^2 fuel, confinement of the secondary charged nuclei decreases the total available gain of energy.

The details of this work will be published in the Journal of Nuclear Energy.

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Coupling Constants and the Mass Levels of Baryons

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February 17, 1959

Recently attempts have been made to explain the observed mass levels of baryons in terms of the self energies which arise from the strong π and K meson interactions, assuming that the baryons are mass degenerate in the absence of these interactions.[1] In the lowest perturbation approximations, it has been shown that the observed mass splitting is not given by the Gell-Mann scheme[2] of global symmetry, in which all the π coupling constants are taken to be equal, if the baryons all have the same relative parity.[1]

In this note, we examine the limitations placed on the coupling constants, of the baryons to the π and K mesons, by requiring that within a certain model the observed masses are obtained correctly. The model at present investigated is the Tamm-Dancoff approximation applied to the Hamiltonian of d'Espagnat and Prentki,[3] assuming scalar K and $(\sigma\cdot\boldsymbol{\nabla})$ π interactions, that all baryons have the same relative parity, and the same mass M_0 in the absence of interaction, and that integrals over momenta are cut-off at a value K_m. This model applied to K^+-P scattering has been found to give reasonable results.[4]

In the single meson approximation, the following system of integral equations is obtained for the mass shifts,†

$$\begin{aligned}
\Delta M(N) &= 3f_{N\pi}^2 I_1(\Delta M(N)) \\
&\quad + 3(g_{N K\Sigma}^2 + g_{N K\Lambda}^2) J_1(\Delta M(N) \\
\Delta M(\Lambda) &= 3f_{\Lambda\pi}^2 I_1(\Delta M(\Lambda)) \\
&\quad + 2(g_{N K\Lambda}^2 + g_{\Xi K\Lambda}^2) J_1(\Delta M(\Lambda)) \\
\Delta M(\Sigma) &= (2f_{\Sigma\pi}^2 + f_{N\pi}^2) I_1(\Delta M(\Sigma)) \\
&\quad + 2(g_{N K\Sigma}^2 + g_{\Xi K\Sigma}^2) J_1 \Delta M(\Sigma) \\
\Delta M(\Xi) &= 3f_{\Xi\pi}^2 I_1(\Delta M(\Xi)) \\
&\quad + (3g_{\Xi K\Sigma}^2 + g_{\Xi K\Lambda}^2) J_1(\Delta M(\Xi)) \qquad (1)
\end{aligned}$$

where $\Delta M(B) = M_0 - M(B)$;

$$I_n(x) = \frac{1}{\pi} \int_0^{K_m} \frac{k^4 dk}{(x+\omega(k))^n} ;$$

$$J_n(x) = \frac{1}{\pi} \int_0^{K_m} \frac{k^3 dk}{(x+W(k))^n} ;$$

and $\omega(k) = \sqrt{1+k^2}$,

$W(k) = \sqrt{\mu_K{}^2 + k^2}$

Consistently with the model and approximation used in obtaining (1), charge renormalisation may be applied to obtain renormalised coupling constants $\bar{g}_i$, $\bar{f}_i$ so that, for example,

$$\begin{aligned}
\bar{f}_{N\pi} &= f_{N\pi}\{1 + 3f_{N\pi}^2 I_2(\Delta M(N)) \\
&\quad + (3g_{\Sigma KN}^2 + g_{\Lambda KN}^2) J_2(\Delta M(N))\}^{-1} \\
&\quad \times \{1 + \tfrac{1}{3} f_{N\pi}^2 I_2(\Delta M(N)) \\
&\quad + 2g_{\Sigma KN}/f_{N\pi}(g_{\Sigma KN} f_{\Sigma\pi} \\
&\quad + f_{\Sigma\Lambda N} g_{\Lambda KN}) J_2(\Delta M(N))\} \qquad (2)
\end{aligned}$$

Using the known values of $M(B)$ and assuming three relations amongst the eight coupling constants, and requiring $\bar{f}_{N\pi}^2 = 0.08$, the five equations (1) and (2) were solved numerically for a range of values of M_0 and for

† Units $\hbar = c = \mu_\pi = 1$ are used throughout.

three values of K_m, 5.51, 6.60 and 7.8. The range of M_0 is limited if it is required that the coupling constants obtained should be real.

The following relations between coupling constants have been investigated:

(1) The Gell-Mann scheme: $f_{N\pi}=f_{\Lambda\pi}=f_{\Sigma\pi}=f_{\Xi\pi}$.

(2) Schemes based on the suggestion (from a consideration of K^+-N scattering)[4] that (a) $g^2_{\Sigma KN}=g^2_{\Lambda KN}$ or (b) $3g^2_{\Sigma KN}=g^2_{\Lambda KN}$, assuming also that (a) $g^2_{\Xi K\Sigma}=g^2_{\Xi K\Lambda}$ or (b) $3g^2_{\Xi K\Sigma}=g^2_{\Xi K\Lambda}$ and writing $f^2_{\Xi\pi}=xf^2_{N\pi}$, solutions have been found for cases (a) and (b) with $x=0.5$, 1.0 and 2.0.

No solution with real f's is possible for the Gell-Mann scheme, confirming the work of Katsumori and Shimoura.[1]

Similarly, no solution was found with $x=2$ and it appears that $f^2_{\Xi\pi}$ cannot be much greater than $f^2_{N\pi}$.

Figures (1) and (2) display the calculated coupling constants as a function of $g^2_{\Lambda KN}$ for the cut-off $K_m=7.8$, and the case $g^2_{NK\Sigma}=g^2_{NK\Lambda}$; $g^2_{\Xi K\Sigma}=g^2_{\Xi K\Lambda}$; $x=0.5$.

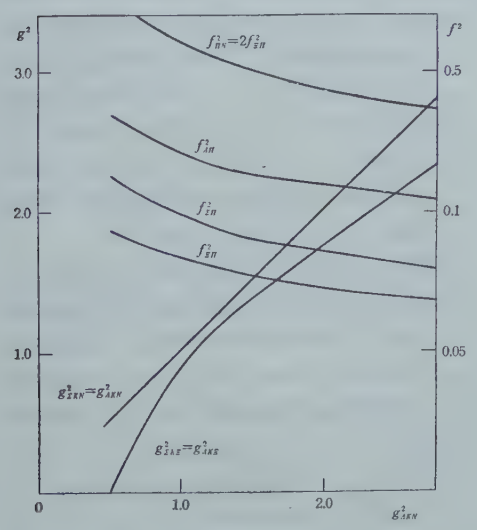

Fig. 1. Unrenormalised constants as a function of $g^2_{\Lambda KN}$ for $K_m=7.8$, $\bar{f}^2_{\pi N}=0.08$.

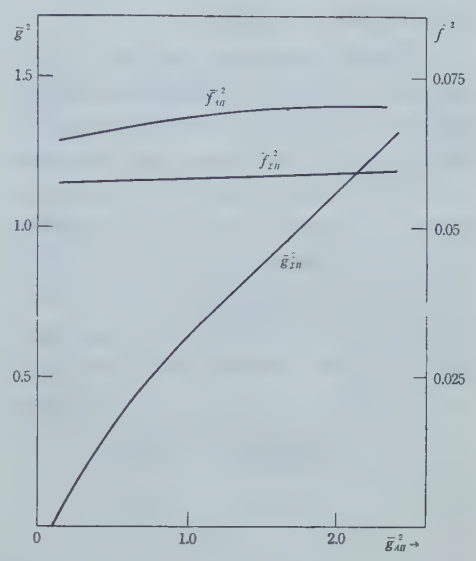

Fig. 2. Renormalised constants as a function of $g^2_{\Lambda KN}$ for $K_m=7.8$, $\bar{f}^2_{N\pi}=0.08$.

The renormalised coupling constants involving Ξ have been suppressed as they are of less interest at present.

Using the same model, the K^+-N elastic scattering cross sections are $g^2_{\Lambda KN}=0.7$, for $K_m=7.8$. Fig. 2 shows consistent with that the corresponding predicted values of $\bar{f}^2_{\Lambda\pi}$, $\bar{f}^2_{\Sigma\pi}$ are 0.066, 0.058, values not inconsistent with the hyperfragment binding energies. Results at $K_m=5.51$ and $K_m=6.6$ and for the other cases mentioned above display very similar features.

A full investigation of the problem with pseudoscalar mesons with γ_5 coupling is in progress, and it is intended to discover whether the resulting coupling constants are consistent with phenomena depending on $\bar{f}_{\Lambda\pi}$, $f_{\Sigma\pi}$ such as hyperfragment binding energies and reactions

$K^- + P = \Sigma^- + \pi^+$, etc.

We wish to thank the staff of the Glasgow University Computing Laboratory for their assistance and advice in the operation of the Deuce Computer.

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Electromagnetic Structure of Nucleons

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March 6, 1959

The r. m. s. radii of charge and a. m. m. of nucleon have been measured by the recent experiments of high-energy electron-proton scattering at Stanford[1] and the experiments of low-energy neutron-atom scattering. Their values are

$$\langle r^2\rangle_1^S = \langle r^2\rangle_1^V \simeq \langle r^2\rangle_2^V \simeq (0.80\times 10^{-13}\text{cm})^2$$

and $\mu^S\langle r^2\rangle_2^S \simeq 0$. (1)

Though many authors have so far calculated these theoretical values, it seems that they do not sufficiently recognize the limitation on the applicability of meson theory. The quantity which sensitively depends on the very inside behaviour of nucleon, such as the magnitude of a. m. m., may receive a large contribution from the region where the theory and approximation methods are very doubtful. However, r. m. s. radii will not be essentially affected by our ignorance of inside behaviour thanks to a weight factor r^2.

The main purposes of the present investigation are as follows.

i) It is known from the analysis of nuclear force that meson theory is valid in the region $r \gtrsim 1/\mu$. So we calculate the distribution functions of charge and a. m. m. in this region in order to supplement the lack of experimental information.

ii) By comparison of our calculation with the experimental r. m. s. radii, we examine whether or not meson theory is semi-quantitatively valid in the region $1/\mu \gtrsim r \gtrsim 1/M$. It is worthwhile to to notice that $\langle r^2\rangle_2^{S,V}$ should be normalized by the *experimental* value of a. m. m. so as to avoid the doubtful contribution from the inside region.

With respect to the outer region of nucleon structure, higher-order effects seem to be less important. Therefore perturbation theory will be reliable at least semi-quantitatively in the region $r \gtrsim 1/M$.

The main contributor to the r. m. s radii of the isoscalar part is probably 3π-state as was pointed out by Chew et al.,[2] but so far no calculation has been

carried out in spite of its importance. We have calculated its lowest order terms of perturbation (g^6-order). The results obtained are as follows.

The distribution functions of charge and a. m. m. respectively are

$$\begin{aligned} 4\pi r^2\rho_1^S(r) &= \int dx A(x)\{\kappa^2(x)r^2 \\ &\quad -2\kappa(x)r\}e^{-\kappa(x)r}, \\ 4\pi r^2\rho_2^S(r) &= \int dx\{B(x)\kappa(x)r \\ &\quad -(2M)^2A(x)r^2\}e^{-\kappa(x)r}, \end{aligned} \tag{2}$$

where

$$dx=\delta(1-\sum_{i=1}^{9}x_i)\prod_{i=1}^{9}dx_i(x_i\geq 0)$$

and $A(x)\geq 0$, $B(x)\geq 0$ and $\kappa(x)\geq 3\mu$ are explicitly obtained. These forms are displayed qualitatively in Fig. 1. The distributions (2) are very favorable (at least qualitatively) to explain the experimental results (1). A lower bound of $\langle r^2\rangle_1^S$ in rough approximation is*

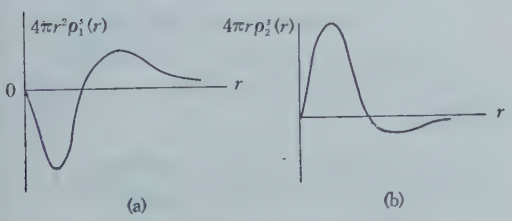

Fig. 1. In (a), the amount of the positive charge is equal to that of negative one. The sign change occurs roughly at $r\sim 1/M$. The peak in (b) is extremely high.

$$\langle r^2\rangle_1^S \gtrsim (0.6\times 10^{-13}\ \mathrm{cm})^2. \tag{3}$$

The numerical evaluations of (2) are extremely tedious because these are eight-fold integrals. Rough numerical evaluations of $\langle r^2\rangle_1^S$, $\langle r^2\rangle_2^S$, etc., (no-cut and cut-off) have been carried out.

We have investigated also the isovector part from our standpoint stated in the beginning. We have calculated the distribution functions by means of static theory and relativistic perturbation up to g^4-order, and examined higher order effects and recoil and s-wave effects. Hyperon effect (closed loop) has also been studied.

Detailed accounts will be reported before long in this journal.

The authors would like to express their deep gratitude to Prof. H. Yukawa for his kind interest and to Profs. S. Hayakawa, G. Takeda and Mr. M. Kato for their valuable discussions and criticisms.

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* We did not separate the contribution from the inner region in this estimation.

On the High Energy Proton ejected in the High Energy $n-d$ Collision

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March 9, 1959

In the previous note[1] we reported some results about high energy $n-d$

elastic scattering. In the present note we shall report the calculated results about high energy protons ejected in high energy $n-d$ collisions. If these protons should be obtained through the process in which the incident neutrons interact with the protons in the deuterons and knock out them, we could use the $n-p$ scattering spin matrix to express the interaction between those nucleons. The scattering spin matrix of spin ½ particles has been analysed by Wolfenstein et al.[2] and related with the phase shifts of nucleon-nucleon scattering by Wright et al.[3] In the present note we use Wright's notation for the scattering spin matrix.

Using the impulse approximation,[4] after some algebraic calculation, we can express the differential cross section and the polarization of high energy proton ejected in the high energy $n-d$ collision as follows:

$$\begin{aligned}I(\theta)=&\tfrac{1}{6}\int d\boldsymbol{k}_{12}\Big[\Big\{[2|N_{13}|^2+2|H_{13}|^2\\&+\tfrac{1}{3}|G_{13}|^2+\tfrac{2}{3}|G_{13}-N_{13}|^2+10|C_{13}|^2\\&+\tfrac{1}{9}|G_{13}-2N_{13}|^2+\tfrac{1}{72}|G_{13}+N_{13}\\&-9B_{13}|^2]|I_{21}^{t}|^2+[2|C_{13}|^2+|H_{13}|^2\\&+\tfrac{1}{3}|G_{13}-2N_{13}|^2\\&+\tfrac{1}{24}|N_{13}+G_{13}+3B_{13}|^2]|I_{21}^{s}|^2\Big\}\\&-\Big\{[2N_{13}{}^*N_{23}+2H_{13}{}^*H_{23}\\&+\tfrac{1}{3}G_{13}{}^*G_{23}+\tfrac{2}{3}(G_{13}{}^*-N_{13}{}^*)(G_{23}\\&-N_{23})+10C_{13}{}^*C_{23}\\&+\tfrac{1}{9}(G_{13}{}^*-2N_{13}{}^*)(G_{23}-2N_{23})\\&+\tfrac{1}{72}(G_{13}{}^*+N_{13}{}^*-9B_{13}{}^*)(G_{23}\\&+N_{23}-9B_{23})]I_{21}^{t*}I_{12}^{t}\\&-[2C_{13}{}^*C_{23}+H_{13}{}^*H_{23}\\&+\tfrac{1}{3}(G_{13}{}^*-2N_{13}{}^*)(G_{23}-2N_{23})\\&+\tfrac{1}{24}(N_{13}{}^*+G_{13}{}^*+3B_{13}{}^*)(N_{23}\\&+G_{23}+3B_{23})]I_{21}^{s*}I_{12}^{s}\Big\}\Big]\delta(E_0-E_1\\&-E_2-E_3),\end{aligned}\tag{1}$$

and

$$\begin{aligned}P(\theta)I(\theta)=&\tfrac{1}{6}\int d\boldsymbol{k}_{12}\Big[\Big\{[\tfrac{9}{2}(N_{13}{}^*C_{13}\\&+N_{13}C_{13}{}^*)-\tfrac{1}{2}(B_{13}{}^*C_{13}+B_{13}C_{13}{}^*)\\&+\tfrac{1}{2}(G_{13}{}^*C_{13}+G_{13}C_{13}{}^*)]|I_{21}^{t}|^2\\&+[\tfrac{9}{6}(N_{13}{}^*C_{13}+N_{13}C_{13}{}^*)\\&+\tfrac{1}{2}(B_{13}{}^*C_{13}+B_{13}C_{13}{}^*)\\&-\tfrac{1}{2}(G_{13}{}^*C_{13}+G_{13}C_{13}{}^*)]|I_{21}^{s}|^2\Big\}\\&-\Big\{[\tfrac{9}{2}(N_{13}{}^*C_{23}+N_{13}C_{23}{}^*)\\&-\tfrac{1}{2}(B_{13}{}^*C_{23}+B_{13}C_{23}{}^*)\\&+\tfrac{1}{2}(G_{13}{}^*C_{23}+G_{13}C_{23}{}^*)]I_{21}^{t*}I_{12}^{t}\\&-[\tfrac{9}{6}(N_{13}{}^*C_{23}+N_{13}C_{23}{}^*)\\&+\tfrac{1}{2}(B_{13}{}^*C_{23}+B_{13}C_{23}{}^*)\\&-\tfrac{1}{2}(G_{13}{}^*C_{23}+G_{13}C_{23}{}^*)]I_{21}^{s*}I_{12}^{s}\Big\}\Big]\\&\times\delta(E_0-E_1-E_2-E_3),\end{aligned}\tag{2}$$

where

$$I_{21}^{t,s*}=\int\psi_{21}^{*t,s}(\boldsymbol{r})\exp\Big[\Big(-i\frac{\boldsymbol{k}_1+\boldsymbol{k}_2}{2}\boldsymbol{r}\Big)\Big]\times\varphi(\boldsymbol{r})\,d\boldsymbol{r},\tag{3}$$

here $\varphi(\boldsymbol{r})$ is the deuteron wave function, $\psi_{21}(\boldsymbol{r})$ is the wave function of final two neutrons, and t, s indicate the singlet or the triplet state of final two neutrons, while 1, 2, and 3 refer to the incident neutron, the neutron and the proton in the deuteron, respectively. B, C, etc., are the coefficients of the $n-p$ scattering spin matrices related with the $n-p$

scattering phase shifts.[3] In derivation of eqs. (1) and (2), we neglect the contribution from the D-state of deuteron wave function.

In order to calculate the B, C, etc., we use the phase shifts of 90 Mev $n-p$ scattering calculated by Gammel and Thaler[5] with the phenomenological potential and by Watari[6] with the meson theoretical potential. These sets of $n-p$ scattering phase shifts can reproduce the same experimental data of free $n-p$ scattering very well although the manner of reproduction are quite different. If we approximate $\psi_{21}^{s's}(\boldsymbol{r})$ by plane waves and use the deuteron wave function suggested by meson theoretical potential[7] for $\varphi(\boldsymbol{r})$, however, the calculated results of eqs. (1) and (2) are somewhat different for the both sets of $n-p$ scattering phase shifts.

As the magnitudes of the corresponding terms owing to the Pauli principle differ for the both sets of $n-p$ scattering phase shifts, these calculated results show the different features.

When the Coulomb interaction is neglected, the study about the protons ejected in $n-d$ collisions is the same as that about the neutrons ejected in $p-d$ collisions. If we had accurate experiments for these collisions, we would obtain more knowledge on the nucleon-nucleon interaction which could not be obtained from free nucleon-nucleon scattering. The rigorous calculations for eqs. (1) and (2) are now in progress.

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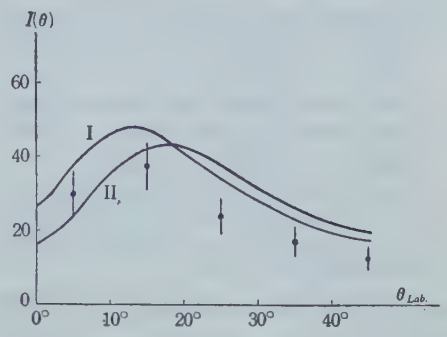

Fig. 1. The differential cross section for high energy proton ejected in the high energy $n-d$ collision.

Curve I is the result calculated by using the 90 Mev $n-p$ scattering phase shifts calculated by Gammel and Thaler[5] with the poenomenological potential. Experimental values are measured by Powell.[8]

Curve II is the result calculated by using the 90 Mev $n-p$ scattering phase shifts calculated by Watari[6] with the meson theoretical potential.

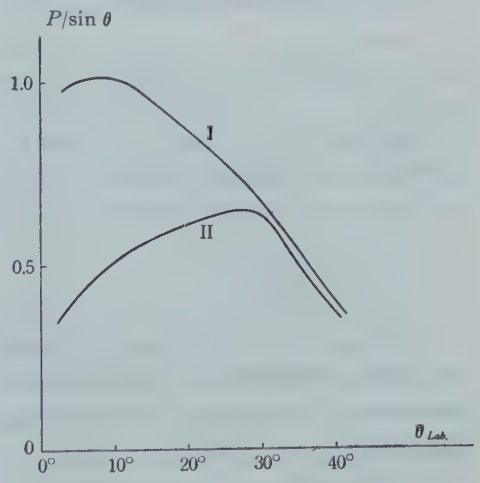

Fig. 2. The polarization for high energy proton ejected in the high energy $n-d$ collision.

Curve I is the results calculated by using the Watari's phase shifts. Curve II is the results calculated by using the Gammel-and Thaler's phase shifts.

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Statistical Nuclei with Repulsive Core Interaction

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The usual statistical theory of nuclei has been developed for the two-body-interaction for nucleons, which is expressed by

$$V_{ik}=J(\boldsymbol{r}_{ik})\,(W+M\mathcal{P}_r+B\mathcal{P}_\sigma +H\mathcal{P}_r\mathcal{P}_\sigma)$$

with

$$J(\boldsymbol{r}_{ik})=-J_0\{\exp(-r_{ik}/r_0)/(r_{ik}/r_0) -\alpha^2\exp(-r_{ik}/\beta r_0)/(r_{ik}/\beta r)\},$$

where $\mathcal{P}_r$ and $\mathcal{P}_\sigma$ are the well-known space- and spin-exchange operators, and W, M, B, H are the weight factors for the Wigner, Majorana, Bartlett and Heisenberg forces respectively; we can assume that $W+M+B+H=1$. The space-dependent part of the interaction, $J(\boldsymbol{r}_{ik})$, has a repulsive core, being a sum of two Yukawa potentials, the attractive and the repulsive, as shown in Fig. 1. J_0 and r_0 are the depth and

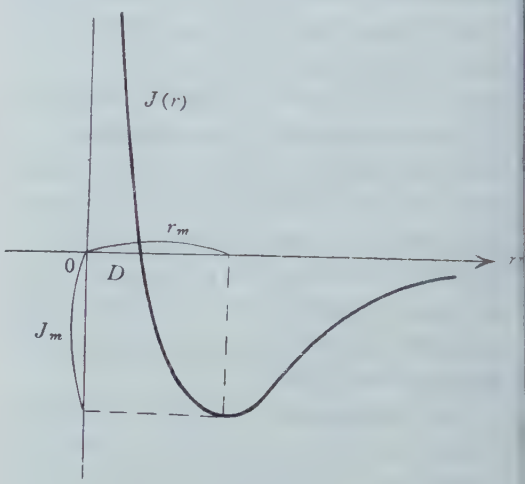

Fig. 1

range of the attractive potential, and $J_0\alpha^2$ and βr_0 are those of the repulsive one; for these constants the conditions

$$0<\beta<1 \quad \text{and} \quad \beta\alpha^2>1$$

must be valid in order to give a real repulsive core.

The expression for the total energy of nuclei may be easily obtained in the same way as in the case of a simple Yukawa attractive potential.[1),2)] If we adopt for the interaction V_{ik} the symmetrical theory of the nuclear forces, i. e. if we put $W=H/2$, $B=M/2$, then the ordinary (not exchange) interaction energy can be eliminated from the total energy, because of the vanishing of its

Table 1

r_0 (10^{-13} cm)	$a'J_0/2\pi$ (Mev)	βr_0 (10^{-13} cm)	α^2	$\beta\alpha^2$	D (10^{-13} cm)	r_m (10^{-13} cm)	$a'J_m/2\pi$ (Mev)
0.6	628	0.36	2.73	1.64	0.445	0.725	41.5
0.8	177	0.28	7.52	2.63	0.418	0.684	40.8
1.0	84.7	0.24	14.1	3.38	0.385	0.638	38.4
1.2	52.1	0.20	26.0	4.34	0.352	0.585	40.8
1.4	35.8	0.20	27.6	3.95	0.320	0.552	38.5

coefficient $a(=W-M/4+B/2-H/2)$, and in this simplified case solutions of the constant density distribution which minimize the whole energy of nuclei become possible. Neglecting for simplicity the Coulomb energies between the protons and also the difference between the proton and the neutron densities, we seeked for such solutions of the constant density as a test of our theory and for the purpose of obtaining some information about the allowable values of the various potential constants.

Now, we equate, for convenience, the theoretical values of the binding energy per nucleon, $-E/A$, and of the radius constant R_0 defined by $R=R_0A^{1/3}$, R and A being the radius and the mass number of any nuclei to the experimental ones,[3]

$$-E/A=8 \text{ Mev}, \quad R_0=1.22\times10^{-13} \text{ cm},$$

then two of the four unspecified parameters r_0, β, α^2 and $a'J_0$ of the interaction remain adjustable, where a' is the coefficient of the exchange interaction energy and is equal to $a'=-W+4M-2B+2H=3W+6B$ in the case of the symmetrical theory. It should be noted that for purely attractive potentials with no repulsive core ($\alpha^2=0$) the value of r_0 must be chosen, in order to conform to the above two experimental data, to be about 3×10^{-13} cm which seems too large as a reasonable range of any single nuclear interaction.[4]

For given values of r_0 and β ($r_0=0.6$, 0.8, 1.0, 1.2 and 1.4×10^{-13} cm; $\beta=0.1$, 0.2, ……, 0.9 for each value of r_0) the remaining two parameters α^2 and $a'J_0$ were calculated, as well as the 'core radius' D, the minimum $-J_m$ and the minimum point r_m of the potential $J(r)$ as defined in Fig. 1. It is found that for each choice of r_0 the core radius D has the maximum value for some value of β; the calculated values of $a'J_0/2\pi$, α^2, $\beta\alpha^2$, D, r_m and $a'J_m/2\pi$ corresponding to these maximum D values are shown in Table 1. It is remarkable that $a'/2\pi$ times the minimum (absolute) values J_m of the potentials are approximately equal, being of about 40 Mev in all these five cases.

A full account of our results has been reported in Soryūsiron-Kenkyū **14**, No. 3 (1957), 237 and **18**, No. 6 (1959), 542 in Japanese. Our next task is to find more precise density distributions corresponding to experiment[3] by taking into account Weizsäcker's inhomogeneity correction term for kinetic energy; such calculations are now proceeding, and further the difference between the proton and the neutron densities will be considered at next stages.

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3) See for example A. E. Glassgold, Revs. Mod. Phys. **30** (1958), 419.
4) Cf. reference 1, § 8.

Errata

Dispersion Relations in Nucleon-Nucleon Scattering

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Prog. Theor. Phys. **19** (1958), 517

The proof in § 5 that $U(\omega)$ tends to adiabatic potential in the static limit is not correct. The replacement of S_F by $-2i\bar{S}$ is not justified. In general (e. g. scalar pion theory or pseudoscalar theory with pseudovector coupling) there appear terms in U which have no static limit. Correct results will be published shortly.

Errata

On the Exchange Magnetic Moments

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Prog. Theor. Phys. **20** (1958), 163

a) In the discussion in § 3, the triton exchange magnetic moment due to pion current, $M^{(\pi)}$, should have the value $+0.13$ n.m. instead of $\simeq+0.3$.
b) In the formula (2), page 167, the sign of $f_6(x)$ should be replaced by plus.
c) As the result of the correction listed in (*b*) the contribution from nucleon current to the triton exchange magnetic moment vanishes exactly, instead of -0.062 n.m. due to our calculational error.
d) In the statement in concluding remarks the value $\simeq+0.24$ should also be corrected to $+0.13$.
e) Correction should be made also in the case of heavy nuclei. The table shown in page 169 should read as follows. (The values calculated when $r_0=0.8$ are listed here for the sake of convenience.)

(i) $j=l+1/2$

r_0	k_0 \ l	0	1	2	3	4	5
1	$3\to\infty$	-0.06	-0.13	-0.17	-0.21	-0.23	-0.26
0.8	$3\to\infty$	-0.25	-0.33	-0.37	-0.41	-0.44	-0.47

(ii) $j=l-1/2$

r_0	k_0 \ l	1	2	3	4	5
1	$3\to\infty$	0.09	0.06	0.03	0.00	-0.02
0.8	$3\to\infty$	0.18	0.20	0.18	0.16	0.16

The authors would like to thank Dr. R. J. Blin-Stoyle and Dr. A. Arking who kindly pointed out the calculational errors.

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Kinematics of the Neutron-Deuteron Scattering

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(Received December 12, 1958)

The kinematics of the elastic neutron-deuteron scattering is studied taking into account the possible spin-orbit coupling in the scattering interaction. The investigation covers the general structure and the parametrization of the scattering matrix, the differential cross section and the polarization, and the behaviour of those parameters characterizing the scattering matrix near zero energy. The present paper will provide a rigorous and explicit basis for the phase shift analysis of the elastic neutron-deuteron scattering.

§ 1. Introduction

A considerable number of theoretical papers are available on the scattering of neutrons by deuterons[1]. However, the spin-orbit coupling such as the tensor force is taken into account in very few of them. Indeed, the tensor force has been considered only in the Born or impulse approximations (which are not applicable to low energy scattering) except in the recent work by Bransden et al.[2]** On the other hand, the spin-orbit coupling may well be important even at low energies because of the rather large deuteron radius. Although the recent series of observations[3] of the spin polarization of scattered neutrons from deuterons in the 2-3 Mev region is not yet conclusive, it is felt that the time has come when one should set out for a large scale computation including the spin-orbit coupling. The present paper aims at presenting the kinematical preliminaries required for such computations. It will also provide the rigorous and explicit basis for the phase shift analysis of the experimental data.

When the spin-orbit coupling is present, the orbital angular momentum l and the spin s fail to be good quantum numbers. This has the effect that the scattering matrix S describing the scattering is of multichannel type. § 2 and § 3 of the present paper are concerned with the study of the structure and the parametriza-

* Also supported by the Nuclear Research Foundation within the University of Sydney.

** Some of our results are also contained in reference 2), which came to our notice after the present work had been finished. Such results should be regarded as included in the present paper for the sake of completeness. The emphasis in reference 2) is on the derivation of the Schroedinger equation using the resonating group method, and not on the detailed study of the kinematical aspects of the problem.

tion of S for the elastic neutron-deuteron scattering. The only assumptions we make are the unitarity and the symmetry of S for given total angular momentum J and parity Π.

In § 4 we give the explicit expressions for the cross section and the polarization in terms of real and independent parameters characterizing S. The behaviours of the parameters near zero energy are studied in § 5 using the derivative matrix formalism for nuclear reactions. This will make it possible to define two scattering lengths which characterize the zero energy scattering completely.

The inelastic events such as three-body disintegration are not taken into account in the present work. We note, therefore, that the parameters characterizing S in this paper will, in general, be complex at higher energies where the inelastic scattering becomes important.

§ 2. Classification of scattering states

The total angular momentm J of the neutron-deuteron system is given by

$$J=l+s, \tag{2·1}$$

where l is the relative orbital angular momentum of neutron and deuteron (*not* the total orbital angular momentum of the three-body system). Since the deuteron has even parity, the parity of the neutron-deuteron system is equal to the parity of l. s is the channel spin defined by

$$s=i+I, \tag{2·2}$$

where i and I are the spins of neutron and deuteron, respectively, with the magnitudes $i=\frac{1}{2}$ and $I=1$. Notice that I is *not* quite the sum of the intrinsic spins of the constituent nucleons in the deuteron because of the D-state admixture in its ground state. The two possible values of s, $\frac{1}{2}$ and $3/2$, are referred to as doublet and quartet, respectively. It is important to notice that this terminology does *not* refer to the three-body spin functions.

Throughout the present paper, the phase of the channel spin function will be fixed by

$$\chi_s^{m_s}=\sum_{m_i}(iIm_im_s-m_i|iIsm_s)\chi_i^{m_i}\chi_I^{m_s-m_i}, \tag{2·3}$$

where the bracket symbol is the Clebsch-Gordan coefficient defined as in Condon and Shortley[4]. This choice agrees with the one used by Simon and Welton[5] in the reduction of spin matrix elements.

For given J and Π there exist, in general, three possible sets of (s, l) which we specify by the numbering ρ as shown in Table 1 for the first few values of J. These three states are coupled to each other to form the scattering eigenstates. The only exception to this general rule occurs for $J=\frac{1}{2}$ where only two substates, one doublet and one quartet, are coupled for each Π.

Table 1 Classification of scattering states

J	$\Pi=+$			$\Pi=-$		
	$\rho=1$	3	2	1	2	3
1/2	$^2S_{1/2}$	—	$^4D_{1/2}$	$^2P_{1/2}$	$^4P_{1/2}$	—
3/2	$^2D_{3/2}$	$^4D_{3/2}$	$^4S_{3/2}$	$^2P_{3/2}$	$^4P_{3/2}$	$^4F_{3/2}$
5/2	$^2D_{5/2}$	$^4D_{5/2}$	$^4G_{5/2}$	$^2F_{5/2}$	$^4F_{5/2}$	$^4P_{5/2}$
7/2	$^2G_{7/2}$	$^4G_{7/2}$	$^4D_{7/2}$	$^2F_{7/2}$	$^4F_{7/2}$	$^4H_{7/2}$

It is stressed at this point that, although the notations look similar, the classification of neutron-deuteron scattering states given in Table 1 is quite different from that of triton states as considered by Derrick and Blatt[6]. In fact, the classification of scattering states is essentially that of a two-body system, the internal structure of the deuteron entering only in the definition of the channel spin s.

§ 3. Parametrization of scattering matrix

Given the total angular momentum J and the parity Π, then, according to the results of § 2, one is to deal with three radial wave functions (multiplied by r) $u_{J\rho}(r)$ ($\rho=1, 2, 3$) describing the relative motion of the neutron and the center of mass of the deuteron. Here, the parity Π is related to ρ by $\Pi=(-1)^{l_\rho}$. For more detailed description on the separation of the radial wave functions we refer to Blatt and Biedenharn[7].

At sufficiently large distances for which $kr \gg \max(l_\rho)$, $u_{J\rho}(r)$ are the superpositions of incoming and outgoing spherical waves:

$$u_{J\rho}(r)=A_\rho^J \exp\{-i(kr-l_\rho\pi/2)\}-B_\rho^J \exp i(kr-l_\rho\pi/2). \tag{3·1}$$

The 3×3 scattering matrix S for given J and Π (equal to the parity of l_ρ) is defined by

$$B_\rho^J=\sum_{\rho'} S_{\rho\rho'}^J A_{\rho'}^J. \tag{3·2}$$

According to Blatt and Biedenharn[7], the matrix S has the following structure:

$$S^{J\Pi}=(U_{J\Pi})^{-1}\exp(2i\Delta_{J\Pi})U_{J\Pi}, \tag{3·3}$$

where $\Delta_{J\Pi}$ is the diagonal matrix

$$(\Delta_{J\Pi})_{\mu\nu}=\delta_\mu^{J\Pi}\delta_{\mu\nu}, \tag{3·4}$$

and $U_{J\Pi}$ is an orthogonal 3×3 matrix. The real parameters $\delta_\mu^{J\Pi}$ ($\mu=1, 2, 3$) in (3·4) are the eigen-phaseshifts. The matrices $U_{J\Pi}$ which generate the linear transformation on the incoming amplitudes to form the scattering eigenstates are isomorphic to three dimensional rotations. Hence, in terms of Euler angles α, β, and γ,

$$U_{JII}=\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\beta & \sin\beta\\ 0 & -\sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\cos\gamma & \sin\gamma & 0\\ -\sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{3·5}$$

Indices JII on the Euler angles are understood.

The physical meanings of the parameters appearing in (3·4) and (3·5) will be clear from the following considerations. Suppose that we are given the ratios of incident waves $A_\rho/A_{\rho'}$. (we suppress indices JII in the following when there is no risk of confusion.) If

$$(A_\rho/A_{\rho'})=(B_\rho/B_{\rho'})=(\sum_\lambda S_{\rho\lambda}A_\lambda)/(\sum_{\lambda'} S_{\rho'\lambda'}A_{\lambda'}), \tag{3·6}$$

then S is diagonal in the sense that it only induces a shift in phase. Substituting (3·3) into (3·6), we get three solutions for $A_\rho/A_{\rho'}$:

$$(A_\rho/A_{\rho'})_\mu=U_{\mu\rho}/U_{\mu\rho'},\ \mu=1,\ 2,\ 3. \tag{3·7}$$

Thus the matrix elements of U specify the necessary mixtures of three substates for given J and II in order to form the scattering eigenstates. To see the meaning of δ_μ in (3·4) as the eigen-phaseshifts, substitute the μ-th solution of (3·7) into (3·1) to get

$$u_\rho^\mu(r)=U_{\mu\rho}\sin(kr-l_\rho\pi/2+\delta_\mu). \tag{3·8}$$

This establishes the statement that δ_μ in (3·4) is the eigen-phaseshift of the μ-th scattering eigensolution charactrized by (3·7). We note an orthogonality relation to be satisfied by the three eigensolutions:

$$\sum_\rho u_\rho^\mu u_\rho^\nu=0,\ \mu\neq\nu. \tag{3·9}$$

This is a direct consequence of the orthogonality of U.

The situation in the special case of $J=\frac{1}{2}$, where there are only two substates for both II, is wellknown in the study of the triplet nucleon-nucleon scattering. Therefore, we shall not go into this special case.

§ 4. Cross section and polarization

The scattering amplitude f may be regarded as a matrix in the six dimensional channel spin space. Elements of this matrix are given by

$$(s'm_s'|f|sm_s)=(\sqrt{\pi}\lambda/i)\sum_{J=1/2}^{\infty}\sum_{l=|J-s|}^{J+s}\sum_{l'=|J-s'|}^{J+s'}(2l+1)^{1/2}(0lsm_s|lsJm_s)$$
$$\times(l's'm_s-m_s'm_s'|l's'Jm_s)\,(l's'|S^{JII}-1|ls)\,Y_l^{m_s-m_s'}(\theta,\varphi). \tag{4·1}$$

Here, Y_l^m is the normalized spherical harmonics defined as in Condon and Shortley[4]. The parity is fixed by $II=(-1)^l=(-1)^{l'}$ for given l and l'. The set $(l,\ s)$ defines the numbering ρ as specified in § 2. In specifying the elements of the matrix S, we have used the fact that they are independent of the z-component of the channel

spin. From this and the symmetry properties of Clebsch-Gordan coefficients, it follows that

$$(s'-m_s'|f|s-m_s)=(-)^{s+s'+m_s+m_s'}(s'm_s'|f|sm_s),$$
$$\text{for}\quad \varphi=0. \tag{4·2}$$

If an automatic computor is available it may be preferable to calculate the cross section and the polarization directly from (4·1) without using the reduced expressions derived below. In such a case, the identity (4·2) will be useful since the values of the amplitudes at $\varphi=0$ only are required for that purpose.

The differential cross section $I_0(\theta)$, when the incident neutrons and the target deuterons are both unpolarized, can be written as[8]

$$I_0(\theta)=(1/6)\,\mathrm{Tr}(f^+f), \tag{4·3}$$

where f^+ is the Hermitian conjugate of f and the trace is taken over six channel spin states. The polarization $P(\theta)$ of the scattered neutron under the same situation is given by[8]

$$I_0(\theta)P(\theta)\boldsymbol{n}=(1/6)\,\mathrm{Tr}(f^+\sigma f), \tag{4·4}$$

where $\boldsymbol{n}$ is the unit vector to the plane of scattering and σ is the Pauli spin operator for the neutron.

The sums over magnetic quantum numbers which arise by introducing (4·1) into (4·3) and (4·4) are of purely kinematical nature. They can be evaluated by using the technique of Blatt and Biedenharn[7] for (4·3) and Simon and Welton[5] for (4·4).

The total cross section is given by

$$\sigma=(2\pi\lambda\!\!\!^{-2}/3)\sum_{J=1/2}^{\infty}\sum_{\Pi=\pm}(2J+1)\,(\sin^2\delta_1^{J\Pi}+\sin^2\delta_2^{J\Pi}+\sin^2\delta_3^{J\Pi}), \tag{4·5}$$

where it is recalled that there are only two eigen-phaseshifts for each Π if $J=\frac{1}{2}$.

The differential cross section can be reduced to the form

$$I_0(\theta)=(\lambda\!\!\!^{-2}/6)\sum_L B_L P_L(\cos\theta), \tag{4·6}$$

where

$$B_L=\sum_{\lambda\mu}\sum_{J_1J_2}\sum_{\Pi_1\Pi_2}\sin\delta_\lambda^{J_1\Pi_1}\sin\delta_\mu^{J_2\Pi_2}\cos(\delta_\lambda^{J_1\Pi_1}-\delta_\mu^{J_2\Pi_2})\,[B_L'(\lambda J_1\Pi_1,\ \mu J_2\Pi_2)]^2, \tag{4·7}$$

and

$$B_L'(\lambda J_1\Pi_1,\ \mu J_2\Pi_2)=\sum_{s=1/2,3/2}\sum_{l_1(\Pi_1)}\sum_{l_2(\Pi_2)}(-1)^{s-1/2}Z(l_1J_1l_2J_2;\ sL)\,(U_{J_1\Pi_1})_{\lambda\rho_1}(U_{J_2\Pi_2})_{\mu\rho_2}. \tag{4·8}$$

Here the sum over $l(\Pi)$ means that it should be taken over l of parity Π only. ρ_1 and ρ_2 take the values 1, 2, and 3 according to the combinations (J_1l_1s) and (J_2l_2s) as defined in § 2. Indices λ and μ refer to the eigensolutions. The explicit expressions for the relevant matrix elements of U in terms of independent para-

meters are found in (3·5). The Z-coefficient in (4·8) differs from the one originally defined by Blatt and Biedenharn[7] by the phase factor $i^{L-l_1+l_2}$ in accordance with the remark by Huby[9].

We observe the symmetry property

$$B_L'(\lambda J_1 \Pi_1,\ \mu J_2 \Pi_2) = B_L'(\mu J_2 \Pi_2,\ \lambda J_1 \Pi_1). \tag{4·9}$$

Further, B_L' vanishes identically unless $|J_1 - J_2| \leq L \leq J_1 + J_2$ and $\Pi_1 \cdot \Pi_2 = (-1)^L$. The last property reduces the sums over Π_1 and Π_2 in (4·7) to a single sum. We also note that the total cross section is given by

$$\sigma = (2\pi\lambda\!\!\!^{-2}/3) B_0. \tag{4·10}$$

The special case $J=\frac{1}{2}$ may be incorporated into the general scheme (4·6) and (4·7) by formally defining as follows:

$$\begin{aligned} &\delta_2^{1/2+} = \delta_3^{1/2-} \equiv 0, \\ &\alpha_{1/2+} = -\gamma_{1/2+} \equiv \pi/2, \\ &\alpha_{1/2-} = \beta_{1/2-} \equiv 0. \end{aligned} \tag{4·11}$$

Turning now to the polarization* $P(\theta)$, (4·4) can be reduced to the form

$$I_0(\theta) P(\theta) = (\lambda\!\!\!^{-2}/\sqrt{3}) \sum_L A_L \bar{P}_L^1(\cos\theta), \tag{4·12}$$

where $\bar{P}_L^1$ is the normalized associated Legendre function[10] and [again with the convention (4·11)]

$$\begin{aligned} A_L = \sum_{\lambda\mu} \sum_{J_1 J_2} \sum_{\Pi_1 \Pi_2} &\sin\delta_\lambda^{J_1\Pi_1} \sin\delta_\mu^{J_2\Pi_2} \sin(\delta_\lambda^{J_1\Pi_1} - \delta_\mu^{J_2\Pi_2}) \\ &\times B_L'(\lambda J_1 \Pi_1,\ \mu J_2 \Pi_2)\, A_L'(\lambda J_1 \Pi_1,\ \mu J_2 \Pi_2). \end{aligned} \tag{4·13}$$

Here B_L' is defined in (4·8) and

$$\begin{aligned} A_L'(\lambda J_1 \Pi_1,\ \mu J_2 \Pi_2) = -\sum_{s_1 s_2} \sum_{l_1(\Pi_1)} \sum_{l_2(\Pi_2)} &(-1)^{J_1+s_1+l_1} \\ &\times [(2l_1+1)(2l_2+1)(2s_1+1)(2s_2+1)(2J_1+1)2J_2+1)]^{1/2} \\ &\times (l_1 l_2 00 | l_1 l_2 L0)\, W(\tfrac{1}{2} s_1 \tfrac{1}{2} s_2;\ 11)\, X(J_1 l_1 s_1;\ J_2 l_2 s_2;\ LL1) \\ &\times (U_{J_1\Pi_1})_{\lambda\rho_1} (U_{J_2\Pi_2})_{\mu\rho_2}, \end{aligned} \tag{4·14}$$

in the same notation as in (4·8). W is the Racah coefficient and X the Fano-Racah coefficient (9-j symbol)[11].

An intermediate expression leading to (4·13) has been checked against the general expression for the polarization obtained by Simon and Welton[5]. They agreed except for some numerical and phase factors. The difference turned out to be due to the different definition of the polarization used by those authors and to Huby's correction[9].

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* The author has since been informed that the polarization in N-D scattering had also been considered by K. Smith (private communication).

It is easily shown that

$$A_L'(\lambda J_1 \Pi_1,\ \mu J_2 \Pi_2) = -A_L'(\mu J_2 \Pi_2,\ \lambda J_1 \Pi_1). \qquad (4\cdot15)$$

The sums over Π_1 and Π_2 again reduce to a single sum. This, together with (4·15), makes the actual evaluation of (4·13) much simpler than it appears.

Before closing this section, it may be worth noting that B_L' and A_L' can both be reduced to the forms

$$\sum_{i,k} C_L^{ik}(J_1 \Pi_1,\ J_2 \Pi_2)\,(U_{J_1\Pi_1})_{\lambda i}(U_{J_2\Pi_2})_{\mu k},$$
$$i,\ k=1,\ 2,\ 3, \qquad (4\cdot16)$$

by explicitly performing the indicated sums in (4·8) and (4·14). The new coefficients C (for both A_L' and B_L') have been tabulated for $L=1,\ 2$ and $J_1,\ J_2 \leq 7/2(+)$.

§ 5. Behaviour of parameters near zero energy

The general behaviour of the parameters characterising S as a function of energy requires the knowledge of the basic interaction causing the scattering. However, their behaviour near zero energy can be determined purely kinematically. The formalism appropriate for this purpose is that of the derivative matrix introduced by Wigner and Eisenbud[12]. This formalism was successfully applied to the nucleon-nucleon scattering with spin-orbit coupling by Blatt and Biedenharn to determine the variations of eigen-phaseshifts and mixture parameters near zero energy.[13] The method has been generalized to a full extent by Delves recently.[14]

Suppose that the scattering interaction vanishes for $r>R$. Then the μ-th eigensolution can be written in the form

$$u_\rho^\mu(r) = U_{\mu\rho}[\cot\delta_\mu F_{l\rho}(r) + G_{l\rho}(r)],\ \text{(for } r \geq R). \qquad (5\cdot1)$$

Here F_l and G_l are the usual regular and irregular solutions of the free wave equation.

Now we introduce another set of linearly independent solutions of the same free wave equation:

$$\begin{aligned}
\mathfrak{H}_l(r) &= G_l(R)F_l(r) - F_l(R)G_l(r),\\
\mathfrak{G}_l(r) &= [F_l'(R) + (l/R)F_l(R)]G_l(r)\\
&\quad -[G_l'(R) + (l/R)G_l(R)]F_l(r)\\
&\equiv H_l(R)G_l(r) - I_l(R)F_l(r),
\end{aligned} \qquad (5\cdot2)$$

where $F_l'(R) = [\partial F_l(r)/\partial r]_{r=R}$, etc. The eigensolutions (5·1) must also be expressed as the linear combinations of these functions, i. e.,

$$u_\rho^\mu(r) = a_\rho^\mu \mathfrak{H}_{l\rho}(r) + b_\rho^\mu \mathfrak{G}_{l\rho}(r),\ \text{(for } r \geq R). \qquad (5\cdot3)$$

The 3×3 derivative matrix $\mathfrak{R}$ (indices $JI\!I$ are understood) is defined by

$$b_\rho^\mu = \sum_{\rho'} \Re_{\rho\rho'} a_{\rho'}^\mu, \quad (\rho, \mu = 1, 2, 3). \tag{5·4}$$

$\Re$ is real and symmetric, and its elements approach certain constants in the zero energy limit.

From (5·1) to (5·4) we have

$$\cot \delta_\mu = -\frac{a_\rho^\mu G_{l\rho}(R) - (\Re a^\mu)_\rho H_{l\rho}(R)}{a_\rho^\mu F_{l\rho}(R) - (\Re a^\mu)_\rho I_{l\rho}(R)}, \tag{5·5}$$

$$U_{\mu\rho}/U_{\mu\rho'} = \frac{a_\rho^\mu G_{l\rho}(R) - (\Re a^\mu)_\rho I_{l\rho}(R)}{a_{\rho'}^\mu G_{l\rho'}(R) - (\Re a^\mu)_{\rho'} I_{l\rho'}(R)}, \tag{5·6}$$

where $(\Re a^\mu)_\rho = \sum \Re_{\rho\rho'} a_{\rho'}^\mu$, etc. Notice that (5·5) actually contains three equations ($\rho = 1, 2, 3$). Two of them can be used to determine two ratios of the amplitudes $a_\rho^\mu/a_{\rho'}^\mu$ (e. g., a_1^μ/a_3^μ and a_2^μ/a_3^μ) in terms of $\Re$-matrix elements and the values of H and I at $r=R$. The third equation and (5·6), then, give $\cot \delta_\mu$ and $U_{\mu\rho}/U_{\mu\rho'}$ as functions of these variables, respectively. We are interested only in the zero energy limits of (5·5) and (5·6). Hence, we follow this procedure only after expanding all quantities in (5·5) and (5·6) in power series in kR and dropping all but the leading terms as $kR \to 0$.

There are two cases to be considered separately:

$$l_1 = l_2 = J - 1/2 \equiv l, \quad l_3 = J + 3/2 \equiv l' = l + 2, \tag{5·7}$$

$$l_1 = l_2 = J + 1/2 \equiv l, \quad l_3 = J - 3/2 \equiv l' = l - 2. \tag{5·8}$$

We first give the answers for (5·8) because this includes the most interesting case (as $kR \to 0$), i. e., where ${}^4S_{3/2}$, ${}^2D_{3/2}$ and ${}^4D_{3/2}$ are coupled together. All matrix elements of $\Re$ refer to zero energy in the following expressions. Let us first define

$$\begin{aligned} K_\rho &= \Re_{\rho\rho} - R/(2l_\rho + 1), \\ x_\pm &= (1/2)[\{2\Re_{12}\Re_{13} - \Re_{23}(\Re_{11} - \Re_{22})\}K_3 \\ &\quad - \Re_{23}\{(\Re_{23}^2 + \Re_{13}^2) \pm D^{1/2}\}][\Re_{12}(\Re_{23}^2 - \Re_{13}^2) \\ &\quad + \Re_{23}\Re_{13}(\Re_{11} - \Re_{22})]^{-1}, \\ y_\pm &= -[K_3 + \Re_{13}x_\pm]/\Re_{23}, \end{aligned}$$

where

$$\begin{aligned} D &= K_3^2\{(\Re_{11} - \Re_{22})^2 + 4\Re_{12}^2\} \\ &\quad + 2K_3\{(\Re_{11} - \Re_{22})(\Re_{23}^2 - \Re_{13}^2) - 4\Re_{12}\Re_{13}\Re_{23}\} \\ &\quad + (\Re_{23}^2 + \Re_{13}^2)^2. \end{aligned}$$

Then, as $kR \to 0$, we obtain

$$\begin{aligned} k \cot \delta_1 \simeq (kR)^{-2l}[(2l-1)!!]^2[K_1 - \Re_{12}\Re_{13}/\Re_{23} \\ + (\Re_{13} - K_3\Re_{12}/\Re_{23})/x_+]^{-1}, \end{aligned}$$

$$k\cot\delta_2 \simeq (kR)^{-2l}[(2l-1)!!]^2[K_2-\mathfrak{R}_{12}\mathfrak{R}_{23}/\mathfrak{R}_{13}+(\mathfrak{R}_{23}-K_3\mathfrak{R}_{12}/\mathfrak{R}_{13})/y_+]^{-1}$$

$$k\cot\delta_3 \simeq (kR)^{-2l'}[(2l'-1)!!]^2 K_3^{-1}, \tag{5·9}$$

where $(2l-1)!!=(2l-1)(2l-3)\cdots\cdots 3\cdot 1$, and

$$U_{11}/U_{12}=\frac{\cos\alpha\cos\gamma-\sin\alpha\cos\beta\sin\gamma}{\cos\alpha\sin\gamma+\sin\alpha\cos\beta\cos\gamma}\simeq x_+/y_+,$$

$$U_{21}/U_{22}=\frac{\sin\alpha\cos\gamma+\cos\alpha\cos\beta\sin\gamma}{\sin\alpha\sin\gamma-\cos\alpha\cos\beta\cos\gamma}\simeq x_-/y_-,$$

$$U_{31}/U_{32}=-\tan\gamma\simeq\mathfrak{R}_{13}/\mathfrak{R}_{23}, \tag{5·10}$$

$$U_{12}/U_{13}=\frac{\cos\alpha\sin\gamma+\sin\alpha\cos\beta\cos\gamma}{\sin\alpha\sin\beta}\simeq(kR)^{-2}(2l-1)(2l-3)y_+,$$

$$U_{22}/U_{23}=\frac{\cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma}{\cos\alpha\sin\beta}\simeq(kR)^{-2}(2l-1)(2l-3)y_-,$$

$$U_{32}/U_{33}=-\tan\beta\cos\gamma\simeq(kR)^2\mathfrak{R}_{23}[(2l-1)(2l-3)]^{-1}K_3^{-1}.$$

Thus the eigen-phaseshifts show the behaviour

$$\begin{aligned}&\delta_1,\ \delta_2\propto k^{(2l+1)},\ \text{i. e., as } l\text{-wave},\\ &\delta_3\propto k^{(2l'+1)},\ \text{i. e., as } l'\text{-wave},\end{aligned} \tag{5·11}$$

as expected in the limit $k\to 0$. For the special case of $J+3/2(+)$ $(l=2,\ l'=0)$, the last equation of (5·9) defines the quartet scattering length a_4 at zero energy:

$$k\cot\delta_3^{3/2+}=-(a_4)^{-1},\ \text{as } k\to 0. \tag{5·12}$$

From (5·1) and (5·10) we also see that the eigensolutions 1 and 2 are predominantly the l-waves whereas the eigensolution 3 is predominantly the l'-wave. Further, the mixture parameters (Euler angles) behave in the following manner:

$$\alpha,\ \gamma\propto\text{const.},\ \beta\propto k^2. \tag{5·13}$$

This is the expected behaviour because α and γ are related to the mixture of two substates with different channel spins but the same l; β, on the other hand, concerns the mixture of two substates with the same channel spin but different orbital angular momenta l and l' $(=l-2)$ [c. f. (3·5)]. All these results are, of course, in agreement with Delves' general results[14].

The other case (5·7) can be treated in the same way leading to similar results corresponding to (5·9) and (5·10). We shall not go into detail in this case except to state that (5·11) and (5·13) remain valid.

The behaviour of the parameters in the special case of $J=\frac{1}{2}$ are wellknown in the study of the triplet nucleon-nucleon scattering[13]. In particular, for $J=\frac{1}{2}(+)$ where $^2S_{1/2}$ is coupled to $^4D_{1/2}$, we have [c. f. (4·11)]

$$\delta_3^{1/2+}\propto k^5,\quad \beta_{1/2+}\propto k^2,$$

and

$$k \cot \delta_1^{1/2+} = -(a_2)^{-1}, \quad k \to 0, \tag{5·14}$$

defines the doublet scattering length a_2 at zero energy.

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σ-π Interaction and Proton Hyperfine Interaction

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Proton hyperfine interactions in aromatic radicals and aromatic ions are calculated by the MO (molecular orbital) method, and the basis of the semi-empirical equaticon, $a_N=Q\rho$, is clarified; here a_N is the splitting constant referred to proton N, ρ is the unpaired electron density at the carbon atom adjacent to proton N, and Q is the semi-empirical constant assumed to be the same for all CH bonds. Two essential approximations in this treatment are: a) the singlet-triplet excitation energies in σ orbitals are replaced by an averaged value ΔE_{AV}, b) all σ orbitals containing CH bonds are transformed into $\bar{\sigma}$ orbitals which are strongly localized to each CH bond, and are approximated to be CH bond orbitals between the *1s* hydrogen atomic orbitals and the carbon sp_2 hybrid orbitals. Then it is shown that the calculation of proton hyperfine splittings in the hypothetical CH fragment leads to the same results as in the case of the entire set of the aromatic system. The above two approximations are shown to be reasonable for aromatic systems.

The calculated value of Q in the present paper is -22.2 gauss, which is quite close to the semi-empirical value -22.5 gauss.

§ 1. Introduction

Important information on unpaired electron distribution in aromatic radicals and aromatic negative ions can be obtained from proton hyperfine splittings in their electron spin resonance spectra in the liquid state. If a_N is the hyperfine splitting constant due to proton N, then a_N is related to the unpaired electron density at the carbon atom adjacent to proton N by the simple equation

$$a_N=Q\rho\,, \tag{1}$$

which is proposed by McConnell[1] semi-empirically. Here Q is the semi-empirical constant whose value is -22.5 gauss[2] and is assumed to be approximately the same for all aromatic CH bonds. The unpaired electron in the aromatic system, to the first, crude approximation, moves in a π MO antisymmetric with respect to the molecular plane. The hyperfine splittings are those arising from in-plane aromatic protons. In attempting to formulate a theory of the relation between the odd electron densities at carbon atoms in aromatic systems and proton hyperfine splittings one is immediately led to a striking paradox. First all dipolar hyperfine interactions average to zero because of rapid tumbling of molecules in the liquid state[3]. Only the contact Fermi hyperfine interaction[4] can contribute to this interaction. If, however, the odd electron density in aromatic system is confined to a π MO, there is zero odd electron density at the in-plane aromatic protons, and the contact hyper-

fine interaction is also zero. Weissman[5] has suggested that the out-of-plane zero point hydrogen vibration might account for the observed couplings. However, calculations by Venkateraman and Fraenkel[6] on this model in the case of *p*-benzosemiquinone ion gave couplings of the magnitude 0.053 gauss, which is too small compared with the observed value of 2.3 gauss. Therefore we may safely rule out this mechanism.

As has been pointed out by many authors,[2,7,8,9] it seems certain that the observed proton hyperfine splittings are mainly due to the σ-π electron exchange interaction. Among them McConnell and Chesnut[2] have discussed the proton hyperfine splittings in terms of a hypothetical CH fragment involved in aromatic systems and showed that the MO method and the VB (valence bond) method can produce almost indentical results. Jarrett[10] has performed a numerical calculation in terms of the CH fragment by the VB method and obtained 28 gauss as the value of Q. The author[11] has calculated the splitting constants of the naphthalene negative ion and *p*-benzosemiquinone ion in the same scheme, and obtained good agreement with the observed values, in which the π MO occupied by the unpaired electron is approximated to be the lowest vacant MO which is constructed by a linear combination of $2p\pi$ atomic orbitals.

McConnel and Chesnut[2] have generalized the theory of the indirect proton hyperfine interaction in the CH fragment to the case of polyatomic π electron system, e.g. large plane aromatic radicals and ions. They have performed this generalization by the use of the MO method and succeeded in the derivation of eq. (1) and in expressing the semi-empirical constant Q in terms of the fundamental molecular integrals. If we look into their theory, in more detail, however, we find that there is a quantitative disagreement between the CH fragment treatment and the general treatment. This disagreement is essentially originated from the trouble in treatment of σ orbitals, of which accurate calculations have not been worked out. Though McConnell's recent studies[12,13] are concentrated on the unpaired electron density of π state, the author thinks it more important to study the σ state to some extent, by which the theory of the proton hyperfine interactions will probably be put a step forward.

§ 2. Fermi contact Hamiltonian

It is allowed to take the strong field approximation in the discussion of the proton hyperfine interactions in aromatic systems, because the proton hyperfine splittings are very small compared with the Zeeman splitting due to the applied field. In this approximation the Fermi contact Hamiltonian which gives the non-vanishing isotropic hyperfine splittings for molecules in solution is

$$K_N = (8\pi g\beta/3)(\mu_1/I)\sum_k \delta(r_{kN}) S_{kz} I_{Nz}, \qquad (2)$$

where β is the Bohr magneton, $\delta(r_{kN})$ is the Dirac delta function of the distance

between electron k and nucleus N, say proton here, μ_1 is the magnetic moment of proton N, and I_{Nz} is the z component of the spin of proton N, in unit $\hbar$. Eq. (2) can also be expressed in terms of the coupling constant for proton N, a_N

$$K_N = g\beta a_N S_z I_{Nz}, \tag{3}$$

where S_z is the total z component of the electron spin angular momentum. Thus if Ψ_G is the ground state electronic wave function,

$$a_N = (8\pi/3)(\mu_1/I)\delta_N \tag{4}$$

in which a_N is given in gauss, the unit of the strength of magnetic field, and

$$\delta_N = \langle \Psi_G | \sum_k \delta(r_{kN}) S_{kz} | \Psi_G \rangle / S_z . \tag{5}$$

We seek to study the relation between a_N and the unpaired electron distribution. In the following the unpaired spin density[12] in a π state

$$\langle \Psi_\pi | \sum_k S_{kz} | \Psi_\pi \rangle / S_z \tag{6}$$

will be introduced instead of the unpaired electron density, because we are interested only in the configurations which give the unpaired spin distribution to the aromatic protons in σ orbitals, as will be discussed below.

§ 3. CH fragment

Now we shall apply the MO method to the hypothetical CH fragment in aromatic systems, in order to discuss the aromatic proton hyperfine splittings. This treatment has first been developed by Weissman[9].

The lowest electronic configuration of the CH fragment, to the first approximation, is that the paired electrons are in the CH σ bonding orbital, and the unpaired electron in a π MO,

$$\psi_0 = A\sigma\sigma\pi\alpha\beta\alpha \tag{7}$$

where the labels for the electron coordinates appear as $\sigma\sigma\pi\alpha\beta\alpha = \sigma(1)\sigma(2)\pi(3)\alpha(1)\cdot\beta(2)\alpha(3)$, and A is the operator of antisymmetrization and normalization. σ is the bonding orbital constructed by a linear combination of u_1, the sp_2 hybrid orbital of carbon and h, *1s* atomic orbital of hydrogen.

$$\sigma = \{2(1+S)\}^{-1/2}(u_1 + h). \tag{8}$$

Here for simplicity the coefficients of u_1 and h are taken to be equal and S is the overlap integral

$$S = \int u_1 h dv . \tag{9}$$

π is the lowest vacant MO of the corresponding aromatic molecule in the case of the aromatic negative ion and the non-bonding MO in the case of the aromatic

radical. It is constructed by taking the linear combination of $2p\pi$ atomic orbitals x_i of carbon atoms

$$\pi=\sum c_i x_i . \tag{10}$$

As we are now interested in a CH fragment, we are allowed to pick up only the contribution to this place from the π MO of eq. (10).

$$\pi=c_i x_i , \tag{11}$$

where x_i is the $2p\pi$ carbon atomic orbital of the CH fragment. Therefore the unpaired electron density in question is

$$\rho=c_i^2 . \tag{12}$$

The configuration expressed in eq. (7) does not contribute to the proton hyperfine interaction because the spins of σ orbital are paired, but an admixture of a doublet excited configuration ψ_1 does.

$$\psi_1=6^{-1/2}A\sigma\sigma^*\pi(2\alpha\alpha\beta-\alpha\beta\alpha-\beta\alpha\alpha) \tag{13}$$

where σ^* is the normalized anti-bonding MO orthogonal to σ:

$$\sigma^*=\{2(1-S)\}^{-1/2}(u_1-h) . \tag{14}$$

The other excited doublet state with the same configuration

$$\psi_1'=2^{-1/2}A\sigma\sigma^*\pi(\alpha\beta\alpha-\beta\alpha\alpha) \tag{15}$$

does not give the hyperfine splitting, so we need not take it into account in the following discussion. The variational parameter k describing the mixing of ψ_1 into ψ_0 is considered to be very small: $k^2\ll 1$

$$\psi_G=\psi_0+k\psi_1 , \tag{16}$$

$$k=-H_{10}/\Delta E_{10} . \tag{17}$$

Here

$$\Delta E_{10}=H_{11}-H_{00} , \tag{18}$$

$$H_{ij}=\int\psi_i H\psi_j\, dv \tag{19}$$

and H is the effective electronic Hamiltonian for the three electrons under discussion, expressed in atomic unit,

$$\left.\begin{aligned} &H=\sum_i H_i+\sum_{i>j}1/r_{ij} ,\\ &H_i=(-1/2)\Delta_i-(4/r_{ci})-(1/r_{Hi})\\ &\quad+(u_2u_2|+(u_3u_3|+\omega(xx|-1/2\{(u_2|u_2+(u_3|u_3+\omega(x|x\} \end{aligned}\right\} \tag{20}$$

where $(pp|$ and $(p|p$ are coulomb and exchange operators respectively due to an electron in the p orbital. In an aromatic system, there exist paired electrons in

the bonding π MO's, whose effect of screening the nuclear attractive potential are included in $\omega(xx|$ and $(-1/2)\omega(x|x$, where ω is the population of π electron in this place.

$$H_{00}=2I_\sigma+I_\pi+J_{\sigma\sigma}+2J_{\sigma\pi}-K_{\sigma\pi},$$
$$H_{11}=I_\sigma+I_{\sigma^*}+I_\pi+J_{\sigma\sigma^*}+J_{\sigma\pi}+J_{\sigma^*\pi}-K_{\sigma\sigma^*}+(1/2)K_{\sigma\pi}+(1/2)K_{\sigma^*\pi}. \quad (21)$$

Here

$$I_p=\int\phi_p(\mu)H_\mu\phi_p(\mu)\,dv(\mu)$$
$$J_{pq}=\int\phi_p(\mu)\phi_p(\mu)(1/r_{\mu\nu})\phi_q(\nu)\phi_q(\nu)\,dv(\mu)\,dv(\nu) \quad (22)$$
$$K_{pq}=\int\phi_p(\mu)\phi_q(\mu)(1/r_{\mu\nu})\phi_q(\nu)\phi_p(\nu)\,dv(\mu)\,dv(\nu)$$

ΔE_{10} of eq. (18) is calculated by means of the electronic Hamiltonian of eq. (20) and the result which contains ω and ρ as parameters is

$$\Delta E_{10}=0.7577+0.0041\omega+0.0899\rho.$$

If we use the following approximations: As usually seen in the aromatic system, $\omega\cong 1$ and $\rho\ll 1$, the second and the third terms may be negligible compared with the first term. Then we can obtain the probable value of ΔE_{10} for the CH fragment not depending upon ω and ρ.

$$\Delta E_{10}\cong 0.758\,(\text{a.u.})=20.6\text{eV}. \quad (23)$$

Next,

$$H_{10}=-(3/\sqrt{6})\int\sigma^*(\mu)\pi(\mu)(1/r_{\mu\nu})\pi(\nu)\sigma(\nu)\,dv(\mu)\,dv(\nu)$$
$$=-\sqrt{6/4}(1-S^2)^{-1/2}(K_{\pi u_1}-K_{\pi h}) \quad (24)$$
$$=-\sqrt{6/4}(1-S^2)^{-1/2}(K_{x_i u_1}-K_{x_i h})\rho.$$

From eq. (5) and eq. (16)

$$\delta_{CH}=(4/\sqrt{6})k\sigma(r_H)\sigma^*(r_H) \quad (25)$$

and by combining with eq. (17) and eq. (24),

$$\delta_{CH}=1/\{2\Delta E_{10}(1-S^2)\}(K_{x_i u_1}-K_{x_i h})(u_1^2(r_H)-h^2(0))\rho. \quad (26)$$

Here approximately

$$u_1^2(r_H)\cong 0$$

and further we may approximate

$$K_{x_i h}\cong S^2K_{x_i u_1},$$

which is obtained from that the h orbital is expanded at the position of carbon atom with its orbitals, then we obtain

$$\delta_{CH}=-1/(2\Delta E_{10})\,K_{x_i u_1}h^2(0)\,\rho\,. \tag{27}$$

We can see from the actual calculation that the above two approximations cancel their errors with each other. Here available molecular integrals are: $S=0.7221$, $K_{x_i u_1}=1.805$ eV and $K_{x_i h}=0.764$ eV.

Therefore we obtain

$$\begin{aligned} Q&=(8\pi/3)\,(\mu_1/I)\,(-1/2)\,(K_{x_i u_1}/\Delta E_{10})\,h^2(0)\\ &=-22.2 \text{ gauss}\,, \end{aligned} \tag{28}$$

which is in good agreement with the semi-empirical value -22.5 gauss. Though our calculated values of ΔE_{10} and $K_{x_i u_1}$ are much larger than those estimated by Jarrett[10] or McConnell[2] from experimental values, it seems that deviding the second by the first in eq. (28) makes the errors owing to the use of the Slater orbitals considerably small. As well known, the Slater orbital is not fully accurate near by the nucleus.

§ 4. General treatment

The electronic state of an aromatic system, to the first approximation, is written as

$$\left.\begin{aligned} \Psi_0&=AO\\ O&=O_\sigma O_\pi\\ O_\sigma&=\sigma_1(1)\,\sigma_2(2)\cdots\sigma_j(\mu)\,\sigma_j(\nu)\cdots\alpha(1)\,\beta(2)\cdots\alpha(\mu)\,\beta(\nu)\cdots\\ O_\pi&=\pi_1(1')\,\pi_1(2')\cdots\pi_i(\tau)\cdots\alpha(1')\,\beta(2')\cdots\alpha(\tau)\,. \end{aligned}\right\} \tag{29}$$

Here the unpaired electron is in a π_i orbital. Some comments are necessary to be made about O_π. It is a good approximation for aromatic negative ions to approximate the π electronic state as a single configuration, AO_π[11]. In this case π_i must be the lowest vacant orbital of the corresponding aromatic molecules. But in the case of aromatic radicals in which the π_i orbital is the non-bonding MO, the above approximation is not necessarily adequate. If we separate the carbon atoms of the aromatic radical into the stared and the unstared ones, as Coulson has done, the π_i orbital is constructed by taking the linear combination of $2p\pi$ atomic orbitals of the stared carbon atoms only. Therefore no proton hyperfine splittings can be obtained theoretically from the protons bound to the unstared carbon atoms. The actual observations sometimes agree with this[14], and other times do not[15]. Generally the configurational interaction between π configurations must be taken into account, about which McConnell has discussed taking allyl radical[2,12,13] as an example. The proton hyperfine interactions of triphenylmethyl radical are very complex[16,17], in

which the configurational interaction between π configurations should be invoked and, moreover, the non-planar structure of this radical which seems certain makes the situation very complicated.

The configurational interaction between π configurations is taken into account as

$$\begin{aligned}\Psi_\pi &= A\{d_i O_\pi(\pi_i) + d_k O_\pi(\pi_k) + \cdots\} \\ &= A\sum d_i O_\pi(\pi_i)\ .\end{aligned} \tag{30}$$

Here $O_\pi(\pi_i)$ is the configuration which has the unpaired electron in π_i. It is needless to say that $O_\pi(\pi_i)$ is the lowest energy configuration. Therefore if we perform the replacement

$$O_\pi \rightarrow \sum d_i O_\pi(\pi_i) \tag{31}$$

in eq. (29), a more general treatment may be possible.

The configuration in eq. (29) does not contribute to the proton hyperfine interaction, so we must consider the admixture of the doublet excited configurations. Among them those which contribute to the proton hyperfine interaction are

$$\begin{aligned}&\Psi_J = AJ \\ &J = \sum d_j \sigma_1(1)\sigma_2(2)\cdots\sigma_j(\mu)\sigma_k^*(\nu)\cdots\pi_i(\tau) \\ &\times\{\cdots 1(/\sqrt{6})\,(2\alpha(\mu)\alpha(\nu)\cdots\beta(\tau) - \alpha(\mu)\beta(\nu)\cdots\alpha(\tau) - \beta(\mu)\alpha(\nu)\cdots\alpha(\tau))\}.\end{aligned} \tag{32}$$

By using the mixing parameter k_J, the ground state wave function is written as

$$\Psi_G = \Psi_0 + \sum_J k_J \Psi_J\ . \tag{33}$$

Considering $k_J^2 \ll 1$, we can use the second order perturbation theory, then

$$k_J = -H_{J0}/\Delta E_{J0}\ , \tag{34}$$

$$\Delta E_{J0} = H_{JJ} - H_{00}\ , \tag{35}$$

$$H_{ij} = \langle \Psi_i | H | \Psi_j \rangle\ , \tag{36}$$

where H is the electronic Hamiltonian. Neglecting k_J^2,

$$\delta_N = -\sum_J 2/(S_Z \Delta E_{J0}) \cdot \langle \Psi_0 | H | \Psi_J \rangle \langle \Psi_J | \sum_k \delta(r_{kN}) S_{kz} | \Psi_0 \rangle\ . \tag{37}$$

Here we use the approximation that all ΔE_{J0}'s are replaced by the average value ΔE_{AV} of them.

$$\Delta E_{J0} \rightarrow \Delta E_{AV} \qquad \text{(approximation I)}.$$

A similar approximation about orbitals can be considered: σ and σ^* orbitals are suitably transformed into $\tilde{\sigma}$ and $\tilde{\sigma}^*$ orbitals, respectively, which are probably made to be localized to each CH bond, by taking linear combinations of σ and σ^* orbitals respectively. In consequence $\tilde{\sigma}$ and $\tilde{\sigma}^*$ may be approximately represented by the bond orbitals discussed in § 3. The above situation is written simply as

$$\left.\begin{aligned}\sigma\to\tilde{\sigma}&\simeq\{2(1+S)\}^{-1/2}(u+h)\\ \sigma^*\to\tilde{\sigma}^*&\simeq\{2(1-S)\}^{-1/2}(u-h)\end{aligned}\right\}\text{(approximation II)}$$

These approximations will be discussed in § 5. Then eq. (37) is

$$\delta_N=-2/(\Delta E_{AF}S_Z)\sum_J\langle\widetilde{\Psi}_0|H|\widetilde{\Psi}_J\rangle\langle\widetilde{\Psi}_J|\sum_k\delta(r_{kN})S_{kz}|\widetilde{\Psi}_0\rangle\,, \tag{38}$$

in which $\widetilde{\Psi}$ is meant by that σ involved in Ψ is replaced by $\tilde{\sigma}$. Each integral of eq. (38) can be simplified by careful consideration of Hamiltonian.

$$\langle\widetilde{\Psi}_J|\sum_k\delta(r_{kN})S_{kz}|\widetilde{\Psi}_0\rangle=\langle\tilde{J}|\sum_k\delta(r_{kN})S_{kz}|\sum_P(-1)^\tau P\tilde{O}\rangle\,, \tag{39}$$

where P is the permutation operator. If the electrons μ and ν are in $\tilde{\sigma}$ which has the hydrogen atomic orbital corresponding to the proton N, the proton hyperfine interaction due to the proton N is arisen from these electrons. Then

$$\langle\widetilde{\Psi}_J|\sum_k\delta(r_{kN})S_{kz}|\widetilde{\Psi}_0\rangle=\langle\tilde{J}|\delta(r_{\mu N})S_{\mu z}+\delta(r_{\nu N})S_{\nu z}|(1-P_{\mu\nu})\tilde{O}\rangle\,. \tag{40}$$

Next let us discuss $\langle\widetilde{\Psi}_0|H|\widetilde{\Psi}_J\rangle$. $\widetilde{\Psi}_0$ and $\widetilde{\Psi}_J$ are the eigen functions to H besides $\sigma-\pi$ exchange terms, and then only $\sigma-\pi$ exchange interactions are left as effective.

$$\langle\widetilde{\Psi}_0|H|\Psi_J\rangle=-\langle\tilde{O}|(1/r_{\tau\mu})+(1/r_{\tau\mu})|(P_{\tau\mu}+P_{\tau\nu})\tilde{J}\rangle\,, \tag{41}$$

where τ electron is to be in a π MO. Accordingly

$$\begin{aligned}\delta_N=&2/(\Delta E_{AF}S_Z)\sum_{\tilde{J}}\langle\tilde{O}|(1/r_{\tau\nu})+(1/r_{\tau\nu})|(P_{\tau\mu}+P_{\tau\nu})\tilde{J}\rangle\\ &\times\langle\tilde{J}|\delta(r_{\mu N})S_{\mu z}+\delta(r_{\nu N})S_{\nu z}|(1-P_{\mu\nu})\tilde{O}\rangle\,.\end{aligned} \tag{42}$$

In order to take into account the unpaired spin density in the π MO to the first approximation, $P_{\tau\mu}$ and $P_{\tau\nu}$ must be expressed in terms of spin operators. $P_{\tau\mu}$ is separated into the spin part and the orbital part,

$$\begin{aligned}P&=_{\tau\mu}P^s_{\tau\mu}P^0_{\tau\mu}\\ P^s_{\tau\mu}&=1/2\cdot(1+4S_\tau S_\mu)\\ &=1/4\cdot\{1+2(S_{\tau x}-iS_{\tau y})(S_{\mu x}+iS_{\mu y})+2(S_{\tau x}+iS_{\tau y})(S_{\mu x}-iS_{\mu y})+4S_{\tau Z}S_{\mu z}\}\,.\end{aligned} \tag{44}$$

Remembering that the spin functions of μ, ν and τ electrons are α, β and α respectively, we get

$$\begin{aligned}&\langle\tilde{O}|P^s_{\tau\mu}S_{\nu z}|\tilde{O}\rangle\\ &=1/2\cdot\langle\alpha\beta\alpha|S_{\nu z}|\alpha\beta\alpha\rangle+2\langle\alpha\beta\alpha|S_{\mu Z}S_{\nu Z}S_{\tau z}|\alpha\beta\alpha\rangle\\ &\langle\tilde{O}|P^s_{\tau\nu}S_{\mu z}|\tilde{O}\rangle\\ &=1/2\cdot\langle\alpha\beta\alpha|S_{\mu z}|\alpha\beta\alpha\rangle+2\langle\alpha\beta\alpha|S_{\mu Z}S_{\nu Z}S_{\tau z}|\alpha\beta\alpha\rangle.\end{aligned}$$

The first two terms of these equations cancel to vanish with each other. Calculating the orbital parts corresponding to the above terms,

$$\langle\tilde{O}|(1/r_{\tau\mu})P^{0}_{\tau\mu}\delta(r_{\nu N})(1-P_{\mu\nu})|\tilde{O}\rangle$$
$$=|\sum d_i|^2\langle\tilde{\sigma}\pi_i|1/r_{\tau\mu}|\pi_i\tilde{\sigma}\rangle\langle\tilde{\sigma}|\delta(r_{\nu N})|\tilde{\sigma}\rangle=|\sum d_i|^2K_{\tilde{\sigma}\pi i}|\tilde{\sigma}(r_H)|^2$$
$$\langle\tilde{O}|(1/r_{\tau\nu})P^{0}_{\tau\nu}\delta(r_{\mu N})(1-P_{\mu\nu})|\tilde{O}\rangle$$
$$=|\sum d_i|^2\langle\tilde{\sigma}\pi_i|1/r_{\tau\nu}|\pi_i\tilde{\sigma}\rangle\langle\tilde{\sigma}|\delta(r_{\mu N})|\tilde{\sigma}\rangle=|\sum d_i|^2K_{\tilde{\sigma}\pi i}|\tilde{\sigma}(r_H)|^2.$$

In the above integrals the terms arisen from $P_{\mu\nu}$ vanish. As the eigen value of $S_{\mu Z}S_{\nu Z}$ is $-1/4$,

$$\delta_N=|\sum d_i|^2(-2/\Delta E_{AV})K_{\tilde{\sigma}\pi i}|\tilde{\sigma}(r_H)|^2(1/S_Z)\langle\alpha|S_{\tau Z}|\alpha\rangle. \tag{45}$$

By combining eq. (45) with eq. (31) and eq. (11),

$$\delta=(-2/\Delta E_{AV})|\sum d_i c_{ij}|^2(1/S_Z)\langle\alpha|S_{\tau Z}|\alpha\rangle K_{\tilde{\sigma}xj}|\tilde{\sigma}(r_H)|^2$$
$$=(-2/\Delta E_{AV})K_{\tilde{\sigma}xj}|\tilde{\sigma}(r_H)|^2\rho. \tag{46}$$

In the last derivation the definition of the unpaired spin density of eq. (6) is used. If we use the approximation II for $K_{\tilde{\sigma}xj}$ and $\tilde{\sigma}(r_H)$ as those for a CH fragment,

$$u(r_H)\simeq 0$$
$$\langle ux_j|x_jh\rangle\simeq S\langle ux_j|x_ju\rangle=SK_{uxj}$$
$$\langle uh|hu\rangle\simeq S^2\langle ux_j|x_ju\rangle=S^2K_{uxj}.$$

Then we can obtain

$$\delta_N=-1/(2\Delta E_{AV})K_{uxj}|h(0)|^2\rho, \tag{47}$$

which is just the same result as that obtained for a CH fragment. Here we shall see in the next section that ΔE_{AV} is equal to ΔE_{10} of the CH fragment.

§ 5. ΔE_{AV} and $\tilde{\sigma}$

As we have seen, it is necessary to know the forms of σ orbitals and the singlet-triplet excitation energies associated with them in order to calculate the exact values of the proton hyperfine spplittings. As to the σ state, we have obtained only the self-consistent orbitals for some simple molecules using the LCAO (linear combination of atomic orbitals) approximation. However, such calculations of aromatic systems in which we are now interested have not been performed and will not be carried out because of the difficulties of actual calculations. On the other byhand, a large amount of information about π MO's of aromatic systems are obtained studying their spectra and others. In calculations with the π electron approximation, the effects of σ electrons are taken into account as core potentials and in many cases the form of the occupied carbon σ orbitals are assumed to be the sp_2 hybrid orbitals. These sp_2 hybrid orbitals are thought to be reasonable ones as to be approximated ones, strongly localized to each chemical bond of aromatic systems, which would be constructed by suitable transformation from the self-consistent σ orbitals, if they could be obtained. In the simple molecule of H_2O[18], the self-con-

sistent orbitals have already been successfully transformed into the equivalent orbitals which are localized to each chemical bond. That is to say, the bond orbitals as shown in approximation II are thought to be a kind of equivalent orbitals. Therefore the self-consistent σ orbitals of aromatic systems may be approximately obtained by a linear transformation starting from the bases of bond orbitals. In this case the interactions between bonds are considered as perturbation.

The above discussion will be illustrated for an example of the benzene molecule. If necessary, the following discussion may be easily generalized to other aromatic hydrocarbons. A bonding orbital and an anti-bonding orbital between carbon and hydrogen are

$$\left.\begin{aligned} w(\mathrm{CH}) &= \{2(1+S)\}^{-1/2}(u+h) \\ w^*(\mathrm{CH}) &= \{2(1-S)\}^{-1/2}(u-h) \end{aligned}\right\} \tag{48}$$

respectively, where u is the carbon sp_2 hybrid orbital and h is the hydrogen $1s$ atomic orbital, Here it is assumed for simplicity that two orbitals are homopolar Similar orbitals between adjacent carbons are

$$\left.\begin{aligned} w(\mathrm{CC}') &= \{2(1+S')\}^{-1/2}(u+u') \\ w^*(\mathrm{CC}') &= \{2(1-S')\}^{-1/2}(u-u') \end{aligned}\right\} \tag{49}$$

where u and u' are sp_2 hybrid orbitals of adjacent carbon atoms, and S' is the overlap integral between them. The four kinds of orbitals of eq. (48) and eq. (49) are bases of linear combination. Here we may take the following linear combinations which are not satisfactory but may be a good approach to obtain the self-consistent σ orbitals of benzene.

$$\left.\begin{aligned} \sigma_j(\mathrm{CH}) &= \sum l_{ji} w_i(\mathrm{CH}) \\ \sigma_j^*(\mathrm{CH}) &= \sum l_{ji}^* w_i^*(\mathrm{CH}) \end{aligned}\right\} \tag{50}$$

and

$$\left.\begin{aligned} \sigma_j(\mathrm{CC}') &= \sum m_{ji} w_i(\mathrm{CC}') \\ \sigma_j^*(\mathrm{CC}') &= \sum m_{ji}^* w_i^*(\mathrm{CC}'). \end{aligned}\right\} \tag{51}$$

The molecular orbitals of eq. (51) which have no hydrogen atomic orbital do not contribute to the proton hyperfine interaction, so we can treat the electrons occupying these orbitals as an effective molecular field. If we here take the interactions between bonds as perturbation, we can estimate the orbital energies of eq. (50), by a way quite similar to what has been done in the semi-empirical calculations in the π electron approximation. Thus the orbital energies of bonding orbitals are, from the lower to the higher,

$$\left.\begin{aligned} \varepsilon_0(\mathrm{CH}) &= \alpha_{\mathrm{CH}} + 2\beta_{\mathrm{CH}} \\ \varepsilon_{\pm 1}(\mathrm{CH}) &= \alpha_{\mathrm{CH}} + \beta_{\mathrm{CH}} \\ \varepsilon_{\pm 2}(\mathrm{CH}) &= \alpha_{\mathrm{CH}} - \beta_{\mathrm{CH}} \\ \varepsilon_3(\mathrm{CH}) &= \alpha_{\mathrm{CH}} - 2\beta_{\mathrm{CH}} \end{aligned}\right\} \tag{52}$$

and those of anti-bonding orbitals are

$$\begin{aligned}
\varepsilon_0^*(\mathrm{CH}) &= \alpha_{\mathrm{CH}}^* + 2\beta_{\mathrm{CH}}^* \\
\varepsilon_{\pm 1}^*(\mathrm{CH}) &= \alpha_{\mathrm{CH}}^* + \beta_{\mathrm{CH}}^* \\
\varepsilon_{\pm 2}^*(\mathrm{CH}) &= \alpha_{\mathrm{CH}}^* - \beta_{\mathrm{CH}}^* \\
\varepsilon_3^*(\mathrm{CH}) &= \alpha_{\mathrm{CH}}^* - 2\beta_{\mathrm{CH}}^* \quad ,
\end{aligned} \tag{53}$$

where α_{CH} and $\alpha_{\mathrm{CH}}{}^*$ are the bond orbital energies of $w(\mathrm{CH})$ and $w^*(\mathrm{CH})$ respcetively, and if H is the electronic molecular field, β_{CH} and β_{CH}^* are

$$\begin{aligned}
\beta_{\mathrm{CH}} &= \int w_i(\mathrm{CH})\,\mathrm{H} w_j(\mathrm{CH})\,dv \\
\beta_{\mathrm{CH}}^* &= \int w_i^*(\mathrm{CH})\,\mathrm{H} w_j^*(\mathrm{CH})\,dv .
\end{aligned} \tag{54}$$

In rough estimation of β_{CH} and β_{CH}^*, they are assumed to be proportional to the overlap integrals between the corresponding bonds,

$$\begin{aligned}
\int w_i(\mathrm{CH})\,w_j(\mathrm{CH})\,dv &\simeq \int w_i^*(\mathrm{CH})\,w_j^*(\mathrm{CH})\,dv \\
&\simeq (1/2)\left(\int uu'\,dv + \int hh'\,dv\right) \\
&= 0.12 .
\end{aligned}$$

Therefore, when we consider that β_π, the resonance integral of benzene, is -2 eV and the corresponding overlap integral is 0.26, we get

$$\beta_{\mathrm{CH}} \simeq \beta_{\mathrm{CH}}^* \simeq -1\mathrm{eV} .$$

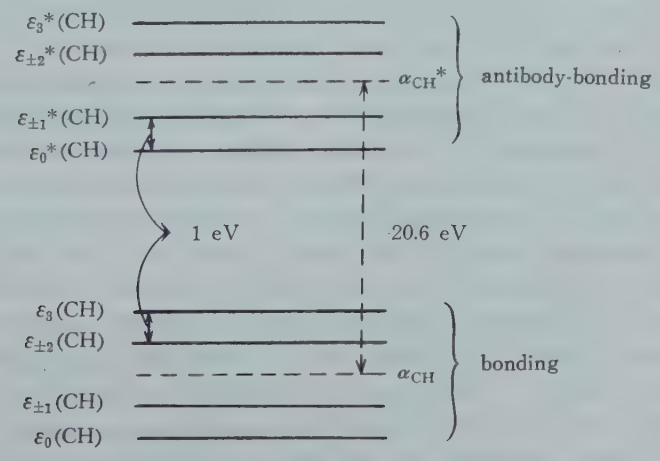

Fig. 1 Orbital energies of $\sigma(\mathrm{CH})$ and $\sigma^*(\mathrm{CH})$ in benzene.

The difference between α_{CH} and α^*_{CH} is 20.6 eV as mentioned in § 3. The above situation is shown in Fig. 1. From it we can say that σ(CH) and σ^*(CH) distribute energetically close about the center of w(CH) and w^*(CH) respectively, and there are two bands which correspond to the bonding and the anti-bonding states respectively. Thus every singlet-triplet excitation in σ orbital is thought to give rise, if it is allowed from the symmetry properties of benzene. Then the meaning of the approximations supposed in § 4 becomes obvious: The probabilities of these excitations are assumed to be equal and the corresponding excitation energies are all averaged to the difference between α_{CH} and α^*_{CH}. At the same time σ(CH) and σ^*(CH) are transformed suitably to bond localized orbitals, $\tilde{\sigma}$(CH) and $\tilde{\sigma}^*$(CH) respectively. Then we can replace the excitation from $\tilde{\sigma}$ to $\tilde{\sigma}^*$ of the CH bond for all excitations from σ to σ^*. This is a substantial meaning of sp_2 hybridization. As has been clarified in the preceding argument, approximations I and II are adequate only if σ(CH) and σ^*(CH) show dense distribution about the center of w(CH) and w^*(CH) respectively, which should be probable in aromatic systems. However, the above conditions are not always satisfied in aliphatic systems about which the more exact treatment will be expected.

§ 6. Discussion and conclusion

We have already had very fine works about proton hyperfine splittings by McConnell and others. In the present article, some improvements on their results were obtained:

McConnell's semi-empirical equation (1) has been proved to be adequate in aromatic systems. And moreover it is apparent that the calculation in the case of a CH fragment leads to the same result as that for the case of the general aromatic system. The result of eq. (28) or eq. (47) is refined and is much simpler compared with that given by McConnell. In this problem the approximations I and II employed in § 4 are essentially important, and they are shown to be reasonable for aromatic systems. We can also say the value of $Q-22.5$ gauss is constant for all aromatic systems, to a good approximation. Therefore, in the calculation of the proton hyperfine interaction for aromatic systems, we are justified to use the hypothetical CH fragment combined with a π MO which has the unpaired electron. The above situation is, however, not always correct for aliphatic systems about which we can conjecture from the fact that Q values of $-CH_2$, $-CH_3$, $C_2H_4^+$ and $-C_2H_5$ are slightly different from each other[19]. In these radicals and ion, we must note that $\sigma-\pi$ interaction and hyperconjugation effect sometimes coexist.

As mentioned earlier, McConnell and Chesnut have succeeded in a theoretical derivation of eq. (1). The present calculation has made the physical background of their work clear and made one step of development of this theory by the use of the knowledge of molecular calculations. Recent efforts of McConnell in the calculation of the proton hyperfine interaction are on the unpaired

spin density under the various approximations of π electron theory, based on the semi-empirical equation (1). This would be natural from the reason that the proton hyperfine splittings give direct information about the unpaired electron densities of radicals and ions. It is, however, more necessary to inquire about σ states and their excitations, because eq. (1) is essentially derived from approximations I and II. The leading object of the present work lies in this point.

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E1 Transition and Nuclear Deformation in the Excited States

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The equilibrium deformation of some heavy nuclei in the excited states have been calculated in the individual particle model of Nilsson. It is found that the deformation remains practically unaltered. With use of Nilsson's eigenfunctions the electric dipole transition probabilities have also been calculated and it has been found that in most of the cases the retardation is maximum when the deformation of the excited state coincides with that of the ground state.

§ 1. Introduction

In the different rotational bands of the same nuclei, it is found that $\hbar^2/2\mathcal{I}$ has appreciably different values, which consequently implies that the moment of inertia for different rotational bands is not the same. This may be due to either a change in the configuration of the nucleons or in the deformation in the excited state. Thus there is a possibility that the deformation may be changed as the nucleus goes from one vibrational state to another. To see this we shall calculate the equilibrium deformation for some odd proton nuclei in their ground states as well as in their excited states. Moreover, the calculation of electric dipole transition probabilities from different bands also gives some information regarding the deformation of the excited states.

§ 2. Determination of equilibrium deformations

The energy eigen values for a single particle moving in a spheroidal potential has been discussed by Moszkowski[1] (1955), Nilsson[2] (1955) and Gottfried[3] (1956). Nilsson has given the eigen values as well as the eigen functions for individula particles by considering an anisotropic harmonic oscillator potential, spheroidal in shape. We have followed his treatment to calculate the total energy for some heavy nuclei.

The total energy E for a system of particles moving in such a potential has been given by Nilsson[2] as

$$E=\tfrac{3}{4}\hbar\omega_0{}^0\sum_i\left[(N_i+\tfrac{3}{2})(1-\tfrac{1}{3}\epsilon^2-\tfrac{2}{27}\epsilon^3)^{-1/3}+\chi\langle R_i\rangle\right]+\tfrac{1}{4}\sum_i U_i \qquad (1)$$

where the summation is carried over to all the nucleons.

The contribution of U_i to the total energy is small and is almost independent

of deformation. ϵ is another deformation parameter defined in terms of the more familiar parameter δ as

$$\epsilon=3\left[\frac{\sqrt{(1+\frac{2}{3}\delta)}-\sqrt{(1-\frac{4}{3}\delta)}}{\sqrt{(1-\frac{4}{3}\delta)}+2\sqrt{(1+\frac{2}{3}\delta)}}\right]. \tag{2}$$

For a particular configuration of nucleons, we have calculated the energy E for various values of η where η is defined by

$$\eta=(\delta/\chi)\left[1-\tfrac{4}{3}\delta^2-16\tfrac{2}{7}\delta^3\right]^{-1/6}. \tag{3}$$

We have neglected U_i in the present investigation, because its expectation value is independent of deformation. $E/\frac{3}{4}\hbar\omega_0{}^0$ for some nuclei has been calculated by varying η from 0 to 10, both for their ground state as well as for their vibrational states.

§ 3. Electric dipole transition probability

The electric dipole transition probability according to the single particle model is given by[4]

$$T(1)=1.5\times10^{14}A^{2/3}E_\tau^3 \tag{4}$$

where E_τ is the energy in Mev.

In strongly deformed nuclei this formula gives the dipole transition probabilities to be very large compared with the actual transition probabilities. This retardation effect has been very satisfactorily explained by Nilsson's model by various authors[5),6)].

The transition probability for an electric dipole gamma transition of energy $\hbar\omega$ is given by[2]

$$T(1)=16\pi/9\cdot1/\hbar(\omega/c)^3B(E1), \tag{5}$$

where

$$B(E1,\ I\to I')=e^2(1-Z/A)^2\hbar/M\omega_0\cdot3/4\pi|\langle I1KK'-K|I1\,I'K'\rangle|^2\cdot|G(E1)|^2 \tag{6}$$

with

$$G(E1)=\sum_{ll'}\langle N'l'|r|Nl\rangle\sqrt{\frac{2l+1}{2l'+1}}\langle l100|l1l'0\rangle$$
$$\cdot\sum_{\Lambda\Lambda'\Sigma\Sigma'}\delta_{\Sigma\Sigma'}a'_{l'\Lambda'}a_{l\Lambda}\langle l1\Lambda K'-K|l1\,l'\Lambda'\rangle. \tag{7}$$

where the symbols are explained in reference 2).

Unlike the previous works we have calculated $E1$ transition probabilities considering various values of η for the excited states but keeping the ground state deformation fixed. The results for individual nuclei has been discussed in the following section.

§ 4. Discussion of the results

In our investigation we have calculated the total energy and the dipole transition probabilities for nuclei $_{63}Eu^{153}$, $_{69}Tm^{169}$, $_{71}Lu^{175}$ and $_{93}Np^{237}$ which are known to be strongly deformed.

(a) Eu^{153}: The intrinsic quadrupole moment Q_0 of Eu^{153} is 7.6 barns from which $\delta \sim .3$ in the ground state. The ground state spin is $5/2+$. This corresponds to orbit No. 27 of Nilsson. The 98 Kev 5/2-state decays to the ground state by $E1$ transition. The experimental retardation factor $\sim 10^4$. The excited state can

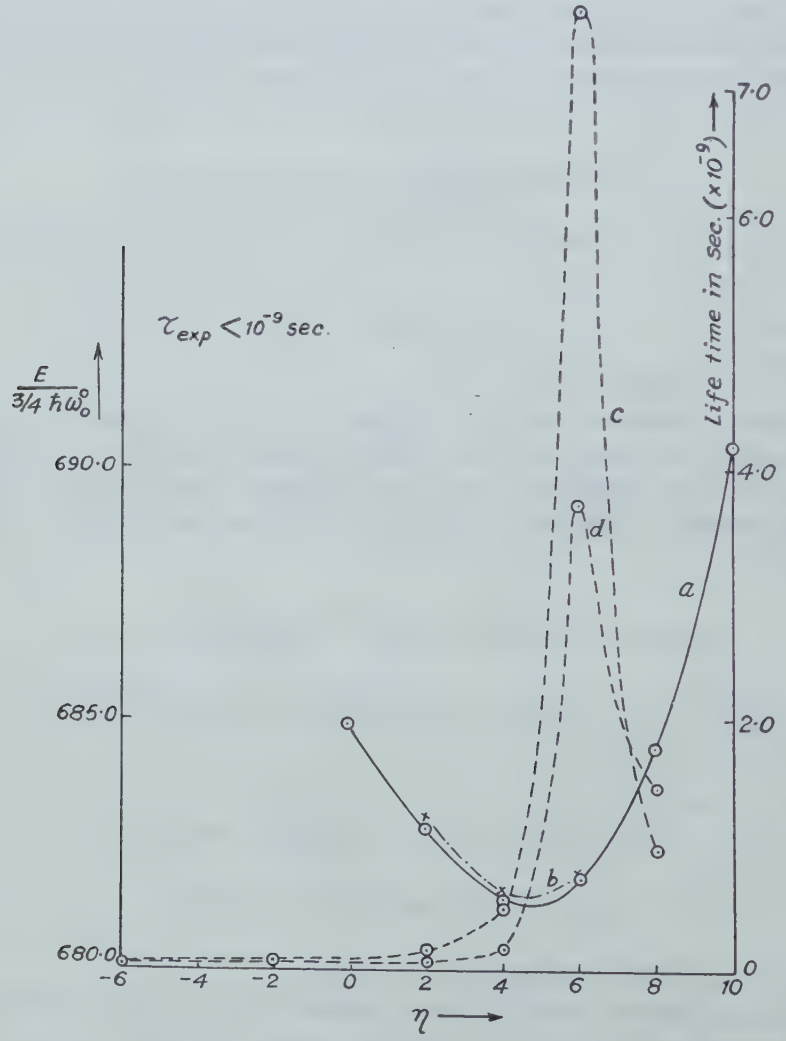

Fig. 1. Curve (a) and (b) give the total energy of Eu^{153} in the ground state and the excited state respectively. Curve (c) corresponds to the lifetime with $\mu=0.55$, while curve (d) is calculated by changing μ to 0.45.

be easily classified to be orbit No. 36 of Nilsson.

With assumption of a configuration which corresponds to $\eta=6$, the total energy for Eu^{153} in $5/2+$ as well as in $5/2-$ state has been plotted in Fig. 1. It will be seen that at $\eta\sim5$ there is a minima in the both cases. Moreover, the difference in energy involved is also nicely explained. The Nilsson's wave function for the ground state is

$$|4\tfrac{5}{2}>=0.24944|442+>+0.22799|422+>-0.94116|443->, \qquad (8a)$$

and for the 98 Kev state

$$|5\tfrac{5}{2}>=a_{52}|552+>+a_{32}|532+>+a_{53}|553->+a_{33}|533->. \qquad (8b)$$

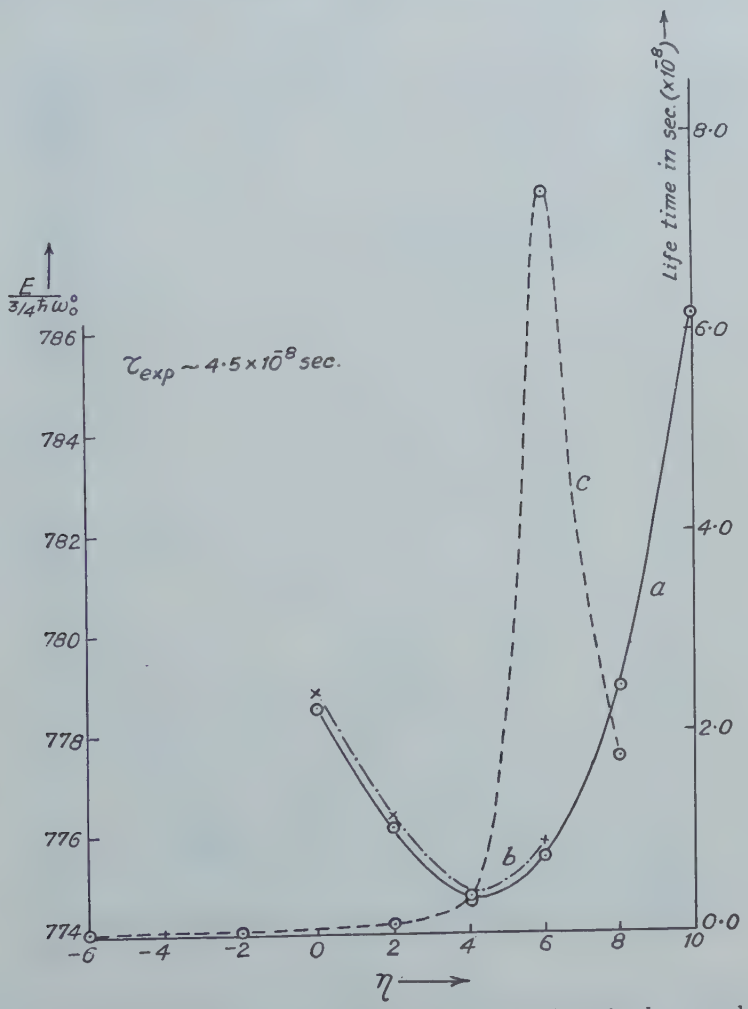

Fig. 2. Curve (a) and (b) are the total energy of Tm^{169} in the ground state as well as in the excited state respectively. Curve (c) shows the variation of lifetime with the deformation of the excited state.

The coefficients appearing in (8b) will be different for different values of η. With use of (8a) and (8b), $T(1)$ is calculated from equations (5), (6) and (7). This is also presented in Fig. 1. In order to investigate the effect of changing the parameter μ (ref. 2) on $E1$ transition probability, we have calculated $T(1)$ both for $\mu=0.45$ and 0.55. It will be found that the dependence of $T(1)$ on μ is rather remarkable.

(b) $_{69}Tm^{169}$: The ground state deformation for Tm^{169} as calculated from intrinsic quadrupole moment is $\delta\sim.28$. The 63 Kev transition occurs when the last proton comes down from orbit No. 35 to 25. As before the total energy for both the states and the lifetime have been presented in Fig. 2 against the deformation of

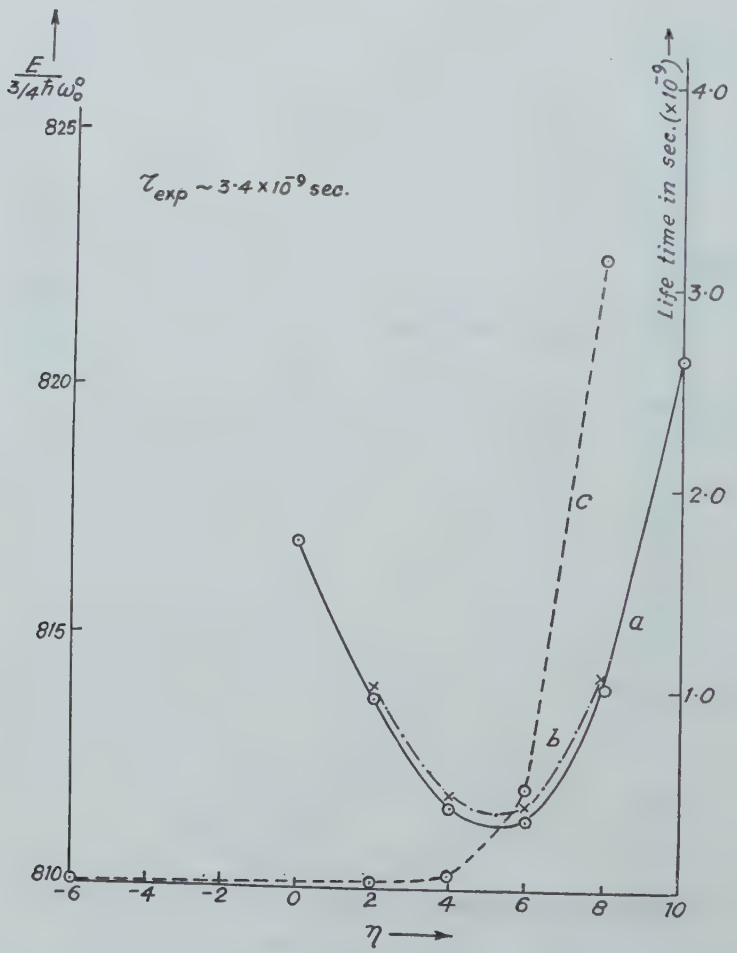

Fig. 3. Curve (a) and (b) correspond to the variation of total energy of Lu^{175} in the ground state and excited state respectively while curve (c) gives the lifetime.

excited state. Almost similar results are obtained as in Eu^{153}.*

(c) $_{71}Lu^{175}$: This has been widely discussed by Chase and Wilets[8]. The 9/2-level has been identified as the orbit No. 32 of Nilsson and the 7/2+ ground state as the orbit No. 25. The total energy curve, Fig. 3, explains nicely the

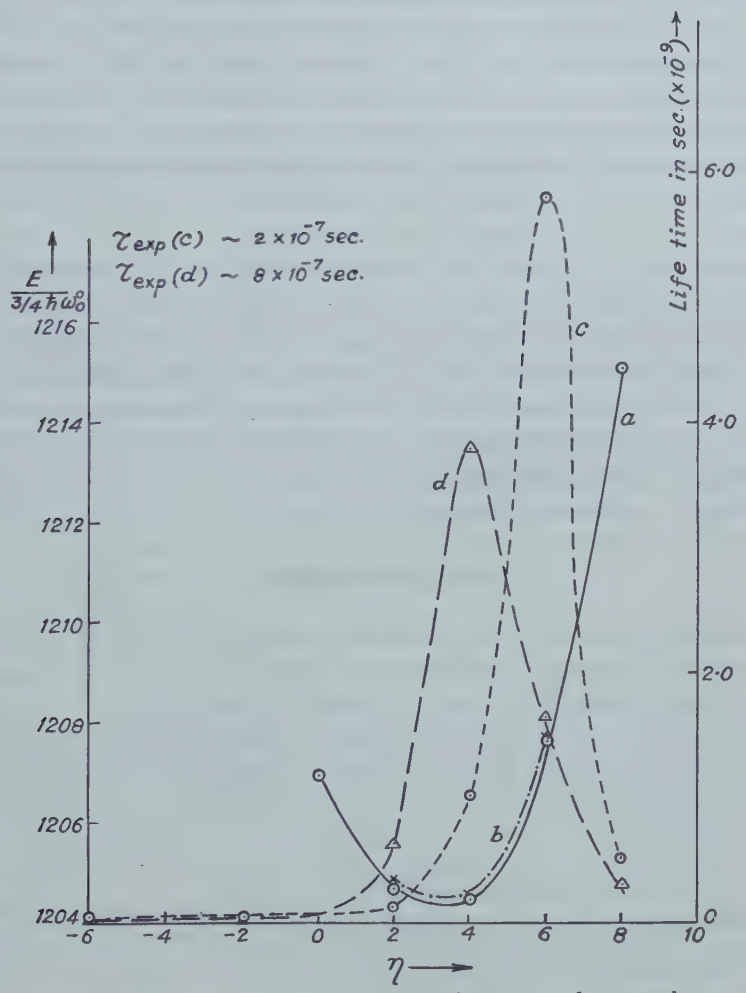

Fig. 4. Curve (a) gives the total energy of Np^{237} in the ground state. In curve (b) the total energy is calculated with one of the protons in either of the 59.6 Kev or 267.5 Kev states. Curve (c) represents the lifetime of 59.6 Kev E1 gamma transition and curve (d) gives the lifetime of 267.5 Kev level. (For curve (d) right-hand ordinate is to be multiplied by 10^{-2}).

* It may be mentioned that this case is slightly different, since orbit No. 25 corresponds to the 316 Kev excited state of Tm^{169}, and we have tentatively assumed that its deformation is the same as that of the ground state. (Refer to Nakasima et al. Prog. Theor. Phys. Japan, Vol. 19, No. 1. p. 40).

9/2-excited state, the corresponding value of η being 6 which coincides with the equilibrium deformation of ground state. But the variation of lifetime with deformation (Fig. 3) shows a different trend. As η is increased the retardation also increases continuously. This may be due to the fact that there are only two terms in $G(E1)$ in this case, which may be purely accidental.

(d) $_{93}Np^{237}$: Unlike the previous cases in this case there are three definitely established rotational bands[7] A, B and C with K values as 5/2, 5/2, and 3/2. We have considered the 59.6 Kev $E1$ transition from band B to A with a corresponding change in Nilsson's orbital from 44 to 55. Figure 4 shows the calculated lifetime with deformation. As in most cases the maximum retardation occurs at $\eta=6$ which corresponds to the ground state deformation. The total energy curve also supports this.

The results for 267.5 Kev level (orbit No. 52) are also presented in Fig. 4. It will be seen that the deformation of the excited state corresponds to $\eta\sim4$.

Thus in the majority of the cases that we have treated it is found that the retardation in the $E1$ transition rate is maximum when the deformation in the excited state coincides with that of the ground state. Moreover, as will be evident from Fig. 1, the $E1$ transition is rather sensitive to the choice of μ, the strength of the $\boldsymbol{l}^2$ term in Nilsson hamiltonian, and often better values of the transition rate can be obtained by adjusting this parameter.

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Meson Decays and Structure of Muon

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Considering the peculiar position of the muon among the elementary particles, it seems reasonable to assume an inner structure for this particle. As it seems we are now reaching a new experimental stage in which the determination of the structure of the muon seems possible, we must consider theoretically how to determine it through the various decay processes. The analyses proposed here are mainly connected to pion and muon decay problems. Assuming the so-called universal Fermi interaction, with V and A couplings, by which the decay processes are realized, our considerations are restricted to the behaviours of the non-local form factor which arise from the muon structure. Also possibilities of the electron structure are discussed. The present analysis sensitively depends on the deviation of the asymmetry parameter. Its deviations from the original local value must be within about 6%, and a value larger than this cannot be explained without considerations of the electron structure and some other sources.

§ 1. Introduction

Up to now any difference of properties between muon and electron, except the remarkable mass difference, has not been discovered. A little before the discovery of pion beta-decay,[1] the contradiction between experiment and theory was strongly suggesting another possibility of distinguishing the muon from the electron, but the recent experimental values are becoming surprisingly close to the theoretical value, the difference between these results being within the experimental errors. Therefore, the problem of explaining why elementary particles so similar to muon and electron have very different masses, is surely one of the most fundamental problems in future.

Now, it seems to be very difficult to get any key for explaining the large mass value of the muon.*** The possibility of explaining the electron mass, through

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** On leave from Research Institute for Fundamental Physics, Kyoto University, Kyoto, Japan.
*** A. Salam and J. Tiomno (preprint) also start from the same opinion. They supposed the existence of a strong interaction of the muon, in order to explain its mass. Although this possibility must be analysed, it seems to be very unnatural to us. We thank them for sending us the manuscript before publication.

the muon, is discussed by some of us elsewhere.[2]

Concerning the decay rate of $\pi \to e+\nu$ and $\pi \to \mu+\nu$, as clarified by Ruderman and Finkelstein[3] and also by one of us (M. T.) and Tokyo group,[4] through a more detailed analysis, only the cases of Fermi interaction with A coupling for Sakata's scheme* and of ps (pv) interaction for direct Yukawa's scheme can explain its small value such as given experimentally. From these results, one can understand the reason why the possibility of decreasing the ratio is due to the mass difference between electron and muon. And at that time, if one had postulated very strictly the so-called universality of weak interactions, one would have predicted the fact that S, T and P coupling cannot be the main terms in β-decays. On the contrary, today, from the agreement between theory and experiment in $\pi-e$ and β-decays, one may conclude that V and A couplings are the main terms in the universal weak interaction between leptons and nucleons, and possibly also other baryons. On the other hand, one could also hope, if the experimental results of the pion decays' ratio would become more precise, to be in position for deciding whether the structure of muon and electron are same or not.

As is known, Lee, Yang[5] and other[6] have considered a structure for the muon in order to explain the experimental deviation of the Michel parameter from the theoretical value. Though the recent experimental values have become close to the value given by the two-component neutrino theory, there still remains some possibility of such attempts owing to the insufficient experimental situations.

The experimental evidences of pion beta-decay have made the circumstances better. However, we still have some doubts about the agreement between experimental and theoretical ratios and we rather believe that the progress of the experimental techniques has been such as to allow the possibility of obtaining information about the muon structure in a more definite way, even in the case in which the departure from the local value is small; in other words, if the experimental accuracy could be improved to another decimal order, then the problem would become a definite one.

When many attempts were made to forbid the pion beta-decay,[7] after the experiment of Anderson and Lattes,[8] we were inclined to the opinion that the ratio should at least be the upper limit value given by Anderson and Lattes and that the departure from the theoretical value could be explained by taking into account the muon structure. This opinion first came to us by noticing the fact that every ratio of decay processes, for instance, $K \to e+\nu / K \to \mu+\nu$, and $K \to \pi+\mu+\nu / K \to \pi+e+\nu$, sensitively depend on the mass ratios of the produced particles.**

Although the present experimental situation has been much improved, such

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* The scheme of pion decay through the nucleon pair was first proposed by S. Sakata, Phys. Rev. **58** (1940), 83, and afterwards used through the famous triangle scheme by J. Tiomno and J. A. Wheeler, Rev. Mod. Phys. **21** (1949), 153. We then use the name of "Sakata's scheme" for this mechanism, while the direct mechanism we call "Yukawa's scheme".

** A similar idea has recently been proposed by K. Fujii and K. Iwata, in press.

improvement does not seem enough to give the values of the pion-decay ratio and also of Michel and asymmetry parameters, in the muon decay, with sufficiently strict accuracy for determining the muon structure. The value of the muon anomalous magnetic moment would become another tool for such an attempt.[9] But the experimental accuracy is also a little too rough to exclude the possibility of a muon spreading of about $m_\mu^{-1} \sim 2$ yukawa.*

The outstanding experimental skill displayed in the pion beta-decay business promises, however, to give better information in the near future. We wish, therefore, to proceed with the analyses of the muon structure within the possibilities of future experimental values.

Our analysis will proceed as follows:

i) If the elementary particles have some structure, this structure will manifest itself through the interactions. We introduce then non-local interactions for the muon and also for the electron.

ii) As we can guess the muon structure through the form factor appearing in the non-local interactions, we must analyse the way of determining it through the various decay processes in which the muon is present.

iii) The baryons also may have some structure related to the strong interactions. In order to separate these problems from the so-called applicability problems in the present field theory, we restrict ourselves to the pion decay ratio, the Michel and asymmetry parameters in the muon decay and kaon decay ratios, i. e., $K \to e + \nu / K \to \mu + \nu$ and $K \to \pi + \mu + \nu / K \to \pi + e + \nu$. Their absolute rates are not discussed.

iv) The analyses are devoted mainly to Sakata's scheme, in which decay processes are realized through the Fermi interaction with V and A couplings, in accordance with the recent experiments. The usual assumption of $V-A$ combination,[10] seems to be unnecessary.

Before a detailed analysis, we build up the formalism of non-local interactions expressing the muon structure (§ 2) and roughly estimate their effects on the pion decay ratio (§ 3) and the muon decay (§ 4). Such analysis is used to determine the muon structure (§ 5) and its validity, and effects on kaon decays are discussed (§ 6). The possibility of a structure for the electron and its relation to the muon structure is considered in the last chapter (§ 7).

§ 2. Formulation of the muon structure

The interactions which represent the lepton structures are only the electromagnetic and weak ones. Restricting ourselves to the decay processes considered here, there is no contradiction with the assumption of Sakata's scheme, in which the decay processes are produced by Fermi interactions between four kinds of

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* 1 yukawa $= 10^{-13}$ cm. As this unit is closely related with the pion Compton length, it is suitable to use the name of Yukawa for it. See M. Taketani, Prog. Theor. Phys. Suppl. **3** (1956), 5.

fermions. For beta-decay, muon-capture and muon-decay, this scheme has been almost established since its proposal by Sakata. For the pion decays, there is no evidence to decide whether they occur through Sakata's or through Yukawa's scheme with the direct couplings.* Even if we restrict ourselves to the former scheme, we cannot decide whether this scheme can be extended to include all baryons or not.**

Considering the possibilities of hyperon-lepton decays, we introduce tentatively the following three kinds of Fermi interactions :

i) $N_1+\mu \rightleftarrows N_2+\nu$

ii) $N_1+e \rightleftarrows N_2+\nu$

iii) $\mu+\nu \rightleftarrows e+\nu$

where N_1, N_2, μ, e and ν are baryons, muon, electron and neutrino, respectively. We assume, moreover, in this scheme, the conservation of electric charge, baryon number and also lepton number.***

The structures of the elementary particles turn these interactions to the non-local ones.[11] The general form factors appearing in the interactions may depend not only on the space-time coordinates of the four kinds of fermions, but also on their properties and also on the coupling types.

In order to clarify the model proposed in this paper, we must restrict it by the following assumptions, which will be introduced step by step.

Assumption I: *The form factor are invariant against any translation of coordinates, as well as Lorentz transformation, and their local limits must exist.*

The translational invariance restricts the form factor, which initially depends on the four coordinates of each fermion, from four to three space-time degrees of freedom. Although the requirement of Lorentz invariance is too severe to construct a consistent theory,[12] we can hope that the future non-local theory will include it in a consistent way and the deviations from the local theory will not be drastic enough to forbid us to maintain our usual images on the elementary particle physics.

Assumption II: *The form factors do not depend on the type of couplings.*

If the form factors are mainly due to an intrinsic structure of the particles, they will appear in the interaction without a strong dependence on the coupling types. Then the change of the type of coupling could be ascribed, approximately, to the change of the numerical value of the coupling constant. After the proposal of Lee and Yang concerning the phenomenological non-local effect on the muon decay, some generalizations were proposed by Bludman and Klein and also by Sirlin in order to include the dependence on the coupling types. But from these generalizations no new results were obtained.

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* A little before the discovery of pion beta-decay, some arguments were put forward on this problem. See S. Ogawa, Prog. Theor. Phys. **13** (1955), 367.

** The discovery of hyperon-muon decay has been reported recently.

*** We define μ^-, e^- and ν as particles and μ^+, e^+ and $\tilde{\nu}$ as anti-particles.

Taking these two assumptions into account the Fermi interaction Lagrangeans can be written as follows:

i) Case $N_1+\mu \rightleftarrows N_2+\nu$. The form factor depends on the structures of N_1, N_2, μ and ν.

$$L_I(\tilde{N}_1 N_2 \tilde{\mu}\nu) = -\int dx_1 dx_2 dx_3 dx_4 F(x_1, x_2, x_3, x_4; N_1 N_2 \mu\nu) \cdot \sum_i \bar{N}_1(x_1) O_i N_2(x_2) \cdot \bar{\mu}(x_3) O_i (C_i + \gamma_5 C_i') \nu(x_4). \quad (1)$$

ii) Case $N_1+e \rightleftarrows N_2+\nu$. The form factor depends on the structures of N_1, N_2, e and ν.

$$L_I(\tilde{N}_1 N_2 \tilde{e}\,\nu) = -\int dx_1 dx_2 dx_3 dx_4 F(x_1, x_2, x_3, x_4; N_1 N_2 e\nu) \cdot \sum_i \bar{N}_1(x_1) O_i N_2(x_2) \cdot \bar{e}(x_3) O_i (C_i + \gamma_5 C_i') \nu(x_4). \quad (2)$$

iii) Case $\mu+\nu \rightleftarrows e+\nu$. The form factor depends only on the lepton structure.

$$L_I(\tilde{e}\,\mu\tilde{\nu}\nu) = -\int dx_1 dx_2 dx_3 dx_4 F(x_1, x_2, x_3, x_4; e\mu\nu\nu) \cdot \sum_i \bar{e}(x_1) O_i \mu(x_2) \cdot \bar{\nu}(x_3) O_i (C_i + \gamma_5 C_i') \nu(x_4). \quad (3)$$

Here N_1, N_2, μ, e and ν are the baryons, muon, electron and neutrino fields, respectively, which depend on the x_i's coordinates; the F's are the form factors. The summation is taken for the five coupling types, i.e., S, V, T, A, P, which correspond to the following operators;

$$O_S = 1, \quad O_V = \gamma_\alpha, \quad O_T = -\frac{i}{2\sqrt{2}}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha),$$
$$O_A = -i\gamma_\alpha\gamma_5, \quad O_P = \gamma_5, \quad (4)$$

C_i and C_i' being the coupling constants, which are not necessarily equal for all cases.

Our next task is to separate these form factors so as to make them correspond to the structure of each particle, this being achieved by the introduction of the following assumptions.

Assumption III: The form factors can be separated in factors mainly corresponding to the baryon pair (N_1, N_2) *and lepton pair* $(\mu\nu)$ *or* $(e\nu)$.

This assumption can be accomplished by reducing the coordinate dependences of the form factors, such as

$$F(x_1, x_2, x_3, x_4; N_1 N_2 \mu\nu) = \int dx' G(x_1, x_2, x'; N_1 N_2) F(x', x_3, x_4; \mu\nu), \quad (5)$$

$$F(x_1, x_2, x_3, x_4; N_1 N_2 e\nu) = \int dx' G(x_1, x_2, x'; N_1 N_2) F(x', x_3, x_4; e\nu), \quad (6)$$

$$F(x_1, x_2, x_3, x_4; e\mu\nu\nu) = \int dx' F(x', x_1, x_4; e\nu) F(x', x_2, x_3; \mu\nu). \qquad (7)$$

As it seems safe to assume that the form factor for the baryon pair is likely to be produced mostly in the strong coupling region, it seems that the separation according to this assumption is very reasonable. And the same reason may assure the next assumption.

Assumption IV : The structure, i.e. the form factor of baryon pair (N_1, N_2) *does not sensitively depend on the leptons.*

We can then take

$$G(x_1, x_2, x'; N_1 N_2) \cong G(x_1 - x_2; N_1 N_2)\, \delta\left(x' - \frac{x_1 + x_2}{2}\right). \qquad (8)$$

As the separation of the form factor of lepton into the ($\mu\nu$) and ($e\nu$) pairs does not seem to be a suitable one, we must consider it more rigorously. But if we expect the form factors to have the same footing through the three cases, the previous assumption is not so unreasonable. The ways of the separation into two lepton pairs are three, as proposed by Lee and Yang, but they are not independent, independent cases being confined to ($\mu^-\tilde{\nu}$) and ($e^-, \tilde{\nu}$) or ($\mu^-\nu$) and ($e^-\nu$) ($\tilde{\nu}$ means antineutrino). Then if the neutrino structure is not so important, the separation can be proposed as above without loss of generality.

Concerning the muon and electron structure, we must discuss the following three cases :

Assumption V–a : The muon structure is more dominant compared to the electron structure.

We can substitute

$$F(x', x_3, x_4; e\nu) \cong \delta(x' - x_3)\, \delta(x_3 - x_4) \qquad (9)$$

for eqs. (6) and (7).

Assumption V–b : The electron structure is more dominant compared to the muon structure.

In the opposite way to (9), we can take

$$F(x', x_3, x_4; \mu\nu) \cong \delta(x' - x_3)\, \delta(x_3 - x_4) \qquad (9')$$

for eqs. (6) and (7).

Assumption V–c . Neither the muon structure nor the electron one are negligible.

As explained in the introduction, there may exist some reasons for taking the assumption V–a. Detailed discussions are, therefore, devoted to this case. But as the other possibilities remain, that is V–b and V–c, we shall discuss them later.

Assuming the canonical formalism, and using eqs. (5), (6), (7), (8) and (9), we have for the first order Hamiltonians :

$$H_I(\tilde{N}_1 N_2 \tilde{\mu}\nu) = (2\pi)^{-2}\int d_1\vec{p}\, d\vec{p}_2\, d\vec{p}_\mu\, d\vec{p}_\nu\, \delta^4(p_1-p_2+p_\mu-p_\nu)$$
$$\cdot \sum_i \bar{N}_1(p_1) O_i N_2(p_2) \cdot \bar{\mu}(p_\mu) O_i (C_i+\gamma_5 C_i')\nu(p_\nu)$$
$$\cdot G[-(p_1-p_2)^2/M_1^2,\ M_2^2/M_1^2]$$
$$\cdot F_{(\mu)}[-(p_\mu-p_\nu)^2/m_\mu^2, -p_\mu^2/m_\mu^2, -p_\nu^2/m_\mu^2], \tag{10}$$

$$H_I(\tilde{N}_1 N_2 \tilde{e}\nu) = (2\pi)^{-2}\int d\vec{p}_1\, d\vec{p}_2\, d\vec{p}_e\, d\vec{p}_\nu\, \delta^4(p_1-p_2+p_e-p_\nu)$$
$$\cdot \sum_i \bar{N}_1(p_1) O_i N_2(p_2) \cdot \bar{e}(p_e) O_i (C_i+\gamma_5 C_i')\nu(p_\nu)$$
$$\cdot G[-(p_1-p_2)^2/M_1^2,\ M_2^2/M_1^2], \tag{11}$$

$$H_I(\tilde{e}\,\mu\tilde{\nu}\,\nu) = (2\pi)^{-2}\int d\vec{p}_e\, d\vec{p}_\mu\, d\vec{p}_\nu\, d\vec{p}_\nu'\, \delta^4(p_e-p_\mu+p_\nu-p_\nu')$$
$$\cdot \sum_i \bar{e}(p_e) O_i \mu(p_\mu) \cdot \bar{\nu}(p_\nu) O_i (C_i+\gamma_5 C_i')\nu(p_\nu')$$
$$\cdot F_{(\mu)}[-(p_\mu-p_\nu)^2/m_\mu^2,\ -p_\mu^2/m_\mu^2,\ -p_\nu^2/m_\mu^2], \tag{12}$$

where

$$G(x_1-x_2;\ N_1N_2) = (2\pi)^{-4}\int dp\, e^{ip(x_1-x_2)} G[-p^2/M_1^2,\ M_2^2/M_1^2],$$

$$F(x_1,\ x_2,\ x_3;\ \mu\nu) = (2\pi)^{-8}\int dk\, dl\, e^{ik(x_1-x_2)+il(x_2-x_3)}$$
$$\cdot F_{(\mu)}[-k^2/m_\mu^2,\ -(k-l)^2/m_\mu^2,\ -l^2/m_\mu^2]. \tag{13}$$

For the decay processes considered here, these first order Hamiltonians are sufficient. Moreover, in this approximation, as we consider the leptons to obey the free Dirac equations, i. e., $-p_\mu^2/m_\mu^2=1$, $-p_\nu^2/m_\mu^2=0$, the form factor for muon becomes

$$F_{(\mu)}[-(p_\mu-p_\nu)^2/m_\mu^2,\ 1,\ 0] \equiv F_{(\mu)}[-(p_\mu-p_\nu)^2/m_\mu^2], \tag{14}$$

which is essentially a one parameter function. We can get, therefore, from the decay processes, information about the behaviour of this function only in the $-(p_\mu-p_\nu)^2/m_\mu^2>0$ region, just opposite to the case of electron-proton scattering.

§ 3. Pion decays

The present evperimental evidences[1) 13)] on both pion decays, i. e., $\pi\to e+\nu$ and $\pi\to\mu+\nu$ are in agreement with the usual theoretical predictions, which take into account similar mechanisms for both cases, regardless of whether direct interaction exists between pion and leptons or not. Both the direct (Yukawa's) scheme and the indirect (Sakata's) scheme give the same formula for the ratio of these decays,

$$R = \alpha\frac{m_e^2}{m_\mu^2}\left[\frac{m_\pi^2-m_e^2}{m_\pi^2-m_\mu^2}\right]^2 = 1.3\cdot 10^{-4}\alpha. \tag{15}$$

The coefficient α expresses the detailed contributions for the ratio, due to: i) non-derivative coupling and derivative coupling effects for Yukawa's scheme and ii) P and A coupling effects of Fermi interaction for Sakata's scheme. This term can be written as

$$\alpha=\frac{|I_{+}^{(e)}|^2+|I_{-}^{(e)}|^2}{|I_{+}^{(\mu)}|^2+|I_{-}^{(\mu)}|^2}, \tag{16}$$

where

$$I_{\pm}^{(\mu)}=F_{(\mu)}[m_\pi{}^2/m_\mu{}^2]\{(g\pm g')+1/m_\mu(f\pm f')\}, \tag{17}$$

if we start with

$$H(\pi\bar{\mu}\nu)=(2\pi)^{-2}\int d\vec{p}_\pi d\vec{p}_\mu d\vec{p}_\nu \delta^4(p_\pi-p_\mu+p_\nu)F_{(\mu)}[-p_\pi{}^2/m_\mu{}^2]$$
$$\cdot\phi(p_\pi)\bar{\mu}(p_\mu)[i\gamma_5(f+\gamma_5 f')+i\gamma_5(i\gamma p_\pi)(g+\gamma_5 g')]\nu(p_\nu) \tag{18}$$

for Yukawa's scheme, where $\phi(p_\pi)$ is pion wave function with momentum p_π and f, f' and g, g' are the non-derivative and derivative couplings constants respectively. Similar expression is given for the electron case.

On the other hand, for the case of Sakata's scheme, we obtain

$$I_{\pm}^{(\mu)}=F_{(\mu)}[m_\pi{}^2/m_\mu{}^2]\cdot\sum_{N_1,N_2}\{(C_A\pm C_A')A(m_\pi{}^2/M_1{}^2,\ m_\pi{}^2/M_2{}^2)$$
$$+1/m_\mu\cdot(C_p\pm C_p')P(m_\pi{}^2/M_1{}^2,\ m_\pi{}^2/M_2{}^2)\}, \tag{19}$$

and the similar expression for the electron case, where the summations are taken on all baryon pairs N_1, N_2 compatible with strangeness conservation, and the functions A, P are the contributions from each pair through A, P couplings respectively.

The possibilities of getting small values for the decay ratio as indicated experimentally are either i) the case in which the non-derivative coupling in Yukawa's scheme, or P coupling in Sakata's scheme, is negligible compared to the other couplings or ii) strong cancellations occur for the electron case in Sakata's scheme. At any rate, it is necessary to retain the derivative coupling or the A coupling of Fermi interaction for the pion decay processes.

Although the present information give no decision as to the mechanism of pion decays and, consequently, cannot exclude the second possibility, it seems to be more suitable to take Sakata's scheme with the first possibility.

If the pion decays are realized through Sakata's scheme, they are closely related with the β-decay phenomena. Although all recent experiments on β-decay have been in favour of V and A couplings, it is still very difficult to exclude S, T, P couplings completely.

The theory with only V and A couplings, however, have two good advantages for the pion decay. As we can discuss the ratio without knowing much about any strong interaction phenomena, the ratio may give some information about the weak

interactions and the structure of leptons. And we must not worry about the competition between $\pi \to e+\nu+\gamma$ and $\pi \to \mu+\nu$ processes, as before, when all experiments were indicating the S and T couplings.

The universality of Fermi interactions with the restricted form is not yet established rigorously, but it seems to be natural and simple to take this standpoint. We assume then:

*Assumption VI: All Fermi interactions of the V, A type are considered universal.**

With this assumption, (15) becomes

$$R=1.3\cdot 10^{-4}[F_{(e)}(m_\pi^2/m_e^2)/F_{(\mu)}(m_\pi^2/m_\mu^2)]^2 \tag{20}$$

and with the assumption *V-a*,

$$R=1.3\cdot 10^{-4}[F_{(\mu)}(m_\pi^2/m_\mu^2)]^{-2}. \tag{21}$$

The muon structure reveals itself through the last factor, i. e., the value of the form factor at a point $m_\pi^2/m_\mu^2=1.7$. Therefore, we can get some information about the form factor from

$$R_{\text{loc.th.}}/R_{\text{exp}}=f(m_\pi^2/m_\mu^2), \quad f(x)\equiv F_{(\mu)}^2(x). \tag{22}$$

Although the experimental value of the ratio R_{exp} is not yet definitely established, we may at least take**

$$R_{\text{exp}}>0.4\cdot 10^{-4},$$

that is,

$$0<f(m_\pi^2/m_\mu^2)\lesssim 3.3. \tag{23}***$$

As far as V and A couplings are concerned, there is no difficulty for pion radiative decay $\pi \to e+\nu+\gamma$. The ratio of this decay to pion-muon decay, R_τ, becomes

$$R_{\tau\,\text{exp}}=f(m_\pi^2/m_\mu^2)\cdot R_{\tau\,\text{loc.th.}}. \tag{24}$$

Taking the theoretical ratio[14]

$$R_{\tau\,\text{loc.th.}}\simeq 10^{-8}$$

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* Strict universal Fermi interactions with V and A couplings have the form of $V-A$. Other possibilities have only a restricted meaning. For instance, the statement that V and A couplings depends on the orders of operators. The proposal here considered is restricted to the charge retaintion order chosen in the previous chapter. Even if we start with V and A couplings for $(\bar{e}O\mu)(\bar{\nu}O'\nu)$ i. e. charge retaintion order, this gives not only V and A couplings, but also S and P couplings for $(\bar{e}O\nu)(\bar{\nu}O'\mu)$ i. e. charge exchange order.

** This value was given by T. Fazzini et al. loc. cit., and it is almost equal to the maximum value given by H. L. Anderson and C. M. G. Lattes, loc. cit.

*** The recent values reported are all surprisingly close to 1.3×10^{-4}. The maximum value is CERN's $(1.4\pm 0.2)\cdot 10^{-4}$. Although the experimental errors are still large and the way of correcting them incomplete, we may safely put the minimum limit for f to 0.8—0.9.

and considering the condition for the form factor (23), the theoretical value is not inconsistent with the experimental value[15]

$$R_{\tau\,\mathrm{exp}} \lesssim 8\cdot 10^{-6}.$$

§ 4. Muon decay

The muon decay is one of the best examples for the consideration of the existing weak interaction theories. No strong interaction intervenes in this process and the radiative effects are at most of the order of a few per cent, the field reactions of the weak interactions being negligible. Muon decay may then give some information about the muon structure, within the uncertainty of the radiative effects. The calculations of the radiative effects worked out by Kinoshita and Sirlin[16] have indicated that the corrections to Michel parameter ρ is about 6%, while the correction to the asymmetry parameter is almost zero. Therefore the asymmetry parameter seems to be more sensitive with respect to the muon structure than ρ is.

The energy and angular distribution of electron is given by

$$dN=\frac{m_\mu{}^5 a}{3\cdot 2^6\pi^4}d\Omega\,dx\,x^2\Big[\{K_1^+-xK_2^++\lambda(1-x)K^-\}$$
$$\mp\lambda\cos\theta\{-K_1^++(2-x)K_2^++1/\lambda(1-x)K^-\}\Big] \tag{25}$$

in virtue of Hamiltonian (12), where

$$x=|\vec{p}_e|/|\vec{p}_{e\,\max}|\,;\quad a=C_v{}^2+C_v{}'^2+C_A{}^2+C_A{}'^2,$$
$$\lambda a=-2(C_A'C_v+C_AC_v') \tag{26}$$

and

$$K_i^\pm=3\int_0^1 du\,u^i[f(xu)\pm f(x-xu)],\quad K^-\equiv K_1^-=K_2^-.$$

In order to see the role of the form factor, we assume it to be so smooth that its derivatives higher than three can be neglected. We have then

$$dN=\frac{m_\mu{}^5 a}{3\cdot 2^6\pi^4}d\Omega\,dx\,x^2\Big[\Big\{(3-2x)f(x/2)+\frac{x^2(5-4x)}{40}f''(x/2)$$
$$+\frac{\lambda x(1-x)}{2}\Big(f'(x/2)+\frac{x^2}{40}f'''(x/2)\Big)\Big\}\mp\lambda\cos\theta\Big\{(1-2x)f(x/2)$$
$$+\frac{x^2(3-4x)}{40}f''(x/2)+\frac{x(1-x)}{2\lambda}\Big(f'(x/2)+\frac{x'}{40}f'''(x/2)\Big\}\Big] \tag{27}$$

for electron spectrum and

$$\Delta\rho=\frac{3}{4}-\rho=-\frac{3}{32}\,\frac{A_1+(1-\lambda)A_2/3}{A_0-(1-\lambda)A_2/24},$$

$$\Delta\xi \equiv \lambda - \xi = \frac{\lambda}{10}\,\frac{B_1 + (1-\lambda)^2 B_2/\lambda}{B_0 - (1-\lambda)B_2/20} \tag{28}$$

for the deviations of Michel and asymmetry parameters from the local values, where

$$A_0 = f - \frac{5}{8}f' + \frac{17}{18}f'' - \frac{71}{1440}f''',$$

$$A_1 = f' - \frac{1}{2}f'' + \frac{23}{180}f''',\quad A_2 = f' - \frac{3}{4}f'' + \frac{23}{80}f''' \tag{29}$$

and

$$B_0 = f - \frac{3}{5}f' + \frac{1}{5}f'' - \frac{1}{21}f''',$$

$$B_1 = f' - \frac{3}{5}f'' + \frac{41}{210}f''',\quad B_2 = f' - \frac{2}{3}f'' + \frac{5}{21}f''',$$

the values of all functions, f's, being taken at the point $m_\mu^2/m_\mu^2 = 1$.

The remarkable consequences deduced from the above formulae are the following:

i) The Michel parameter depends on the coupling constants through λ in virtue of the form factor, in opposition to the local V and A theory. But if we expand it in terms of the deviation from the $V-A$ theory, i. e., $1-\lambda$, we get

$$\rho \sim \frac{3}{4}\left(1 + \frac{1}{8}\,\frac{A_1}{A_0}\right) + \frac{1}{32}(1-\lambda)\,\frac{A_2}{A_0}\left(1 + \frac{1}{8}\,\frac{A_1}{A_0}\right). \tag{30}$$

As we can estimate the first correction to be of the order of a few percent, the contribution of the deviation from the $V-A$ theory then becomes negligible.

ii) The correction to the asymmetry parameter is proportional to λ, as

$$\xi \sim \lambda\left[\left(1 + \frac{1}{10}\,\frac{B_1}{B_0}\right) + \frac{1}{200}(1-\lambda)\,\frac{B_1}{B_0}\,\frac{B_2}{B_0}\right]. \tag{31}$$

Then ξ is sensitive to the initial value λ, and gives a good support to the $V-A$ theory, but not to the two-component neutrino theory.[17]

iii) Using for $F_{(\mu)}$ the usual functions

$$F_{(\mu)}(-k^2/m_\mu^2) = \begin{cases} [1+\epsilon^2 k^2/m_\mu^2]^{-1}, \\ [1+\epsilon^2 k^2/2m_\mu^2]^{-2}, \\ \exp[-\epsilon^2 k^2/m_\mu^2], \end{cases} \qquad f(x) = \begin{cases} [1-\epsilon^2 x]^{-2}, & \text{Yukawa} \\ [1-\epsilon^2 x/2]^{-4}, & \text{Exponential} \\ \exp(\epsilon^2 x), & \text{Gauss} \end{cases} \tag{32}$$

with small values for ϵ^2, we have*

$$\Delta\rho \simeq -\frac{1}{8}\left(1 - \frac{\lambda}{4}\right)\frac{f'}{f} < 0, \quad \rho > \frac{3}{4},$$

$$\Delta\xi \simeq \frac{1}{20}(1+\lambda^2)\,\frac{f'}{f} > 0, \quad \xi < \lambda. \tag{33}$$

* Similar conclusion for ρ was taken by Lee and Yang, loc. cit. The tendency of $\xi < \lambda$ is different from Sirlin's calculation, loc. cit.

The last conclusion seems to be inconsistent with the experimental tendencies which favours $\Delta\rho \gtrsim 0$ and $\Delta\xi > 0$. The possibilities of removal of this descrepancy are the following: the first is to include the radiative corrections which may cancel out $\Delta\rho$, that is, making $\Delta\rho \simeq 0$. The next possibility is to take into account the higher derivative terms, like f'' and f''' in order to realize the usual tendencies. As the former possibility seems to be too accidental, we should consider the latter. If this procedure would be correct, the form factor could be restricted by more detail. The next chapters are devoted to this procedure.

§ 5. Analyses of the form factor

We discuss here the way of determining the form factor from data given by the decay processes raised before. If we restrict the behavior of the form factor to the energy region between 0 and m_π, the information available are: i) condition of local limit, ii) Michel parameter, iii) asymmetry parameter and iv) $\pi \to e+\nu / \pi \to \mu+\nu$ ratio.

The experimental data about the last three points are not yet definite and the values reported recently may change to some extent. The data presently available are the following:

ρ_{exp}: 0.68 ± 0.02,[18] 0.67 ± 0.05,[19] 0.74 ± 0.02,[20a] 0.79 ± 0.04.[20b]

ξ_{exp}: 0.87 ± 0.12,[21] 0.909 ± 0.144,[22] 0.95 ± 0.06,[20b] 0.97 ± 0.05,[20b] 0.89 ± 006.[24]

$R_{\text{exp}} \cdot 10^4$: >0.4,[1] 0.9 ± 0.4,[13] 1.24 ± 0.08,[20c] 1.3 ± 0.2,[20d] 1.4 ± 0.2.[20e]

Judging from these experimental circumstances, we are obliged to see how the behavior of the form factor changes if we make the sufficient allowances for these three data, as the aim of this paper is to predict the possibility of the muon structure when the experiments would give the definite values for them. It seems sufficient to take, for the allowance regions, the domains

$$0.65 \lesssim \rho \lesssim 0.85,\ 0.85 \lesssim \xi \lesssim 1.15,\ 0.5 \cdot 10^{-4} \lesssim R \lesssim 1.6 \cdot 10^{-4}.$$

In order to simplify the interpretation, the $V-A$ combination which makes $\lambda=1$ is chosen in the following analyses, but this is done only for the sake of convenience and sometimes this choice may become unsuitable for the interpretation of the results as discussed afterwards.

If we use the following definitions of the deviations for the case of $V-A$ combination

$$\Delta\rho = 0.75 - \rho,\ \Delta\xi = 1 - \xi,\ \Delta r = R_{\text{ioc.th.}}/R_{\text{exp}} - 1, \tag{34}$$

the above mentioned regions correspond to

$$-0.10 \lesssim \Delta\rho \lesssim +0.10,\ -0.15 \lesssim \Delta\xi \lesssim +0.15,\ -0.20 \lesssim \Delta r \lesssim +1.6. \tag{35}$$

It may be sufficient to assume that the form used here is effective at most up to its third-order derivative and there is no data from which we can get infor-

mation beyond this restriction. The validity of our approximation then depends on whether the form factor behaves so smoothly that its derivatives higher than the third can be negligible:

Assumption VII: The form factor here proposed should be effective at most up to its third-order derivative.

If one accepts this assumption, one can get the following equations from the four conditions stated before.

$$\begin{aligned} 1 &= f(0) = f\cdot(1-X+0.50Y-0.17Z), \\ \Delta\rho &= -(0.09X-0.05Y+0.01Z)/(1-0.63X+0.21Y-0.05Z), \\ \Delta\xi &= +(0.10X-0.06Y+0.02Z)/(1-0.60X+0.20Y-0.05Z), \\ \Delta\gamma &= +(1.69X-0.26Y+0.22Z)/(1-X+0.50Y-0.17Z), \end{aligned} \tag{36}$$

where

$$f=f(1),\ X=f'(1)/f(1),\ Y=f''(1)/f(1),\ Z=f'''(1)/f(1).$$

Then if we take the conditions (35), we can know the allowance regions of X, Y, Z. Though the approximation used here depends on the assumption VII, one must bear in mind that they only give the rough tendencies of the form factor.

In order to judge the general tendencies, we select the suitable values distributed within these regions and indicate them in Figs. 1, 2, and 3.

We retain $\Delta\rho=0$ throughout all Figures, because the deviation of the Michel parameter cannot be fixed without definite knowledge about the radiative corrections.

The "reasonable" curves thus obtained are three curves corresponding to the value $\Delta\rho=0$ and $\Delta\xi=0$ with the three different values of $\Delta r(1b,\ 2b,\ 3b)$. They

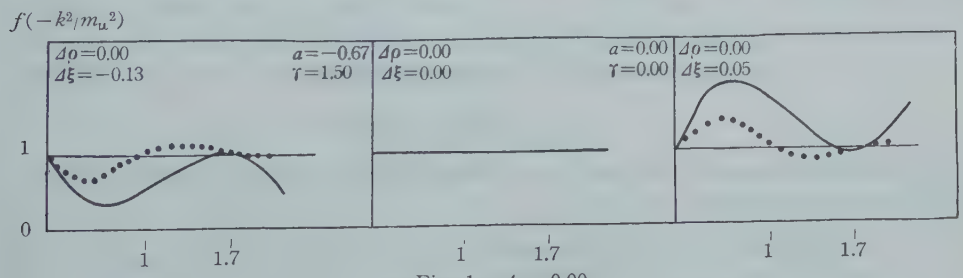

Fig. 1. $\Delta r=0.00$

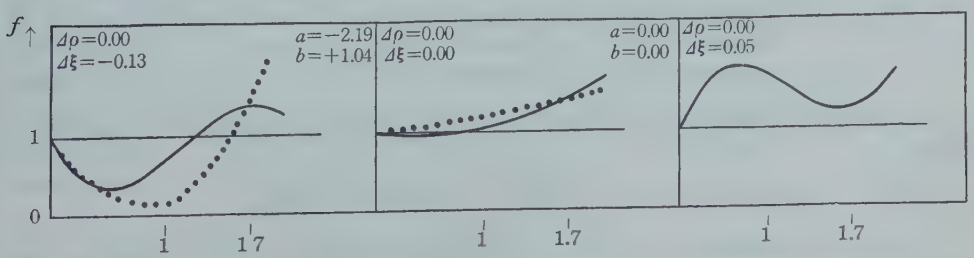

Fig. 2. $\Delta r=0.40$

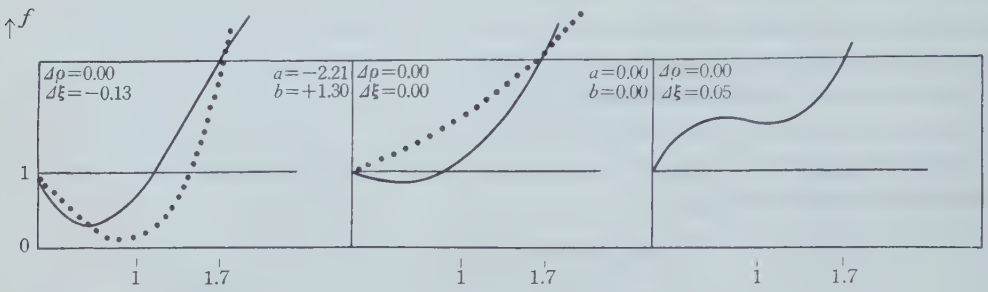

Fig. 3. $\Delta r=1.60$

Behaviors of square of the form factor $f(-k^2/m_\mu{}^2)=F^2(-k^2/m_\mu{}^2)$, corresponding the deviations of three values Δr, Δp and $\Delta\xi$, defined bs eqs. (34). The ordinate axis corresponds to the values of square of the form factor and the abscissas the values of $-k^2$ in unit of $m_\mu{}^2$. The solid lines are phenomenological ones determined from our analysis, and the dotted lines are approximated analytical curves. The straight line is the standard local limit.

show that there is some possibility for changing the pion-decays ratio without destroying the muon decay spectrum predicted theoretically. We could approximate them by choosing either the exponential, Yukawa or Gauss type for the form factor within the experimental errors. But to retain the least deviations from $\Delta\rho=0$, and $\Delta\xi=0$, it is more suitable to take the Gauss type, that is, we take

$$f(x)=\begin{cases}1,\\ \exp[0.20\,x],\\ \exp[0.56\,x],\end{cases}\qquad \text{for}\quad R=\begin{cases}1.3\cdot10^{-4},\\ 0.9\cdot10^{-4},\\ 0.5\cdot10^{-4},\end{cases} \tag{37}$$

which correspond to

$$F_{(\mu)}(-k^2/m_\mu{}^2)=\begin{cases}1,\\ \exp[-0.10\,k^2/m_\mu{}^2],\\ \exp[-0.28\,k^2/m_\mu{}^2],\end{cases} \tag{38}$$

the corresponding spreads* in space-time being 0, $1.0\,m_\pi^{-1}\fallingdotseq1.4$ yukawa, $1.7\,m_\pi^{-1}\fallingdotseq2.4$ yukawa, respectively.

All other curves can be interpreted as some deviations from these standard curves. As the tendencies of their deviations are common to each Figure, they can be approximated by the following formula with some correction factors $Z(x)$,

$$f(x)=Z(x)\exp[\epsilon^2 x], \tag{39}$$

where ϵ^2 takes the values 0, 0.20 and 0.56, corresponding to Fig. 1, 2 and 3 respectively.

All curves appearing in Fig. 3 can be realized by taking

* We define the spread as $\langle r^2\rangle=-6dF_{(\mu)}(-k^2/m^2{}_\mu)/d(-k^2/m^2{}_\mu)_{k^2=0}$, but this does not mean that the muon has such a spread.

$$f_3(x) \doteqdot [1+ax+bx^2]\exp[0.56\,x], \tag{40}$$

the exceptions being the cases for $\Delta\tilde{\xi} \gtrsim +0.06$, which will be discussed later. We also plot these approximate curves, dotted with the coefficients a and b which were chosen so as to make minimize the deviations from the Δ_l^{ρ} and $\Delta\tilde{\xi}$ values taken from the beginning.

The use of similar correction factors, as applied to Fig. 3, becomes not good for certain part of the curves in Fig. 2 which resembles the behavior of a cubic function. Judging from the nature of the approximations used here, it may be more convenient to approximate with quadratic functions, because we may expect all third derivatives to be very small. Then we also take

$$f_2(x) \doteqdot [1+ax+bx^2]\exp[0.20\,x] \tag{41}$$

for all curves in Fig. 2.

Since the standard curve in Fig. 1 corresponds to the local case, i. e. the straight line, the other curves are very sensitive to the values of Δ_l^{ρ} and $\Delta\tilde{\xi}$. The rough behavior of all curves appear as sine-like. The functions which best fit them are

$$f_1(x) \doteqdot 1+A\sin\omega x\cdot e^{-\tau x}. \tag{42}$$

Although they are very curious, we indicate them in Fig. 1 by the dotted lines.

As indicated in Figs. 1, 2, and 3, the approximation used becomes poor in the region of $\Delta\tilde{\xi}>0$. One can read from the Figs. that the variations of curves are sensitive to the variations of $\Delta\tilde{\xi}$, specially in its positive side (but not to Δ_l^{ρ}). In order to understand the circumstances, we add three curves corresponding to $\Delta\tilde{\xi}=0.05$. We also point out the region of validity of our approximation in Fig. 3. From this region, it is safe to restrict the deviations $\Delta\tilde{\xi}$ into

$$\Delta\tilde{\xi} \lesssim 0.06. \tag{43}$$

We are then obliged to conclude that the large deviations as $\Delta\tilde{\xi} \simeq 0.13$ (which corresponds to the value $\tilde{\xi}=0.87$) cannot be realized by the above considerations. Even if the experiment about the asymmetry parameter is very difficult, we hope that more experimental data must be accumulated.

should consider some other reasons. The following consideration is one of the possibilities: The statement is to change the C_A/C_V ratio from -1 as in the β-decay. From (27), we have $\tilde{\xi} \simeq \lambda[1-B_1/10B_0]$. Supposing the correction $B_1/10B_0$ would be the maximum value, i. e. 0.06 from (43), and $\tilde{\xi}=0.87$, then we would get $\lambda \simeq 0.93$. With use of the definition of λ and (taking) the two-component neutrino theory, $C_A=C_A'$ and $C_V=C_V'$, this gives

$$C_A/C_V \simeq -1.5$$

which is not inconsistent with $C_A/C_V \simeq -1.2$ for β-decay. If we accept the usual

prescription of the renormalization theory, C_A/C_V of β-interaction is less than the corresponding quantity for $\mu-e$ interaction, because the axial vector coupling of nucleon pair will become less than in the case of $\mu-e$ interaction, owing to the renormalization effect ($C_{A\,\mathrm{ren.}} \lesssim C_{A\,\mathrm{unren.}}$) of the strong interaction.

§ 6. Kaon decays

In considering the kaon decay problems, we apply the same line of thought. Comparing it to the pion and muon decays, the present experimental knowledge about kaon decays is rather crude. Although we cannot expect any detailed information about the muon structure, it may be worthwile to know the asymptotic behaviour of the form factor previously discussed, as any small variation of the form factor in the region from 0 to m_π will be amplified in the region where the kaon decays are considered, i. e. from m_K-m_π to m_k. The experimental values for the kaon decays, if their accuracies would be improved, could then give a good criterion for the analysis of the muon structure.

The considerations are here restricted to the $K\to e+\nu/K\to\mu+\nu$ and the $K\to\pi+\mu+\nu/K\to\pi+e+\nu$ ratios, as a similar argument used in the case of pion decays, concerning the strong interaction effects applied here.

i) *$K\to e+\nu$ to $K\to\mu+\nu$ ratio*

Taking Sakata's scheme of Fermi interactions with V and A couplings, we have

$$R_{K_2}=\frac{K\to e+\nu}{K\to\mu+\nu}=f^{-1}(m_K^2/m_\mu^2)\frac{m_e^2}{m_\mu^2}\left[\frac{m_K^2-m_e^2}{m_K^2-m_\mu^2}\right]^2=2.4\cdot10^{-5}f^{-1}(21.8), \quad (44)$$

which is similar to the pion case.

If we extrapolate the form factor obtained in the previous chapter, its maximum value becomes about 10^2 (or 10^5) for the case of $R=0.9\times10^{-4}$ (or $R=0.5\times10^{-4}$). That is, we have

$$10^{-7}\ (\text{or } 10^{-10}) \lesssim R_{K_2} \lesssim 10^{-5}, \quad (45)$$

which is not inconsistent with the present experimental knowledge.

ii) *$K\to\pi+\mu+\nu$ to $K\to\pi+e+\nu$ ratio*

Assuming Sakata's scheme with Fermi interaction, the matrix element for muon decay is given by

If the such large deviation of the asymmetry parameter would be correct, we

$$M_{(\mu)}=\sum_{N_1,N_2}[G(p_K,\ p_\pi;\ M_1M_2)\,p_K^{\alpha}+G'(p_K,\ p_\pi;\ M_1M_2)\,p_\pi^{\alpha}]\cdot J^{\alpha}_{(\mu)}$$
$$\cdot\,\delta^4(p_K-p_\pi-p_\mu-p_\nu)\,F_{(\mu)}(p_\mu,\ p_\nu;\ m_\mu), \quad (46)$$

where

$$J^{\alpha}_{(\mu)}=[p_K{}^0\,p_\pi{}^0]^{-1/2}\bar{\nu}(-p_\nu)\gamma_\alpha(C_\nu+\gamma_5C_\nu')\mu(p_\mu),$$

the summation is taken over all baryon pairs (N_1, N_2) which have the total

strangeness equal to -1, and the unknown strong pair effects appearing through G and G'. The total probability per unit time is

$$P\sim\int dp_\mu{}^0 dp_\pi{}^0\cdot f\cdot\Big[m_K(2p_\mu{}^0 p_\nu{}^0+m_K(p_\pi{}^0-M))|H|^2+m_\mu{}^2(M-p_\pi{}^0)|G'|^2 -2m_\mu{}^2 p_\nu{}^0 R_e(G'H)\Big], \tag{47}$$

where

$$p_\nu{}^0=m_k-p_\pi{}^0-p_\mu{}^0,\ M=(m_K{}^2+m_\pi{}^2-m_\mu{}^2)/2m_K \text{ (pion maximum energy)},$$
$$H=G+G',\ G=\sum_{N_1,N_2}G(p_K,\ p_\pi;\ M_1M_2),\quad G'=\sum_{N_1,N_2}G'(p_K,\ p_\pi;\ M_1M_2),$$
$$f=F_{(\mu)}^2[1+2m_K{}^r(M-p_\pi{}^0)/m_\mu{}^2]. \tag{48}$$

Since we cannot get the exact value for the ratio, the following assumptions are introduced:

i) Concerning the form factor, we suppose it to be a monotonic function in the region $1\lesssim 1+2m_K(M-p_\pi{}^0)/m_\mu{}^2\lesssim 12$, and we replace it by the average value.

ii) G and G' are almost independent of p_π.

We have then*:

$$R_{K_3}=\frac{K\to\pi+\mu+\nu}{K\to\pi+e+\nu}\simeq f(6)[0.79-0.33\,Re(\eta)+0.08|\eta|^2], \tag{49}$$

where

$$\eta=G'/(G+G').$$

As the perturbation calculation gives

$$G(M_1,\ M_2)\simeq G'(M_1,\ M_2), \tag{50}$$

we can estimate $\eta\simeq 1/2$. The ratio becomes

$$R_{K_3}\simeq 0.64f(6). \tag{51}$$

If we take the maximum value of $f(6)$ as ~ 3 (or ~ 30) for $R=0.9\cdot 10^{-4}$ (or $R=0.5\cdot 10^{-4}$), we have

$$0.6\lesssim R_{K_3}\lesssim 1.9 \text{ (or } 19.2). \tag{52}$$

This result is still consistent with the experimental value $R_{\exp}\simeq 1.16^{+0.74}_{-0.44}$.[23] (The case of $R=0.5\cdot 10^{-4}$ gives too large a value, so that this case must be excluded).

§ 7. Possibility of the electron structure

In the previous chapters, we have assumed the existence of a structure for the muon and discussed the way how it could be determined from the meson decay

* Our result is a little different from the result of R. F. Streater and J. C. Taylor, Nucl. Phys. **7** (1958), 276. Their result is $0.92-0.42\,R_e(\eta)+0.08|\eta|^2$ for the second factor of (49), a modification which replaces (51) by $R_{K_3}\simeq 0.73f(6)$.

processes. Since the problem is not, strictly speaking, a definite one, still depending on the future experimental results, it may give, at least, a possible answer to the questions raised. We cannot of course, deny the rather probable case of an agreement between the experimental and the usual theoretical local values. On this point, we can consider the two standpoints: either the muon structure may be too small to be experimentally detected, or the muon structure may be masked by some other effects, for example, radiative corrections or by the electron structure.

Although the assumption *V-a* which takes into account only the muon structure seems to be preferable as stated in the introduction, we cannot reject the opposite possibility, in which the electron structure plays the principal role (assumption *V-b*). This last possibility, is somewhat akin to the classical picture of Lorentz. And we also must consider the third possibility given by assumption *V-c*.

For the *V-b* case, the form factor appearing in muon decay must be replaced by $F_{(e)}(-k^2/m_e^2)$ instead of $F_{(\mu)}(-k^2/m_\mu^2)$, in virtue of (9′). For the ratio of the pion decays, we have

$$R_{\text{loc.th.}}/R_{\text{exp}}=f_{(e)}(m_\pi^2/m_e^2). \tag{53}$$

As the effect of the form factor is opposite, if we take the Gauss type of the form factor as the case of *V-a*, we have

$$R_{\text{loc.th.}}<R_{\text{exp}}. \tag{54}$$

In the case in which we consider the electron structure, the way to realize the same experimental possibilities as before, i. e.,

$$0.38\lesssim f_{(e)}(m_\pi^2/m_e^2)\lesssim 1, \tag{55}$$

is to introduce a stronger form factor, such as

$$F_{(e)}(-k^2/m_e^2)=\exp[-\epsilon_e^2 k^4/2m_e^4]. \tag{56}$$

But the effect of this form factor for the muon decay will be opposite to the former case.

For the *V-c* case, the effect on the pion decay ratio becomes

$$F_{(e)}(m_\pi^2/m_e^2)/F_{(\mu)}(m_\pi^2/m_\mu^2),$$

while for the muon decay, the effects will appear through

$$F_{(e)}(-k^2/m_e^2)\cdot F_{(\mu)}(-k^2/m_\mu^2).$$

Summarizing the above effects, through their rough estimates, we may conclude as follows:

1) If the pion decay ratio would become larger than the usual theoretical value, one should consider two possibilities: either the smooth (simple) electron structure or the singular complex muon structure would be dominant. The contributions to the muon decay would be opposite from one use to the other.

2) If the pion decays ratio would become in agreement with the usual theoretical value, one should also consider two possibilities: either the both structures are negligibly small or both of them are almost the same. The spectrum of the muon decay could decide which of them is correct, because, in the latter possibility, the deviations of the parameter might be doubled.

3) If the pion decay ratio would become smaller than the usual theoretical one, two possibilities should again be considered: either the smooth (simple) muon structure or the singular complex muon structure would be dominant. The contributions to the muon decay would be opposite from each other.

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On the Origin of the Electron Mass

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Some possibilities for understanding the dynamical origin of the electron mass are examined. As the present formulation of quantum-electrodynamics does not give, within its region of validity, a satisfactory answer to this problem, two possibilities for obtaining the electron mass, starting from a zero bare mass, are presented. The first one is a nonlocal modification of quantum-electrodynamics, which may be closely related to the electron structure. The other is to get the electron mass from the masses of other elementary particles through weak interactions, because at the high energy region discussed here, the weak interactions become more predominant compared to the electromagnetic interaction. These possibilities can be examined by improvements in experiments. The former depends on the experimental accuracies of the electron moment and the latter is related to the degree of coexistence of S, P, T couplings in coexistence with the main V and A couplings in β-decays and/or to the degree of the deviation from the V-A combination in the muon-electron interaction.

§ 1. Introduction

The idea of the electromagnetic origin of the electron mass has been much considered since Lorentz proposed it in his famous book[1]. The present quantum-electrodynamics, however, seems to be out of this line of thought. The thought that the electromagnetic mass of the electron is only due to its charge has been believed for a long time. But the present calculation shows that any electromagnetic energy of the electron cannot be obtained without the help of the intrinsic electron mass, that is, the electromagnetic interaction is qualitatively too different to realize the whole mass of the electron. This is easily demonstrated as follows.

For the Lagrangian of the electron without bare mass and taking the Lagrangean of the interacting system as electromagnetic field,

$$\bar{L}=-\int dx\,[\bar{e}(x)\gamma_\alpha\partial_\alpha e(x)-i\varepsilon\bar{e}(x)\gamma_\alpha A_\alpha(x)e(x)+\tfrac{1}{2}\partial A_\beta(x)/\partial x_\alpha\cdot\partial A_\alpha(x)/\partial x_\beta \quad (1)$$

where e and A_α are the field quantities of the electron and the electromagnetic field respectively, and ε means charge, we see that this Lagrangian is invariant not only

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under the gauge transformation, but also under the following transformation, with an arbitrary constant, β often found in the literature[2]:

$$e(x) \to \exp(i\beta\gamma_5) \cdot e(x), \tag{2}$$

while the desired self-energy term

$$\delta\bar{L} = -\delta m \int dx \bar{e}(x) e(x) \tag{3}$$

is not invariant under this transformation. In other words the electron without the bare mass cannot get any field reaction.

One more difficult point is the following: supposing the electron to have a very small bare mass m_0 compared with the true mass m_e ($m_0 \ll m_e$) and using perturbation theory, we get the radiative correction

$$\delta m^{(\alpha)} \simeq \alpha(3/2\pi) m_0 \log(\lambda/m_0), \tag{4}$$

where $\alpha = \varepsilon^2/4\pi$ is the fine structure constant and λ the cut-off energy. Then in order to get the electron mass, we are obliged to use the theory up to the energy $\lambda \simeq m_0 \exp(2\pi m_e/3\alpha m_0)$, a value far beyond the limit of the validity of quantum-electrodynamics, the limit being usually taken as $\lambda_{crit} \simeq m_e \exp(3\pi/\alpha)$.[3]

It seems, then, that in order to obtain a satisfactory answer for this problem, we must look for some modification of the present quantum-electrodynamics. The proposals presented here are restricted to some possibilities within the framework of the present quantum field theory.

§ 2. Non-local electromagnetic interaction as the mass origin

The first possibility is some modification of the usual formulation of quantum-electrodynamics by considerations of non-locality. Though such a modification would be needed from more general and profound considerations, we shall here be satisfied with the introduction of non-locality only in connection with the interaction between the electron and the electromagnetic field.

We then propose to replace such an interaction by the non-local one[4]

$$\bar{L}_I = i\varepsilon \int dx_1 dx_2 dx_3 \bar{e}(x_1) \gamma_\alpha F(x_1 - x_3, x_2 - x_3) \cdot \exp\left[-i\varepsilon \int_{x_2}^{x_1} dx_\beta A_\beta(x) \right] e(x_2) \cdot A_\alpha(x_3), \tag{5}$$

where the form factor is generally given by

$$F_{rs}(x_1 - x_3, x_2 - x_3) = \delta_{rs} S(x_1 - x_3, x_2 - x_3) - (\gamma_\alpha)_{rs} \frac{\partial}{\partial x_{3\alpha}} V(x_1 - x_3, x_2 - x_3), \tag{6}$$

r and s being the spinor indices. S and V are now some scalar functions.

The requirement of gauge invariance restricts only S to*

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* An alternative condition which we could take is $\partial S/\partial x_{3\alpha} = 0$, but we consider it too queer to use.[4]

$$S(x_1-x_3,\ x_2-x_3)=\delta^4(x_1-x_3)\delta^4(x_2-x_3). \tag{7}$$

The modified interaction then becomes

$$\bar{L}_I=i\varepsilon\int dx\,\bar{e}(x)\gamma_\alpha e(x)A_\alpha(x)$$
$$-\frac{i\varepsilon}{2}\int dx_1dx_2dx_3\bar{e}(x_1)\gamma_\alpha\gamma_\beta e(x_2)F_{\alpha\beta}(x_3)\exp\left[-i\varepsilon\int_{x_2}^{x_1}dx_\tau A_\tau(x)\right]V(x_1-x_3,\ x_2-x_3), \tag{8}$$

where $F_{\alpha\beta}$ is the electromagnetic force.

The self-mass of the electron will be produced by the last term, as this term destroys the invariance under the transformation (2), while the first term is again invariant. Since the main contribution to the radiative mass comes from the term similar to Pauli's, its magnitude is closely related to the experimental accuracy of the anomalous magnetic moment.

In the lowest order approximation, the contribution to the mass comes only from the cross-terms of both interaction terms, and it is given by

$$\delta m^{(m)}\simeq\alpha\frac{3}{8\pi}\int dadbdc\frac{ac}{a-c}\log\frac{a}{c}\cdot\left\{v(a,b,c)+v(b,c,a)\right\}, \tag{9}$$

where v is the spectrum function, i.e.

$$V(x_1-x_3,\ x_2-x_3)=(2\pi)^{-8}\int dkdl\cdot\exp[ik(x_1-x_3)+il(x_2-x_3)]v(k^2,\ l^2,\ (k+l)^2),$$

$$v(k^2,\ l^2,\ (k+l)^2)=\int dadbdc\frac{abc}{(k^2+a)\,(l^2+b)\,((k+l)^2+c)}v(a,\ b,\ c). \tag{10}$$

Though our aim is to point out some qualitative possibility of accounting for the elactron mass in terms of the interaction (8) we can roughly estimate the magnitude of $\delta m^{(m)}$. If we take the magnitude of the new term at a very low photon energy (which is equivalent to the introduction of Pauli's term),

$$\int da\,db\,dc\,v(a,\ b,\ c)=\delta/m_e, \tag{11}$$

as large as the possible deviation of the experimental value of the electron moment from the theoretical one, for instance, $10,^{-5}$* and replace the form factor to some cut-off factor with the cut-off energy λ, we have

$$\delta m^{(m)}\simeq\alpha\frac{3}{8\pi}\frac{\delta}{m_e}\lambda^2. \tag{12}$$

* The present situation of the experimental and theoretical values of the electron moment μ is as follows:

$(\mu_e/\mu_0)_{\text{th.}}=1.0011596^{5)}$, $(\mu_e/\mu_0)_{\text{exp}}=1.001146\pm0.000012$, $1.001165\pm0.000011.^{6)}$

This gives the whole electron mass by using the cut-off energy $\lambda \simeq 10^4$ Mev, a value which has not of unreasonable order of magnitude and is more familiar to the classical electron theory. The smaller δ becomes, the greater the cut-off energy must be taken. This means that we have a possibility of deducing the electron mass without destroying the usual successes of quantum-electrodynamics.

§ 3. Weak interactions as the mass origin

The second possibility is to take into account all existing interactions of the electron. If the electron, having no bare mass, cannot get any reaction mass through the local electromagnetic interaction, we must consider the possibility of getting it from the masses of other elementary particles with which the electron interacts through weak interactions, that is, the nucleon (baryons) can "share" its mass with the electron through the β-interaction, and the muon can do it through the muon-electron interaction. The assumption of the pre-existence of the nucleon and muon masses is made for the sake of simplicity. The possibility of explaining the nucleon mass through the pion-interaction still remains, but the application of a similar argument to the muon seems to us to be rather questionable.* Some of us treated the problem that the large mass difference between the muon and the electron may be closely related to the inner structure of the leptons.[7]

If we introduce weak interactions and assume the pre-existence of the nucleon (baryons) and the muon masses, the whole system is no more invariant under the transformation (2), which means the revival of the possibility of getting the radiative mass of the electron. But the possibility depends on the conditions imposed on the weak interactions.

For the purpose of getting such conditions, it is convenient to separate the field quantities of all elementary particles in eigenstates of the chirality:

$$\Psi = \Psi^{(+)} + \Psi^{(-)}, \qquad \Psi^{(\pm)} = (1 \pm \gamma_5)/2 \cdot \Psi. \tag{13}$$

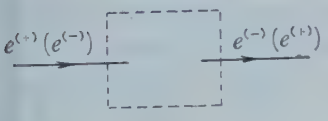

Fig. 1.

Then the invariance (or non-invariance) of the bilinear quantities of spinors under the transformation (2) corresponds to no chirality change (or chirality change).

The desired reactive mass term, as indicaten in Fig. 1, belongs to the case of a change of chirality. In this diagram, the chirality must change an odd number of times as the electron changes to other particles, i.e. to muon and neutrino, or to electron itself, through weak interactions.** It

* A. Salam and J. Tiomno favor the opposite standpoint and assumd the existence of a strong interaction for the muon. If such interaction does exist, why it is not detected by the cosmic ray phenomena seems to be a problem. The authors express much thanks to them for sending their manuscript before publication.

** We take the triangle scheme in which the electron and the muon appear together with the neutrinos. The electron cannot be changed to the muon without the mediation of the neutrino. For this scheme, it is more suitable to take the charge exchange order $(\bar{\mu}O\nu)$ $(\bar{\nu}O'e)$ rather than the charge retaintion order $(\bar{\mu}Oe)$ $(\bar{\nu}O'\nu)$ usually considered in the literature.

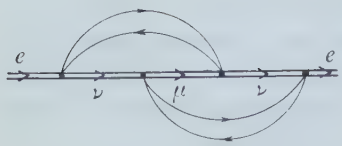

Fig. 2.

is sufficient to focus our attention concerning the chirality changes only to the "course" starting from the initial electron, which does not include any closed loop; this we call the "associated line". We give one example in Fig. 2, the associated line being denoted by the double line.

The chirality changes in the associated line are connected not only to the interactions, but also to the muon propagator.* This propagator can be splitted, according to the chirality change, as follows:

$$\langle T(\mu^{(\pm)}(x)\overline{\mu^{(\pm)}}(y))\rangle_0=\frac{1}{2}\cdot\frac{1\pm\gamma_5}{2}\gamma\partial\Delta_F(x-y;\ \mu)\equiv\frac{1}{2}K_0^{(\pm)}(x-y;\ \mu)$$

and (14)

$$\langle T(\mu^{(\pm)}(x)\overline{\mu^{(\mp)}}(y))\rangle_0=-\frac{1}{2}\cdot\frac{1\pm\gamma_5}{2}m_\mu\Delta_F(x-y;\ \mu)\equiv\frac{1}{2}K_1^{(\pm,\mp)}(x-y;\ \mu),$$

where K_0 and K_1 are the chirality conserving and chirality non-conserving parts of the muon propagator, respectively.

All interactions, i.e., the electromagnetic interaction and the weak interactions with the five couplings can be classified in much the same way as the propagators.

Then we have the following conditions for obtaining the reactive mass:

$$\begin{aligned}&N_S+N_V+N_T+n_{\mu 1}=\text{odd},\\&N_S+N_V+N_T+N_V+N_A=\text{even},\end{aligned}\qquad(15)$$

where N_x is the number of times the weak interaction with x-coupling appears at the vertices on the associated line, and $n_{\mu 1}$ gives the number of K_1 propagators of the muon.

From these conditions, we can easily draw the following conclusions:

i) When the weak interactions with the restricted form as stated in the footnote in p. 821 have only V and A couplings, the electron must change to the muon an odd number of times in the intermediate states in order to get the non-zero reactive mass. And if the muon also has no bare mass, V and A theory cannot give any mass for the lepton.

ii) When the weak interactions have only S, T and P couplings, the circumstances are similar to the above ones.

iii) The possibility of getting the electron mass, without the intervention of the muon along the associated line, is restricted to the coexistence of V, A and S, T, P couplings. Here the electron mass is given from the sources of the masses

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* The electron mass appears in the propagator, owing to the renormalization technique. But this effect is completely cancelled out by the counter self-mass term. Therefore we may consider neither the K_1-part, nor the center self-mass term of the electron. This argument also can apply to the reactive part of the other elementary particle mass.

of other elementary particles, nucleon (baryons) and muon, which intervene in the closed loops.

In order to realize the first two cases, the assumption of a four-component neutrino is necessary, while the last case holds either in a four-component neutrino theory or in a two-component one.

Though we cannot draw any quantitatively definite answer owing to the present unsatisfactory situation in quantum field theory, we try to estimate the above result by using perturbation theory.

From the recent experimental results of beta and pion decays, the allowable ratios of the mixtures of S, T and P coupling to the main V and A couplings would be

$$|C_S|,\ |C_S'|,\ |C_T|,\ |C_T'|,\ |C_P|,\ |C_P'| \lesssim 10^{-2}\cdot(|C_V|,\ |C_V'|,\ |C_A|,\ |C_A'|), \tag{16}$$

where the coupling constants C_X and C_X' are defined through

$$H_\beta = \sum_X (\bar{P}\, O_X\, N)\,(\bar{e} O_X (C_X + \gamma_5 C_X')\,\nu) + \text{h.c.} \tag{17}$$

as usual in the literature. The main contribution to the electron mass comes from the case iii), the effective square of the coupling constant being proportional to 10^{-100} $\text{erg}^2\cdot\text{cm}^6$ rather than to the usual one, i.e. $10^{-98}\,\text{erg}^2\cdot\text{cm}^6$. Thus the reactive mass of the electron through the β-interaction is given by

$$\delta m^{(N)} \simeq \frac{2}{(2\pi)^4}\tilde{g}^2 M_n \lambda^4 \log\frac{\lambda}{M_n},$$

$$\tilde{g}^2 = -\tfrac{1}{4}\Big[(C_V C_S + C_V' C_S') - 3(C_T C_A + C_T' C_A')\Big]. \tag{18}$$

If we extend this consideration so as to include the whole contributions of baryon's, a pair of baryons with masses M_1 and M_2 gives

$$\delta m^{(B_1,B_2)} \simeq \frac{2}{(2\pi)^4}\Big[\tilde{g}_1^{\,2} M_1 \lambda^4 \log\frac{\lambda}{M_1} + \tilde{g}_2^{\,2} M_2 \lambda^4 \log\frac{\lambda}{M_2}\Big],$$

$$\tilde{g}_1^{\,2} = -\tfrac{1}{8}\Big[(C_V C_S + C_V' C_S') - (C_A C_P + C_A' C_P') - 3(C_T C_V + C_T' C_V') - 3(C_T C_A + C_T C_A')\Big], \tag{19}$$

$$\tilde{g}_2^{\,2} = -\tfrac{1}{8}\Big[(C_V C_S + C_V' C_S') + (C_A C_P + C_A' C_P') + 3(C_T C_V + C_T' C_V') - 3(C_T C_A + C_T C_A')\Big].$$

The reactive mass of the electron is also given through the muon-electron interaction. The ratio of the S, T and P couplings to the main V and A couplings of this interaction depend on the electron spectrum of the muon decay.

The small deviation of the Michel parameter from 0.75[8] shows that the interaction could be

$$H_\mu=(\bar{e}\gamma_\alpha\mu)\,(\bar{\nu}\gamma_\alpha(D_V+\gamma_5D_V')\nu)-(\bar{e}\gamma_\alpha\gamma_5\mu)\,(\bar{\nu}\gamma_\alpha\gamma_5(D_A+\gamma_5D_A')\nu)+\text{h.c.} \tag{20}$$

and the experimental accuracy of the asymmetry parameter[9] does not forbid the difference between D_V and $-D_A$ within 10% to 50%.[7] If we rewrite the Hamiltonian in the order we are here considering, we have

$$\begin{aligned}H_\mu=&(\bar{\nu}\mu)\,(\bar{e}(C_S+\gamma_5C_S')\nu)+(\bar{\nu}\gamma_5\mu)\,(\bar{e}\gamma_5(C_P+\gamma_5C_P')\nu)\\&+(\bar{\nu}\gamma_\alpha\mu)\,(\bar{e}\gamma_\alpha(C_V+\gamma_5C_V')\nu)-(\bar{\nu}\gamma_\alpha\gamma_5\mu)\,(\bar{e}\gamma_\alpha\gamma_5(C_A+\gamma_5C_A')\nu)+\text{h.c.}\end{aligned} \tag{21}$$

with

$$\begin{aligned}&C_S=-C_P=\pm(D_V+D_A),\qquad C_S'=-C_P'=\pm(D_V'+D_A'),\\&C_V=-C_A=\mp\tfrac{1}{2}(D_V-D_A),\qquad C_V'=-C_A'=\mp\tfrac{1}{2}(D_A'-D_A'),\end{aligned} \tag{22}$$

where the upper and lower signs correspond respectively to the assumptions of commutation and anti-commutation of the fields. Then we can take

$$\begin{aligned}&|C_S|,\ |C_S'|,\ |C_P|,\ |C_P'|\lesssim 10^{-1}(|C_V|,\ |C_V|,\ |C_A'|,\ |C_A'|),\\&|C_T|,\ |C_T'|\simeq 0\end{aligned} \tag{23}$$

within the experimental accuracy. The contribution to the electron mass is given by

$$\begin{aligned}&\delta m^{(\mu)}\simeq\frac{2}{(2\pi)^4}\tilde{f}^2m_\mu\lambda^4\log\frac{\lambda}{m_\mu},\\&\tilde{f}^2=-\tfrac{1}{8}\Big[(C_VC_S+C_V'C_S')+(C_AC_P+C_A'C_P')\Big],\end{aligned} \tag{24}$$

the effective square of the coupling being $\tilde{f}^2\simeq 10^{-99}\,\text{erg}^2\cdot\text{cm}^6$ instead of $10^{-98}\,\text{erg}^2\cdot\text{cm}^6$.

The contributions to the electron mass from the cases i) and ii) become of higher order and sensitively depend on the existence of the four-component neutrino. The assumption of the four-component neutrino gives rise to the reactive mass for the neutrino. For instance, the case iii) gives

$$\begin{aligned}&\delta m_\nu^{(N)}\simeq\frac{2}{(2\pi)^4}\Big[g'^2M_n\lambda^4\log\frac{\lambda}{M_n}+f'^2m_\mu\lambda^4\log\frac{\lambda}{m_\mu}\Big],\\&g'^2=-\tfrac{1}{4}(C_VC_S-C_V'C_S'),\qquad\qquad\text{for }\beta\text{-coupling},\\&f'^2=-\tfrac{1}{8}\Big[(C_VC_S-C_V'C_S')+(C_AC_P-C_A'C_P')\Big],\quad\text{for }\mu\text{-e-coupling},\end{aligned} \tag{25}$$

which becomes zero in the case of the complete polarization of the neutrino.

§ 4. Semi-quantitative estimates of the electron mass and discussions

The main aim of this paper is only to show the possibilities of getting the reactive mass of the electron through the elementary interactions. Since the order of its magnitude depends fundamentally on the validity of quantum field theory in the high energy regions, we are still on speculative grounds. If, nevertheless, we dare to indicate some rough estimates, we must discuss every possible case according to the probable limitations of field theory.

The applicability of the present quantum-electrodynamics seems to be assured experimentally upto 10^3 Mev, but around $10^4 \sim 10^5$ Mev, a region which is now being considered mainly in connection to the electron-nucleon scattering[9]. Unfortunately, if quantum field theory would break down in this region, we could not get a satisfactory answer for the reactive origin of the electron mass from the usual formulation, because quantum-electrodynamics cannot give any contribution and the contributions from weak interactions become very small. Nevertheless, the modifications of field theory would remove the difficulty from two aspects. The non-local modification of quantum-electrodynamics would give a good answer as sketched in § 2. And even if the contributions from the weak interaction were so small as to give the experimentally undetectable mass to the electron as to the neutrino for instance, 10^2 eV[18], the new quantum-electrodynamics might add sufficient contributions only to the electron mass so as to distinguish it from the neutrino's. This is only an expectation, but it seems to be a probable one, because there is no difference between the electron and the neutrino except by the charge.

Fortunately, if quantum field theory holds over 10^5 Mev, or still has some meaning for such energies, the usual quantum-electrodynamics would be much modified by the interventions of weak interactions*, the contribution to the electron mass from the latter becomes stronger than the former at energies of $\lambda \gtrsim \alpha^{1/2}/\sqrt{g}$, where g means the order of magnitude of the weak coupling. And from (18), (19) and (24), we can see that the whole electron mass can be explained only by weak interactions with the cut-off energy $\lambda \simeq 10^6$ Mev. While the contribution to the neutrino mass is quadratically proortional to the deviation from the two-component assumption. If the two-compopnent (left-handed) neutrino is realized in nature, the weak interactions give the direct answer to the origin of the electron mass, at the region where weak interactions become predominant over the electromagnetic one.

The arguments presented here depend on many assumptions. The important one is the necessity of a modification of quantum-electrodynamics or the existence of the S, T and P couplings in the β-decay or the muon decay. Unfortunately, if these cases would be completely eliminated, the question of the origin of the

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* One of the authors presented this possibility: M. Taketani, Report at the Meeting of the Physical Society of Japan, Kanazawa, 1957.
Also a similar argument has been proposed by D. I. Blohincev, Nuovo Cimento **9** (1959), 925.

electron mass would become mysterious problem than ever. The small deviation from the two-component neutrino and the existence of the muon or nucleon mass are also important assumptions. We believe that more profound considerations are necessary for these problems.

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Global Symmetry and a Connection in Iso-Space

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Yang and Mills' idea of generalized rotation group in iso-space is extended in such a way that one of the bose fields, which are newly introduced there, can be indentified with π-meson field. In our theory iso-space becomes to have some correlation with ordinary space-time. Gell-Mann's global symmetry results from our theory and the type of interaction between π-meson and hyperon is restricted to pseudo-vector coupling.

§ 1. Introduction

Recently, some tentative theories have been proposed by many authors to explain various characteristic features of the phenomena inclusively, in which the so-called strange particles take part, and there the analogy to electrodynamics plays an important role generally. For instance, Schwinger[1] proposed to consider that π- and K-meson fields characterize the nucleonic charge and the hyperonic charge respectively, in a similar manner as photon field does the electric charge. Further he expressed every kind of charge by a third member of a set of Pauli type matrices.[2] This corresponds to the situation that electromagnetic field specializes the third axis in iso-space. Such an idea includes also the universality of the coupling constants between bosons and baryons, and the universality is treated in a more phenomenological way by Gell-Mann[3]. Here we may emphasize that Gell-Mann's global symmetry means the universality of the coupling constant between π-meson and baryons, and that it requires a more strong symmetry than charge independence.

For π-meson field, the correlation between electromagnetic field and the gauge transformation, which is another important character of electrodynamics, is followed by Yang and Mills[4]. But they failed to establish the parallelism, as the boson field, which was introduced by the existence of their generalized rotation group in iso-space, has the vector character in both Minkowski's space and iso-space. The introduction of the generalized rotation in iso-space is equivalent giving mapping group of the representation space of the group which is generated in iso-space, as will be seen in § 2. In the following we call such a mapping Yang and Mills' *connection of iso-space*.* As far as the connection is considered to be independent

* R. Utiyama[5] already pointed out that, considering from the point of view of invariance theory, Yang and Mills' theory had distinct formal similarity to general relativity theory.

in any sense of Minkowski's space, we cannot make any further progress in this direction.

Such attempts following the example of electrodynamics have their origin in the facts that electromagnetic field is most familiar to us and that it has resulted in brilliant success. Further, it seems to be natural that we expect for boson field to play some similar role to electromagnetic field.

On the other hand, there are provisional theories to make iso-space correlate to Minkowski's one by some possibilities. Klein[6] attempted to take the degree of freedom of isotopic spin out of Kaluza-Klein's unified field theory in a five dimensional space. Further, Okabayashi[7] recently proposed to construct a field theory in such a way that there exists some correlation between our ordinary space-time and iso-space, by analysing the characteristic features in the weak and strong interactions.

In this article, we want to try to improve Yang and Mills' connection of iso-space by taking into account the possibility of the correlation mentioned above. Section 2 is devoted to explain briefly the concept of connection[8] which is fundamental in differential geometry and to rewrite Yang and Mills' theory in a more convenient form to the latter extension. In § 3 we propose to modify Yang and Mills' connection in such a way that π-meson field plays the desired role. In the general theory of relativity the concept of connection is fully used to construct unified theories[9]. The possibility to extend our theory in such a direction is discussed in the Appendix.

§ 2. Connection of iso-space

The concept of *connection* may be understood as follows. First, a representation space $E(P)$ of a certain group $\mathcal{G}$, which may be a spinor-space or a tensor-space, is attached at each space time point P. Second, a mapping from $E(P)$ to $E(P+dP)$ is given, where $P+dP$ is a point in the neighborhood of P. The mapping is required to form a mapping group G.* Namely, the connection is characterized by the space $E(P)$ and the mapping group G. For example, if $E(P)$ is a vector space and G is the affine group, the connection is an *affine conection* and the mapping is called a parallel-displacement, and if $E(P)$ is projective space and G is the projective group, the connection is a *projective connection*.

In Yang and Mills' case, $E(P)$ is the space of iso-spinor $\psi(P)$ and G is given by a rotation group in the iso-space. The connection is defined by

$$\psi'(P+dP)=\psi(P)+\Gamma_\mu(P)\psi(P)dx^\mu,$$

where

$$\Gamma_\mu(P)=iH_\mu^\alpha(P)\tau_\alpha, \tag{1}$$

* G happens to be identical with $\mathcal{G}$ in many cases, for example, in Yang and Mills' case. But generally they may be different from each other as will be seen in the next section.

and the generating operators are $i\tau_\alpha$'s. Here x^μ is the coordinates of point P and $\psi'(P+dP)$ is the image of $\psi(P)$ at $P+dP$ by the mapping. The connection of a vector space is determined usually by that of the spinor space, by taking account of the situation that $V_\alpha \equiv \psi^* \tau_\alpha \psi$ is a iso-vector. Then the connection of V_α is given by

$$V'_\alpha(P+dP) = V_\alpha(P) + \Gamma^\beta_{\mu\alpha}(P)\, V_\beta(P)\, dx^\mu,$$

where (2)

$$\Gamma^\beta_{\mu\alpha}(P) = \tfrac{1}{2}\,\mathrm{spur}\,[\tau_\beta(\tau_\alpha \Gamma_\mu(P) - \Gamma_\mu(P)\tau_\alpha)].$$

On the other hand, in order to discuss Yang and Mills' generalized charge independence, we must consider the rotation of iso-space which may be different at each space time point,

$$\psi(P) \to S(P)\psi(P).$$

This rotation transforms Γ_μ also, and the connection for the transformed spinor $S\psi$ becomes

$$(S\psi')_{p+dp} = (S\psi)_p + \Gamma^{(s)}_\mu(P)\,(S\psi)_p\, dx^\mu,$$

where $\Gamma^{(s)}_\mu$ is the transformed Γ_μ. $S^{-1}(P+dP)S(P)$ is to have the form

$$\mathbf{1} + iH^\alpha_\mu(P)\tau_\alpha dx^\mu,$$

as it is also a infinitesimal rotation operator, and we get the following formula

$$\Gamma^{(s)}_\mu = S(\Gamma_\mu - iH^\alpha_\mu \tau_\alpha)S^{-1}. \tag{3}$$

(We omit P in the following.) Then, if we define iso-covariant derivative as

$$\nabla_\mu \psi = \partial\psi/\partial x_\mu - \Gamma_\mu \psi \qquad \text{(for spinor)}$$

and

$$\nabla_\mu V_\alpha = \partial V_\alpha/\partial x_\mu - \Gamma^\beta_{\mu\alpha} V_\beta \qquad \text{(for vector)},$$

$\nabla_\mu\psi$ and $\nabla_\mu V_\alpha$ are the spinor and the vector of iso-space respectively. The quantities $\bar\psi\gamma_\mu\nabla_\mu\psi$ and $\bar\psi\psi$ (where $\bar\psi = \psi^*\gamma_4$) are invariant with respect to the transformation $S(P)$. Further we have another invariant,

$$R = R_{\mu\nu}R_{\mu\nu}, \tag{4}$$

where $R_{\mu\nu}$ is the so-called Riemann Christoffel tensor. [Appendix A] $R_{\mu\nu}$ is given explicitly as

$$R_{\mu\nu} = \partial\Gamma_\mu/\partial x^\nu - \partial\Gamma_\nu/\partial x^\mu + \Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu. \tag{4}'$$

Thus we have the Lagrangian, which is invariant in Yang and Mills' sense,

$$\mathcal{L} = -(1/4f^2)R - \bar\psi[\gamma_\mu\nabla_\mu + M]\psi,$$

where f, M are constants.

If we put

$$\Pi_\mu^\alpha(P) = f\chi_\mu^\alpha(P),$$

the coupling between the χ_μ^α-field and ψ-field becomes $f\chi_\mu^\alpha\bar{\psi}\gamma_\mu\tau_\alpha\psi$, and χ_μ^α must be a vector field with vector coupling and f is the coupling constant. The χ_μ^α-field and $1/f\cdot R_{\mu\nu}$ correspond respectively to $\boldsymbol{b}_\mu$-field and $F_{\mu\nu}$ in Yang and Mills' article.

§ 3. An extension of the connection of iso-space

In the previous section, we used spinor space or vector space with respect to isotopic spin as $E(P)$ and the mapping group G belonged to the rotation group in iso-space which was just the group $\mathcal{G}$. However, if we take into account Okabayashi and Klein's idea, we must use the direct product $E^I(P)\times E^M(P)$ as $E(P)$, where $E^I(P)$ and $E^M(P)$ are the representation spaces in iso-space and Minkowski's spce respectively, and the mapping group G should be different from $\mathcal{G}$ contrary to the previous example. Here we must assume that G is invariant as a whole under proper Lorentz transformation. In the following, we restrict the group G more strictly so that it commutes with proper Lorentz transformation. (This means that each element of G is invariant under the transformation.) Further, for instance, the quantity which has the spinor character in iso-space and the vector character in ordinary space we name *spinor-vector*.

In order that, except for the simple rotation group in iso-space, there exists the mapping group G which commute with proper Lorentz transformation, $E^M(P)$ must be the space of the spinor fields with half-integer spin. Because, as to the quantities with integer spin there is no matrix which commute with a proper Lorentz group, contrary to the spinor case where γ_5 matrix has this property. Further, scalar-spinor fields are also excluded, as the group G then cannot have any correlation with iso-space.

Now, we must treat not only spinor quantity but also tensor quantities. In the previous section, it has been assumed that the connection of vector quantity is induced by the one of spinor field. Usually, such a fusion technique is fully used, but it may be considered to be only conventional from the point of view of the theory of connection.* Therefore, we try to give their connections independently and to set up some correlation between the connections in such a way that the desired physical results are obtained. Then, the relations established by the fusion technique may be disturbed by the connection, but here we do not enter into this situation.

Thus we assume that only spinor-spinor fields have some generalized connection mentioned above and that the connection of other physical quantities are given by the simple rotation group in iso-space as in § 2. Then the following three groups are permissible for spinor-spinor fields,

---

* For instance, Einstein and Mayer started from a projection technique instead of fusion technique.[10]

G_R: the group whose generating operator is $i\tau_\alpha(1-\gamma_5)$,

G_L: the group whose generating operator is $i\tau_\alpha(1+\gamma_5)$,

G_T: the group whose generating operator is $i\tau_\alpha, i\tau_\alpha\gamma_5$.

If we use G_R, Γ_μ has the form

$$\Gamma_\mu = i\Pi_\mu^\alpha \tau_\alpha(1-\gamma_5). \tag{5}$$

But this form is not kept invariant under the generalized rotation $S(P)$. Because the transformed $\Gamma_\mu^{(s)}$ contains the term which is proportional to τ_α but does not accompany the factor $(1-\gamma_5)$, as will be seen by repeating the similar calculation as in the previous section. Therefore, this group is excluded as long as we require the invariance of the theory under the generalized rotation $S(P)$. Similarly, G_L is excluded, too. Accordingly, for spinor-spinor fields we use G_T which is the direct sum of G_R and G_L. Then the connections are written as follows,

$$\psi'(P+dP) = \psi(P) + L_\mu(P)\psi(P)dx^\mu,$$

(for spinor-spinor)

$$\psi'(P+dP) = \psi(P) + \Gamma_\mu(P)\psi(P)dx^\mu,$$

(for spinor-scalar, spinor-vector, and etc.)

$$V'_\alpha(P+dP) = V_\alpha(P) + \Gamma^\beta_{\mu\alpha}(P)\, V_\beta(P)dx^\mu,$$

(for vector-scalar, vector-vector, etc.)

where

$$L_\mu = i\theta_\mu^\alpha \tau_\alpha + i\Lambda_\mu^\alpha \tau_\alpha\gamma_5,$$

and

$$\Gamma_\mu = i\Pi_\mu^\alpha \tau_\alpha.$$

Next we must set up a correlation between these connections. We assume as

$$\theta_\mu^\alpha = \Pi_\mu^\alpha$$

and

$$\Gamma^\beta_{\mu\alpha} = \tfrac{1}{2}\,\mathrm{spur}\,[\tau_\beta(\tau_\alpha\Gamma_\mu - \Gamma_\mu\tau_\alpha)].^* \tag{6}$$

The covariant derivative is to be defined as follows,

$\nabla_\mu\psi = \partial\psi/\partial x^\mu - L_\mu\psi$ (for spinor-spinor),

$\nabla_\mu\psi = \partial\psi/\partial x^\mu - \Gamma_\mu\psi$ (for spinor-scalar, spinor-vector, etc.),

$\nabla_\mu V_\alpha = \partial V_\alpha/\partial x^\mu - \Gamma^\beta_{\mu\alpha}V_\beta$ (for vector-scalar, vector-vector, etc.).

---

* The first equation shows that this correlation is a projection of L_μ to Γ_μ and the second equation has the same meaning as in § 2.

$\Gamma_\mu\psi$, $\Gamma_\mu V_\alpha$ are iso-spinor and iso-vector, respectively.

We can construct the so-called Riemann-Christoffell tensor in terms of the quantities L_μ, Γ_μ and $\Gamma^3_{\mu\alpha}$. But, for our purpose, it is more relevant to use a new vector-scalar field φ^α covariantly defined by

$$\Lambda^\alpha_\mu = g\nabla_\mu\varphi^\alpha \tag{7}$$

instead of the vector-vector field Λ^α_μ, as will be seen soon. As to the Π^α_μ-field, we cannot give any iso-covariant equation similar to the equation (7), as Π^α_μ is not iso-vector. Thus the invariants which involve φ^α, $\chi^\alpha_\mu (=f^{-1}\Pi^\alpha_\mu)$ and their first derivatives are

$$R, \quad \nabla_\mu\varphi^\alpha\nabla_\mu\varphi^\alpha, \quad \varphi^\alpha\varphi^\alpha,$$

and we can take Lagrangian $\mathcal{L}$ as

$$\mathcal{L} = -\frac{R}{4f^2} - \frac{1}{2}[\nabla_\mu\varphi^\alpha\nabla_\mu\varphi^\alpha + m^2\varphi^\alpha\varphi^\alpha] - \bar{\psi}[\gamma_\mu\nabla_\mu + M]\psi.$$

Here, R is given by Equation (4).

The interaction between φ^α and ψ is given by

$$ig\frac{\partial\varphi^\alpha}{\partial x^\mu}\bar{\psi}\gamma_\mu\gamma_5\tau_\alpha\psi + ig\Gamma^\alpha_{\mu 3}\varphi^3\bar{\psi}\gamma_\mu\gamma_5\tau_\alpha\psi,$$

and the first term has the same structure as the ordinary coupling between nucleon and π-meson. Therefore, we can identify the φ^α-field with π-meson field. Further, ψ may be considered as the field operator of every kind of hyperon which is rearranged to iso-doublet. Thus, we get also the global symmetry. We have no physical field to be identified with the χ^α_μ-field which is vector in Minkowski's space and has no definite tensor character in iso-space. Therefore, the coupling constant f may be considered to be so small that the χ^α_μ-field is not observed. Then the second term of the above interaction can also be neglected, as $\Gamma^\alpha_{\mu\beta}$ is proportional to f.

§ 4. Discussion

The invariance under the generalized rotation in iso-space introduces the π-meson field in a similar way as the second kind gauge transformation is correlated to the electromagnetic field. There the following points are essential. First, the group of the mapping G has some correlation with Minkowski's space, even if it commutes with proper Lorentz transformation. Second, the correlation between the connections of spinor and vector field is released from the so-called fusion technique. There are many kinds of possibility to assume the correlation, and the one used in the previous section is only an example adequate for our physical purpose.

It may be emphasized that our theory not only gives a universal coupling constant for the interactions between π-meson and baryons, (i. e. global spmmetry),

but also restrict the type of the interaction to pseudo-vector coupling. It may be considered as a support to our theory that it unifies three characteristic features of the interactions of π-mesons, though they are not yet established experimentally.

As to the χ_μ^α field, the mass term is not permissible in the Lagrangian, because it is not invariant under the transformation $S(P)$. Therefore the range of the effect of the χ_μ^α field must be as long as the electromagnetic field, and the effect must be observed macroscopically, if the coupling constant is not zero.

In the previous section the mapping group G_R and G_L are excluded by the reason why they do not allow the generalized rotation invariance. If we are contented with the ordinary rotation in iso-space with constant parameter S instead of $S(P)$, they are also permissible, and we can include the interaction of β-decay of hyperons into our scheme as follows. We begin by assuming

i) electron and neutrino form an iso-doublet $\psi^{(e)}$ and its mapping group is given by G_R,

ii) all hyperons are iso-doublets[8] and their mapping group is G_T,

iii) there is no quantity which has the mapping group G_L. (Here μ-meson cannot be taken into account.)

Then the connection coefficient for hyperons and electron-neutrino doublet are given by

$$L_\mu^{(h)}=i\,\Pi_\mu^\alpha\tau_\alpha+i\,\Lambda_\mu^\alpha\tau_\alpha\gamma_5$$

and

$$L_\mu^{(e)}=i\sum_\mu^\alpha\tau_\alpha(1-\gamma_5).$$

Further, we assume

$$\Pi_\mu^\alpha=\sum_\mu^\alpha,$$

and rewrite $\Lambda_\mu^\alpha+\Pi_\mu^\alpha$ as Φ_μ^α. Then $L_\mu^{(h)}$ can be rearranged as

$$L_\mu^{(h)}=i\sum_\mu^\alpha\tau_\alpha(1-\gamma_5)+i\Phi_\mu^\alpha\tau_\alpha\gamma_5,$$

and $L_\mu^{(e)}$ becomes a projection of $L_\mu^{(h)}$. If we put

$$\Phi_\mu^\alpha=g\cdot\partial\varphi^\alpha/\partial x^\mu$$

and

$$\sum_\mu^\alpha=f\,A_\mu^\alpha,$$

φ^α can be identified with the π-meson field as in the previous section and A_μ^α is a vector-vector field. The A_μ^α-field may be interpreted as the similar field introduced by Schwinger[2] and Feynman and Gell-Mann[11)12)]. If the interactions of the β-decay of hyperons are considered as the secondary effect through the intermediary of the A_μ^α field, they must be of V-A type and have the same coupling constant:

$$\frac{f^2}{M^2}\,[\bar{\psi}\gamma_\mu(\mathbf{1}-\gamma_5)\tau_\alpha\psi]\cdot[\bar{\psi}^{(e)}\gamma_\mu(\mathbf{1}-\gamma_5)\tau_\alpha\psi^{(e)}].$$

Before concluding this section, we may remark that we can easily connect this theory to the general theory of relativity. For this purpose, we may extend the mapping group G in such a way that G is a direct product of a proper Lorentz group G_{L0} and some group G_c which is commutable with G_{L0}. Then we can apply Fock's theory to this G_{L0}, as will be shown in Appendix B. Here we may remark that in this case the group G does not commute with the proper Lorentz group.

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Appendix A

For completeness, we here give the derivation of the Riemann-Christoffel tensor and explain its geometrical meaning,

The geometrical meaning of the Riemann-Christoffel tensor is as follows. If we displace a spinor $\psi(P)$ in accordance with the rule of a connection and make a round along the circuit as shown in Fig. I, starting from a point P, $P\to P+\delta P\to P+\delta P+dP\to P+dP\to P$, the finally obtained spinor, $\psi'(P)$, is generally different from the initial one. We can express the transformation as

$$\psi'(P)=(\mathbf{1}+\tfrac{1}{2}R_{\mu\nu}\cdot\delta x^{[\nu}dx^{\mu]})\psi(P).$$

Tensor $R_{\mu\nu}$ thus defined is the so-called Riemann-Christoffel tensor,

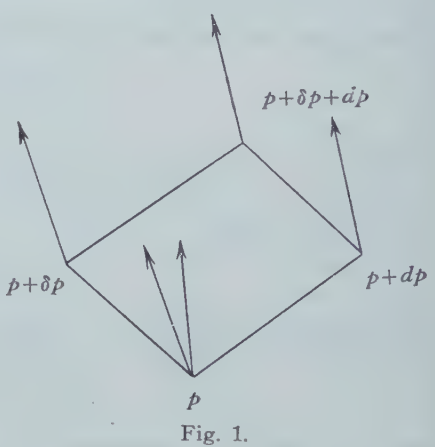

Fig. 1.

The spinor $\psi'(P+dP)$ at the point $P+dP$ can be expressed by the spinor at the point P as

$$\psi'(P+dP)=\psi(P)+d\psi(P).$$

Then, Equation (1) which defines the connection coefficient $\Gamma_\mu(P)$ can be interpreted as

$$d\psi(P)=(\partial\psi(P)/\partial x^\mu-\Gamma_\mu(P)\psi(P))\,dx^\mu.$$

If we displace $\psi(P+\delta P+dP)$ to P through the following path, $P+\delta P+dP\to P+dP\to P$, we get

$$\psi(P)+d\psi(P)+\delta\psi(P)+d\delta\psi(P),$$

and if we displace it to P through another path, $P+\delta P+dP\to P+\delta P\to P$, we get

$$\psi(P)+\delta\psi(P)+d\psi(P)+\delta d\psi(P).$$

Therefore, the difference between these quantities is just $\frac{1}{2}R_{\mu\nu}\psi\delta^{[\nu}xdx^{\mu]}$ and is given by

$$d\delta\psi-\delta d\psi=d(\partial\psi/\delta x^{\mu}-\Gamma_{\mu}\psi)\delta x^{\mu}-\delta(\partial\psi/\partial x^{\nu}-\Gamma_{\nu}\psi)dx^{\nu}$$
$$=\tfrac{1}{2}[\partial\Gamma_{\mu}/\partial x^{\nu}-\partial\Gamma_{\nu}/\partial x^{\mu}+\Gamma_{\mu}\Gamma_{\nu}-\Gamma_{\nu}\Gamma_{\mu}]\delta x^{[\nu}dx^{\mu]}\psi.$$

Thus we obtain the expression in Equation (4)′.

Appendix B

In this appendix we want to show it is possible to make our theory melt into Fock's theory about the spinor field in the general theory of relativity. Then it is necessary to use $G_c\times G_{I.0}$ as the mapping group, as was mentioned in § 4, The concerning connection coefficient L_{μ} may be written as

$$L_{\mu}=L_{\mu}^{(c)}+L_{\mu}^{(o)}$$

where $L_{\mu}^{(o)}$ and $L_{\mu}^{(c)}$ are the connection coefficients which correspond to G_c and $G_{I.0}$ respectively, and it is sufficient to give the correlation between the L_{μ} and the connection coefficient in the Riemannian space, the so-called Riemann-Christoffel's symbol $\{^{\lambda}_{\mu\nu}\}$,

The generalized Dirac matrix $\gamma_{\mu}{}^{0}$ attached to an oblique coordinate in the theory of spinor in Riemannian space, which is defined by the commutation relation

$$\{\gamma_{\mu}^{0}\gamma_{\nu}^{0}\}=2g_{\mu\nu},$$

is a function of space-time. Here $g_{\mu\nu}$ is the metric tensor of our space-time. If we make the connection in the Riemannian space following Fock's idea that $\bar{\psi}\gamma_{\mu}^{0}\psi$ is connected as a vector in the space, we must put

$$(\bar{\psi}\gamma_{\mu}^{0}\psi)'_{p+dp}=(\bar{\psi}\gamma_{\mu}^{0}\psi)_{p}+\{^{\lambda}_{\mu\nu}\}(\bar{\psi}\gamma_{\lambda}^{0}\psi)_{p}dx^{\nu},$$
$$(\bar{\psi}\gamma_{\mu}^{0}\psi)'_{p+dp}=\bar{\psi}'(P+dP)\gamma_{\mu}^{0}(P+dP)\psi'(P+dP),$$

and

$$\psi'(P+dP)=\psi(P)+L_{\mu}\psi(P)dx^{\mu}.$$

Therefore, L_{μ} must satisfy the equation

$$\partial\gamma_{\mu}^{0}/\partial x^{\nu}+\gamma_{\mu}^{0}L_{\nu}+\gamma_{4}L_{\nu}^{*}\gamma_{4}\gamma_{\mu}^{0}=\{^{\lambda}_{\mu\nu}\}\gamma_{\lambda}^{0}.$$

Since $L_{\mu}^{(o)}$ is commutable with $\gamma_{4}\gamma_{\mu}^{0}$ and the relations

$$L_{\mu}^{(c)*}=-L_{\mu}^{(c)}$$

and

$$\gamma_{4}L_{\mu}^{(o)*}\gamma_{4}=-L_{\mu}^{(o)}$$

hold, we get

$$\partial\gamma_\mu^0/\partial x^\nu+[\gamma_\mu^0 L_\nu^{(\sigma)}]=\{{}^{\lambda}_{\mu\nu}\}\gamma_\lambda^0$$

$L_\mu^{(\sigma)}$ is determined by this equation uniquely. In the above discussion it is essential that $L_\mu^{(\sigma)}$ is commutable with $\gamma_4\gamma_\mu$, because otherwise the above equation is not satisfied. And this means that $L_\mu^{(\sigma)}$ is commutable with G_{L0}.

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Use of Antisymmetrized Wave Function for Deuteron Stripping Reaction*

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The general S matrix of nuclear reaction is derived by the method of second quantization in order to take account of antisymmetrization of the total system. As an example of the general theory, a deuteron stripping reaction is treated. The ampliude obtained by this method is compared with those obtained by other theories in this reaction. The amplitude of this reaction is discussed for a special target which seems to demonstrate the effect of antisymmetrization most clearly. In the case of $O^{17}(d, p)O^{18}$ the effect of antisymmetrization becomes comparable with the direct part.

§ 1. Introduction

Although many theories of nuclear reaction have hitherto been proposed, most of them have not taken of antisymmetrization of a total wave function based on the Fermi-Dirac statistics except for some cases in which light nuclei were treated. One of the reasons why they did not take account of the Fermi-Dirac statistics is that impractically complicated and tedious calculations were required; in other words, there was no formalism which is suitable to treat the antisymmetrization. So far as we know, however, there are no reasons to justify the neglect of the antisymmetrization.

The first aim of this paper is, therefore, to establish a general formulation of nuclear reactions which is suitable to treat the antisymmetrization. In our method we treat the nucleon as a nucleon field and postulate adequate commutation relations among creation and annihilation operators, so that we can introduce automatically the effect of the antisymmetrization into a reaction S matrix without tedious considerations. This method is familiar to us in treating meson scattering at low

* Based in part upon the thesis submitted by M. Soga to Tokyo University of Education in 1958, in partial fulfillment of the requirements for the degree of D. Sc.

energies by Chew and Low.[1)]

Since Butler proposed the theory of deuteron stripping reaction[2)], numerous theoretical studies and analyses[3)–34)] have deen performed on the deuteron stripping or pick up reactions. They calculated the angular distribution of stripped off nucleons from the external region by the Born approximation neglecting the antisymmetrization, and reproduced the forward peak of the angular distribution. As to the backward shape of the angular distribution Owen and Madansky[24] proposed the heavy particle stripping theory which gave a good agreement with $B^{11}(d, n)C^{12}$ experimental data. They antisymmetrized a loosely bound neutron of $B^{11}$with the neutron of the incoming deuteron. A question arises from their treatment of the final states; one of the states comes from a direct stripping process, the other from a heavy particle stripping process. In the former state the residual nucleus is approximated by the target nucleus + the loosely bound proton, and in the latter by the core of the target nucleus + the bound deuteron. As they are not exactly the same with each other, we cannot expect great interference to occur between two processes. Owen and Madansky avoid this inconsistency by using an arbitrary parameter, but we cannot understand what their parameter means. Recently, French[27)] proposed an approximate formula for an exchange amplitude. He treats the antisymmetrization in the configuration space, but he neglects a part of the total interactions.

The Second aim of this paper is to apply our general formula of nuclear reaction to a deuteron stripping reaction and to discuss the effect of antisymmetrization. If the deuteron energy concerned is not so low that deuteron stripping processes occur mainly in an external region, we may consider antisymmetrization only among a nucleon of the deuteron and loosely bound nucleons of the target. When the deuteron energy is so high that the Born approximation is applicable, or so low that the resonance peaks are observable, we must consider the antisymmetrization of the total system.

In Section 2, we derive the general formula of nuclear reaction including elastic and inelastic scatterings which take account of the antisymmetrization. In Section 3, we apply this method to the deuteron stripping reaction generally. In Section 4, we compare the results obtained by our method with the results by other methods. In Section 5, we apply this method to deuteron stripping reaction of a special target nucleus at moderate energies. We select a target nucleus composed of a hard core and a loosely bound nucleon, which seems most clearly to demonstrate the effect of antisymmetrization on the angular distribution of (d, p) and (d, n) reactions.

The method proposed in this may be applied to many other types of reactions and may be the first step of refinement for the theory of nuclear reactions.

§ 2. General formulation

We assume for simplicity that the total system concerned is composed of N

identical particles. The initial state is composed of two systems of A particles and B particles respectively and the final state is composed of two systems of A' particles and B' particles, where $A+B=A'+B'=N$. The interaction among them is assumed to be written as the sum of two-body interactions which depends only on the relative coordinates and can be expanded in Fourier series. The total Hamiltonian of the system can be written as

$$H=\sum_{i=1}^{N}-\frac{\hbar^2}{2M}\Delta_i+\frac{1}{2}\sum_{i\neq j}^{N}\sum_{j}^{N}V(\boldsymbol{x}_i-\boldsymbol{x}_j)\ . \tag{2·1}$$

As was mentioned in Section 1, it is convenient to introduce the second quantization formalism. In this formalism the nucleon field $\Psi(\boldsymbol{x})$ and the Hamiltonian $\mathcal{H}$ are given by

$$\Psi(\boldsymbol{x})=\sum_{\boldsymbol{k}_n\mu_n}a_{\boldsymbol{k}_n\mu_n}\frac{1}{\sqrt{\Omega}}e^{i\boldsymbol{k}_n\boldsymbol{x}}u(\mu_n)\ , \tag{2·2}$$

$$\mathcal{H}=\sum_{\boldsymbol{k}_n\mu_n}\omega_{\boldsymbol{k}_n}a^*_{\boldsymbol{k}_n\mu_n}a_{\boldsymbol{k}_n\mu_n}+\frac{1}{2}\sum_{q}v_q\sum_{\substack{\boldsymbol{k}_n\boldsymbol{k}_{n'}\\ \mu_n\mu_{n'}}}a^*_{\boldsymbol{k}_n+q\mu_n}a^*_{\boldsymbol{k}_{n'}-q\mu_{n'}}a_{\boldsymbol{k}_{n'}\mu_{n'}}a_{\boldsymbol{k}_n\mu_n}\ , \tag{2·3}$$

where

$$\omega_{\boldsymbol{k}_n}=\frac{\hbar^2\boldsymbol{k}_n^2}{2M}\ ,\qquad v_q=\frac{1}{\Omega}\int v(\boldsymbol{x})e^{-i\boldsymbol{q}\boldsymbol{x}}d\boldsymbol{x}\ , \tag{2·4}$$

$a^*_{\boldsymbol{k}_n\mu_n}$ and $a_{\boldsymbol{k}_n\mu_n}$ are creation and annihilation operators respectively of the nucleon with momentum $\boldsymbol{k}_n$ and spin μ_n, and Ω is the normalization volume.

An eigenstate of this Hamiltonian (2·3) may have any particle number, so that systems of A particles, B particles, A' particles and B' particles are all eigenstates of the same Hamiltonian (2·3). Therefore, we have

$$[\mathcal{H}-E_\alpha(A)]\Psi_\alpha(A)>_0=0\ , \tag{2·5}$$

where A is a particle number, α represents a state of this system and $>_0$ a vacuum state. In (2·5) we denote by $\Psi_\alpha(A)>_0$ the state whose total momentum is zero, and now by $\Psi_{K\alpha}(A)>_0$ the state whose total momentum is $\boldsymbol{K}$ and internal state is α. As the center-of-mass of $\Psi_\alpha(A)>_0$ can be separated completely, we can construct $\Psi_{K\alpha}(A)>_0$ from $\Psi_\alpha(A)>_0$ as follows:

$$\Psi_{K\alpha}(A)>_0=\Pi_n a^*_{K/A+\boldsymbol{k}_n\mu_n}a_{\boldsymbol{k}_n\mu_n}\Psi_\alpha(A)>_0\ , \tag{2·6}*$$

which satisfies the Schrödinger equation

$$[\mathcal{H}-E_{K\alpha}(A)]\Psi_{K\alpha}(A)>_0=0\ . \tag{2·7}$$

---

* Equation (2·6) means that

$$e^{i\boldsymbol{K}\boldsymbol{R}}\frac{1}{\sqrt{A!}}\sum_P(-)^{\delta_P}P\sum_{\boldsymbol{k}_1\cdots\boldsymbol{k}_A}\Psi_\alpha(\boldsymbol{k}_1\cdots\boldsymbol{k}_A)\,e^{i\boldsymbol{k}_1\boldsymbol{r}_1}\cdots e^{i\boldsymbol{k}_A\boldsymbol{r}_A}$$

$$=\frac{1}{\sqrt{A!}}\sum_P(-)^{\delta_P}P\sum_{\boldsymbol{k}_1\cdots\boldsymbol{k}_A}\Psi_\alpha(\boldsymbol{k}_1\cdots\boldsymbol{k}_A)\ \ e^{i\boldsymbol{k}+(\boldsymbol{K}/A)\cdot\boldsymbol{r}_1}\cdots e^{i\boldsymbol{k}_A+(\boldsymbol{K}/A)\cdot\boldsymbol{r}_A}$$

Here we consider a wave packet with regard to the center-of-mass of the state $\Psi_{K\alpha}(A)>_0$ in the configuration space. Let us denote by $c(\boldsymbol{K})$ a weight of the total momentum $\boldsymbol{K}$, then the wave packet is represented by

$$g(\boldsymbol{R},\zeta;t)=\sum_{\boldsymbol{K}}c(\boldsymbol{K})\Psi_\alpha(\zeta)e^{i\boldsymbol{K}\boldsymbol{R}}e^{-iE_{K\alpha}(A)t/\hbar}, \tag{2·8}$$

where $\boldsymbol{R}$ means the coordinate of the center-of-mass and ζ are the internal coordinates of the particles relative to it. It is natural to postulate that at $t\to-\infty$ the wave packet is very far from the origin and its internal wave function is localized around the center-of-mass; i.e.

$$g(0,\zeta;-\infty)=0. \tag{2·9}$$

Then we have

$$\sum_{\boldsymbol{K}}c(\boldsymbol{K})e^{-iE_{K\alpha}(A)t/\hbar}=0 \quad (t\to-\infty). \tag{2·10}$$

In the case when, besides the system $\Psi_{K\alpha}(A)>_0$, there is another system $\Psi_\beta(B)>_0$ whose center-of-mass is at the origin, the wave packet is expected to be

$$\Phi(t)=\sum_{\boldsymbol{K}}c(\boldsymbol{K})\Psi_{K\alpha}(A)\Psi_\beta(B)>_0 e^{-i(E_{K\alpha}(A)+E_\beta(B))t/\hbar}. \tag{2·11}$$

In order to show that (2·11) is the case, we first note that $\Phi(t)$ satisfies the following time dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Phi(t)-\mathcal{H}\Phi(t)=K(t), \tag{2·12}$$

where we define $V_{K\alpha}$ by

$$[\mathcal{H},\Psi_{K\alpha}(A)]=E_{K\alpha}(A)\Psi_{K\alpha}(A)+V_{K\alpha}, \tag{2·13}$$

and we have

$$K(t)=-\sum_{\boldsymbol{K}}c(\boldsymbol{K})V_{K\alpha}\Psi_\beta(B)>_0 e^{-i(E_{K\alpha}(A)+E_\beta(B))t/\hbar}. \tag{2·14}$$

Next we show that $K(t)$ vanishes at $t\to-\infty$. If $V_{K\alpha}$ is a slow varying function of the momentum $\boldsymbol{K}$ in a region where $c(\boldsymbol{K})$ is large, $\bar{V}_{K\alpha}\Psi_\beta(B)>_0$ can be represented by $V_\alpha\Psi_\beta(B)>_0$ approximately, and we have

$$K(t)\simeq-\sum c(\boldsymbol{K})e^{-iE_{K\alpha}(A)t/\hbar}\bar{V}_\alpha\Psi_\beta(B)>_0 e^{-iE_\beta(B)t/\hbar}. \tag{2·15}$$

From the property of (2·10), we see that $t\to-\infty$ $K(t)$ becomes zero.

We consider a state in which a system of A particles with the total momentum $\boldsymbol{K}$ comes from infinity and approaches to a system of B particles at the origin and is scattered. We define a state $\Psi_i^{(\pm)}>_0$ by the boundary condition that there are an incident wave $\Psi_{K\alpha}(A)\Psi_\beta(B)>_0$ and an outgoing or incoming wave at infinity. If we write

$$\Psi_i^{(\pm)}>_0=\Psi_{K\alpha}(A)\Psi_\beta(B)>_0+\chi^{(\pm)}, \tag{2·16}$$

then $\chi^{(\pm)}$ represents a scattered wave. $\Psi_i^{(\pm)}>_0$ satisfies the Schrödinger equation

$$[\mathcal{H}-(E_{K\alpha}(A)+E_{\beta}(B))]\Psi_i^{(\pm)}>_0=0\,. \tag{2·17}$$

From the equations (2·13), (2·16) and (2·17) we have

$$\Psi_i^{(\pm)}>_0=\Psi_{K\alpha}(A)\,\Psi_{\beta}(B)>_0 - \frac{1}{\mathcal{H}-E_{K\alpha}(A)-E_{\beta}(B)}\,V_{K\alpha}\Psi_{\beta}(B)>_0\,. \tag{2·18}$$

Similarly, we have

$$\Psi_f^{(\pm)}>_0=\Psi_{K'\alpha'}(A')\,\Psi_{\beta'}(B')>_0 - \frac{1}{\mathcal{H}-E_{K'\alpha'}(A')-E_{\beta'}(B')}V_{K'\alpha'}\Psi_{\beta'}(B')>_0\,, \tag{2·19}$$

for the final state which consists of a system of A' particles and a system of B' particles. The S matrix of $B_{\beta}(A_{K\alpha},\ A'_{K'\alpha'})B'_{\beta}$ reaction is defined by references 35) and 36)

$$S(A_{K\alpha},B_{\beta};\,A'_{K'\alpha'},B'_{\beta'})=<\Psi_f^{(-)},\Psi_i^{(+)}>_{00}\,. \tag{2·20}$$

From Equations (2·18) and (2·19) we have

$$\Psi_i^{(+)}>_0=\Psi_i^{(-)}>_0-2\pi i\delta(\mathcal{H}-E_{K\alpha}(A)-E_{\beta}(B))\,V_{K\alpha}\Psi_{\beta}(B)>_0\,, \tag{2·21}$$

$$\Psi_f^{(-)}>_0=\Psi_f^{(+)}>_0+2\pi i\delta(\mathcal{H}-E_{K'\alpha'}(A')-E_{\beta'}(B'))\,V_{K'\alpha'}\Psi_{\beta'}(B')>_0\,. \tag{2·22}$$

Therefore the S matrix becomes

$$S(A_{K\alpha},B_{\beta}\,;\,A'_{K'\alpha'},B'_{\beta'})=<\Psi_f^{(-)},\ \Psi_i^{(-)}>_{00} -2\pi i\delta(E_{K'\alpha'}(A')+E_{\beta'}(B')-E_{K\alpha}(A)-E_{\beta}(B))\,T_{K\alpha}(\beta), \tag{2·23}$$

$$T_{K\alpha}(\beta)\equiv<\Psi_f^{(-)},\ V_{K\alpha}\Psi_{\beta}(B)>_{00}\,, \tag{2·24}$$

or

$$S(A_{K\alpha},B_{\beta}\,;\,A'_{K'\alpha'},B'_{\beta'})=<\Psi_f^{(+)},\ \Psi_i^{(+)}>_{00} -2\pi i\delta(E_{K\alpha}(A)+E_{\beta}(B)-E_{K'\alpha'}(A')-E_{\beta'}(B'))\,T^{\dagger}_{K'\alpha'}(\beta'), \tag{2·25}$$

$$T^{\dagger}_{K'\alpha'}(\beta')\equiv<\Psi_{\beta'}(B')\,V^{\dagger}_{K'\alpha'}\Psi_i^{(+)}>_{00}\,, \tag{2·26}$$

where $\dagger$ represents a hermite adjoint. Moreover, since the initial and the final states with the same boundary conditions are orthonormal[35], we have $\langle\Psi_f^{(-)},\ \Psi_i^{(-)}\rangle_{00}=\langle\Psi_f^{(+)},\ \Psi_i^{(+)}\rangle_{00}=\delta_{fi}$. Thus we arrive at the final form of S matrix:

$$S(i\to f)=\delta_{fi}-2\pi i\delta(E_f-E_i)\,T_{K\alpha}(\beta) \tag{2·27}$$

$$T_{K\alpha}(\beta)\equiv<\Psi_{\beta'}(B')\,\Psi^{\dagger}_{K'\alpha'}(A')\,V_{K\alpha}\Psi_{\beta}(B)>_{00} -<\Psi_{\beta'}(B')\,V^{\dagger}_{K'\alpha'}\frac{1}{\mathcal{H}-E_f-i\epsilon}V_{K\alpha}\Psi_{\beta}(B)>_{00}\,, \tag{2·28}$$

or

$$S(i\to f)=\delta_{fi}-2\pi i\delta(E_i-E_f)\,T^{\dagger}_{K'\alpha'}(\beta')\,, \tag{2·29}$$

$$T^{\dagger}_{K'\alpha'}(\beta') \equiv <\Psi_{\beta'}(B')\, V^{\dagger}_{K'\alpha'} \Psi_{K\alpha}(A)\, \Psi_{\beta}(B)>_{00}$$

$$- <\Psi_{\beta'}(B')\, V^{\dagger}_{K'\alpha'} \frac{1}{\mathcal{H}-E_i-i\epsilon} V_{K\alpha} \Psi_{\beta}(B)>_{00}, \tag{2·30}$$

where $E_i = E_{K\alpha}(A) + E_{\beta}(B)$ and $E_f = E_{K'\alpha'}(A') + E_{\beta'}(B')$. From Equations (2·27), (2·28), (2·29) and (2·30) we have

$$<\Psi_{\beta'}(B')\, \Psi^{+}_{K'\alpha'}(A')\, V_{K\alpha} \Psi_{\beta}(B)>_{00} = <\Psi_{\beta'}(B')\, V^{\dagger}_{K'\alpha'} \Psi_{K\alpha}(A)\, \Psi_{\beta}(B)>_{00}. \tag{2·31}$$

This equation means that the so-called initial interaction and the final interaction give the same result for the Born approximation. Generally, we can represent $\Psi_{\lambda}(N)>_0$ in such a form as

$$\Psi_{\lambda}(N)>_0 = \sum_{\substack{k_1\cdots k_N \\ \mu_1\cdots \mu_N}} \varphi_{\lambda}(\boldsymbol{k}_1\cdots\boldsymbol{k}_N;\, \mu_1\cdots\mu_N)\, a^{*}_{k_1\mu_1}\cdots a^{*}_{k_N\mu_N}>_0, \tag{2·32}$$

and $V_{K\alpha}(A)$, $V_{K'\alpha'}(A')$ or the vacuum expectation values in Equations (2·28) and (2·30) can be calculated explicitly. Consequently Equations (2·27) and (2·29) give the general S matrix of the reaction including both elastic and inelastic scatterings taking account of antisymmetrization.

§ 3. Deuteron stripping reaction

As an example we apply the general method above mentioned to the deuteron stripping reaction. For the sake of comparison with other theories we assume that the two-body interaction among the nucleons depends only on the relative distance of a pair of nucleons. In this simplified case we can treat proton and neutron as distinguishable particles. Therefore we introduce the proton field and the neutron field separately:

$$\Psi_P(\boldsymbol{x}) = \sum_{k_n\mu_n} a_{k_n\mu_n} \frac{1}{\sqrt{\Omega}} e^{i\boldsymbol{k}_n\boldsymbol{x}} u(\mu_n), \tag{3·1}$$

$$\Psi_N(\boldsymbol{y}) = \sum_{k_n\mu_n} b_{k_n\mu_n} \frac{1}{\sqrt{\Omega}} e^{i\boldsymbol{k}_n\boldsymbol{y}} u(\mu_n), \tag{3·2}$$

and postulate the commutation relations as follows:

$$\begin{aligned} &\{a^{*}_{k\mu}, a_{k'\mu'}\} = \{b^{*}_{k\mu}, b_{k'\mu'}\} = \delta_{kk'}\,\delta_{\mu\mu'}, \\ &\{a^{*}_{k\mu}, a^{*}_{k'\mu'}\} = \{a_{k\mu}, a_{k'\mu'}\} = 0, \\ &\{b^{*}_{k\mu}, b^{*}_{k'\mu'}\} = \{b_{k\mu}, b_{k'\mu'}\} = 0, \\ &\{a^{*}_{k\mu}, b^{*}_{k'\mu'}\} = \{a^{*}_{k\mu}, b^{*}_{k'\mu'}\} = 0, \end{aligned} \tag{3·3}$$

where $a_{k\mu}$ refers to proton and $b_{k\mu}$ to neutron. The total Hamiltonian is given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \tag{3·4}$$

$$\mathcal{H} = \sum_{k_n\mu_n} \omega_{k_n} a^{*}_{k_n\mu_n} a_{k_n\mu_n} + \sum_{k_n\mu_n} \omega_{k_n} b^{*}_{k_n\mu_n} b_{k_n\mu_n}, \tag{3·5}$$

$$\mathcal{H}_I=\tfrac{1}{2}\sum_q v_q \sum_{k_n\mu_n}\sum_{k_n'\mu_n'}(a^*_{k_n+q\mu_n}a^*_{k_n'-q\mu_n'}a_{k_n'\mu_n'}a_{k_n\mu_n}$$
$$+b^*_{k_n+q\mu_n}b^*_{k_n'-q\mu_n'}b_{k_n'\mu_n'}b_{k_n\mu_n}$$
$$+2a^*_{k_n+q\mu_n}b^*_{k_n'-q\mu_n'}b_{k_n'\mu_n'}a_{k_n\mu_n}). \tag{3·6}$$

As the ground state of deuteron is approximated by a pure S^3 state, we obtain for the state of deuteron with the total momentum $\boldsymbol{K}$ and the z component μ_D of spin

$$\Psi_{K\mu_D}(2)>_0=\sum_{\mu_P}\sum_{\mu_N}(\tfrac{1}{2}\tfrac{1}{2}\mu_N\mu_P|1\mu_D)\sum_k\chi(k)\,a^*_{K/2+k\mu_P}b^*_{K/2-k\mu_N}>_0\,, \tag{3·7}$$

and for the energy

$$E_K(2)=\tfrac{1}{2}\omega_K+E_D\,, \tag{3·8}$$

where $\chi(k)$ is the Fourier component of the deuteron wave function, $-E_D$ is the deuteron binding energy and $k=|\boldsymbol{k}|$.

Now we calculate the interaction term $V_{K\mu_D}$ according to the definition (2·13). The commutation relations (3·3) lead to the following results:

$$[\mathcal{H}_0,\ \Psi_{K\mu_D}(2)]=\tfrac{1}{2}\omega_K\Psi_{K\mu_D}(2)$$
$$+\sum_{\mu_N\mu_P}(\tfrac{1}{2}\tfrac{1}{2}\mu_N\mu_P|1\mu_D)\sum_k\chi(k)\,\omega_k a^*_{(K/2)+k\mu_P}b^*_{(K/2)-k\mu_N}\,, \tag{3·9}$$

$$[\mathcal{H}_I,\ \Psi_{K\mu_D}(2)]=\sum_{\mu_N\mu_P}(\tfrac{1}{2}\tfrac{1}{2}\mu_N\mu_P|1\mu_D)\sum_k\chi(k)$$
$$\times\{v_{(K/2)+k\mu_P}b^*_{(K/2)-k\mu_N}+a^*_{(K/2)+k\mu_P}w_{(K/2)-k\mu_N}\}\,, \tag{3·10}$$

where v and w are defined by

$$v_{k\mu}\equiv[\mathcal{H}_I,\ a^*_{k\mu}]\,,$$
$$w_{k\mu}\equiv[\mathcal{H}_I,\ b^*_{k\mu}]\,. \tag{3·11}$$

Thus the form $V_{K\mu_D}$ becomes

$$V_{K\mu_D}=\sum(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\sum\chi(k)$$
$$\times\{(2\omega_k-E_a)\,a^*_{(K/2)+k\mu_P}b^*_{(K/2)-k\mu_N}$$
$$+v_{(K/2)+k\mu_P}b^*_{(K/2)-k\mu_N}+a^*_{(K/2)+k\mu_P}w_{(K/2)-k\mu_N}\}\,. \tag{3·12}$$

If $V_{k\mu_I}\Psi_{0\mu_A^I}(A)>_0$ does not depend so sensitively on the momentum $\boldsymbol{K}$, we shall be able to write the initial state in such a form as shown by Equation (2·18). Writing $V_{K\mu_D}\Psi_{0\mu_A^I}(A)>_0$ explicitly and examining each term we can easily obtain the condition under which Equation (2·15) is established. The condition is that both the target system $\Psi_{0\mu_A^I}(A)>_0$ and the deuteron system $\Psi_{K\mu_D}(2)>_0$ are so strongly localized around the origin and the center-of-mass of the deuteron, respectively, that their wave functions occupy very wide domain in the momentum space. As this condition seems to be realized by the system under consideration we can express the initial state in the following form,

$$\Psi_i^{(\pm)}>_0=\sum_{\mu_P\mu_N}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\sum_k\chi(\boldsymbol{k})\,a^*_{(\boldsymbol{K}/2)+k\mu_P}b^*_{(\boldsymbol{K}/2)-k\mu_P}\Psi^{I_A}_{0\mu_A}(A)>_0$$

$$-\frac{1}{\mathcal{H}-(1/2)\omega_K-E_D-E_0(A)\mp i\epsilon}V_{K\mu_D}\Psi^{I_A}_{0\mu_A}(A)>_0\,, \tag{3·13}$$

where

$$[\mathcal{H}-E_0(A)]\Psi^{I_A}_{0\mu_A}(A)>_0=0\,. \tag{3·14}$$

Similar result is also true for the final state. When the emitted particle is proton and residual nucleus is left in a state m, we have

$$\Psi_f^{(\pm)}>_0=a^*_{k_P\mu_f}\Psi^{I_B}_{m\mu_B}(A+1)>_0$$

$$-\frac{1}{\mathcal{H}-\omega_{k_P}-E_m(A+1)\mp i\epsilon}V_{k_P\mu_f}\Psi^{I_B}_{m\mu_B}(A+1)>_0\,, \tag{3·15}$$

where

$$[\mathcal{H}-E(A+1)]\Psi^{I_B}_{m\mu_B}(A+1)>_0=0\,, \tag{3·16}$$

and the definition of $v_{k_P\mu_f}$ is the same as Equation (3·11). According to the definition (2·20) of S matrix we can construct S matrix of (d, p) reaction from Equations (3·13) and (3·15), and we obtain

$$S(D_{K\mu_D}, A^{I_A}_{0\mu_A}; P_{k_P\mu_f}, B^{I_B}_{m\mu_B})$$
$$=\delta_{fi}-2\pi i\delta(\omega_{k_P}+E_m(A+1)-\tfrac{1}{2}\omega_K-E_D-E_0(A))\,T_{K\mu_D}(\mu_A)\,, \tag{3·17}$$

$$T_{K\mu_D}(\mu_A)\equiv<\Psi^{I_B}_{m\mu_B}(A+1)\,a_{k_P\mu_f}V_{K\mu_D}\Psi^{I_B}_{0\mu_A}(A)>_{00}$$

$$-<\Psi^{I_B}_{m\mu_B}(A+1)\,v^\dagger_{k_P\mu_f}\frac{1}{\mathcal{H}-\omega_{k_P\mu_f}-E_m(A+1)-i\epsilon}V_{K\mu_D}\Psi^{I_A}_{0\mu_A}(A)>_{00}\,, \tag{3·18}$$

or

$$S(D_{K\mu_D}, A^{I_A}_{\mu_A}; P_{k_P\mu_f}, B^{I_B}_{m\mu_B})$$
$$=\delta_{fi}-2\pi i\delta(\tfrac{1}{2}\omega_K+E_D+E_0(A)-\omega_{k_P}-E_m(A+1))\,T^\dagger_{k_P\mu_f}(\mu_B)\,, \tag{3·19}$$

$$T^+_{k_P\mu_f}(\mu_B)\equiv<\Psi^{I_B}_{m\mu_B}(A+1)\,v^\dagger_{k_P\mu_f}\Psi_{K\mu_D}(2)\,\Psi^{I_A}_{0\mu_A}(A)>_{00}$$

$$-<\Psi^{I_B}_{m\mu_B}(A+1)\,v^\dagger_{k_P\mu_f}\frac{1}{\mathcal{H}-\tfrac{1}{2}\omega_k-E_D-E_0(A)-i\epsilon}V_{K\mu_D}\Psi^{I_A}_{0\mu_A}(A)>_{00}\,. \tag{3·20}$$

Equations (3·17) and (3·19) represent all the processes that occur in the (d,p) reaction, but hereafter we should like to omit the so-called "compound process" and treat only the Born term which gives a direct rearrangement reaction. We consider only the cases when the Born approximation is valid.

In accordance with the shell model we assume that the neutron is captured in one of the shell states of the target nucleus. Then the state of the residual

nucleus can be expanded into a shell state of the captured neutron and the state of the target nucleus,

$$\Psi^{I_B}_{m\mu_B}(A+1)>_0=\sum_{m\mu}(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\sum_{k}\varphi_{1m}(\boldsymbol{k})\,b^*_{k\mu}\,\Psi^{I_A}_{0\mu_A}(A)>_0\,, \tag{3·21}$$

where l and m are the orbital angular momentum and its z component, and $\frac{1}{2}$ and μ are the spin angular momentum and its z component, of a shell wave function $\varphi_{ljj_z}(\boldsymbol{r},\sigma)$. I_A and μ_A are the total spin and its z component of the target nucleus and I_B and μ_B are of the residual nucleus. The Born term then becomes

$$M_I\equiv\sum_{m\mu}(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\sum\varphi_{lm}(\boldsymbol{k})<\Psi^{I_A}_{0\mu_A}(A)\,b_{k\mu}\,a_{k_P\mu_f}V_{K\mu_D}\Psi^{I_A}_{0\mu_A}(A)>_{00}\,, \tag{3·22}$$

$$M_F\equiv\sum_{m\mu}(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\sum\varphi^*_{lm}(\boldsymbol{k})<\Psi^{I_A}_{0\mu_A}(A)\,b_{k\mu}v^\dagger{}_{k_P\mu_f}\Psi_{K\mu_D}(2)\,\Psi^{I_A}_{0\mu_A}(A)>_{00}\,, \tag{3·23}$$

where the suffixes I and F mean the initial and the final interaction formalism respectively. In order to obtain the final result we only need to calculate the expectation value of the initial interaction or the final interaction by the target state $\Psi^{I_A}_{0\mu_A}(A)>_0$.

§ 4. Comparison with other theories

In order to compare the amplitude obtained in the preceding section with those of other theories we write the initial interaction and the final interaction explicitly as

$$b_{k\mu}\,a_{k_P\mu_f}V_{K_nD}\equiv\sum_{\mu_P\mu_N}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\sum\chi(\boldsymbol{k}')\,\{(2\omega_{k'}-E_D)\,b_{k\mu}\,a_{k_P\mu_f}$$
$$\times a^*_{(\boldsymbol{K}/2)+\boldsymbol{k}'\mu_P}b^*_{(\boldsymbol{K}/2)-\boldsymbol{k}'\mu_N}+\sum_q v_q b_{k\mu}a_{k_P\mu_f}a^*_{(\boldsymbol{K}/2)+\boldsymbol{k}'+q\mu_P}$$
$$\times\sum_{k_n\mu_n}(a^*_{k_n-q\mu_n}\,a_{k_n\mu_n}+b^*_{k_n-q\mu_n}\,b_{k_n\mu_n})\,b^*_{(\boldsymbol{K}/2)-\boldsymbol{k}'\mu_N}$$
$$+\sum_q v_q b_{k\mu}a_{k_P\mu_f}a^*_{(\boldsymbol{K}/2)+\boldsymbol{k}'\mu_P}b^*_{(\boldsymbol{K}/2)-\boldsymbol{k}'-q\mu_N}$$
$$\times\sum_{k_n\mu_n}(a^*_{k_n+q\mu_n}\,a_{k_n\mu_n}+b^*_{k_n\mu_n}\,b_{k_n\mu_n})\}\,, \tag{4·1}$$

and

$$b_{k\mu}\,v^\dagger{}_{k_P\mu_f}\Psi_{K\mu_D}(2)\equiv\sum_{\mu_P\mu_N}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\sum_{k'}\chi(\boldsymbol{k}')$$
$$\times\{\sum_q v_q b_{k\mu}\sum_{k_n\mu_n}(a^*_{k_n\mu_n}\,a_{k_n-q\mu_n}+b^*_{k_n\mu_n}\,b_{k_n-q\mu_n})$$
$$\times a_{k_P+q\mu_f}\,a^*_{(\boldsymbol{K}/2)+\boldsymbol{k}'\mu_P}b^*_{(\boldsymbol{K}/2)-\boldsymbol{k}'\mu_N}\}\,, \tag{4·2}$$

where $q=|\boldsymbol{q}|$.

First we consider the final interaction formalism M_F. Using the explicit form of the interaction (4·2) and putting for the target system the vacuum state** with the spin I_A and its z component μ_A, we obtain

$$M_F=\sum_{m\mu}(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu|1\mu_D)\,(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\sum_q v_q\chi(|\boldsymbol{k}_P+\boldsymbol{q}-(\boldsymbol{K}/2)|)\varphi^*_{lm}(\boldsymbol{K}-\boldsymbol{k}_P)\ . \tag{4·3}$$

The Butler amplitude[2] M^B is obtained from M_F if we divide $\varphi^*_{lm}(\boldsymbol{K}-\boldsymbol{k}_p)$ into two parts, one of which is related to the internal region and the other to the external region,

$$\varphi^*_{lm}(\boldsymbol{K}-\boldsymbol{k}_P)=\underset{(\mathrm{iut})}{\varphi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}_P)+\underset{(\mathrm{ext})}{\varphi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}_P)\ , \tag{4·4}$$

$$\underset{(\mathrm{iut})}{\varphi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}_P)=\frac{1}{\sqrt{\Omega}}\int_{|\boldsymbol{r}|<R}\varphi^*_{lm}(\boldsymbol{r})e^{-i\boldsymbol{K}-\boldsymbol{k}_P\cdot\boldsymbol{r}}d\boldsymbol{r}\ .$$

$$\underset{(\mathrm{ext})}{\varphi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}_P)=\frac{1}{\sqrt{\Omega}}\int_{|\boldsymbol{r}|>R}\varphi^*_{lm}(\boldsymbol{r})e^{-i\boldsymbol{K}-\boldsymbol{k}_P\cdot\boldsymbol{r}}d\boldsymbol{r}\ . \tag{4·5}$$

where R is the radius separating the two regions, and we are interested only in the external region. Namely,

$$M^B=\sum_{m\mu}(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu|1\mu_D)\,(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\underset{(\mathrm{ext})}{\varphi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}_P)\sum_q v_q\chi(|\boldsymbol{k}_P+\boldsymbol{q}-(\boldsymbol{K}/2)|)\ . \tag{4·6}$$

Moreover if the outgoing proton comes from the external region, the operator of the outgoing proton $a_{k_P\mu_f}$ is exchanged by $\sum_{k'_P(\mathrm{ext})}u^*_{k_P}(\boldsymbol{k}'_P)a_{k'_P\mu_f}$ and we obtain the direct stripping amplitude of Nagasaki's theory[22],

$$M^N=\sum_{m\mu}(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu|1\mu_D)\,(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\sum_{(\mathrm{ext})}u^*_{k_P}(\boldsymbol{k}'_P)\underset{(\mathrm{ext})}{\varphi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}'_p)\sum_q v_q\chi(|\boldsymbol{k}'_P+\boldsymbol{q}-(\boldsymbol{K}/2)|)\ , \tag{4·7}$$

where

$$\underset{(\mathrm{ext})}{u^*_{k_P}}(\boldsymbol{k}'_P)=\frac{1}{\Omega}\int_{|\boldsymbol{r}|>R}e^{-i\boldsymbol{k}_P-\boldsymbol{k}'_P\cdot\boldsymbol{r}}d\boldsymbol{r}\ . \tag{4·8}$$

As it is easily seen from Equation (4·3) that the Butler amplitude is obtained by neglect of the interaction between the deuteron and the target, it is natural to consider that the final proton wave comes from the external region. Experimental distribution of protons at small angles seems to be explained better by Nagasaki's amplitude than by the Butler amplitude.

If the target $\Psi_{0\mu_A^{I_A}}(A)>_0$ can be divided into a proton and the rest in the fol-

** By the vacuum state, we mean a state $\Psi>_0$ where the operator Ψ commutes with the interaction term and with any other particle operators.

lowing way,

$$\Psi^{I_A}_{0\mu_A}(A)>_0=\sum_{l'm'\mu'j'j'_z I_r\mu_r}(l'\tfrac{1}{2}m'\mu'|j'j'_z)\,(j'I_rj_z'\mu_r|I_A\mu_A)$$
$$\times\sum_{k'}\phi_{l'm'}(\boldsymbol{k}')\,a^*_{k'\mu'}\,\Psi^{I_r}_{r\mu_r}(A-1)>_0, \tag{4·9}$$

and equate $\Psi^{I_r}_{r\mu_r}(A-1)>_0$ with a vacuum state** with the spin I_r and its z component μ_r, we obtain French's amplitude[27],

$$M^F=\sum_{\mu_P\mu_N m\mu}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\sum_{l'm'\mu'j'j_z'I_r}(l'\tfrac{1}{2}m'\mu'|j'j_z')\,(j'I_rj_z'\mu_r|I_A\mu_A)$$
$$\times\sum_{l''m''\mu''j''j_z''}(l''\tfrac{1}{2}m''\mu''|j''j_z'')\,(j''I_rj_z''\mu_r|I_A\mu_A)$$
$$\times\sum_{k'}\chi(k')\sum_{k}\phi^*_{lm}(\boldsymbol{k})\sum_{k''}\phi^*_{l'm'}(\boldsymbol{k}'')\sum_{k'''}\phi_{l''m''}(\boldsymbol{k}''')\sum_{q}v_q$$
$$\times<a_{k'\mu'}\,b_{k\mu}\sum_{k_n\mu_n}(a^*_{k_n\mu_n}a_{k_n-q\mu_n}+b^*_{k_n\mu_n}b_{k_n-q\mu_n})$$
$$\times a_{k_P+q\mu_f}\,a^*_{(K/2)+k'\mu_P}\,b^*_{(K/2)-k'\mu_N}\,a^*_{k'''\mu''}>_{00}. \tag{4·10}$$

However, French neglects the proton-proton interaction, and then M^F takes a simpler form

$$M^F=M^F(\text{direct})+M^F(\text{exchange}), \tag{4·11}$$

$$M^F(\text{direct})=\sum_{m\mu}(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu|1\mu_D)\,(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\varphi^*_{lm}(\boldsymbol{K}-\boldsymbol{k}_P)\sum_q v_q\chi(|\boldsymbol{k}_P+\boldsymbol{y}-(\boldsymbol{K}/2)|), \tag{4·12}$$

$$M^F(\text{exchange})=-\sum_{\mu_P m\mu}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu|1\mu_D)\,(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)$$
$$\times\sum_{l'm'j'j'_zI_r}(l'\tfrac{1}{2}m'\mu_P|j'j_z')\,(j'I_rj_z'\mu_r|I_A\mu_A)$$
$$\times\sum_{l''m''j''}(l''\tfrac{1}{2}m''\mu_f|j''j_z')\,(j''I_rj_z'\mu_r|I_A\mu_A)$$
$$\times\sum_q v_q\sum_{\boldsymbol{k}}\chi(|(\boldsymbol{K}/2)-\boldsymbol{k}+\boldsymbol{q}|)\,\phi_{l''m''}(\boldsymbol{k}_P+\boldsymbol{q})$$
$$\times\varphi^*_{lm}(\boldsymbol{k})\,\phi^*_{l'm'}(\boldsymbol{K}-\boldsymbol{k}+\boldsymbol{q}). \tag{4·13}$$

The M^F(exchange) term (4·13) is equivalent to G amplitude of French's notation.

Next we consider the initial interaction formalism M_I. If the target is composed of a loosely bound proton and a hard core of the spin I_c and its z component μ_c,

$$\Psi^{I_A}_{0\mu_A}(A)>_0=\sum_{m'\mu'}(l'\tfrac{1}{2}m'\mu'|j'j_z')\,(j'I_cj_z\mu_c|I_A\mu_A)$$

** ibid.

$$\times\sum_{k'}\phi^{*}_{l'm'}(\boldsymbol{k}')\,a^{*}_{k'\mu'}\,\Psi^{I_c}_{0\mu_c}(A-1)>_0 . \tag{4·14}$$

M_I becomes

$$\begin{aligned}M_I=&\sum_{m\mu m'\mu' m''\mu''}(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)\,(l'\tfrac{1}{2}m'\mu'|j'j_z')\\&\times(j'I_cj_z'\mu_c|I_A\mu_A)\,(l'\tfrac{1}{2}m''\mu''|j'j_z')\,(j'I_cj_z'\mu_c|I_A\mu_A)\\&\times\sum_{k}\varphi^{*}_{lm}(\boldsymbol{k})\sum_{k'}\phi^{*}_{l'm'}(\boldsymbol{k}')\sum_{k''}\phi_{l''m''}(\boldsymbol{k}'')\\&\times<\Psi^{I_c}_{0\mu_c}(A-1)\,a_{k'\psi}\,b_{k\mu}\,a_{k_P\mu_f}\,V_{K\mu_D}\,a^{*}_{k''\mu''}\,\Psi^{I_c}_{0\mu_c}(A-1)>_{00} .\end{aligned}\tag{4·15}$$

Here we put for $\Psi^{I_c}_{0\mu_c}(A-1)>_0$ a vacuum state** with the total spin I_c and its z component μ_c, and obtain

$$\begin{aligned}M_I=&\sum_{m\mu m'\mu' m''\mu''\mu_P\mu N}(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)\,(l'\tfrac{1}{2}m'\mu'|j'j_z')\\&\times(j'I_cj_z\mu_c|I_A\mu_A)^2(l'\tfrac{1}{2}m''\mu''|j'j_z')\,(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|\mu_D)\\&\times\sum_{k}\varphi^{*}_{lm}(\boldsymbol{k})\sum_{k'}\phi^{*}_{l'm'}(\boldsymbol{k}')\sum_{k''}\phi_{l''m''}(\boldsymbol{k}'')\sum_{k'''}\chi(\boldsymbol{k}''')\\&\times<a_{k'\psi}\,b_{k\mu}\,a_{k_P\mu_P}\{(2\omega_{k'''}-E_D)\,a^{*}_{(\boldsymbol{K}/2)+k'''\mu_P}\,b^{*}_{(\boldsymbol{K}/2)-k'''\mu_N}\\&+\sum_{q}v_q a^{*}_{(\boldsymbol{K}/2)+k'''+q\mu_P}\sum_{k_n\mu_n}(a^{*}_{k_n-q\mu_n}\,a_{k_n\mu_n}+b^{*}_{k_n-q\mu_n}\,b_{k_n\mu_n})\,b_{(\boldsymbol{K}/2)-k'''\mu_N}\\&+\sum_{q}v_q a^{*}_{(\boldsymbol{K}/2)+k'''\mu_P}\,b^{*}_{(\boldsymbol{K}/2)-k'''-q\mu_N}\sum_{k_n\mu_n}(a^{*}_{k_n+q\mu_n}\,a_{k_n\mu_n}+b^{*}_{k_n+q\mu_n}\,b_{k_n\mu_n})\}\\&\times a^{*}_{k''\mu''}>_{00} .\end{aligned}\tag{4·16}$$

Using the equation of ground state of deuteron $(2\omega_k-E_D(\chi(k)=\sum v_q\chi(|\boldsymbol{k}+\boldsymbol{q}|)$, (4·16) becomes

$$M_I=M_I(\text{direct})+M_I(\text{exchange}),\tag{4·17}$$

$$\begin{aligned}M_I(\text{direct})=&\sum_{m\mu m'\mu'}(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu|1\mu_D)\,(l\tfrac{1}{2}m\mu|jj_z)\\&\times(jI_Aj_z\mu_A|I_B\mu_B)\,(l'\tfrac{1}{2}m'\mu'|j'j_z')^2(j'I_cj_z'\mu_c|I_A\mu_A)^2\\&\times\sum_{q}v_q\varphi^{*}_{lm}(\boldsymbol{K}+\boldsymbol{q}-\boldsymbol{k}_P)\sum_{k'}\phi^{*}_{l'm'}(\boldsymbol{k}')\,\phi_{l'm'}(\boldsymbol{k}'+\boldsymbol{q})\\&\times\{\chi(|\boldsymbol{k}_P-(\boldsymbol{K}/2)-\boldsymbol{q}|)+\chi(|\boldsymbol{k}_P-(\boldsymbol{K}/2)|)\} ,\end{aligned}\tag{4·18}$$

$$\begin{aligned}M_I(\text{exchange})=&\sum_{\mu_P m\mu m' m''}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu|1\mu_D)\,(l\tfrac{1}{2}m\mu|jj_z)\,(jI_Aj_z\mu_A|I_B\mu_B)\\&\times(l'\tfrac{1}{2}m'\mu_P|j'j_z')\,(l'\tfrac{1}{2}m''\mu_f|j'j_z')\,(j'I_cj_z'\mu_c|I_A\mu_A)^2\\&\times\sum_{q}v_q\sum_{k'}\varphi^{*}_{lm}(\boldsymbol{K}+\boldsymbol{q}-\boldsymbol{k}')\,\phi^{*}_{l'm'}(\boldsymbol{k}')\,\phi_{l''m''}(\boldsymbol{k}_P+\boldsymbol{q})\\&\times\{\chi(|\boldsymbol{k}'-(\boldsymbol{K}/2)-\boldsymbol{q}|)+\chi(|\boldsymbol{k}'-(\boldsymbol{K}/2)|)\} .\end{aligned}\tag{4·19}$$

In the nominations of Owen and Madansky M_I (direct) is the direct stripping

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** ibid.

amplitude, but M_I(exchange) does not correspond to the heavy particle stripping amplitude.

§ 5. Angular distribution of emitted particles

If the energy of the incident deuteron is low, the Born approximation is not valid inside the internal region. Therefore we only consider the external region by the Born term. For simplicity we treat a special target composed of a loosely bound neutron and a hard core. This target nucleus seems to demonstrate the effect of antisymmetrization most clearly. Moreover, we assume that the spin of the hard core is zero. Therefore, the target and the residual nucleus may be expressed by

$$\Psi^{I_A}_{0\mu_A}(A)>_0=\sum_{m\mu}(l\tfrac{1}{2}m\mu|I_A\mu_A)\sum_k\varphi_{lm}(\boldsymbol{k})\,b^*_{k\mu}\Psi^0_{00}(A-1)>_0\,, \tag{5·1}$$

$$\Psi^{I_B}_{m\mu_B}(A+1)>_0=\sum_{m'\mu'm''\mu''j_zj_z'}(l'\tfrac{1}{2}m'\mu'|jj_z)\,(l''\tfrac{1}{2}m''\mu''|j'j_z')\,(jj'j_zj_z'|I_B\mu_B)$$
$$\times\sum_{k'}\varphi_{l'm'}(\boldsymbol{k}')\sum_{k''}\varphi_{l''m''}(\boldsymbol{k}'')\,a^*_{k''\mu''}\,b^*_{k'\mu'}\,\Psi^0_{00}(A-1)>_0\,, \tag{5·2}$$

$$\Psi^{I_B'}_{m\mu_B}(A+1)>_0=\sum_{m_l\mu'm''\mu''j_zj_z'}(l'\tfrac{1}{2}m'\mu'|jj_z)\,(l''\tfrac{1}{2}m''\mu''|j'j_z')\,(jj'j_zj_z'|I_B'\mu_B')$$
$$\times\sum_{k}\varphi_{l'm'}(\boldsymbol{k}')\sum\varphi_{l''m''}(\boldsymbol{k}'')\,b^*_{k''\mu''}\,b^*_{k'\mu'}\,\Psi^0_{00}(A-1)>_0\,, \tag{5·3}$$

where $\Psi^{I_B}_{m\mu_B}(A+1)>_0$ is the residual nucleus formed by a (d, n) reaction and $\Psi^{I_B'}_{m\mu_B}(A+1)>_0$ by a (d, p) reaction. In the final interaction formalism the amplitude of the Born term for (d, n) and (d, p) reactions is given by

$$M(D_{K\mu_D}, A^{I_A}_{0\mu_A}; N_{k_N\mu_f}, B^{I_B}_{m\mu_B})$$
$$=\sum_{\mu_P\mu_Nm\mu'm''\mu''j_zj_z'}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu|I_A\mu_A)\,(l'\tfrac{1}{2}m'\mu'|jj_z)$$
$$\times(l''\tfrac{1}{2}m''\mu''|j'j_z')\,(jj'j_zj_z'|I_B\mu_B)$$
$$\times\sum_{k'''}\chi(\boldsymbol{k}''')\sum_{k(\mathrm{ext})}\varphi_{lm}(\boldsymbol{k})\sum_{k'(\mathrm{ext})}\varphi^*_{l'm'}(\boldsymbol{k}')\sum_{k''(\mathrm{ext})}\phi^*_{l''m''}(\boldsymbol{k}'')\sum_q v_q\sum_{k'_N(\mathrm{ext})}u^*_{k'_N}(\boldsymbol{k}_N')$$
$$\times<b_{k'\mu'}\,a_{k''\mu''}\sum_{k_n\mu_n}(a^*_{k_n\mu_n}a_{k_n-q\mu_n}+b^*_{k_n\mu_n}b_{k_n-q\mu_n})$$
$$\times b_{k'_N+q\mu_f}\,a^*_{(\boldsymbol{K}/2)+k'''\mu_P}\,b^*_{(\boldsymbol{K}/2)-k'''\mu_N}\,b_{k''\mu}>_{00}\,, \tag{5·4}$$

and
$$M(D_{K\mu_D}, A^{I_A}_{0\mu_A}; P_{k_P\mu_f}, B^{I'_B}_{m\mu'_B})$$
$$=\sum_{\mu_P\mu_nm\mu'm''\mu''j_zj_z'}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu|I_A\mu_A)\,(l'\tfrac{1}{2}m'\mu'|jj_z)$$
$$\times(l''\tfrac{1}{2}m''\mu''|j'j_z')\,(jj'j_zj_z'|I_B'\mu_B')$$
$$\times\sum_{k'''}\chi(\boldsymbol{k}''')\sum_{k(\mathrm{ext})}\varphi_{lm}(\boldsymbol{k})\sum_{k'(\mathrm{ext})}\varphi^*_{l'm'}(\boldsymbol{k}')\sum_{k''(\mathrm{ext})}\varphi^*_{l''m''}(\boldsymbol{k}'')\sum_q v_q\sum_{k'_P(\mathrm{ext})}u^*_{k'_P}(\boldsymbol{k}_P')$$
$$\times<b_{k\mu'}\,b_{k''\mu''}\sum_{k_n\mu_n}(a^*_{k_n\mu_n}a_{k_n-q\mu_n}+b^*_{k_n\mu_n}b_{k_n-q\mu_n})$$
$$\times a_{k'_P+q\mu_f}\,a^*_{(\boldsymbol{K}/2)+k'''\mu_P}\,b^*_{(\boldsymbol{K}/2)-k'''\mu_N}\,b^*_{k\mu}>_{00}\,, \tag{5·5}$$

where $\underset{(\mathrm{ext})}{\varphi_{lm}}(\boldsymbol{k})$, $\underset{(\mathrm{ext})}{\phi_{lm}}(\boldsymbol{k})$ and $\underset{(\mathrm{ext})}{u_{l}}(k')$ are Fourier components for the external region defined in Equation (4·5). Now for simplicity, we assume $I_A=j=j'$ and $l=l'=l''$. This is the case with such a target as O^{17} and a residual nucleus as F^{18} or O^{18}. Therefore we obtain the following final results:

$$
\begin{aligned}
&M(D_{K\mu_D},\ A^{I_A}_{0\mu_A};\ N_{k_N\mu_f},\ B^{I_B}_{m\mu_B})\\
&\quad=-\sum_{\mu_P}{}'(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu_f|1\mu_D)\,(l\tfrac{1}{2}m\mu_f|I_Aj_z')\,(I_AI_A\mu_A\mu_A|I_B\mu_B)\\
&\quad\times\sum_{k'_N(\mathrm{ext})}u_{k_N}(\boldsymbol{k}_N')\sum_q v_q\chi(|(\boldsymbol{K}/2)-\boldsymbol{k}_N'-\boldsymbol{q}|)\,\underset{(\mathrm{ext})}{\phi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}_N')\\
&\quad+\sum_{\mu_P\mu_Nmm'm''j_z}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu_f|I_A\mu_A)\,(l\tfrac{1}{2}m'\mu_N|I_Aj_A)\\
&\quad\times(l\tfrac{1}{2}m''\mu_P|I_Aj_z')\,(I_AI_Aj_zj_z'|I_B\mu_B)\sum_{k'_N(\mathrm{ext})}u_{k_N}(\boldsymbol{k}_N')\sum_q v_q\\
&\quad\times\sum_{k'}\chi(|(\boldsymbol{K}/2)-\boldsymbol{k}'|)\,\underset{(\mathrm{ext})}{\varphi_{lm}}(\boldsymbol{k}_N'+\boldsymbol{q})\,\underset{(\mathrm{ext})}{\varphi^*_{lm'}}(\boldsymbol{k}')\,\underset{(\mathrm{ext})}{\phi^*_{lm''}}(\boldsymbol{K}-\boldsymbol{k}'+\boldsymbol{q})\\
&\quad\sum_{\mu_Pm\mu m'm''j_z}{}'(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu_f|1\mu_D)\,(l\tfrac{1}{2}m\mu|I_A\mu_A)\,(l\tfrac{1}{2}m'\mu|I_Aj_z)\\
&\qquad\times(l\tfrac{1}{2}m''\mu_P|I_Aj_z')\,(I_AI_Aj_zj_z'|I_B\mu_B)\sum_{k_N'}u^*_{k_N}(\boldsymbol{k}_N')\sum_q v_q\\
&\qquad\times\chi(|\boldsymbol{K}/2-\boldsymbol{k}_N'-\boldsymbol{q}|)\sum_{\boldsymbol{k}}\underset{(\mathrm{ext})}{\varphi_{lm}}(\boldsymbol{k})\,\underset{(\mathrm{ext})}{\varphi^*_{lm'}}(\boldsymbol{k}+\boldsymbol{q})\,\underset{(\mathrm{ext})}{\phi^*_{lm''}}(\boldsymbol{K}-\boldsymbol{k}_N'-\boldsymbol{q})\\
&\quad+\sum_{\mu_P\mu_Nmm'm''j_z}(\tfrac{1}{2}\tfrac{1}{2}\mu_P\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu_f|I_A\mu_A)\,(l\tfrac{1}{2}m'\mu_N|I_Aj_z)\\
&\qquad\times(l\tfrac{1}{2}m''\mu_P|I_Aj_z')\,(I_AI_Aj_zj_z'|I_B\mu_B)\sum_{k_N'(\mathrm{ext})}u_{k_N}(\boldsymbol{k}_N')\sum_q v_q\\
&\qquad\times\sum_{k'}\chi(|\boldsymbol{K}/2-\boldsymbol{k}'+\boldsymbol{q}|)\,\underset{(\mathrm{ext})}{\varphi_{lm}}(\boldsymbol{k}_N'+\boldsymbol{q})\,\underset{(\mathrm{ext})}{\varphi^*_{lm'}}(\boldsymbol{k}')\,\underset{(\mathrm{ext})}{\phi^*_{lm''}}(\boldsymbol{K}-\boldsymbol{k}'+\boldsymbol{q})\ ,
\end{aligned}
\tag{5·6}
$$

and

$$
\begin{aligned}
&M(D_{K\mu_D},\ A^{I_A}_{0\mu_A};\ P_{k_P\mu_f},B^{I'_B}_{m'\mu'_B})\\
&\quad=\sum_{\mu_N}(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu_N|I_Aj_z)\,(I_AI_A\mu_A j_z|I_B'\mu_B')\\
&\quad\times\sum_{k_P'(\mathrm{ext})}u^*_{k_P}(\boldsymbol{k}_P')\sum_q v_q\chi(|\boldsymbol{k}_P'+\boldsymbol{q}-(\boldsymbol{K}/2)|)\,\underset{(\mathrm{ext})}{\varphi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}_P')\\
&\quad-\sum_{\mu_N}(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu_N|I_Aj_z)\,(I_AI_Aj_z\mu_A|I_B'\mu_B')\\
&\quad\times\sum_{k_P'(\mathrm{ext})}u^*_{k_P}(\boldsymbol{k}_P')\sum_q v_q\chi(|\boldsymbol{k}_P'+\boldsymbol{q}-(\boldsymbol{K}/2)|)\,\underset{(\mathrm{ext})}{\varphi^*_{lm}}(\boldsymbol{K}-\boldsymbol{k}_P')\\
&\quad+\sum_{\mu_Nm\mu m'm''j_z}{}'(\tfrac{1}{2}\tfrac{1}{2}\mu_f\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu|I_A\mu_A)\,(l\tfrac{1}{2}m'\mu|I_Aj_z)\\
&\quad\times(l\tfrac{1}{2}m''\mu_N|I_Aj_z')\,(I_AI_Aj_zj_z'|I_{B'}\mu_{B'})\sum_{k'_P(\mathrm{ext})}u^*_{k_P}(\boldsymbol{k}_P')\sum_q v_q\\
&\quad\times\chi(|\boldsymbol{k}_P'+\boldsymbol{q}-(\boldsymbol{K}/2)|)\sum_{\boldsymbol{k}}\underset{(\mathrm{ext})}{\varphi_{lm}}(\boldsymbol{k})\,\underset{(\mathrm{ext})}{\varphi^*_{lm'}}(\boldsymbol{k}+\boldsymbol{q})\,\underset{(\mathrm{ext})}{\varphi^*_{lm''}}(\boldsymbol{K}-\boldsymbol{k}_P'-\boldsymbol{q})
\end{aligned}
$$

$$-\sum_{\mu_N m\mu m' m'' j_z}(l\tfrac{1}{2}\mu_f\mu_N|1\mu_D)\,(l\tfrac{1}{2}m\mu|I_A\mu_A)\,(l\tfrac{1}{2}m'\mu_N|I_A j_z)$$
$$\times(l\tfrac{1}{2}m''\mu|I_A j_z')\,(I_A I_A j_z j_z'|I_B'\mu_B')\sum_{k_P'\ (\mathrm{ext})}u^*_{k_P'}(k_P')\sum_q v_q$$
$$\times\chi(|k_P'+q-(K/2)|)\sum_{k\ (\mathrm{ext})}\varphi_{lm}(k)\,\varphi^*_{lm''}(k+q)\,\varphi^*_{l m'}(K-k_P'-q)\ . \tag{5·7}$$

In Equation (5·6) the 1st term gives the amplitude by Nagasaki's theory, the 2nd term gives the exchange amplitude by n-p force, the 3rd and the 4th give the direct and the exchange amplitudes by n-n force. In Equation (5·7), if we use the symmetry property of Clebsh-Gordan coefficient

$$(j_1 j_2 m_1 m_2|JM)=(-)^{j_1+j_2-J}(j_2 j_1 m_2 m_1|JM)\,, \tag{5·8}$$

the 1st and the 2nd terms give the amplitude by Nagasaki's theory, the 3rd and the 4th give the amplitude by the force between the proton of the incoming deuteron and a neutron of a target. Therefore, if the value of $2I_A-I_B$ is even, (d, p) stripping reactions cannot occur by the mechanism which we consider here and if the value is odd the amplitude is larger by factor 2 than the ordinary one.

As easily seen from Equations (5·6) and (5·7) the ampliudes of (d, n) and (d, p) reactions for such a target as O^{17} are different even if $I_B=I_B'$. This makes the angular distributions of neutron and proton to be of different shapes. The differential cross section of the stripping reaction can be expressed by,

$$\frac{da}{d\Omega}\propto\int\sum_{\mu_A\mu_D\mu_f\mu_B(\mu_{B'})}|M\cdot\delta(\omega_{k_N(k_P)}+E_0(A+1)-\tfrac{1}{2}\omega_K-E_D-E_0(A)|^2$$
$$\times d\omega_{k_N(k_P)}\,. \tag{5·9}$$

As an illustration we calculate the angular distribution of $O^{17}(d, p)O^{18}$ stripping reaction. In this case $I_A=5/2$, $I_B=0$ and $l=2$. Even in the simplest case of the two-body force of δ type, calculation of the exchange amplitude is not easy without the assumption that the spread of the deuteron wave function is negligibly small as compared with the radius of the target. The accompanying figures show the angular distribution curve of (1) the pure direct term, (2) the pure exchange term, (3) the cross term of direct and exchange amplitude and (4) the sum of (1), (2) and (3). The incident energy of deuteron is assumed to be 10 MeV (Fig. 1) and 4 MeV (Fig. 2), the binding energy of loosely bound neutron is assumed to be 4.143 MeV and then the final energy of emitted proton is 11.943 MeV and 5.943 MeV respectively. The nucler radius of O^{17} is assumed to be $3.96\cdot10^{-13}$ cm. From the result of the calculation we see that the exchange effect is of comparable order of magnitude with the direct part and modifies the shape of the angular distribution.

§ 6. Discussions

In the preceding sections we derived the general S matrix taking account of

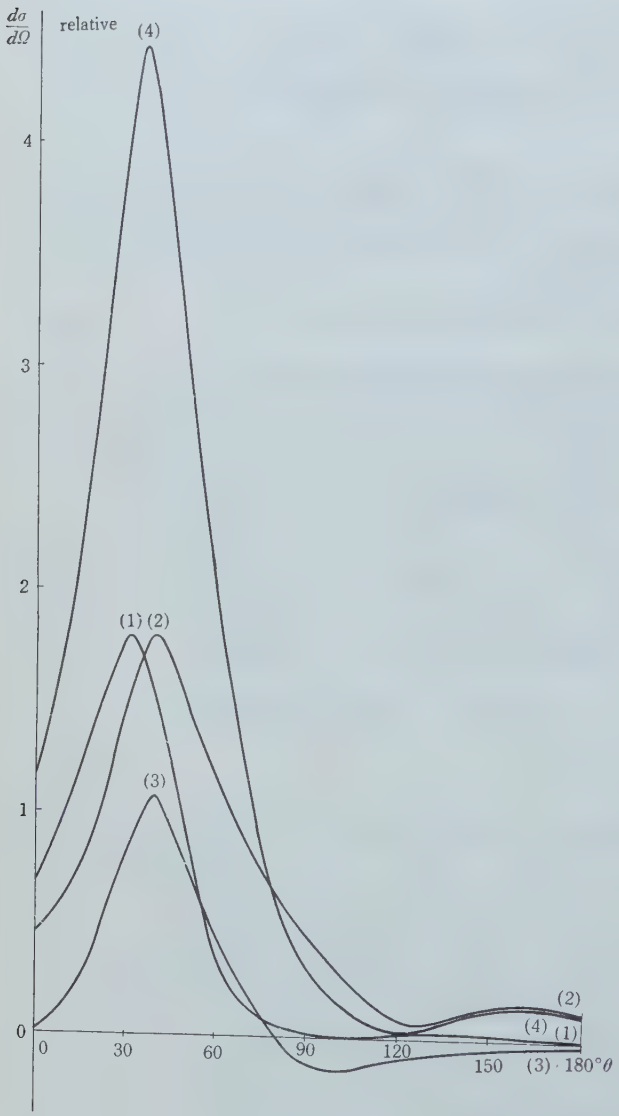

Fig. 1 $O^{17}(d, p)O^{18}$

(1)	Direct Term	E_D=10MeV
(2)	Exchange Term	E_P=11.943MeV
(3)	Cross Term	$R=3.96\cdot10^{-13}$cm
(4)	Total	

antisymmetrization of the total system and applied the method to the deuteron stripping reaction and showed by an example that the effect of antisymmetrization is appreciable. We took only the Born term into account but this procedure is not exactly correct even if we are interested in the external region, since the wave in the internal region may modify the wave in the external region. In order to refine the theory we should treat T matrix more exactly. One of the procedures to improve the theory may be to treat "compound process"[21,22] by considering the antisymmetrization of the formal theory of the nuclear reaction. The most important point of this procedure is what kind of nuclear model we should select for the composite particles in the initial, intermediate and final states.

As to the deuteron stripping reaction at moderate energies, the Butler and the Nagasaki amplitudes reproduce fairly well the general features of the angular distribution of stripped off nucleons. This seems to show that the approximation mentioned in Section 5 is fairly good. Therfore we selected a special target, for which both the Butler type amplitude and the exchange amplitude seem to be of comparable order of magnitude. Our result of calculation of $O^{17}(d, p)O^{18}$ shows that the exchange effect is appreciable. At about 10 MeV of the incident energy of the deuteron the angular distribution of emitted proton shows a forward peak as in the case of the pure Butler type, but the

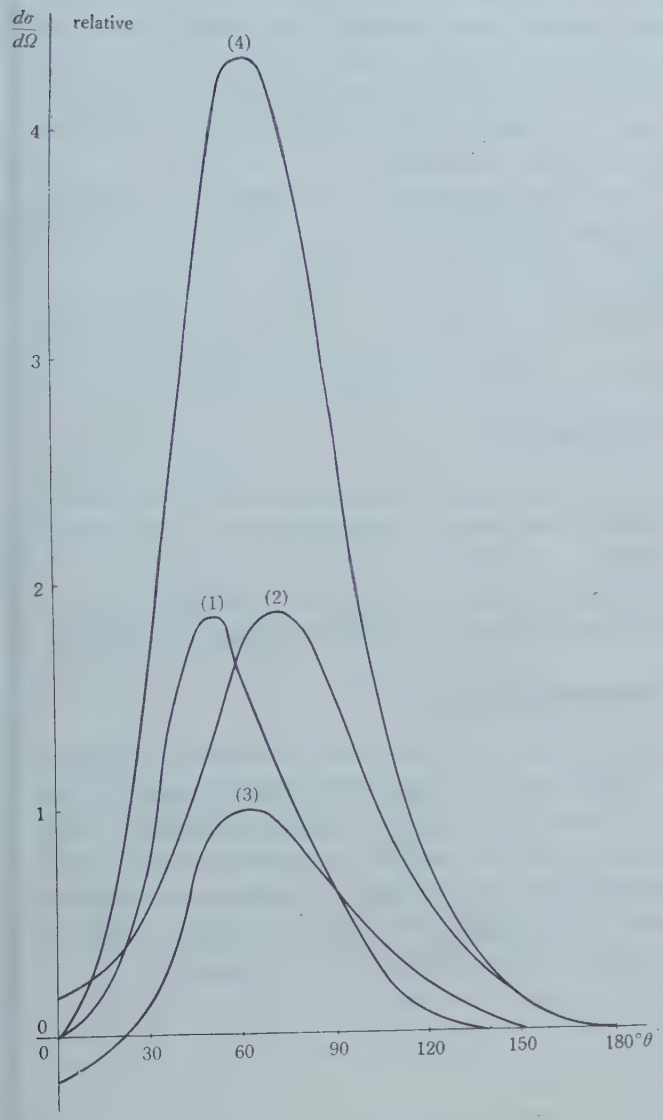

Fig. 2 $O^{17}(d, p)O^{18}$

(1) Direct Term E_D=4MeV
(2) Exchange Term E_P=5.943MeV
(3) Cross Term $R=3.96\cdot10^{-13}$cm
(4) Total

position of the peak is shifted to the large angle part by about 10°. The position of the peak depends on the radius R, i.e. a small R gives the peak at a large angle. The absolute value of the cross section is changed by about factor $\lesssim 2$ by the exchange effect. This absolute value also depends on the radius R, that is, a small R gives a large value. The same features are also seen at 4MeV of the incident energy of the deuteron. The peak of the angular distribution of emitted proton is shifted to the large angle part by about 20° or so in comparison with the case of 10 MeV. In both case, we must mention, the exchange amplitude modifies the backward component of the total cross section curve, while the pure Butler amplitude falls down rather in the forward angle region. And, if the radius R becomes larger, the exchange amplitude becomes of a large value in relative to the Butler one.

If the pure Butler amplitude is considered the wave of the emitted proton consists of $j_2(|\boldsymbol{K}-\boldsymbol{k}|\cdot r)$ only, where $\boldsymbol{K}$ and $\boldsymbol{k}$ are the wave numbers of deuteron and proton. On the other hand, the exchange amplitude gives three kinds of waves of the emitted proton, namely $j_0(|\boldsymbol{K}-\boldsymbol{k}|\cdot r)$, $j_2(|\boldsymbol{K}-\boldsymbol{k}|\cdot r)$ and $j_4(|\boldsymbol{K}-\boldsymbol{k}|\cdot r)$; which suggests the possibility of the spin flip reactions[29] and the modification of the angular distribution at low incident energies. The $O^{17}(d, p)O^{18}$ reaction is one

of the most typical reactions which is suitable to test the effect of antisymmetrization. Our method is also applicable to other target nuclei which have several extra nucleons. When the incident energy of the deuteron is so low that the final proton wave can be approximated by the distorted wave by means of the cloudy crystal ball model[37], Equation (2·24) shows that the transition matrix is approximated by the distortion of the final wave without the distortion of the initial one. This is because we use the exact interaction $V_{K\alpha}$.

Our method is most useful to treat the exchange character of two body interactions. In this case the interaction Hamiltonian is modified to

$$\mathcal{H}_I=\tfrac{1}{2}\sum_q v_q \sum_{k_n k_{n'}} \sum_{\substack{\mu_n \mu_{n'} \mu_{n''} \mu_{n'''} \\ \tau_n \tau_{n'} \tau_{n''} \tau_{n'''}}} <\mu_n'', \tau_n'' ; \mu_n''', \tau_n''' | P(\tau_x, \tau_y ; \sigma_x, \sigma_y) |$$

$$|\mu_n', \tau_n' ; \mu_n, \tau_n > a^*_{k_n+q\,\mu_n''\tau_n''}\, a^*_{k_{n'}-q\,\mu_n'''\tau_n'''}\, a_{k_{n'}\mu_{n'}\tau_{n'}}\, a_{k_n\mu_n\tau_n} ,$$

$$\Psi(x)=\sum_{k_n \mu_n \tau_n} a_{k_n \mu_n \tau_n} \frac{1}{\sqrt{\Omega}} e^{ik_n x}\, u(\mu_n)\chi(\tau_n) ,$$

where τ is the charge spin operator, σ is the normal spin operator and $P(\tau_x, \tau_y ; \sigma_x, \sigma_y)$ is the exchange operator of two-body interactions. The procedure of treating this term will not be changed substantially.

The method proposed in this paper may be applied to many other types of reactions and may give the first step for refinement of the theory of nuclear reactions.

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The Pion-Nucleon Dispersion Relation and Small *P*-Wave Phase Shifts

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The behavior of the small p-wave phase shifts are investigated, with use of the integrated forms of the G.M.O. dispersion relations, which are free from the ambiguity due to the principal integral of the total cross sections but are sensitive to the real part of the amplitude at low energies ($<$100Mev). For the (3/2, 3/2) amplitude the result from Puppi's analysis is used besides the experimental data, which greatly reduces the ambiguity due to the lack of precise experimental knowledge on the (3/2, 3/2) phase shift at very low energies. It is shown that Lomon-Chiu's data on the small p-wave phase shifts are definitely in contradiction with the dispersion relation for the isospin even amplitude. This conclusion is not altered even if every one of the small p-wave phase shifts is varied by 100%. The disparity is removed, however, if we use rather large negative δ_{13} at low energies ($<$100Mev). On the other hand, for the isospin odd amplitude, there is no such disparity irrespective of such alternative choices of the small p-wave phase shifts with the coupling constant of 0.07~0.10.

§ 1. Introduction and summary

In the pion physics at low energies the small p-wave phase shifts are yet quite obscure both theoretically and experimentally, so it is very interesting to investigate into this problem. In this article a crude information is given on the behavior of the small p-wave phase shifts, with use of the integrated form of the G.M.O. dispersion relation,[1] which can be easily obtained by integrating the G.M.O. relation multiplied by the factor $\tilde{\xi}^{-2}$ from $\tilde{\xi}=0$ to $\tilde{\xi}=L$ with variable $\tilde{\xi}$, the lab. sys. pion momentum.

In this integrated form, the integral of the real part of the scattering amplitude is expressed as the sum of the coupling constant term and the integral of the total cross sections. As the integrand of the cross section integral is singular only logarithmically at $\tilde{\xi}=L$, the integral is expected to be fairly insensitive to the detailed shape of the total cross sections. Further, the convergence of this integral is very good. On the other hand the real part integral is sensitive to the behavior of the phase shifts at low energies. Therefore, we can study in what extent the behavior of the small p-wave phase shifts may be compatible with the dispersion relations, if we can avail ourselves of the more reliable information about (3/2, 3/2) phase shift and s-wave phase shifts empirically or theoretically.

Detailed analysis is given in section 3. The sources of the experimental data used in the present analysis are the following: Total cross sections from Anderson

et al.[2] (below 400 Mev) and Cool et al.[3] (from 0.4 to 2 Bev). The small p-wave phase shifts and the s-wave phase shifts from Lomon and Chiu.[4] The s-wave scattering lengths from Salzman and Schnitzer.[5] For the (3/2, 3/2) phase shift, the result from Puppi's analysis with Chew's static theory[7] is used besides Anderson's[8] empirical formula. L, the upper limit of the integral of the real part, is set equal to 300 Mev in this analysis. The results are as follows:

1) There remains a certain discrepancy for the isospin even amplitude* so far as we use Lomon-Chiu's small p-wave phase shifts extrapolated to the threshold from about 100 Mev by assuming $\delta \propto \eta^3$. η is the pion momentum in c.m. system. This conclusion is not affected even if every one of the small p-wave phase shifts is changed by 100% in the energy range below 300Mev.

2) The discrepancy can be removed, however, if we adopt alternative δ_{13}, which is negative and rather large in magnitude at low energies; δ_{13} is assumed to be $-(0.05\sim0.1)\eta^3$ below 100 Mev.

3) For the isospin odd amplitude there appears no such disparity irrespective of the wide variation of small p-wave phase shifts with the coupling constant of 0.07 $\sim$0.10.

§ 2. The integral form of the dispersion relation

In this section we derive a relation which is more suitable for investigating the behavior of small p-wave phase shifts at low energies, from the G.M.O. dispersion relation. The guiding principle for deriving such a relation is to avoid the ambiguity originated in the principal integral of the total cross sections in the G.M.O. relation and to make the resultant relation competently sensitive to the phase shifts at low energies. The principal integral in the original relations is very sensitive to the detailed shape of the total cross sections near the resonance and, further, it yields unwanted error of 10$\sim$20 percent in numerical integration.[9]

We start from G.M.O. dispersion relations for the isospin even amplitude,

$$h^e(\tilde{\xi}) = \frac{8\pi g^2}{m}\frac{\tilde{\xi}^2}{\tilde{\xi}^2+1} + \frac{2}{\pi}\tilde{\xi}^2 P\int_0^\infty \frac{d\tilde{\xi}'}{\tilde{\xi}'^2-\tilde{\xi}^2} f^e(\tilde{\xi}'), \tag{1}$$

and for the isospin odd amplitude,

$$h^o(\tilde{\xi}) = -16\pi g^2\frac{\tilde{\xi}^2}{\tilde{\xi}^2+1} + \frac{2}{\pi}\tilde{\xi}^2 P\int_0^\infty \frac{d\tilde{\xi}'}{\tilde{\xi}'^2-\tilde{\xi}^2} f^o(\tilde{\xi}'), \tag{2}$$

with

$$f^e(\tilde{\xi}) = (\sigma^+ + \sigma^-)_{\tilde{\xi}}, \quad f^o(\tilde{\xi}) = \frac{(\sigma^- - \sigma^+)_{\tilde{\xi}}}{\sqrt{1+\tilde{\xi}^2}}$$

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* The isospin even (odd) amplitude means the amplitude which is even (odd) in exchanging the initial and final pion isospin indices.

$$h^e(\tilde{\xi}) = \frac{1}{m}\mathrm{Re}[D^e(\tilde{\xi}) - D^e(0)],$$

$$h^o(\tilde{\xi}) = \frac{1}{m\sqrt{1+\tilde{\xi}^2}}[\mathrm{Re}\,D^o(\tilde{\xi}) - \sqrt{1+\tilde{\xi}^2}\,\mathrm{Re}\,D^o(0)].$$

Here $\tilde{\xi}$ is the pion momentum in the laboratory system, and $D^e(\tilde{\xi})$ and $D^o(\tilde{\xi})$ are the isospin even and odd amplitudes of the pion-nucleon scattering, respectively. σ^+ and σ^- are the total $\pi^+ - p$ and $\pi^- - p$ cross section, respectively. m is the nucleon rest mass and g is the small coupling constant, renormalized but unrationalized. The symbol P stands for the principal integral. The unit $\hbar = c = \mu(\textit{pion rest mass}) = 1$ is used throughout in this article. Multiplying (1) [or (2)] by the weight-function $F(\tilde{\xi}) = (2/\pi)\tilde{\xi}^{-2}$,* we integrate both sides of (1) [or (2)] from $\tilde{\xi}=0$ to $\tilde{\xi}=L$. Then, we have the following relations:

$$\frac{2}{\pi}\int_0^L \frac{d\tilde{\xi}}{\tilde{\xi}^2} h^e(\tilde{\xi}) = -f^e(0) + \frac{16g^2}{m}\tan^{-1}L + \frac{2}{\pi^2}\int_0^\infty d\tilde{\xi}\,\frac{f^e(\tilde{\xi})}{\tilde{\xi}}\ln\left|\frac{L+\tilde{\xi}}{L-\tilde{\xi}}\right|, \tag{3}$$

and

$$\frac{2}{\pi}\int_0^L \frac{d\tilde{\xi}}{\tilde{\xi}^2} h^o(\tilde{\xi}) = -f^o(0) - 32g^2\tan^{-1}L + \frac{2}{\pi^2}\int_0^\infty d\tilde{\xi}\,\frac{f^o(\tilde{\xi})}{\tilde{\xi}}\ln\left|\frac{\tilde{\xi}+L}{\tilde{\xi}-L}\right|. \tag{4}$$

There are several characteristics in these relations. First, the integral of the total cross sections are fairly convergent at high energies, so the ambiguity coming from the unknown high energy region is very small. Second, the integrands of the integrals are singular at $\tilde{\xi}=L$ only logarithmically, so the integrals are not sensitive to the detailed shape of the total cross sections. On the other hand, the integrals of the real part are sensitive to the behavior of the phase shifts at low energies. The use of Chew's static theory for $(3/2, 3/2)$ amplitude greately reduces the ambiguity of the present analysis (see next section).

§ 3. Discussion with experimental data.

Next we shall investigate how the small p-wave phase shifts may be compatible with relations (3) and (4). It is desirable to make the parameter L appropriately large to obtain a less ambiguous conclusion from the present experimental data. To make L smaller brings out the larger ambiguity as it should be from the derivation of the relations. In this paper we choose L to be 3, which corresponds to about 300 Mev of the pion kinetic energy in lab. sys. We shall study the problem for the isospin even and odd amplitude separately.

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* In general, the weight-function $F(\tilde{\xi})$ is somewhat arbitrary, except that F must not be more singular than $\tilde{\xi}^{-2}$ at the threshold. The choice of $F(\tilde{\xi})$ in this paper makes the integrated relation simplest, and further minimizes the ambiguity of the cross section integral. The relations (3) and (4) are also obtainable[12] by combining Gilbert's new dispersion relation[10] with G.M.O. relation[6].

i) *Isospin even amplitude.*

First we shall investigate the right side of (3), which is set to I_R^e

$$I_R^e \equiv \frac{16}{m} g^2 \tan^{-1} L - f^e(0) + \frac{2}{\pi^2} \int_0^\infty d\xi \, \frac{f^e(\xi)}{\xi} \ln \left| \frac{L+\xi}{L-\xi} \right| . \tag{3'}$$

The first term of (3′) is easily calculated with $m=6.725$ and $L=3$, and the result is

$$-2.97\, g^2. \tag{5}$$

$f^e(0)$, sum of the total cross sections at the threshold, can be expressed as

$$f^e(0) = (\sigma^+ + \sigma^-)_{\text{threshold}} = (8\pi/3)\,(\alpha_1^2 + 2\alpha_3^2), \tag{6}$$

where α is the s-wave scattering length and the suffix 1 or 3 refers to the isospin singlet or triplet state. The latest data by Salzman and Schnitzer[5] are

$$\alpha_1 = 0.167 \pm 0.01, \quad \alpha_3 = -0.10 \pm 0.01. \tag{7}$$

These α's give

$$f^e(0) = 0.41 \pm 0.06. \tag{8}$$

The integral in (3′), say H^e, is the main part of I_R^e. To facilitate the discussion, we separate the high energy part from the integral which corresponds to the domain of the integration above 2 Bev ($\xi = 15 = M$), where the total cross sections are not known experimentally. Further, we must take special care in integrating over the point $\xi = L$, where the integrand is singular logarithmically. Thus we separate the integral into three parts, namely :

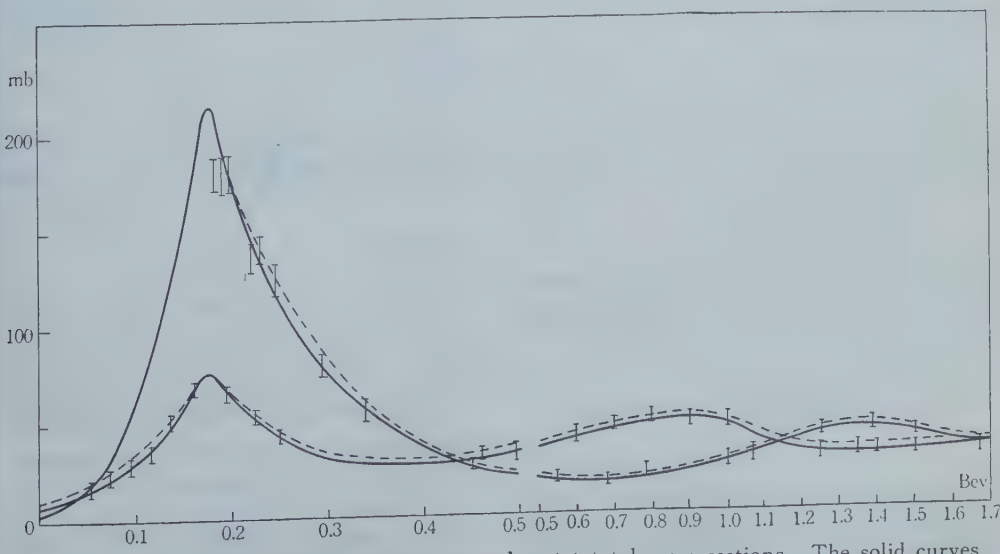

Fig. 1. Energy dependence of the $\pi^+ + p$ and $\pi^- + p$ total cross sections. The solid curves refer to the experimental cross sections[9,10], and the dashed curves are those modified so as to reduce the disparity within the experimental errors.

$$H^e = H_1^e + H_2^e + V^e,$$

$$H_1^e = \frac{2}{\pi^2}\int_0^{L-\Delta} d\tilde{\xi}\,\frac{f^e(\tilde{\xi})}{\tilde{\xi}} \ln\left|\frac{\tilde{\xi}+L}{\tilde{\xi}-L}\right| + \frac{2}{\pi^2}\int_{L-\Delta}^{M} d\tilde{\xi}\,\frac{f^e(\tilde{\xi})}{\tilde{\xi}} \ln\frac{\tilde{\xi}+L}{\tilde{\xi}-L}, \tag{9}$$

$$H_2^e = \frac{2}{\pi^2}\int_{L-\Delta}^{L+\Delta} d\tilde{\xi}\,\frac{f^e(\tilde{\xi})}{\tilde{\xi}} \ln\left|\frac{\tilde{\xi}+L}{\tilde{\xi}-L}\right|,$$

$$V^e = \frac{2}{\pi^2}\int_M^{\infty} d\tilde{\xi}\,\frac{f^e(\tilde{\xi})}{\tilde{\xi}} \ln\left|\frac{\tilde{\xi}+L}{\tilde{\xi}-L}\right|.$$

H_1^e is numerically integrable with suitable choice of Δ. In this paper Δ is chosen as 0.1. The experimental data are taken from the results of Anderson et al.[2] (below 400 Mev) and of Cool et al.[3] (from 0.4 to 2 Bev). Above resonance the statistics is not so good and there may be errors of about 10 per cent in these data. The corresponding error to H_1^e is estimated from the modified curves in Fig. (1).

The result thus obtained is

$$H_1^e = 4.02 \pm 0.21. \tag{10}$$

H_2^e is analytically integrable with the Taylor expansion of the integrand in powers of $(\tilde{\xi}-L)$ and leads to a function of $f^e(L)$, its first and second derivatives with sufficient accuracy. $f^e(L)$ and its derivatives estimated from the above data (see Fig. 2) are

$$f(L) = 5.1 \pm 0.5, \quad f'(L) = -5.1,$$

$$f''(L) = 7.3.$$

These values lead to

$$H_2^e = 0.35 \pm 0.04. \tag{11}$$

The ambiguity associated with these derivatives gives rise to certain amount of error, namely,

$$\delta H_2^e = -0.000068\,\delta f'^e + 0.0001\,\delta f''^e. \tag{12}$$

The smallness of the factors in (12) ensures that H_2^e is extremely insensitive to the detailed shape of the total cross sections.

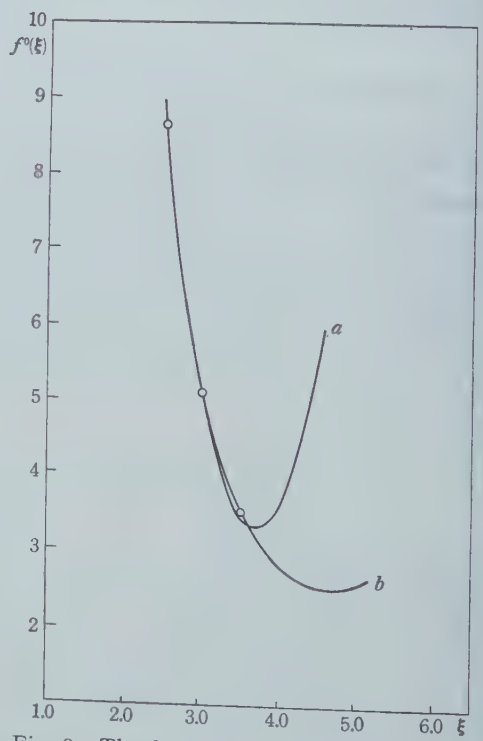

Fig. 2. The first and second derivatives of $f(\xi)$ at $\xi = L$ are estimated by assuming the form $f(\xi) = a\xi^2 + b\xi + c$, near $\xi = 3$. The best fit is obtained with $a = 3.64$, $b = -27.0$, and $c = 53.3$. This parabola (curve a) is shown together with the cross section curve b.

The high energy contribution V^e can be easily estimated by assuming the constant cross sections above 2 Bev. The value of $f^e(\tilde{\xi})$ at 2 Bev is, according to Cool et al.,

$$f^e(2\text{ Bev}) = (60 \pm 6)_{\text{mb}},$$

which leads to

$$V^e = 0.25 \pm 0.03. \tag{13}$$

The fairly good convergence of the integral ensures that V^e does not increase by so much amount unless the cross sections increase considerably with energy of pion. In fact, if $f^e(\tilde{\xi})$ increase linearly up to $2f^e$ (2 Bev) at 20 Bev and then stays constant above 20 Bev, the increment of V^e is about 20% of the value of (13). The total cross sections would rather decrease with increasing energy, for the forces with shorter range would play an essential role at very high energies. Then, we believe (13) may be an over-estimate.

From (5), (8), (10), (11), and (13) we have

$$\begin{aligned} I_R^e &= 2.97g^2 - (0.41 \pm 0.06) + (4.02 \pm 0.21) \\ &\quad + (0.35 \pm 0.04) + (0.25 \pm 0.03) \\ &= 2.97g^2 + 4.21 \pm 0.34. \end{aligned} \tag{14}$$

Next we examine the left-hand side of (3) :

$$I_L^e = \frac{2}{\pi}\int_0^L \frac{d\tilde{\xi}}{\tilde{\xi}^2} h^e(\tilde{\xi}) = \frac{2}{\pi m}\int_0^L \frac{d\tilde{\xi}}{\tilde{\xi}^2}\{\text{Re}D^e(\tilde{\xi}) - \text{Re}D^e(0)\}. \tag{15}$$

The phase shifts expansion of the forward scattering amplitude D^e is given by[10]

$$D^e(\tilde{\xi}) = \frac{8\pi E}{\eta}\sum_{l=0}^{\infty}[la_{l-}^e + (l+1)a_{l+}^e], \tag{16}$$

where E and η are the center of mass total energy and meson momentum, respectively. The even amplitude a^e is related to both the isospin singlet and triplet amplitudes by

$$a^e = \tfrac{1}{3}(a_1 + 2a_3). \tag{17}$$

Each of these amplitudes is expressed by a relevant phase shift as $a = e^{i\delta}\sin\delta$. The suffix $l_\pm$ in (16) refers to the state $j = l \pm \frac{1}{2}$.

If we restrict ourselves to keep only s and p waves in the expansion (16), the integral (15) can be reduced to

$$I_L^e = P_3^e + P_1^e + S^e \tag{18}$$

with

$$P_3^e = \frac{32}{3}\int_0^L \frac{d\tilde{\xi}}{\eta^2\tilde{\xi}}\sin 2\delta_{33}, \tag{19a}$$

$$P_1^e=\frac{8}{3}\int_0^L\frac{d\xi}{\eta^2\xi}(\sin 2\delta_{11}+2\sin 2\delta_{31}+2\sin 2\delta_{13}), \tag{19b}$$

$$S^e=\frac{8}{3}\int_0^L\frac{d\xi}{\xi}\left\{\frac{1}{\eta^2}(\sin 2\delta_1+2\sin 2\delta_3)-\frac{2}{\xi}\left(1+\frac{1}{m}\right)(\alpha_1+2\alpha_3)\right\}. \tag{19c}$$

P_3^e represents the contribution from the $(3/2, 3/2)$ p-state, P_1^e the contribution from the other p-states, and S^e from s-states.

At first we discuss P_3^e, which is the largest of all terms in (18). Using Anderson's empirical formula,[8] we have the following value of P_3^e,

$$P_3^e(Anderson)=7.51. \tag{20}$$

However, we need rather precise information about the phase shifts from 150 Mev to the threshold in order to obtain a reliable result because of the denominator $\eta^2\xi$ in the integrand of P_3^e. Then, it is better to calculate P_3^e, theoretically, by means of Chew's static theory, since the theory is satisfactory on the $(3/2, 3/2)$ scattering phenomena below and near the resonance and further contribution from the above resonance is only few per cent of P_3^e. According to Puppi et al.[6] the $(3/2, 3/2)$ phase shift is given by

$$\tan\delta_{33}=\frac{4\eta^3}{3\omega^*}g^2\left(1-\frac{\omega^*}{\omega_0}\right),\qquad \omega^*=\omega+\frac{\eta^2}{2m}, \tag{21}$$

with

$$g^2=0.095\pm0.006,\qquad \omega_0=2.22\pm0.20.$$

The corresponding value of P_3^e is

$$P_3^e=7.72\pm0.40. \tag{22}$$

$P_3^e(Anderson)$ falls within the errors given above.

Next we examine the integral S^e. The latest data on s-wave phase shifts are given by Lomon-Chiu,[11] which are shown in Fig. 3, with extrapolation down to the threshold. From these data we have the following value of S^e,

$$S^e=-0.20. \tag{23}$$

However, the scattering lengths obtained by the above extrapolation are $\alpha_1=0.144$ and $\alpha_3=-0.130$, which differ appreciably from the values by Salzman-Schnitzer (7). Therefore we calculate alternatively the value of S^e by modifying these curves of phase shifts so as to give the scattering lengths by Salzman-Schnitzer at the threshold. The modified curves are also shown in Fig. 3. These curves give the following value of S^e,

$$S^e_{\mathrm{Mod}}=-0.53. \tag{24}$$

Further, if Anderson's phase shifts are modified to give Salzman's scattering lengths, we obtain the value of S^e about -0.5. So we believe the true value of S^e falls within the range $-0.80\sim-0.20$, i.e.,

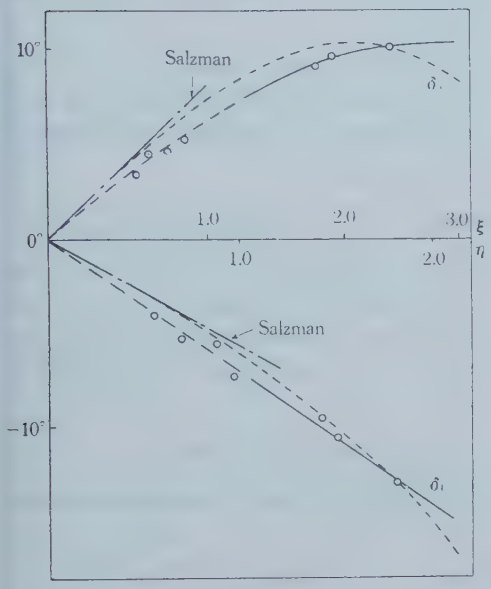

Fig. 3. S-wave phase shifts δ_1 and δ_3 as functions of poin momentum. The solid curves refer to Lomon-Chiu's phase shifts extrapolated to the threshold, and the dashed curves are those modified so as to give the scattering lengths by Salzman and Schnitzer at the threshold.

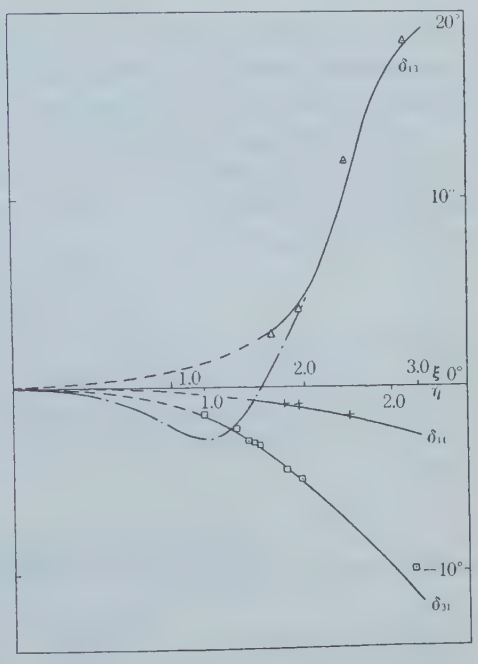

Fig. 4. Small p-wave phase shifts, δ_{13}, δ_{31}, and δ_{11}, as functions of pion momentum. The solid curves refer to Lomon-Chiu's phase shifts extrapolated to the thereshold by assuming the form of $\delta \propto \eta^3$. The dot-dashed curve of δ_{13} is that modified so as to give large negative δ_{13} at low energies; the form $\delta_{13} = -0.10\eta^3 + 2.7\eta^5$ is assumed for $\eta < 1$.

$$S^e = -0.50 \pm 0.3. \tag{25}$$

Now we are in the stage to see in what extent the behavior of the small p-wave phase shifts may be compatible with the dispersion relations. From (14), (18), (22), and (25) we have

$$\begin{aligned} P_1^e &= I_R^e - (P_3^e + S^e) \\ &= -3.0 + 2.97 g^2 \pm 1.0. \end{aligned} \tag{26}$$

The value of (26) with the reasonable value of coupling constant must be equal to that of the integral (19b) of the small p-wave phase shifts.

At first we examine the consistency of the data on these phase shifts with the experimental data known up to date. The most recent data on these phase shifts are these reported by Lomon-Chiu,[11] though their reliability is not yet so certain. If each of these phase shifts is extrapolated to the threshold by assuming $\delta \propto \eta^3$ (see Fig. 4), the integral (19b) is calculated to give

$$P_1^e = -0.11. \tag{27}$$

There is a disparity of $2.6\pm1.0\sim2.7\pm1.0$ between (27) and (26). Above disparity is not removed even by varying one of the phase shifts by 100% to decrease the discrepancy, leaving other parameters unchanged.

Next we change δ_{13} so artificially that it is negative at low energy ($\delta=-0.10\eta^3+2.7\eta^5$ for $\eta<1$) and changes its sign so as to connect with Lomon's curve at about 180 Mev (see Fig. 4). There are some supports as to such a modification. The following result,

$$\delta_{13}/\eta^3=-0.055\pm0.06, \tag{28}$$

is reported at Genèva Conference (1958) as the phase shift at 41.5 Mev. And further, Chiu and Lomon insist on the unambiguously positive value of δ_{13} at 150 and 170 Mev. The above modified phase shift, others remaining unchanged, gives $P_1^e=-1.5$. The discrepancy is barely vanished. Above results are summarized in Table 1.

Table 1. The disparity Δ between (26) and P_1^e, calculated with variety of the small p-wave phase shifts are shown.

Coupling constant	$I_R^e-(P_3^e+S^e)$	Lomon's phase shifts extrapolated with $\delta\propto\eta^3$		One of phase shifts 100% modified		Negative δ_{13} below 100 MeV	
		P_1^e	Δ	P_1^e	Δ	P_1^e	Δ
0.10	-2.7 ± 1.0	-0.1	2.6 ± 1.0	-0.9	1.8 ± 1.0	-1.5	1.2 ± 1.0
0.07	-2.8 ± 1.0	-0.1	2.7 ± 1.0	-0.9	1.9 ± 1.0	-1.5	1.3 ± 1.0

ii) *Isospin odd amplitude.*

At first we examine the right side of (4):

$$I_R^o=-32g^2\tan^{-1}L-f^o(0)+\frac{2}{\pi^2}\int_0^\infty d\xi\frac{f^o(\xi)}{\xi}\ln\left|\frac{L+\xi}{L-\xi}\right|. \tag{4'}$$

The first term in (4′) is just $2m$ times as large in magnitude as the corresponding term in (3′). With $L=3$, it becomes

$$-40.0g^2. \tag{29}$$

The second term can be written as, except the sign,

$$f^o(0)=\frac{(\sigma^--\sigma^+)}{\sqrt{1+\xi^2}}\bigg|_{\xi=0}=\frac{8\pi}{3}(\alpha_1^2-\alpha_3^2). \tag{30}$$

Using the data by Salzman et al. (7), we have the following value as $f^o(0)$,

$$f^o(0)=0.15\pm0.04. \tag{31}$$

The integral in (4′) converges more rapidly than the corresponding integral in (3′), for it has the additional factor $\sqrt{1+\xi^2}$ in the denominator of the integrand. The similar analysis as in the previous subsection yields the result,

$$H^{o}\equiv\frac{2}{\pi^2}\int_0^\infty d\xi\frac{f^{o}(\xi)}{\xi}\ln\left|\frac{\xi+L}{\xi-L}\right| \\ =-0.57\pm0.03. \tag{32}$$

The contribution from the high energy region (above 2 Bev) is quite negligible. From (30), (31), and (32) we have

$$I_R^{o}=-0.72\pm0.07-40.0g^2. \tag{33}$$

Next, we examine the left side of (4),

$$I_L^{o}=\frac{2}{\pi}\int_0^L\frac{d\xi}{\xi^2}h^{o}(\xi)=P_3^{o}+P_1^{o}+S^{o} \tag{34}$$

with

$$P_3^{o}=-\frac{16}{3}\int_0^3\frac{d\xi}{\xi\eta^2\sqrt{1+\xi^2}}\sin2\delta_{33}, \\ P_1^{o}=\frac{8}{3}\int_0^3\frac{d\xi}{\xi\eta^2\sqrt{1+\xi^2}}(\sin2\delta_{11}-\sin2\delta_{31}+2\sin\delta_{13}), \tag{35} \\ S^{o}=\frac{8}{3}\int_0^3 d\xi\left\{\frac{1}{\xi\eta^2\sqrt{1+\xi^2}}(\sin2\delta_1-\sin2\delta_3)-\frac{2}{\xi^2}\left(1+\frac{1}{m}\right)(\alpha_1-\alpha_3)\right\}.$$

Chew's static theory, by the use of (21), results the fairly accurate value of P_3^{o},

$$P_3^{o}=-2.89\pm0.14. \tag{36}$$

Use of the Anderson's formura gives also -2.89 as the numerical value of P_3^{o}, though this very coincidence is only by chance. S^{o} is estimated quite analogously as S^{e} and is led to

$$S^{o}=-1.23\pm0.1. \tag{37}$$

From Lomon-Chiu's data on small p-wave phase shifts, extrapolated naively down to the threshold, we obtain the following value as P_1^{o},

$$P_1^{o}=0.51. \tag{38}$$

And thus, from (35), (36), and (37), we have

$$I_R^{o}=-3.61\pm0.3,$$

which, combined with (33), gives

$$g^2=0.072\pm0.01.^{*)} \tag{39}$$

This in good agreement with the authorized value 0.08. It is interesting to see how the coupling constant changes if we use the alternative sets of phase shifts,

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* (39) dose not involve the error corresponding to the ambiguity associated with small p-wave phase shifts.

as has been discussed for the isospin even amplitude. If we set δ_{13} equal to zero, P_1^0 decreases by 50% in magnitude and the corresponding value of coupling constant is 0.079 ± 0.01. The variation of one of other small p-wave phase shifts by 100% gives rise to smaller change in the coupling constant. The choice of negative δ_{13} at low energies ($\lesssim 100$ Mev) results $P_1^0 = -0.61$, and correspondingly $f^2 = 0.10 \pm 0.01$. Thus, in this case, contrary to the isospin even case, we have the reasonable coupling constant irrespective of the wide variation of the small p-wave phase shifts.

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Characterization of (Y, Z)- and (Λ, Σ)-Modes

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New assignment of isotopic spin and the concepts of K-mesonic charge and of K-mesonic charge conjugation are introduced in order to characterize (Y, Z)- and (Λ, Σ)-modes. If the selection rule $|\Delta I|=1/2$ is assumed for weak interactions, suitable combinations of them can be expressed in terms of (Y, Z)-mode, and they are invariant under K-mesonic charge conjugation, and under the rotation in isotopic spin space, owing to the new assignment of isotopic spin. Then in decay processes neutral K-mesons appear always in (K_1^0, K_2^0)-mode. Strong interactions are naturally invariant under the rotation. Therefore the newly assigned isotopic spin can be treated as an absolutely conserved quantity in an approximation. The invariancy of strong interactions under K-mesonic charge conjugation is discussed. It is shown that characteristic features of K-mesonic charge can be united well with the model proposed in a previous article[1]. Thus, some close correlation between the three selection rules about hyperonic charge, isotopic spin and space pariy, which characterize the strength of interactions is achieved.

§ 1. Introduction

Special emphasis was expressed by the present author on the fact that there exists some kind of correlation between two conservation laws of hyperonic charge and ordinary space parity[1]. While both conservation laws hold in strong interactions, as for weak interactions it is possible to take the standpoint that neither of them is conserved in the zeroth approximation where the contributions from various kinds of cloud surrounding bare particles are neglected. In the article (A), it was shown that we could construct a tentative model which gives the desired correlation between the two conservation laws. The degree of freedom of isotopic spin, I, whose selection rule also seems useful to characterize the strength of interactions, was laid up there. The selection rule about isotopic spin was first proposed to be $\Delta I=\pm\frac{1}{2}$ for weak interactions[2]. But some objections[3] compelled us to take into account other weaker interactions with various values of ΔI. However, recently the experimental value[4] of the branching ratio between the decays $\theta_1^0\to\pi^+ +\pi^-$ and $\theta_1^0\to 2\pi^0$, which was one of the important objections to the simplest selection rule $\Delta I=\pm\frac{1}{2}$, was revised, and the objection shows a tendency to disappear. Further, it may have some heuristic significance to assume the selection rule $\Delta I=\pm\frac{1}{2}$ in order to find a comprehensive but tentative model which describes characteristic features of interactions between elementary particles, even if there remain some

objections*. Then there are three kinds of selection rules which characterize the discrimination between strong and weak interactions: selection rules about hyperonic charge, isotopic spin and ordinary space parity. These selection rules are treated independently in the conventional theory, but it is desirable to correlate them with each other. In this article a kind of characterization of the (Λ, Σ)- and (Y, Z)-modes is proposed as a stepping stone to correlate the selection rule about isotopic spin with the one about hyperonic charge.

The concerned correlation was already taken into account by the spurion method[6], but it is not sufficient as spurion is not physical entity in any sense and the interactions which are obtained by eliminating spurion variables have no clear prospect. On the other hand, searching for higher symmetry than charge independence, it was proposed[7),8)] to rearrange Λ and $\boldsymbol{\Sigma}$ which are a scalar and a vector in isotopic spin space into two spinors in the space, $Y=(\Sigma^+, Y^0)$ and $Z=(Z^0, \Sigma^-)$, where $Y^0=1/\sqrt{2}\cdot(\Lambda^0-\Sigma^0)$ and $Z^0=1/\sqrt{2}\cdot(\Lambda^0+\Sigma^0)$. As will be shown in the Appendix, if we use (Y, Z)-mode in stead of (Λ, Σ)-mode, some combinations of the interactions obtained by the spurion method can be simply expressed in terms of them, and there remains no unphysical variable.

At this stage, we may ask the physical significance of the modes Y^0 and Z^0. (Y^0, Z^0) is nothing but a different description of the same physical entity as (Λ^0, Σ^0). The similar situation has already been seen as to neutral K-mesons, that is, there are two ways of description, $(K^0, \widetilde{K}^0)$ and (K_1^0, K_2^0). The $(K^0, \widetilde{K}^0)$-mode is a suitable one to production processes and characterized by hyperonic charge. On the other hand, K_1^0 and K_2^0 are the modes which appear in decay processes, and they are characterized by C- or PC-parity. Further, there is a definite relation between hyperonic charge and the operation of particle-antiparticle conjugation, which was fully used in the previous article (A). By the analogy to the case of neutral K-mesons, we may expect the existence of such operators that characterize the (Y^0, Z^0)- and (Λ^0, Σ^0)-modes, and of some relation between them. Moreover, the physical processes in which the concerned particles participate will be classified into two classes in such a way that the (Y^0, Z^0)-mode is realized in the interactions belonging to the one of the classes and (Λ^0, Σ^0)-mode in the others. In the next section such a conjecture will be assured. There a new assignment of isotopic spin and the concepts of K-mesonic charge and of K-mesonic charge conjugation are introduced, and then (Y^0, Z^0)- and (Λ^0, Σ^0)-modes are characterized by them. Further, a new standpoint about the conservation of isotopic spin will be proposed. In § 3 the connection of the present scheme with the model proposed in the article (A) will be explained.

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* The remaining objections may be taken into account in some corrections. In the analysis of strange particles the effects of cloud of various fields are not trivial, and can change the behaviour of bare particles considerably. For instance, the proportion of parity violation may be varied extraordinarily[5].

§ 2. Characterization of (Y, Z)- and (Λ, Σ)-modes

Corresponding to the situation that the isotopic spins of Y and Z states are different from the ones of Λ and Σ states, isotopic spins of elementary particles must be reassigned before everything. In the following this newly defined isotopic spin is denoted by $\mathcal{I}$ in order to be discriminated from the originally defined one, I. As to nucleon, Ξ, Y and Z particles, they may be described as doublets with respect to $\mathcal{I}$, as is usually done. In order to determine $\mathcal{I}$ of K-mesons, we use the *hyperonic charge symmetry** which was already pointed out in the article (A). The symmetry is as follows: The following combination of the strong K-meson interactions which were proposed by d'Espagnat and Prentki[12],

$$\frac{g}{\sqrt{2}}\{\bar{N}K\Lambda+\bar{N}\boldsymbol{\tau}\boldsymbol{\Sigma}K+i\bar{\Xi}\tau_2K^*\Lambda+i\bar{\Xi}\boldsymbol{\tau}\boldsymbol{\Sigma}\tau_2K^*+\text{herm. conj.}\}, \tag{1}$$

where

$$K=\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \quad \text{and} \quad i\tau_2K^*=\begin{pmatrix} K^{0*} \\ -K^{+*} \end{pmatrix}=\begin{pmatrix} \widetilde{K}^0 \\ -K^- \end{pmatrix}, \tag{2}$$

can be rearranged in the from

$$g\{\bar{N}YK^0+\bar{N}ZK^+-\bar{\Xi}YK^-+\bar{\Xi}Z\widetilde{K}^0+\text{herm. conj.}\}. \tag{3}$$

The interactions which appear in the above combination are invariant under the rotation in $\mathcal{I}$ space, if we consider four K-mesons as four scalar fields in $\mathcal{I}$ space.** As for π-mesons, we may consider it as $\mathcal{I}$-triplet as usual. Present experimental results do not seem to permit such symmetry where both hyperonic charge symmetry and global symmetry hold simultaneously, as analysed by Pais[9], hence we must choose either of them. Hyperonic charge seems to us more fundamental, and the theory in which hyperonic charge plays a fundamental role, for instance, the tentative model proposed in (A), seems to introduce hyperonic charge symmetry automatically. Therefore, we accept this symmetry here as a kind of zeroth approximation.***

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* This symmetry was also proposed independently by Pais[9], and Sakurai[10] who named it "cosmic symmetry". But this name does not seem to fit, as there may be such a more highly symmetrical world as proposed in another article[11] and "cosmos" is the largest world. K-mesons have non-zero hyperonic charge and may be considered as such boson fields that discriminate the hyperonic charge of hyperons. Therefore our nomination may be more suitable.

In the theory in which hyperonic charge is taken as a fundamental quantity such a symmetry is considered as a natural result. But for the assignment of $\mathcal{I}$ it is sufficient as a starting point to assume that $\mathcal{I}$ is conserved at least in the interactions where hyperonic charge is conserved.

** The similar assigment was also proposed by many authors[9)5)].

*** At the present stage, we cannot see to what extent this symmetry describes our nature well. The phenomenological results obtained by this symmetry (cosmic symmetry) are analyzed by Sakurai[10].

Electric charges of elementary particles are expressed by the relation,

$$Q=I_3+\tfrac{1}{2}\eta, \tag{4}$$

where η is hyperonic charge. This relation must be modified when we use $\mathcal{I}$ instead of I. In order to get a similar relation, we must introduce new parameter $\tilde{\xi}$, *K-mesonic charge*, in such a way that the new relation

$$Q=\mathcal{I}_3+\tfrac{1}{2}\eta+\tfrac{1}{2}\tilde{\xi} \tag{5}$$

holds. The assignment of K-mesonic charge to elementary particles is shown in

TABLE 1

	Ξ		N		Y		Z		π			K		$\widetilde{K}$	
	Ξ^0	Ξ^-	p	n	Σ^+	Y^0	Z^0	Σ^-	π^+	π^-	π^0	K^+	K^0	$\widetilde{K}^0$	$-K^-$
Q	0	−1	+1	0	+1	0	0	−1	+1	−1	0	+1	0	0	−1
$\mathcal{I}_3$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	+1	−1	0	0	0	0	0
η	−1	−1	+1	+1	0	0	0	0	0	0	0	+1	+1	−1	−1
$\tilde{\xi}$	0	0	0	0	+1	+1	−1	−1	0	0	0	+1	−1	+1	−1
$\mathcal{K}$-conj	n	p	Ξ^-	Ξ^0	Σ^-	Z^0	Y^0	Σ^+	π^-	π^+	$-\pi^0$	$-K^-$	$\widetilde{K}^0$	K^0	K^+

Table I. The following characters of K-mesonic charge may be noticed. First, together with hyperonic charge, it discriminates four K-mesons. Then K-mesons may be described by real fields, as they are $\mathcal{I}$-singlet states. Secondly, hyperons with non-zero hyperonic charge have K-mesonic charge zero and *vice versa*.

Owing to this new parameter, Σ^+, Y^0, Z^0 and Σ^- states can be characterized by the signs of $\mathcal{I}_3$ and $\tilde{\xi}$, as will be immediately seen from Table I. Then Λ and Σ^0 states can be written as

$$\Psi_{\Lambda^0}=\frac{1}{\sqrt{2}}\{\Psi(+\tfrac{1}{2},\,-1)+\Psi(-\tfrac{1}{2},\,+1)\}$$

and

$$\Psi_{\Sigma^0}=\frac{1}{\sqrt{2}}\{\Psi(+\tfrac{1}{2},\,-1)-\Psi(-\tfrac{1}{2},\,+1)\},$$

where the arguments in Ψ on the right-hand side refer to the parameters $\mathcal{I}_3$ and $\tilde{\xi}$. These expressions suggest that Λ^0 and Σ^0 states are characterized by the parity with respect to *K-mesonic charge conjugation* operation, $\mathcal{K}$, which is defined phenomenologically as such an operation that changes the signs of $\mathcal{I}_3$ and $\tilde{\xi}$. Further it is convenient to get $\mathcal{K}$-conjugation accompanied with the inversion of the sign of hyperonic charge, as will be shown in the following. The most preferable recipe to $\mathcal{K}$-conjugation is listed in Table I, and the formulation of the operator will be explained in the next section. Here it may be emphasized that this operation does

not transform the "particle" state into the "antiparticle" state. Thus we arrive at *a characterization of* (Λ, Σ) *and* (Y, Z)*-modes, and a similar complementarity between them to the case of neutral K-mesons.* K-mesonic charge and $\mathcal{K}$ operation correspond to hyperonic charge and particle-antiparticle conjugation operation $\mathcal{C}$, respectively.

In weak K-meson-interactions we already know that neutral K-mesons should appear in the (K_1^0, K_2^0)-mode. Owing to our definition of the $\mathcal{K}$-conjugation which change K^0 into $\widetilde{K}^0$, not into K^-, neutral K-mesons are realized only in the desired mode in the $\mathcal{K}$-invariant combinations of the spurion interactions of K-mesons which are given in the Appendix: In the interactions $H^{(1)}(K)_i$, $(i=1, 2, \cdots, 5)$, and $H^{(k)}(K)_1 (k=2, 3)$, the spinor biproducts multiplied to K_0 are the same combinations as the ones in the corresponding $\widetilde{K}^0$-meson interactions $H^{(1)}(\widetilde{K})_i$ and $H^{(k)}(\widetilde{K})_1$, contrary to the strong K-meson interactions, equation (3). Further, as to these interactions, the K-meson interactions are transformed into the corresponding $\widetilde{K}$-meson interactions by $\mathcal{K}$-conjugation, and *vice versa.* The remaining spurion interactions of K-mesons, $H^{(k)}(K)_j$ and $H^{(k)}(\widetilde{K})_j$, where $j \neq 1$ and $k=2, 3, 4$, are not invariant under $\mathcal{P}$-rotation. On the other hand, the interactions $H^{(k)}(K)_1$ and $H^{(k)}(\widetilde{K})_1 (k=2, 3, 4)$ are invariant under the rotation, and, as to $H^{(1)}(K)_i$ and $H^{(1)}(\widetilde{K})_i$, we can construct such linear combinations that are invariant under $\mathcal{P}$-rotation, as will be seen in the Appendix. Therefore, the behaviour in decay processes of neutral K-mesons suggests that *we should impose the invariances under* $\mathcal{P}$*-rotation and* $\mathcal{K}$*-conjugation on the weak K-meson interactions.*

All the other $\mathcal{P}$ independent interactions can also be combined into such forms that are invariant under $\mathcal{K}$-conjugation: The strong K-meson interaction, equation (3), is invariant under $\mathcal{K}$-conjugation.* The following combinations of $\mathcal{P}$-invariant π-meson interactions** are also invariant under $\mathcal{K}$-conjugation.

Strong interactions:

$$\bar{N}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})N+\bar{\Xi}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\Xi \tag{6}$$

and

$$\bar{Y}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})Y+\bar{Z}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})Z \tag{7}$$

Weak interaction:

$$\bar{N}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})Y+\bar{\Xi}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})Z, \tag{8}$$

which is introduced in the Appendix. If we accept the combinations (3) and (7) simultaneously, we encounter Pais' criticism[9]. To avoid this criticism various ways may be considered. Such interactions that disturb the symmetrical behaviour of Y

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* This interaction is also invariant under such an alternative $\mathcal{K}$-conjugation that is not accompanied with the inversion of the sign of hyperonic charge.

** The other $\mathcal{P}$-independent forms disturb the conservation law of electric charge.

and Z states without violating the $\mathcal{I}$-independence may be introduced. An interesting alternative way may be to formulate a theory in which $\mathcal{K}$-invariance holds only in weak interactions (and strong K-meson interactions)*. In all the concerned strong interactions $\mathcal{K}$-invariance guarantees that they can be expressed in both modes (Λ, Σ) and (Y, Z).** Therefore the alternative suggests such a standpoint that the (Y, Z) is a suitable mode to strong interactions and the (Λ, Σ)-mode is proper to weak interactions in this zeroth approximation.

All interactions that can give rise to the $\Lambda^0-\Sigma^0$ mass-splitting must be $\mathcal{K}$-invariant, as Λ^0 and Σ^0 states have good $\mathcal{K}$-parity. Further it may be desirable that the mass-splitting is so induced that Σ^+, Σ^0 and Σ^- form an I-triplet (at least in some approximation). Because that selection rule with respect to the ordinary isotopic spin I describes phenomena considerably well. For this purpose, it is necessary for the concerned interactions to be expressible in terms of both (Λ, Σ)- and (Y, Z)-modes.

§ 3. Discussions

In the preceding section, (Λ, Σ)- and (Y, Z)-modes were characterized by introducing K-mesonic charge and the operation of K-mesonic charge conjugation. The strange maze that the ordinary isotopic spin I was conserved in strong interactions and that it changed by $\pm\frac{1}{2}$ in weak interactions would be solved, if the (Y, Z)-mode was taken as a more fundamental one than (Λ, Σ)-mode and if the newly assigned isotopic spin $\mathcal{I}$ was considered as an *absolutely* conserved quantity in such an approximation as proposed here.*** If we require that *all* interactions must be invariant under $\mathcal{K}$-conjugation, it imposes on them a stronger symmetry than $\mathcal{I}$-independence and leads to global symmetry and hyperonic charge symmetry simultaneously. Therefore, in order to accept the above proposition, we must add such $\mathcal{I}$-independent interactions that violate global symmetry, or impose the $\mathcal{K}$-invariance on all interactions *except for* charge independent π-meson interactions. Further, the selection rule $\Delta I=\pm\frac{1}{2}$ is yet very questionable. Taking these situations**** into account, it will be of little significance to compare the results obtained from our scheme with experiments directly. However, (Y, Z)-description and $\mathcal{I}$-independence requirement may be taken as a better model in a zeroth approximation

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* This alternative method may be realized in such a model as proposed in (A). Further the coupling constant of the charge independent K-meson interactions seems to be smaller than the corresponding one of π-mesons. It may be desirable in order to explain the smallness of Λ-Σ mass-splitting in comparison with the other hyperon mass-splittings.

** Here it may be noticed that in this alternative the I-independence of strong interactions of Λ and Σ is disturbed, and the deviation may be rather useful in some questions.

*** It may be noticed that $\mathcal{I}$-independence gives a stronger selection rule than $\Delta I=\pm 1/2$ to weak interactions.

**** At these points, electromagnetic interaction and cloud of various fields may play important roles.

than the conventional ones. Because the ordinary isotopic spin I is not absolutely conserved and a kind of *pseudo*-quantum number which is conserved in some class of interactions, and we must explain why it changes by $\pm\frac{1}{2}$ in the interactions with $\Delta\eta=\pm 1$. Here it may be noticed that the interactions with higher values of ΔI are weaker than the ones with $\Delta I=\pm\frac{1}{2}$, even if they are necessary. Further, we are enforced to introduce K-mesonic charge and $\mathcal{K}$-conjugation operation, if we use the (Y, Z) mode at all. Provided that such a scheme as explained here cannot fit experiments by all means, we should abandon (Y, Z)-description itself, even if the mode is convenient to formulate an inclusive scheme of elementary particles.

Although these questions exist, it is interesting that the present scheme can be easily connected to the previously proposed model[1]. The characters of the two parameters η and $\tilde{\xi}$ picked out in the preceding section just correspond to the ones of the eigenvalues of the operators $\zeta_2\gamma_5$ and $\zeta_3\gamma_5$ introduced in the article (A). Namely, there we proposed to consider (N, Ξ) and (Y, Z) as eigenstates of $\zeta_2\gamma_5$ and $\zeta_3\gamma_5$, respectively. Further the hyperonic charge symmetry suggested that the tensors associated to K-mesons are characterized by projection operators $(1\pm\zeta_2\gamma_5)/2$ and $(1\pm\zeta_3\gamma_5)/2$. [See Equation (4·17) in (A)]. The expression of electric charge (5) can be translated into the operator relation

$$Q=\mathcal{I}_3+\tfrac{1}{2}\zeta_2\gamma_5+\tfrac{1}{2}\zeta_3\gamma_5. \tag{9}$$

and its expectation values give the relation (5) correctly as to hyperons. The appearance of the two non-commuting operators $\zeta_2\gamma_5$ and $\zeta_3\gamma_5$ does not bring in any difficulty in *quantized* form. In quantized theory, for instance, the gauge transformation of spinor field is given by $\Psi\to e^{i\Omega\alpha}\Psi e^{-i\Omega\alpha}$, where $\boldsymbol{Q}=\int\overline{\Psi}Q\Psi d^4x$, and either of $\zeta_2\gamma_5$ or $\zeta_3\gamma_5$ disappears in $\boldsymbol{Q}$. As to boson fields, the formulation of gauge transformation is impossible until we understand the fundamental significance of the phenomenological results obtained in the article (A) and formulate free boson fields completely.

It may be emphasized that the relation (9) can be used for leptons too, if it is taken into account that the type of weak interactions is $(V-A)$ and if we reassign the spinor associated to neutrino as Ψ_η. Such a neutrino assignment is possible owing to the massless character, as pointed out in the article (A). Then the discrimination of hyperons from leptons may be attributed to the existence of the degree of freedom of isotopic spin, and the two conservation laws of the number of leptons and of hyperons are guaranteed automatically. Such situation is of interest from the standpoint of the fundamental symmetry[11], and may be taken as another support to consider $\mathcal{I}$ more fundamental than I.

By the above correspondences between $\tilde{\xi}$, η and $\zeta_i\gamma_5$'s the $\mathcal{K}$-conjugation operator can be represented as

$$\mathcal{K}\sim\mathcal{I}_1\cdot\zeta_1. \tag{10}$$

Then the complementarity between (Λ, Σ)- and (Y, Z)-modes is expressed in the operator relations

$$\{\tilde{\xi}_{0p},\ \mathcal{K}\}=0,$$

$$\{\eta_{0p},\ \mathcal{K}\}=0$$

and

$$\{\vartheta_3,\ \mathcal{K}\}=0. \tag{11}$$

These relations correspond to the relation

$$\{\eta_{0p},\ \mathcal{A}\}=0$$

in the article (A). The appearance of the operator ζ_1 (or $\zeta_1\gamma_5$) in $\mathcal{K}$-operator and the character of spinor associated to neutrino might give some light on the future development.

We may point out that the arbitrariness in combining the spurion interactions of K-mesons, which is pointed out in the Appendix, is removed by the existence of the projection operator $(1\pm\zeta_3\gamma_5)/2$. For instance, the tensor G_{K^0} associated to K^0-meson has the operator $(1+\zeta_3\gamma_5)/2$ on the right. Therefore the spinor Φ in the interaction $\bar{\Psi}G_{K^0}\Phi$ must be Y. If we resolve $\bar{\Psi}$ into $\bar{\Psi}((1+\zeta_3\gamma_5)/2+(1-\zeta_3\gamma_5)/2)$, the possible interactions are $\bar{Y}YK^0$ and $\bar{Z}YK^0$, and the latter does not conserve electric charge. Therefore, the concerned interaction is restricted to $\bar{Y}YK^0$. Further it may be noticed that the assignment $G_\pi\sim\zeta_2+\gamma_5$ forbids the direct $\bar{\Xi}\Xi\pi$ interaction and disturbs the $\mathcal{K}$-invariance.

In the previous article, the types of interactions were not taken into account. Before connecting the present scheme with the previous one (A) more concretely, we must make a little modification of the latter scheme and find out the matrices associated to derivative operations. This problem will be treated in a forthcoming article separately.

Appendix

In this appendix, it will be shown that some combinations of the so-called spurion interactions can be expressed in terms of the (Y, Z)-mode. In the beginning, the following features of spurion, which was introduced by d'Espagnat[6], may be summarized: Spurions $x=\binom{0}{1}$ and $y=\binom{1}{0}$ can be interpreted as a first kind and a second kind spinor in I-space respectively, as there is a relation $y=i\tau_2 x=i\tau_2 x^*$ between them. Therefore there hold $|\Delta\eta|=1$ and $|\Delta I|=\frac{1}{2}$ simultaneously in the *spurion interactions* which involve x or y and are invariant under rotation and reflection in I-space. Further, either spurion must be so chosen in a spurion interaction that electric charge is conserved there. This situation corresponds to the differential expression of the relation (4), $\Delta Q=\Delta I_3+\frac{1}{2}\Delta\eta=0$, and the fact that x and y have $I_3=-1$ and $+1$ respectively. For instance, electric charge is conserved in a spurion interaction $\bar{N}(\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\boldsymbol{\Sigma}))x$, but not in $\bar{N}(\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\boldsymbol{\Sigma}))y$.

The processes with $|\Delta\eta|=1$ in which a π-meson is absorbed are (Λ or Σ) $+\pi\rightarrow$(N or Ξ). There are three kinds of spurion interaction* which decribe (Λ or Σ) $+\pi\rightarrow N$;

* In the following, hermitean conjugate terms are omitted.

$$
\begin{aligned}
H^{(1)}(\pi)_1 &= -i\frac{1}{\sqrt{2}}\bar{N}(\tau\cdot(\boldsymbol{\pi}\times\boldsymbol{\Sigma}))x \\
&= \bar{P}(\pi_+\Sigma_0-\pi_0\Sigma_+)+\frac{1}{\sqrt{2}}\bar{n}(\pi_+\Sigma_- -\pi_-\Sigma_+), \\
H^{(1)}(\pi)_2 &= \frac{1}{\sqrt{2}}\bar{N}(\boldsymbol{\pi}\cdot\boldsymbol{\Sigma})x \\
&= \frac{1}{\sqrt{2}}\bar{n}(\pi_+\Sigma_- +\pi_-\Sigma_+ +\pi_0\Sigma_0),
\end{aligned}
\qquad (A\cdot 1)
$$

and

$$
\begin{aligned}
H^{(1)}(\pi)_3 &= \frac{1}{\sqrt{2}}\bar{N}(\tau\cdot\boldsymbol{\pi})\Lambda x \\
&= \bar{P}\Lambda\pi_+ -\frac{1}{\sqrt{2}}\bar{n}\Lambda\pi_0.
\end{aligned}
$$

As will be easily seen, the only combination of them that can be rewritten by Y-mode is

$$
-H^{(1)}(\pi)_1+H^{(1)}(\pi)_2+H^{(1)}(\pi)_3=\bar{N}(\tau\boldsymbol{\pi})Y. \qquad (A\cdot 2)
$$

Further, any combination cannot be expressed by using the Z-mode instead of Y-mode, and it corresponds to the situation that the concerned spurion is x, that is, to the conservation law of electric charge. In a similar manner, we obtain the linear combination

$$
-H^{(2)}(\pi)_1+H^{(2)}(\pi)_2+H^{(2)}(\pi)_3=\bar{\Xi}(\tau\cdot\boldsymbol{\pi})Z, \qquad (A\cdot 3)
$$

where

$$
\begin{aligned}
H^{(2)}(\pi)_1 &= -i\frac{1}{\sqrt{2}}\bar{\Xi}(\tau(\boldsymbol{\pi}\times\boldsymbol{\Sigma}))y \\
&= \frac{1}{\sqrt{2}}\bar{\Xi}_0(\pi_-\Sigma_+-\pi_+\Sigma_-)+\bar{\Xi}_-(\pi_0\Sigma_- -\pi_-\Sigma_0), \\
H^{(2)}(\pi)_2 &= \frac{1}{\sqrt{2}}\bar{\Xi}(\boldsymbol{\pi}\cdot\boldsymbol{\Sigma})y \\
&= \frac{1}{\sqrt{2}}\bar{\Xi}_0(\pi_+\Sigma_- +\pi_-\Sigma_+ +\pi_0\Sigma_0)
\end{aligned}
\qquad (A\cdot 4)
$$

and

$$
\begin{aligned}
H^{(2)}(\pi)_3 &= \frac{1}{\sqrt{2}}\bar{\Xi}(\tau\boldsymbol{\pi})\Lambda y \\
&= \frac{1}{\sqrt{2}}\bar{\Xi}_0\Lambda\pi_0+\bar{\Xi}_-\Lambda\pi_-.
\end{aligned}
$$

In this case any combination of interactions $H^{(2)}(\pi)_i$ cannot be rewritten by the Y-mode. The interactions (A·2) and (A·3) were already used by Takeda and Kato[5]. (The operators $(1\pm\gamma_5)\gamma_\mu$ matter nothing to the present discussions.)

There are many kinds of K- or $\widetilde{K}$-reactions with $|\Delta\gamma|=1$. We begin with the interactions where Λ or Σ particles participate. The spurion interactions which describe the reactions (Λ or Σ)$+K\to$(Λ or Σ) are

$$
\begin{aligned}
H^{(1)}(K)_1 &= -i\frac{1}{\sqrt{2}}x^*(\boldsymbol{\tau}\cdot(\bar{\boldsymbol{\Sigma}}\times\boldsymbol{\Sigma}))K \\
&= (\bar{\Sigma}_0\Sigma_- - \bar{\Sigma}_+\Sigma_0)K_+ + \frac{1}{\sqrt{2}}(\bar{\Sigma}_-\Sigma_- - \bar{\Sigma}_+\Sigma_+)K_0, \\
H^{(1)}(K)_2 &= \frac{1}{\sqrt{2}}x^*(\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\Sigma})K \\
&= \frac{1}{\sqrt{2}}(\bar{\Sigma}_+\Sigma_+ + \bar{\Sigma}_-\Sigma_- + \bar{\Sigma}_0\Sigma_0)K_0, \\
H^{(1)}(K)_3 &= \frac{1}{\sqrt{2}}x^*(\boldsymbol{\tau}\cdot\boldsymbol{\Sigma})K\Lambda \\
&= \bar{\Sigma}_+\Lambda K_+ - \frac{1}{\sqrt{2}}\bar{\Sigma}_0\Lambda K_0, \qquad \text{(A·5)} \\
H^{(1)}(K)_4 &= \frac{1}{\sqrt{2}}\bar{\Lambda}x^*(\boldsymbol{\tau}\cdot\boldsymbol{\Sigma})K \\
&= \bar{\Lambda}\Sigma_- K_+ - \frac{1}{\sqrt{2}}\bar{\Lambda}\Sigma_0 K_0
\end{aligned}
$$

and

$$
\begin{aligned}
H^{(1)}(K)_5 &= \frac{1}{\sqrt{2}}x^*\bar{\Lambda}\Lambda K \\
&= \frac{1}{\sqrt{2}}\bar{\Lambda}\Lambda K_0.
\end{aligned}
$$

There is no combination of them which can be expressed only by Y- or Z-mode without Z- or Y-mode respectively. Further, there are several kinds of combination which can be expressed in terms of Y- and Z-modes. For instance, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}\{-H^{(1)}(K)_1 + H^{(1)}(K)_2 + H^{(1)}(K)_3 + H^{(1)}(K)_4 + H^{(1)}(K)_5\} \\
&\qquad = (\bar{Y}Z)K_+ + (\bar{Y}Y)K_0, \qquad \text{(A·6·1)} \\
&\frac{1}{\sqrt{2}}\{H^{(1)}(K)_1 + H^{(1)}(K)_2 - H^{(1)}(K)_3 - H^{(1)}(K)_4 + H^{(1)}(K)_5\}
\end{aligned}
$$

$$=-(\bar{Y}Z)K_{+}+(\bar{Z}Z)K_{0} \qquad (A\cdot 6\cdot 2)$$

or

$$H^{(1)}(K)_{2}+H^{(1)}(K)_{5}=\frac{1}{\sqrt{2}}\{(\bar{Y}Y)+(\bar{Z}Z)\}K_{0}. \qquad (A\cdot 6\cdot 3)$$

Similarly, for $(\Lambda \text{ or } \Sigma)+\tilde{K}\rightarrow(\Lambda \text{ or } \Sigma)$, we have

$$\left.\begin{aligned}
H^{(1)}(\tilde{K})_{1}&=-i\frac{1}{\sqrt{2}}\,y^{*}(\tau(\bar{\boldsymbol{\Sigma}}\times\boldsymbol{\Sigma}))\tilde{K}\\
&=\frac{1}{\sqrt{2}}(\bar{\Sigma}_{+}\Sigma_{+}-\bar{\Sigma}_{-}\Sigma_{-})\tilde{K}_{0}-(\bar{\Sigma}_{-}\Sigma_{0}-\bar{\Sigma}_{0}\Sigma_{+})K_{-},\\
H^{(1)}(\tilde{K})_{2}&=\frac{1}{\sqrt{2}}\,y^{*}(\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\Sigma})\tilde{K}\\
&=\frac{1}{\sqrt{2}}(\bar{\Sigma}_{+}\Sigma_{+}+\bar{\Sigma}_{-}\Sigma_{-}+\bar{\Sigma}_{0}\Sigma_{0})\tilde{K}_{0},\\
H^{(1)}(\tilde{K})_{3}&=\frac{1}{\sqrt{2}}\,y^{*}(\tau\bar{\boldsymbol{\Sigma}})\tilde{K}\Lambda=\frac{1}{\sqrt{2}}\bar{\Sigma}_{0}\Lambda\tilde{K}_{0}-\bar{\Sigma}_{-}\Lambda K_{-},\\
H^{(1)}(\tilde{K})_{4}&=\frac{1}{\sqrt{2}}\,\bar{\Lambda}y^{*}(\tau\boldsymbol{\Sigma})\tilde{K}=\frac{1}{\sqrt{2}}\,\bar{\Lambda}\Sigma_{0}\tilde{K}_{0}-\bar{\Lambda}\Sigma_{+}K_{-}
\end{aligned}\right\} \qquad (A\cdot 7)$$

and

$$H^{(1)}(\tilde{K})_{5}=\frac{1}{\sqrt{2}}\,y^{*}\bar{\Lambda}\Lambda\tilde{K}=\frac{1}{2}\,\bar{\Lambda}\Lambda_{0}\tilde{K}.$$

The combinations corresponding to (A·6) are

$$\frac{1}{\sqrt{2}}\{H^{(1)}(\tilde{K})_{1}+H^{(1)}(\tilde{K})_{2}-H^{(1)}(\tilde{K})_{3}-H^{(1)}(\tilde{K})_{4}+H^{(1)}(\tilde{K})_{5}\}$$

$$=(\bar{Z}Y)K_{-}+(\bar{Y}Y)\tilde{K}_{0} \qquad (A\cdot 8\cdot 1)$$

$$\frac{1}{\sqrt{2}}\{-H^{(1)}(\tilde{K})_{1}+H^{(1)}(\tilde{K})_{2}+H^{(1)}(\tilde{K})_{3}+H^{(1)}(\tilde{K})_{4}+H^{(1)}(\tilde{K})_{5}\}$$

$$=-(\bar{Z}Y)K_{-}+(\bar{Z}Z)\tilde{K}_{0} \qquad (A\cdot 8\cdot 2)$$

or

$$H^{(1)}(\tilde{K})_{2}+H^{(1)}(\tilde{K})_{5}=\frac{1}{\sqrt{2}}\{(\bar{Z}Z)+(\bar{Y}Y)\}\tilde{K}_{0}. \qquad (A\cdot 8\cdot 3)$$

There remain six groups of interactions $H^{(4)}(K)_{j}, H^{(i)}(K)_{j}$ $(i=2, 3, 4,$ and $j=1, 2, 3, 4)$. Here we explicitly write down only $H^{(2)}(K)_{j}$:

$$
\left.\begin{aligned}
&H^{(2)}(K)_1=(\bar{N}N)\,(x^*K)=(\bar{N}N)\,K_0,\\
&H^{(2)}(K)_2=(\bar{N}K)\,(x^*N)=(\bar{p}K_+ +\bar{n}K_0)\,n,\\
&H^{(2)}(K)_3=(\bar{N}\boldsymbol{\tau}N)\,(x^*\boldsymbol{\tau}K)=2\bar{p}nK_+-(\bar{p}p-\bar{n}n)\,K_0\\
\text{and}\quad &H^{(2)}(K)_4=(\bar{N}\boldsymbol{\tau}K)\,(x^*\boldsymbol{\tau}N)=(2\bar{p}p+\bar{n}n)\,K_0-\bar{p}nK_+.
\end{aligned}\right\}\qquad (A\cdot 9)
$$

$H^{(3)}(K)_j$ is obtained from $H^{(2)}(K)_j$ by substituting Ξ in place of N, and $H^{(4)}(K)_j$'s are interactions for $N+\widetilde{K}\to\Xi$, for example, $H^{(4)}(K)_1=(\bar{N}\Xi)\,(y^*K)$. $H^{(i)}(\widetilde{K})_j$'s are the interactions which are obtained by the substitutions $N\leftrightarrow\Xi$ and $x\leftrightarrow y$. The $\mathcal{S}$-invariant interactions among them are only $K^{(i)}(K)_1$ and $K^{(i)}(\widetilde{K})_1 (i=2, 3, 4)$.

As will be easily checked, the combinations

$$
\left.\begin{aligned}
&H^{(1)}(\pi)_j+H^{(2)}(\pi)_j \quad j=1, 2, 3),\\
&H^{(1)}(K)_j+H^{(1)}(\widetilde{K})_j \quad (j=1, 2, \cdots, 5),\\
&H^{(i)}(K)_1+H^{(i)}(\widetilde{K})_1 \quad (i=2, 3),\\
\text{and}\quad &H^{(4)}(K)_1-H^{(4)}(\widetilde{K})_1
\end{aligned}\right\}\qquad (A\cdot 10)
$$

are invariant under the $\mathcal{K}$ conjugation, if we use the recipe listed in Table I. Further, neutral K-mesons appear in the mode K_1^0 in the interactions $H^{(1)}(K)_j+H^{(1)}(\widetilde{K})_j$ $(j=2, 5)$ and $H^{(i)}(K)_1+H^{(i)}(\widetilde{K})_1 (i=2, 3)$, and in the other mode K_2^0 in the interactions $H^{(1)}(K)_j+H^{(1)}(\widetilde{K})_j (j=1, 3, 4)$. It may be noticed that $\mathcal{S}$-dependent interactions $H^{(i)}(K)_j$ and $H^{(i)}(\widetilde{K})_j (i=2, 3, 4$ and $j\neq 1)$ are not interchanged with each other by $\mathcal{K}$-conjugation.

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The Elastic Scattering of High Energy Neutron by Deuteron

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We investigated the elastic scattering of high energy neutron by deuteron, using the two-body scattering amplitudes and the deuteron wave functions calculated making use of the meson theoretical potential.

We found that

(1) the impulse approximation can well reproduce the experimental result of the differential cross section in the region $\theta \lesssim 90°$ near 100 Mev neutron energy,

(2) the phase shifts of nucleon-nucleon scattering and the deuteron wave function derived by using meson theory may give the best fit among others, and

(3) the impulse approximation will also be expected to well reproduce the polarization at small angles. The experiment of the polarization may serve as a further test for discriminating various sets of phase shifts of nucleon-nucleon scattering.

§ 1. Introduction

In connection with the problem of n-d scattering, Chew[1] first presented the impulse approximation. The fundamental assumptions underlying in this approximation are as follows: (1) the collision time is very short when it is compared with the internal period of the deuteron; (2) the wave function of the incident neutron at one of the two scattering centers in the deuteron is not appreciably perturbed by the presence of another center. This is due to the diffuse structure of the deuteron. Then the three-body problem is reduced to a superposition of the two-body problem. But, since the perturbation of the incident wave is not so small, the Born approximation may not be used. The amplitude of n-d scattering is therefore, described as the sum of the "true" two-body scattering amplitudes. This is the basic idea of the impulse approximation.

The merit of this approximation is that two-body scattering amplitudes are treated as the empirical quantities. Besides these quantities, we have only to adopt some form as the momentum distribution of the target nucleus. In fact, the remarkable developments in the scattering theories in recent years were brought out by this simplification. As many experimental analyses on the two-body scattering have shown, the potential acting in a nucleon-nucleon system is not so simple. Not only that its radial part is not monotonic as inferred from the low energy data, but also it depends on the states of the two particles. This fact made it difficult directly to introduce the two-body potential to the many-particle problems.

But in the impulse approximation, this difficulty is overcome, since we describe the two-body interaction not with the potential itself but with the experimental phase shifts. Of course, these experimental phase shifts of the two-body scattering are determined only with the aid of some assumed phenomenological or meson theoretical potential. In calculations of the many-body problems, however, the direct use of the two-body potential is considerably laborious compared with the use of the two-body phase shifts. (The main reason for this difference is found in the neglect of the non-diagonal part of the "t-matrix" in the latter case.* In the two-body scattering, the t-matrix always has only diagonal elements since the momentum is conserved in the center of mass system of the two-body. In the many-particle system, however, this is not the case. As for the application of the impulse approximation, theoretical justification of this point has not been tried so far. This will be discussed in a forthcoming paper.)

After the impulse approximation had been developed by Chew and his collaborators[2], it was applied to the nuclear many-body problems. Watson and others[3] applied it to nuclear scattering problems. Riesenfeld and Watson[4] showed that phase shifts of the elementary scattering are related to the optical potential in nucleon-nucleus scattering. Brueckner and others[5] applied this approximation to the theory of nuclear structure and succeeded in explaining the nuclear saturation. These theories owe their successes to the tractability of the impulse approximation.

However, the applicability of this approximation to many-body problems must be examined in many respects. Part of this examination, e.g. the examination of the many particle scattering correction, was discussed by several authors, but the validity of this approximation has not been examined yet in its simplest aspect. This means that the validity of the impulse approximation is not examined yet in n-d scattering,the simplest of the many-body problems. Qualitative examination shows that this approximation is reasonable at least when it is applied to n-d scattering, as it was noted by Chew. Then it seems important to examine quantitatively the validity of this approximation in n-d scattering, because if this approximation should be found to be quantitatively unsuccessful even in its application to the n-d scattering, there would be no hope in its application to the many-body problem.

Unfortunately in those days when Chew put forward the idea of the impulse approximation, we had not any knowledge on the phase shifts of the elementary scattering, although experimental results on the cross section of the elementary scattering were being accumulated. This made it impossible to test the idea of the impulse approximation in the nucleon-deuteron scattering. (Chew[6] tried to test this approximation in n-d scattering using the known data on the cross section,

* The non-diagonal part of the "t-matrix" or the "off-energy shell" contribution in many-body scattering is neglected altogether in some papers. In other papers it is taken into account partly by assuming some form to the off-energy shell contribution.

but he could not use the phase shifts of the two-body scattering, hence his trial was incomplete. Comparison of Chew's result with that obtained by using the phase shifts was made in our previous note.[7])

In recent years the data on p-n or p-p scattering were analysed by using the phenomenological[8] or the semi-phenomenological potential[9] in the energy region up to 310 Mev and by using the meson theoretical potential[10,11,12] up to 150 Mev. On the other hand, Wolfenstein et al.[13,14] analysed the scattering spin matrix between particles of spin 1/2, and Wright[15] expressed its coefficients by the phase shifts of nucleon-nucleon scattering. Now we can check the validity of the impulse approximation, making use of these results. Naturally, the first attack should be made to the reproduction of the experimental result of the high energy nucleon scattered by deuteron.

Here we should note the following. We now have a few sets of the phase shifts.[8~12),16] It is expected that analysis of the nucleon-deuteron scattering may serve to the discrimination of the nature of these sets. Thus the purpose of the present paper is laid on the following two points: (1) the experimental check of the impulse approximation in the nucleon-deuteron scattering and (2) the discrimination among a few sets of phase shifts. In the previous note,[7] we found that the impulse approximation can reproduce surprisingly well the angular distribution of the elastic scattering of proton by deuteron.

The difference between the phase shifts calculated by making use of the meson theoretical potential[10,11,12] and the phase shifts calculated by using the phenomenological[8] or semi-phenomenological potential[9] was discussed in detail by Otsuki et al.[10,11,12] They showed that different sets of phase shifts of nucleon-nucleon scattering calculated by various authors could reproduce the experimental data (cross section and polarized cross section) in quite differerent ways. (The main difference between the phase shifts analysed meson theoretically[10,11,12] and those analysed phenomenologically[8] is found especially in the ${}^3P_J(J=0, 1, 2)$, ${}^3F_2$, ${}^3S_1$, ${}^3D_1$-state phase shifts and the mixing parameters $\varepsilon_J(J=1, 2)$. At 90 Mev nucleon energy, which is of interest to us, the dominant contributions to the cross sections of the nucleon-nucleon scattering as well as the n-d scattering are determined by the aboved phase shifts and the above mixing parameters.

In spite of this situation in the nucleon-nucleon scattering, we pointed out in our previous note[7] that the calculated values of the differential cross section of the n-d scattering give similar results for the two sets of phase shifts, given by separate authors. Nevertheless, in high energy n-d scattering the interference terms between n-n and n-p scattering and the nature of the sticking factor is important when we compare the calculated differential cross section divided by the sticking factor. The manner of interference and its magnitude are quite different for various sets of phase shifts.* The reason why we obtained quite similar results for both sets of phase shifts is found in the fact that the sticking factor acts to

* See Fig. 6 of the present paper.

mask the difference between the various sets of phase shifts.* For the purpose of discriminating phase shifts, then, the reproduction of the differential cross section of the elastic p-d scattering would not be suitable. So we added the calculation of the polarization.

It might be expected that in the polarization the effect of the sticking factor is weak and the interference terms between n-n and n-p scattering would appear explicitly. Then the calculated result of the polarization may, when it is compared with the experimental one, serve as a tool to decide the more reasonable sets of phase shifts of nucleon-nucleon scattering. Strictly speaking, at large scattering angles, where the effect of the sticking factor is reduced and the interference effect becomes important, the impulse approximation itself is certainly not applicable. However, we are encouraged by the surprisingly good result of reproducing the differential cross section to large angles on the basis of the impulse approximation. This was the motive of our calculation of the polarization. The result of the polarized cross section is reported in another note[17] by one of the authors. The present paper is devoted to the detailed descriptions of these two previous reports.[7,17] Theoretical investigations of the impulse approximation, for example, the investigation of the effect of the non-diagonal part of the t-matrix, etc., are now in progress.

§ 2. Scattering spin matrix

Wolfenstein and Ashkin[13] and Dalitz[14] gave the expression for the scattering spin matrix between particles with spin 1/2 in terms of some quantities which are invariant with respect to the space rotation and reflection and time reversal. It is

$$\begin{aligned} M&=M_S\Lambda_S+M_T\Lambda_T \\ M_S&=B \end{aligned} \tag{2·1}$$

and

$$\begin{aligned} M_T=&C(\sigma_1+\sigma_t)\boldsymbol{n}+D(\sigma_1-\sigma_t)\boldsymbol{n}+N(\sigma_1\cdot\boldsymbol{n})\,(\sigma_t\cdot\boldsymbol{n}) \\ &+\tfrac{1}{2}G\{(\sigma_1\cdot\boldsymbol{k})\,(\sigma_t\cdot\boldsymbol{k})+(\sigma_1\cdot\boldsymbol{p})\,(\sigma_t\cdot\boldsymbol{p})\}+\tfrac{1}{2}H\{(\sigma_1\cdot\boldsymbol{k})\,(\sigma_t\cdot\boldsymbol{k})-(\sigma_1\cdot\boldsymbol{p})\,(\sigma_t\cdot\boldsymbol{p})\}, \end{aligned}$$

where Λ_S and Λ_T are the projection operators into the singlet and the triplet spin states of the two-body spin wave functions. B, C, D, H, G and N are functions of the incident energy and the scattering angle. Naturally, the expressions for these letters are different for the scattering between like particles and for that between unlike particles. The suffices 1 and t refer to the incident and the target nucleons respectively. $\boldsymbol{n}, \boldsymbol{k}$ and $\boldsymbol{p}$ are three mutually perpendicular unit vectors defined by

$$\boldsymbol{n}=\frac{\boldsymbol{k}_i\times\boldsymbol{k}_f}{|\boldsymbol{k}_i\times\boldsymbol{k}_f|}, \qquad \boldsymbol{k}=\frac{\boldsymbol{k}_f-\boldsymbol{k}_i}{|\boldsymbol{k}_f-\boldsymbol{k}_i|}, \qquad \boldsymbol{p}=\frac{\boldsymbol{k}_f+\boldsymbol{k}_i}{|\boldsymbol{k}_f+\boldsymbol{k}_i|},$$

* There we adopted a fixed form as the sticking factor.

where $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are the momenta of the incident and scattered nucleons in the center of mass system, respectively. Wolfenstein[18] showed that the term D in the scattering spin matrix cannot be present if the nuclear force satisfies the requirement of charge symmetry. Taking the direction of the incident beam to be z-axis and that of $\boldsymbol{n}$ to be y-axis in the right-hand system, and designating the angle between $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ to be θ, M is expressed as*

$$M_T=\tfrac{1}{2}G+C(\sigma_{1y}+\sigma_{ty})+(N-\tfrac{1}{2}G)\sigma_{1y}\sigma_{ty}$$
$$+\tfrac{1}{2}H[(\sigma_{1x}\sigma_{tx}-\sigma_{1z}\sigma_{tz})\cos\theta-(\sigma_{1z}\sigma_{tx}+\sigma_{1x}\sigma_{tz})\sin\theta]. \qquad (2\cdot1')$$

B, C, etc., in the scattering spin matrix were expressed in terms of the nucleon-nucleon scatering phase shifts by Wright[15] and Stapp et al.[19] The relations between Wright's expression and that of Stapp and others were discussed by Goldberger et al.[20] We shall use Wright's expression throughout this paper.**,***

§ 3. Cross section for n-d elastic scattering

In this section we reproduce the impulse approximation originally put forward by Chew. Our aim in this section is to give discussions on some approximations which will be adopted later to reduce some complicated formulas to the final tractable forms. These approximations are based on the physical intuition. Here after we name the incoming neutron 1, and the neutron and the proton in the deuteron 2 and 3, respectively.

The differential cross section for the n-d elastic scattering is given by

$$I(\theta)=\frac{(2m/3)^2}{4\pi^2\hbar^4}\frac{1}{6}\sum_{i,f=1}^{6}|\mathcal{H}_{f,i}|^2, \qquad (3\cdot1)$$

$$\mathcal{H}_{f,i}=(\Phi_f(1,23),(V_{12}+V_{13})(1-P_{12}Q_{12})\Psi). \qquad (3\cdot2)$$

---

* The authors are indebted to Dr. T. Sawada for discussions on Wright's expression.

** In the preprint which was sent to us from Dr. J. Sawicki in the course of our finishing touch of the present paper, the n-d scattering amplitude is expressed in terms of the n-n and n-p scattering amplitudes in the form of Stapp et al. We wish to thank for his sending his manuscript to us before publication and for his communications. In the course of the present work, Dr. L. S. Singh also informed us of his work on the n-d scattering.

*** The differential cross section and the polarized cross section for the nucleon-nucleon scattering can be expressed, using (2·1) and (2·1′) as

$$I(\theta)=\frac{\sum_i[(M\chi_i)^\dagger(M\chi_i)]}{\sum_i\chi_i^\dagger\chi_i}=1/4\cdot\{|B|^2+8|C|^2+2|H|^2+2|N|^2+|N-G|^2\}$$

and

$$P(\theta)I(\theta)=\frac{\sum_i[(M\chi_i)^\dagger(\sigma_1M\chi_i)]}{\sum_i\chi_i^\dagger\chi_i}=NC^*+CN^*(\text{in direction of }\boldsymbol{n}),$$

where χ_i are the two-body spin wave functions.

Here the following notations are used.

Φ_f: the final free state of the system,

V_{ij}: the interaction between the particles i and j,

P_{12} and Q_{12}: the space and the spin exchange operators of the particles 1 and 2,

Ψ: the exact solution of the three-particle scattering problem.

According to the basic idea of the impulse approximation, which was described in the beginning of this paper, (3·2) is approximated by

$$\mathcal{H}_{f,i} \simeq \sum_{m,n} (\Phi_f(1,23), \phi_m)\,(\phi_m, (V_{12}+V_{13})(1-P_{12}Q_{12})\psi_n)\,(\phi_n, \Phi_i(1,23)). \qquad (3\cdot3)$$

Here

$$\phi_m = (1/2\pi)^{3/2\times3} \exp\left[i(\boldsymbol{k}_1\boldsymbol{r}_1+\boldsymbol{k}_2\boldsymbol{r}_2+\boldsymbol{k}_3\boldsymbol{r}_3)\right]\chi_m\ ,$$

$$\phi_n = (1/2\pi)^{3/2\times3} \exp\left[i(\boldsymbol{k}_1{}^0\boldsymbol{r}_1+\boldsymbol{k}_2{}^0\boldsymbol{r}_2+\boldsymbol{k}_3{}^0\boldsymbol{r}_3)\right]\chi_n\ ,$$

$$\psi_n = (1/2\pi)^{3/2\times2} \exp\left[i\boldsymbol{k}_s{}^0\boldsymbol{r}_s\right]\cdot\exp\left[i(\boldsymbol{k}^0{}_i+\boldsymbol{k}_j{}^0)\frac{\boldsymbol{r}_i+\boldsymbol{r}_j}{2}\right]\varphi_{(1/2)(\boldsymbol{k}_i{}^0-\boldsymbol{k}_j{}^0)}(\boldsymbol{r}_i-\boldsymbol{r}_j)\chi_n\ ,$$

$$\Phi_{i,f} = \exp\left[i\boldsymbol{k}_{i,f}(\boldsymbol{r}-\tfrac{1}{2}(\boldsymbol{r}_2+\boldsymbol{r}_3))\right]\psi_0(\boldsymbol{r}_2-\boldsymbol{r}_3\,;\sigma_2,\sigma_3)\chi_{t,f}\ ,$$

$$|\boldsymbol{k}_i| = 2/3\cdot|\boldsymbol{k}_{lab}|\ ,$$

$\varphi_{(1/2)(\boldsymbol{k}_i{}^0-\boldsymbol{k}_j{}^0)}(\boldsymbol{r}_i-\boldsymbol{r}_j)$: the solution of the scattering problem of the two particles i and j.

$\psi_0(\boldsymbol{r}_2-\boldsymbol{r}_3\,;\sigma_2,\sigma_3)$: the deuteron wave function.

χ's are well-known spin functions of the three-particle system: four of them are quartet states and the remaining two are doublet states which are symmetric about 2 and 3. $\sum$ means the integration over momenta and the sum over spin states.

After explicit calculations of (3·3), $(\Psi_f(1,23), \phi_m)$ multiplied by $(\phi_n, \Phi_i(1,23))$ constitutes the sticking factor. Then the scattering matrix becomes very simple and tractable: it is represented by the sticking factor times the sum of the scattering amplitudes of two-body interactions. (See (3·20).)

To reach this simple form, however, some additional assumptions should be introduced. First of all, we neglect the contribution from the D-state of deuteron. This neglect is justified because of the fact that the D-state probability of deuteron is very small. Hulthén et al.[21] obtained by the variational method that the D-state probability of deuteron is 4%. Iwadare et al[22] estimated it to be 6.4% making use of the meson theoretical potential. In any case, the contribution from the D-state of deuteron is expected to be very small even at forward direction. The neglect of the contribution from the D-state of deuteron is the first step of simplifying the expression (3·3). Of course, this neglect is not basic to the impulse approximation, but terms to be computed are much reduced without introducing a serious error by this neglect. This makes the final form very simple together

with further approximations which will be introduced later. We shall give numerical estimates of the contribution from the D-state in § 5.

After this first subsidiary assumption, (3·3) becomes

$$\mathcal{H}_{f,i}=\sum_{m,n}(\Phi_f', \phi_m')\,(\phi_m, (V_{12}+V_{13})(1-P_{12}Q_{12})\psi_n)\,(\phi_n'\Phi_i') \ . \tag{3·4}$$

Here

$$\phi_m'=\phi_m/\chi_m \ ,$$

$$\phi_n'=\phi_n/\chi_n \ ,$$

and

$$\Phi_i{}_f'=\exp\left[i\boldsymbol{k}_{i,f}(\boldsymbol{r}_1-\tfrac{1}{2}(\boldsymbol{r}_2+\boldsymbol{r}_3))\right]\psi_0(|\boldsymbol{r}_2-\boldsymbol{r}_3|) \ .$$

Next approximation we wish to introduce is the neglect of the contribution from $\sum_{m\,n}(\Phi_f', \phi_m')\,(\phi_m', V_{13}P_{12}Q_{12}\psi_n')\,(\phi_n', \Phi_i')$ in calculating the forward scattering. We neglect this term because this term can never be expressed by the phase shifts of the two-body scattering and this makes estimating of the term not easy. The neglect of this term may perhaps not lead to a serious error, since the contribution from this term is mainly expected to the backward. This was examined by Wu and Ashkin[23] by using the Born approximation.

On the contrary, a part of the contribution from $\sum_{m,n}(\Phi_f', \phi_m')\,(\phi_m, V_{12}P_{12}Q_{12}\psi_n)$ $\times(\phi_n', \Phi_i')$ can be described by the phase shift of the two-body scattering. Hence we take account of this part, although this part also is expected to give rise to the backward scattering. Then we start from the approximated form for $\mathcal{H}_{f,i}$:

$$\mathcal{H}_{f,i}=\sum_{m,n}(\Phi_f, \phi_m')\,(\phi_m, (V_{12}(1-P_{12}Q_{12})+V_{13})\psi_n)\,(\phi_n', \Phi_i) \ . \tag{3·5}$$

The interaction $V_{12}(1-P_{12}Q_{12})$ can be expressed as

$$V_{12}(1-P_{12}Q_{12})=V^T{}_{12}(1-P_{12})\Lambda_T+V^S{}_{12}(1+P_{12})\Lambda_S \ , \tag{3·6}$$

where Λ_T and Λ_S are the triplet and singlet projection operators, respectively.

Then

$$\begin{aligned}(\phi_m, (V_{12}(1-P_{12}Q_{12})\psi_n)&=(\chi_m|\,(\phi_m'|\,V^T{}_{12}(1-P_{12})\,|\psi_n')\Lambda_T\\&+(\phi_m'|\,V^S{}_{12}(1+P_{12})\,|\psi_n')\Lambda_S|\chi_n) \ ,\end{aligned} \tag{3·7}$$

where

$$\psi_n'=\psi_n/\chi_n \ .$$

At first sight, one may think that

$$\left(-\frac{4\pi\hbar^2}{m}\right)M_{12}=(\phi_m'|\,V^T{}_{12}(1-P_{12})\,|\psi_n')\Lambda_T+(\phi_m'|\,V^S{}_{12}(1+P_{12})\,|\psi_n')\Lambda_S \tag{3·8}$$

would be the scattering spin matrix of two like particles 1 and 2, which was given by (2·1). In order to identify the expression (3·8) with the scattering spin

matrix (2·1), however, one must introduce further assumptions, to which we shall proceed.

For this purpose, we shall calculate the spatial part of the matrix element in $\mathcal{H}_{f,i}$ in a more explicit form. After some obvious calculations, which we do not describe here,

$$\begin{aligned}
&\sum_{m,n}(\Phi_f', \phi_m')\,(\phi_m', V_{12}^{T,S}\psi_n')\,(\phi_n', \Phi_i') \\
&=\int d\boldsymbol{k}_2(1/2\pi)^{3/2}\int d\boldsymbol{r}_{23}''\exp\left[i\frac{2\boldsymbol{k}_2+\boldsymbol{k}_f}{2}\boldsymbol{r}_{23}''\right]\psi_0(|\boldsymbol{r}_{23}''|) \\
&\quad\cdot(1/2\pi)^{3/2}\int d\boldsymbol{r}_{12}'\exp\left[-i\frac{\boldsymbol{k}_f-\boldsymbol{k}_2}{2}\boldsymbol{r}_{12}'\right]V_{12}^{T,S}(\boldsymbol{r}_{12}';\sigma_1,\sigma_2)\,\varphi_{(1/2)\cdot(\boldsymbol{k}_i-(\boldsymbol{k}_2+\boldsymbol{k}_f-\boldsymbol{k}_i))}(\boldsymbol{r}'_{12}) \\
&\quad\cdot(1/2\pi)^{3/2}\int d\boldsymbol{r}_{23}\exp\left[-i\frac{2(\boldsymbol{k}_2+\boldsymbol{k}_f)-\boldsymbol{k}_i}{2}\boldsymbol{r}_{23}\right]\psi_0(|\boldsymbol{r}_{23}|) \qquad (3\cdot9)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m,n}(\Phi_f', \phi_m')\,(\phi_m', V_{12}^{T,S}P_{12}\psi_n')\,(\phi_n', \Phi_i') \\
&=\int d\boldsymbol{k}_2(1/2\pi)^{3/2}\int d\boldsymbol{r}_{23}''\exp\left[i\frac{2\boldsymbol{k}_2+\boldsymbol{k}_f}{2}\boldsymbol{r}''_{23}\right]\psi_0(|\boldsymbol{r}''_{23}|) \\
&\quad\cdot(1/2\pi)^{3/2}\int d\boldsymbol{r}'_{12}\exp\left[-i\frac{\boldsymbol{k}_f-\boldsymbol{k}_2}{2}\boldsymbol{r}'_{21}\right]V_{12}^{T,S}(\boldsymbol{r}'_{12};\sigma_1,\sigma_2)\,\varphi_{1/2\cdot(\boldsymbol{k}_i-(\boldsymbol{k}_2+\boldsymbol{k}_f-\boldsymbol{k}_i))}(\boldsymbol{r}'_{12}) \\
&\quad\cdot(1/2\pi)^{3/2}\int d\boldsymbol{r}_{13}\exp\left[-i\frac{2(\boldsymbol{k}_2+\boldsymbol{k}_f)-\boldsymbol{k}_i}{2}\boldsymbol{r}_{13}\right]\psi_0(|\boldsymbol{r}_{13}|)\,, \qquad (3\cdot10)
\end{aligned}$$

where

$$\boldsymbol{r}_{ij}=\boldsymbol{r}_i-\boldsymbol{r}_j\,.$$

Now we shall introduce a fundamental assumption that only one component, say $\boldsymbol{k}_{\mathrm{II}}$, is important of $\boldsymbol{k}_2$ in the interaction factors;

$$\int d\boldsymbol{r}_{12}'\exp\left[-i\frac{\boldsymbol{k}_f-\boldsymbol{k}_2}{2}\boldsymbol{r}'_{12}\right]V_{12}^{T,S}(\boldsymbol{r}'_{12};\sigma_1,\sigma_2)\,\varphi_{(1/2)\cdot(\boldsymbol{k}_i-(\boldsymbol{k}_2+\boldsymbol{k}_f-\boldsymbol{k}_i))}(\boldsymbol{r}'_{12}) \qquad (3\cdot11)$$

and

$$\int d\boldsymbol{r}_{12}'\exp\left[-i\frac{\boldsymbol{k}_f-\boldsymbol{k}_2}{2}\boldsymbol{r}'_{21}\right]V_{12}^{T,S}(\boldsymbol{r}'_{12};\sigma_1,\sigma_2)\,\varphi_{(1/2)\cdot(\boldsymbol{k}_i-(\boldsymbol{k}_2+\boldsymbol{k}_f-\boldsymbol{k}_i))}(\boldsymbol{r}'_{12}) \qquad (3\cdot12)$$

in (3·9) and (3·10), respectively. This assumption was originally introduced by Chew.[1] Meanwhile, we shall discuss this assumption. We treat here the three-body problem in the center of mass system. The initial wave number $\boldsymbol{k}_i$ of the incident particle 1 is equal to parallel and opposite to $\boldsymbol{k}_2+\boldsymbol{k}_3$:

$$\boldsymbol{k}_i=-(\boldsymbol{k}_2+\boldsymbol{k}_3)\,. \qquad (3\cdot13)$$

$\boldsymbol{k}_2$ is most likely parallel to $\boldsymbol{k}_3$ with equal magnitudes. Hence we take $\boldsymbol{k}_{\text{II}}$ as

$$\boldsymbol{k}_{\text{II}}=-(1/2)\boldsymbol{k}_i'\simeq\boldsymbol{k}_2\simeq\boldsymbol{k}_3\,. \tag{3·14}$$

Besides this assumption, we introduce another important assumption that we are interested in the forward scattering;

$$\boldsymbol{k}_i\simeq\boldsymbol{k}_f\,. \tag{3·15}$$

By introducing this assumption, the off-energy shell contribution from (3·11) and (3·12) is dropped. Consequently, (3·11) and (3·12) can be expressed in terms of the phase shifts of the two body scattering. Besides, we can calculate (3·11) and (3·12), using the phase shifts of two body scattering of just equal incident energy to the energy of the incident particle in the three body scattering. This is due to the relation,

$$\frac{\boldsymbol{k}_f-\boldsymbol{k}_{\text{II}}}{2}\simeq\frac{\boldsymbol{k}_i-(\boldsymbol{k}_{\text{II}}+\boldsymbol{k}_f-\boldsymbol{k}_i)}{2}=(3/4)\boldsymbol{k}_i=(1/2)\boldsymbol{k}_{\text{lab}}. \tag{3·16}$$

The two-body scattering angle is related to the three-body scattering angle by $\sin(1/2)\theta_{\text{2-body}}=(4/3)\sin(1/2)\theta_{\text{3-body}}$.

After introducing the assumption (3·14), and (3·15), eqs. (3·9) and (3·10) are further simplified;

$$\text{Eq. (3·9)}=(1/2\pi)^{3/2}\int d\boldsymbol{r}_{12}'\exp\left[-i\frac{\boldsymbol{k}_f-\boldsymbol{k}_{\text{II}}}{2}\boldsymbol{r}_{12}'\right]V^{T,S}{}_{12}(\boldsymbol{r}_{12}';\sigma_1,\sigma_2)$$

$$\cdot\varphi_{(1/2)\cdot(\boldsymbol{k}_i-\boldsymbol{k}_{\text{II}})}(\boldsymbol{r}_{12}')\int d\boldsymbol{r}_{23}\exp\left[-i\frac{\Delta\boldsymbol{k}}{2}\boldsymbol{r}_{23}\right]|\psi_0(|\boldsymbol{r}_{23}|)|^2,$$

and

$$\text{Eq. (3·10)}=(1/2\pi)^{3/2}\int d\boldsymbol{r}'_{12}\exp\left[-i\frac{\boldsymbol{k}_f-\boldsymbol{k}_{\text{II}}}{2}\boldsymbol{r}'_{21}\right]V_{12}{}^{T,S}(\boldsymbol{r}_{12}';\sigma_1,\sigma_2)$$

$$\cdot\varphi_{(1/2)(\boldsymbol{k}_i-\boldsymbol{k}_{\text{II}})}(\boldsymbol{r}'_{12})\int d\boldsymbol{r}_{23}\exp\left[-i\frac{\Delta\boldsymbol{k}}{2}\boldsymbol{r}_{23}\right]|\psi_0(|\boldsymbol{r}_{23}|)|^2,$$

where

$$\Delta\boldsymbol{k}=\boldsymbol{k}_f-\boldsymbol{k}_i\,.$$

Hence

$$\left(-\frac{m}{4\pi\hbar^2}\right)(\mathcal{K}_{12})_{f,i;m,n}=(\chi_m|M_{12}|\chi_n)S^{1/2}(\Delta\boldsymbol{k}). \tag{3·17}$$

Here M_{12} is (3·8), but now it is the scattering spin matrix of two particles 1 and 2, which is given by (2·1). $S(\Delta\boldsymbol{k})$ is the so-called sticking factor. $S^{1/2}(\Delta\boldsymbol{k})$ is the Fourier transform of the deuteron position probability amplitude:

$$S^{1/2}(\Delta\boldsymbol{k})=(2\pi)^{-3/2}\int d\boldsymbol{r}\,\exp\left[-i\frac{\Delta\boldsymbol{k}}{2}\boldsymbol{r}\right]\cdot|\psi_0(|\boldsymbol{r}|)|^2\,. \tag{3·18}$$

Similarly,

$$(-m/4\pi\hbar^2)\,(\mathcal{H}_{13})_{f,i\,;\,m,n}=(\chi_m|M_{13}|\chi_n)\,S^{1/2}(-\Delta\boldsymbol{k})\,, \tag{3·19}$$

where

$$M_{13}=\{(\phi_m'|V^T{}_{13}|\psi_n')\,\Lambda_T+(\phi_m'|V^S{}_{13}|\psi_n')\,\Lambda_S\}\times\left(-\frac{m}{4\pi\hbar^2}\right).$$

Finally, the differential cross section (3·1) becomes

$$I(\theta)=\frac{1}{6}\cdot\frac{16}{9}\sum_{i=1}^{6}[(M_{12}+M_{13}|\chi_i)^\dagger(M_{12}+M_{13}|\chi_i)]S(\Delta\boldsymbol{k}). \tag{3·20}$$

Here 16/9 comes from the reduced mass effect. After some calculations, (3·20) can be expressed as

$$\begin{aligned}I(\theta)=&\frac{1}{6}\cdot\frac{16}{9}\Big\{10CC^*+2HH^*+\frac{9}{8}(BB^*+GG^*)+\frac{25}{8}NN^*\\&-\frac{7}{8}(N^*G+NG^*)+\frac{1}{8}(B^*(N+G)+B(N^*+G^*))\Big\}S(\Delta\boldsymbol{k}),\end{aligned} \tag{3·21}$$

where $B=B_{12}+B_{13}$, $C=C_{12}+C_{13}$ and so forth. B_{12}, C_{12}, etc., are the coefficients of n-n scattering spin matrix and B_{13}, C_{13} etc. the coefficients of n-p scattering spin matrix.

If we adopt some form as the sticking factor, then the calculation is made straightforwardly.

§ 4. Polarized cross section of n-d scattring

The polarized cross section of n-d scattering is given by

$$P(\theta)\,I(\theta)=\frac{16}{9}\cdot\frac{1}{6}\sum_i[(M_{12}+M_{13}|\chi_i)^\dagger(\sigma_1M_{12}+\sigma_1M_{13}|\chi_i)]S(\Delta\boldsymbol{k}) \tag{4·1}$$

under just the same assumptions as described in § 3.

In terms of the coefficients of the two-body scattering spin matrix, eq. (4·1) becomes

$$\begin{aligned}P(\theta)\,I(\theta)=&\frac{16}{9}\cdot\frac{1}{6}\{(1/2)\,(B^*C+C^*B)+(1/2)\,(G^*C+C^*G)\\&+(9/2)\,(N^*C+C^*N)\}S(\Delta\boldsymbol{k}),\\&\qquad\text{(in direction of }\boldsymbol{n}).\end{aligned} \tag{4·2}$$

where B, C etc. are given in the previous section (§ 3).

§ 5. Sticking factor

In the previous sections, we showed that the elastic cross section of the *n-d* scattering and the polarized cross section can be expressed as a product of the sum of the scattering amplitudes of the two-body scattering and the sticking factor. To calculate the scattering amplitudes of the two-body scattering, we require the knowledge on the phase shifts. In the present paper, we adopt the phase shifts analysed by Tamagaki et al. and that by Gammel and Thaler. Naturally, we should assume the compatible form as the deuteron wave function in the sticking factor with the phase shifts which we intend to adopt. In this sense, we assume the deuteron wave function appearing in the review article of Iwadare et al.[22]

$$\left.\begin{aligned} &S\text{-state}: \\ &\quad u=1.039\exp(-0.328x)-1.392\exp(-2.360x)\,, \\ &D\text{-state}: \\ &\quad w=0.02624\{1+3/(0.328x)+3/(0.328x)^2\}\exp(-0.328x) \\ &\qquad -(1.298/x^2)\exp(-0.962x), \end{aligned}\right\} \quad (5\cdot1)$$

with the normalization

$$\int_0^\infty (u^2+w^2)\,dx=1,$$

where

$$x=(\mu c/\hbar)\,r=0.7067r\times10^{13}\text{cm}^{-1}\,. \qquad (\mu:\ \text{the pion mass})$$

This wave function is based on the meson theoretical potential. It is compatible with the phase shifts by Tamagaki et al. However, before showing the numerical results we shall examine the following problems.

(1) Is the presence of the hard core in the deuteron wave function sensitive to the results?

(2) Is the neglect of the effect of the *D*-state in the deuteron wave function sensitive to the results?

(3) Hence, how much is the reliability of our numerical results of the cross sections (3·21) and (4·2), obtained by using the deuteron wave function suggested meson theoretically?

(4) Is the assumed form of the *S*-state of the deuteron wave function sensitive to the results?

The results of numerical calculations of the cross sections will be given in the next section.

(1) *Effect of the hard core*

Before showing the explicit numerical example of the sticking factor with the hard core and without it in the deuteron wave function, we shall briefly examine

the effect of the hard core. The best way for it will be to inquire about the following: At what place does the product

$$j_0(\Delta kr/2)\times|u(x)|^2 \tag{5·2}$$

takes the maximum value for given Δk? Here Δk is given by

$$\Delta k=(4/3)K_{Lab}\sin\theta/2\,,$$

where θ is the scattering angle. We shall take the case of 190 Mev d-p scattering.* This is equivalent to 95 Mev p-d scattering. At $\theta=10°$, (5·2) takes the maximum value at $r=1.7\times10^{-13}$ cm, at $\theta=45°$ at $r=1.5\times10^{-13}$ cm and at $\theta=90°$, it takes the maximum value at 1.1×10^{-13} cm. Since the radius of the hard core is expected to be about 0.4×10^{-13} cm or so, the presence of the hard core will give only small effect to the sticking factor at the energy under consideration. Although, for the backward scattering, the hard core may give some effect, our concern is limited mainly to the forward direction, as discussed in the previous sections.

Next we shall estimate the effect of the hard core directly. It will be proved that the above discussion is the case. Since the meson theoretical deuteron wave function is somewhat complicated, we shall estimate the effect of the hard core assuming the wave function resembling the meson theoretical one. This is as follows.

(I) *The deuteron wave function with the hard core*[24].

We adopt the following form as the deuteron wave function with the hard core.

$$u(r)=\begin{cases}N\{e^{-\alpha(r-r_c)}-e^{-\beta(r-r_c)}\}\,, & r\geqq r_c\,,\\ 0\,, & r\leqq r_c\,.\end{cases} \tag{5·3}$$

Here

$$N^2=(\alpha\beta/2\pi)\,(\alpha+\beta)/(\alpha-\beta)^2\,.$$

r_c is the core radius and it is taken to be 0.4×10^{-13} cm. We shall adopt the following constants.

$$\alpha^{-1}=4.31\times10^{-13}\,\text{cm},\quad \beta=(3/\rho')\,(1+(4/9)\alpha\rho')^{-1}=3.050\times10^{13}\,\text{cm}^{-1},$$

$$\rho'=\rho-2r_c=0.90\times10^{-13}\,\text{cm}\,,$$

ρ: the triplet effective range, 1.7×10^{-13} cm.

On using the wave function (5·3), the square root of the sticking factor becomes

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* After completion of the manuscript, experiments on high energy p-d scattering at several laboratories were reported.

$$S^{1/2}(\Delta k)\cong\frac{8\pi N^2}{\Delta k}(1+2\alpha r_c(1+\alpha r_c))\tan^{-1}\frac{\Delta k}{4\alpha}$$
$$-2(1+(\alpha+\beta)r_c(1+\frac{(\alpha+\beta)}{2}r_c))\tan^{-1}\frac{\Delta k}{2(\alpha+\beta)}$$
$$+(1+2\beta r_c(1+\beta r_c))\tan^{-1}\frac{\Delta k}{4\beta}\,. \qquad (5\cdot4)$$

(II) *The deuteron wave function without the hard core*

For comparison with (5·2), we take the deuteron wave function obtained by putting $r_c=0$ in (5·2) :

$$u(r)=N(e^{-\alpha r}-e^{-\beta r})\,,$$
$$N^2=\alpha\beta(\alpha+\beta)/2\pi(\alpha-\beta)^2\,. \qquad (5\cdot5)$$

Constants are $\alpha^{-1}=4.31\times10^{-13}$ cm, $\beta=(3/\rho)\,(1+(4/9)\alpha\rho)^{-1}=1.502\times10^{13}$ cm^{-1}. The square root of the sticking factor based on (5·5) is obtained by putting $r_c=0$ in (5·4) :

$$S^{1/2}(\Delta k)=\frac{8\pi N^2}{\Delta k}\left\{\tan^{-1}\frac{\Delta k}{4\alpha}-2\tan^{-1}\frac{\Delta k}{2(\alpha+\beta)}+\tan^{-1}\frac{\Delta k}{4\beta}\right\}. \qquad (5\cdot6)$$

We shall designate the square of (5·4) be $S_{\rm I}$ and that of (5·6) be $S_{\rm II}$. The result of comparison of $S_{\rm I}$ with $S_{\rm II}$ is tabulated in Table I. For $\theta\lesssim120°$ the difference between $S_{\rm I}$ and $S_{\rm II}$ is about 5% at most.

Table I Comparison of sticking factors with the hard core ($S_{\rm I}$) and without the hard core ($S_{\rm II}$). $S_{\rm I}$ is given by (5·4) and $S_{\rm II}$ is given by (5·6).

θ	$S_{\rm I}$	$S_{\rm II}$	$\left(1-\frac{S_{\rm II}}{S_{\rm I}}\right)\times100$
0°	1.000	1.000	0.00
10°	0.942	0.929	1.38
30°	0.596	0·577	3.19
45°	0.374	0.358	4.28
60°	0.231	0.219	5.20
90°	0.0951	0.0922	3.05
120°	0.0467	0.0484	−3.64
180°	0.0250	0.0286	−14.4

The above estimate was made by using the wave function different from the meson theoretical one. But the difference would not be so serious for the estimation of the hard core effect. (See discussions in (4) of this section). Hence we can conclude that the presence of the hard core in the deuteron wave function is

not important for the calculation of the n-d scattering cross section near and below 100 Mev neutron energy. The presence of the hard core gives the additive contribution at forward directions, but the negative contribution at backward directions.

(2) *Effect of the D-state in the deuteron wave function*

As discussed in § 2, we neglect the D-state in the deuteron wave function because the D-state probability is about 6% or so. If we take into account the S-D coupling term, the expression for the cross section is not so simple as (3·20) or (4·1). Nevertheless, we can estimate the contribution from the S-D coupling term.

The wave function from the D-state of the deuteron is in general a linear sum of terms, whose angular parts are characterized by $Y_{2M}(\theta, \varphi)$. Here M is the z-component of the angular momentum and takes the value from -2 to $+2$. However, for the S-D coupling term, we should only take into account the term that is characterized by $M=0$. This is due to the conservation of the z-component of the angular momentum. Hence we only consider the terms whose spatial part is given by

$$\int e^{-\Delta k/2 \cdot r \cos\theta} u(r) w(r) Y_{00} Y_{20} Y_{20} d\Omega dr = -\sqrt{5} \int j_2\left(\frac{\Delta k r}{2}\right) u(r) w(r) dr . \quad (5\cdot7)$$

Here j_2 is the spherical Bessel function of the order 2. In Fig. 2, the absolute value of (5·7) is compared with the square root of the sticking factor corresponding to the S-state deuteron wave function of (5·1). (5·7) is estimated also using the meson theoretical wave function (5·1). From this figure, one can expect that the contribution from the S-D coupling term is very small at small angles. The ratio of the absolute value of (5·7) (Curve M_{S-D} in Fig. 1) to the square root of the sticking factor corresponding to the S-S term (Curve M_{S-S} in Fig. 1) is only 0.0445 at 30°, 0.105 at 45° and 0.366 at 60°. The "sticking factor" for the S-D term must be multiplied by the Clebsch-Gordan coefficients, which take values of $(1/10)^{1/2}$ or $-(4/10)^{1/2}$. This coefficients thus pull down the curve M_{S-D} about 40%~60%. Total number of the S-D terms is just twice the S-S terms. Of these S-D terms, two-thirds are multiplied by $(1/10)^{1/2}$ and the remaining one-third terms are multiplied by $-(4/10)^{1/2}$. Hence a rough estimate of the upper limit of the ratio of the contribution from the S-D coupling term to that from the S-S term is given by

$$\begin{aligned} &\lesssim 2\{(1/10)^{1/2} \times (2/3) + |-(4/10)^{1/2} \times (1/3)|\} (M_{S-D}/M_{S-S}) \\ &= 0.42 (M_{S-D}/M_{S-S}) . \end{aligned} \quad (5\cdot8)$$

Hence the error will not be more than about 1.9% at 30°, 4.4% at 45° and 15% at 60°, when we drop the S-D coupling terms from our numerical calculation. As the sign of the expression (5·7) shows, the contribution from these terms makes

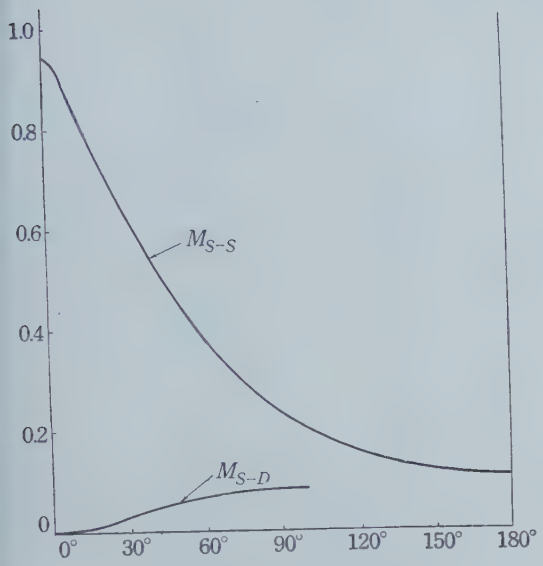

Fig. 1. Effect of the D-state.
M_{S-S}; the square root of the sticking factor. This is given by (3·18). As $\phi_0(|\boldsymbol{r}|)$ in (3·18), we assume u of (5·1).
M_{S-D}; the effect of the S-D coupling term. This is given by (5·7). u and w are given by (5·1).

the calculated cross section smaller.

(3) *Reliability of using the S-state wave function of* (5·1)

From the considerations made in (1) and (2), we can estimate the reliability of using only the S-state wave function of (5·1) to calculate n-d scattering. In Table II, S_{S-S} is the sticking factor by making use of the S-state wave function given by (5·1). By definition of M_{S-S}, S_{S-S} is the square of M_{S-S}. S'_{S-S} is estimated by the square of M'_{S-S}, which is given by the relation

$$M'_{S-S}=M_{S-S}(1-(5\cdot8))\left(1+\frac{S_1^{1/2}-S_{11}^{1/2}}{S_1^{1/2}}\right). \qquad (5\cdot9)$$

Hence, $(1-S'_{S-S}/S_{S-S})\times100$ shows the upper bound of the error, arising from the neglect of the effect of the hard core and the D-state in the deuteron wave function. In the forward directions, the hard core correction is larger than the D-state correction, thus $S'_{S-S}>S_{S-S}$. But for $\theta\gtrsim30°$, the D-state correction overcomes the hard core correction, hence $S'_{S-S}<S_{S-S}$. This table shows that the error is not greater than 1% for $\theta\lesssim30°$, and not greater than 10% for $40°\lesssim\theta\lesssim60°$. But at 90° the error may amount to about 26% when we use S_{S-S} to calculate the n-d scattering cross section.

Table II The sticking factor S_{S-S} obtained by using the deuteron wave function (5·1) based on the meson theoretical potential. In calculating S_{S-S}, the hard core effect and the effect of the D-state in the deuteron wave function is neglected. S'_{S-S} is the upper bound of the sticking factor, if we take account of the effects of the hard core and the S-D coupling terms. S'_{S-S} is given by the square of M'_{S-S} which is defined by (5·9)

θ	S	S'	$\frac{S_{S-S}-S'_{S-S}}{S_{S-S}}\times100$
0°	0.8761	0.8761	0.00
10°	0.6849	0.6912	−0.92
30°	0.4182	0.4518	+0.57
45°	0·2454	0.2344	+4.48
60°	0.1455	0.1312	+9.83
90°	0.05378	0.03976	+26.07

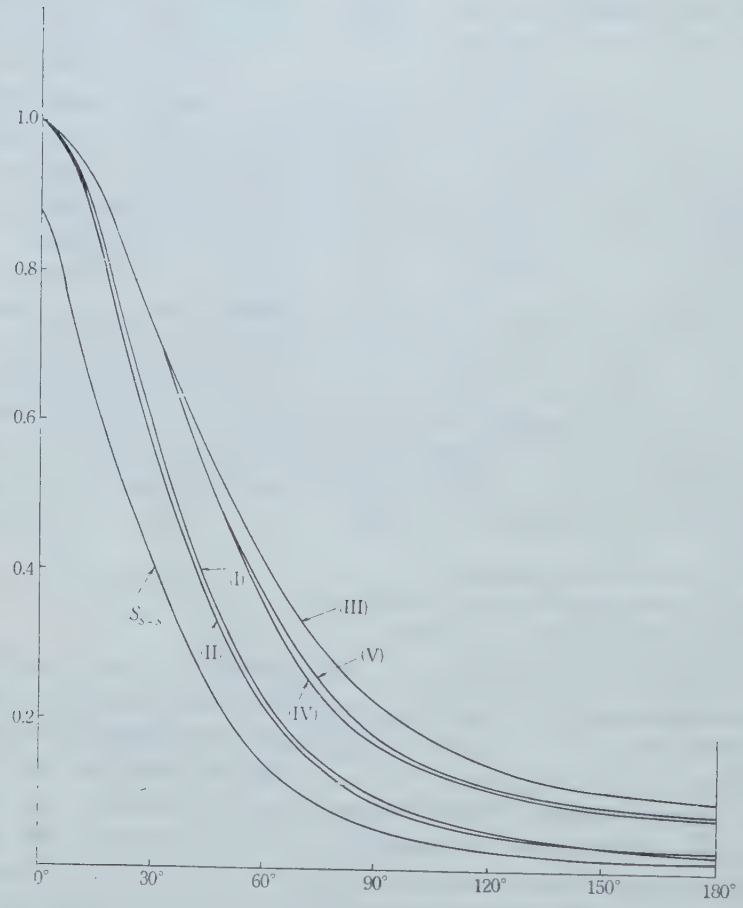

Fig. 2. Comparison of various sticking factors for 95Mev n-d scattering.

(I) ; The sticking factor obtained from the Feshbach type wave function (5·3) in the text. This includes the hard core in the deuteron wave function.

(II) ; The sticking factor obtained from the Feshbach type wave function (5·5) in the text. This does not include the hard core in the deuteron wave function.

(III) ; The sticking factor which we adopted in the previous note.[7]

(IV) ; The sticking factor adopted by Chew.[25]

(V) ; The sticking factor obtained from the wave function due to Hulthén and Sugawara.[21]

S_{s-s} ; The sticking factor obtained from the meson theoretical wave function u in (5·1). Here the S-state probability is taken to be 93.6% whereas in (I)～(V) this is assumed to be 100%.

(4) *Various forms of the deuteron wave functions*

Although our final numerical calculations are performed by using the wave function (5·1), we shall examine various deuteron wave functions assumed by several authors. The form of these wave function are the same as (5·5) but constants are different from each other. We shall compare various sticking factors calculated from various deuteron wave functions.

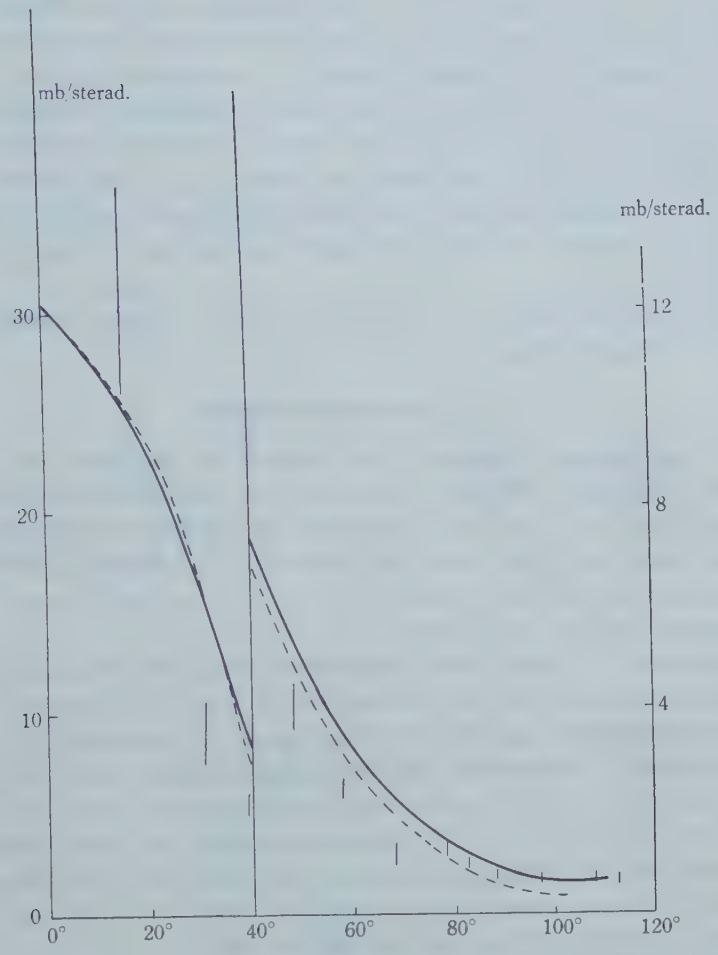

Fig. 3. Comparison of our numerical result with the experimental one (95 Mev *p-d* scattering). The solid curve shows the result obtained by using the sticking factor S_{S-S}. The dotted curve shows the result obtained by using the sticking factor S'_{S-S}.

(III) *The wave function which we have adopted in our previous report*[7]

Constants are

$$\alpha=0.323\times10^{13}\ \mathrm{cm}^{-1}, \qquad \beta=7\alpha=2.261\times10^{13}\ \mathrm{cm}^{-1}.$$

(IV) *The wave function adopted by Chew*[25]

$$\alpha=0.323\times10^{13}\ \mathrm{cm}^{-1}, \qquad \beta=5.476\alpha=1.76\times10^{13}\ \mathrm{cm}^{-1}.$$

(V) *The wave function adopted by Hulthén and Sugawara*[21]

$$\alpha=0.2317\times10^{13}\ \mathrm{cm}^{-1}, \qquad \beta=5.751\alpha=1.332\times10^{13}\ \mathrm{cm}^{-1}.$$

These sticking factors together with the sticking factor obtained by using the

meson theoretical wave function (5·1) are shown in Fig. 2. From this figure one sees that Curves (I) and (II) are very similar and Curves (III), (IV) and (V) are very similar to each other. Curve S_{s-s} is considerably lower than others. One reason is that in Curve S_{s-s}, the S-state probability is taken to be 0.936, whereas in other curves this is assumed to be 1. Adjusting this factors one will see that Curve (I) or (II) is very similar to Curve S_{s-s}. This justifies the estimation made in (1), where the hard core effect is estimated using (I) and (II) instead of using the proper wave function based on the meson theoretical potential with the hard core.

This figure shows also that if we do not use the proper sticking factor, the error amounts to large values.

§ 6. Numerical results

We shall calculate the differential cross section and the polarized cross section for n-d elastic scattering at 90Mev. This numerical result will be compared with the experimental results of the elastic scattering of 190 Mev deuteron by proton.[20] At such high energy as 90Mev, we can disregard the difference between n-d and p-d scatterings except for very small angles. Also we can neglect the energy difference between 90 Mev and 95 Mev.

As discussed in § 3, under some simplifying conditions the cross section and the polarized cross section can be expressed by the product of the sum of the two-body scattering amplitudes and the sticking factor. To calculate the sum of the scattering amplitudes of nucleon-nucleon scattering, we use the phase shifts found by Tamagaki et al[10)~12)]. As the deuteron wave function in the sticking factor, we use the form given by (5·1). Thus the phase shifts and the sticking factor we used are compatible. Contributions from the S-D and D-D coupling terms and the hard core in the deuteron wave function is neglected. This is justified by discussions developed in detail in the previous section.

(a) *Differential cross sections*

Fig. 3 shows the comparison of our numerical result with the experimental one. Our result can reproduce the experimental one very well. (The good fit at large angle may, however, be accidental in view of our fundamental assumption.)

In Fig. 3 the solid curve shows the result obtained by using the sticking factor S_{s-s}. The dotted curve shows the result obtained by using the sticking factor S'_{s-s}. The agreement of the numerical result with the experimental one is not bad for the simplifying assumptions introduced in § 2. In what follows, we shall argue that our result may be the best attainable one, if we stand on the impulse approximation, which is expressed by eq. (3·20)

Equation (3·20) is expressed as the product of the sum of the amplitudes of the two-body scattering and the sticking factor. As seen in the previous section, our sticking factor S_{s-s} shows the lowest values among others for all angles. Then if we adopt other sticking factors without varying the two body scattering amplitudes,

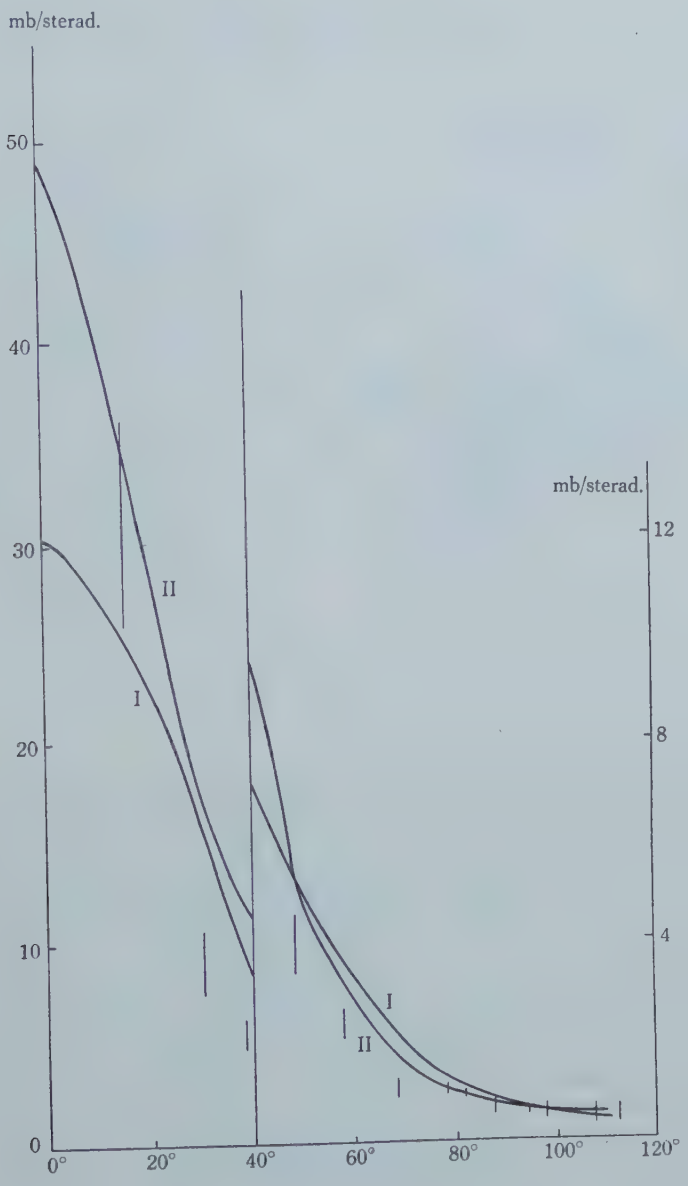

Fig. 4. The comparison of our numerical results and the experimental one.
Curve I: The sum of the two-body scattering amplitudes are calculated using the phase shifts analysed meson theoretically.
Curve II: The sum of the two-body scattering amplitudes are calculated by using the phase shifts analysed by Gammel and Thaler. In both curves, the sticking factor is S_{s-s} in the text. S_{s-s} is obtained by using the deuteron wave function compatible with the meson theoretical potential.

the discrepancy between the experimental and the numerical results becomes more serious.

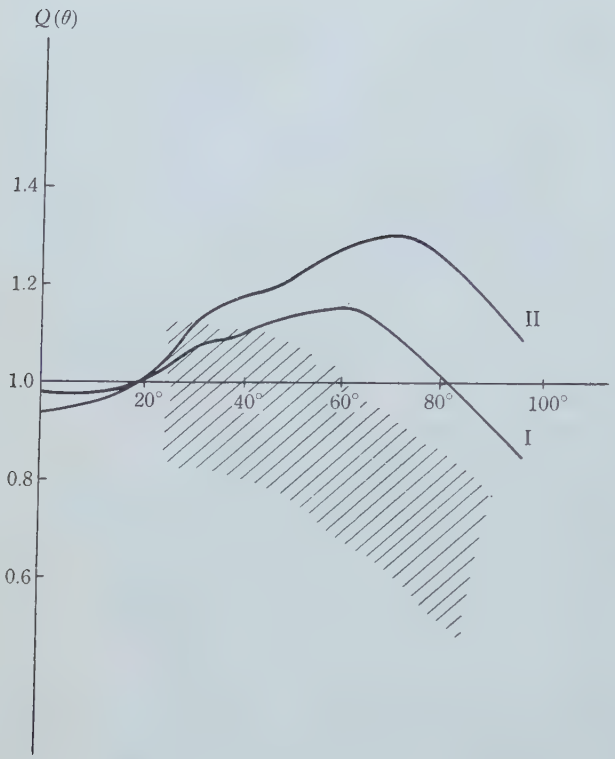

Fig. 5. Plot of $Q(\theta)$ defined by (6·1) in the text.

$Q(\theta)$ gives the measure of the interference. By definition, the numerical value of this quantity is free from the sticking factor. Hence this quantity directly discriminates good two-body phase shifts from various ones.

Curve I is the calculated value of $Q(\theta)$ by using the phase shifts suggested by meson theory. Curve II shows the calculated value of $Q(\theta)$ by using the phase shifts analysed by Gammel and Thaler. The shadowed region shows the experimental result.

Next we shall examine the two-body scattering amplitude with the fixed sticking factor. For comparison, we choose the phase shifts of two-body scattering analysed by Gammel and Thaler[8]. Gammel and Thaler* got their set of phase shifts using the entirely phenomenological potential, whereas Tamagaki et al. got their set of phase shifts standing on the meson theoretical potential. In spite of the difference of the potentials which both groups used to calculate the phase shifts and the difference of values between both sets of phase shifts, the calculated results of the *n-d* elastic cross section are very similar in their gross aspects and both can reproduce the experimental result very well. See Fig. 4. Naturally, the magnitude of the interference term of *n-p* and *n-n* scattering are quite different for all angles for both sets of phase shifts. But it is the sticking factor that makes both results so similar. The sticking factor acts to mask the difference of this interference term for both sets.

Then the best way of comparison will be to use the quantity

* Gammel and Thaler say that they used Blatt-Biedenharn's formulae to calculate the nuclear phase shifts. But when we follow Blatt-Bidenharn's formulae we find the signs of the mixing parameters ϵ_J different from theirs, that is, opposite to their signs of ϵ_J. Some difference between the calculated values of this paper and of our previous note[7] comes partly from this point, partly from slight numerical errors contained in the previous note and finally the sticking factor adopted. These points are corrected in the present paper.

$$Q(\theta)=\frac{I(\theta)_{n-d}/[(16/9)S(\Delta k)]}{I(\theta)_{nn}+I(\theta)_{np}}. \qquad (6\cdot1)$$

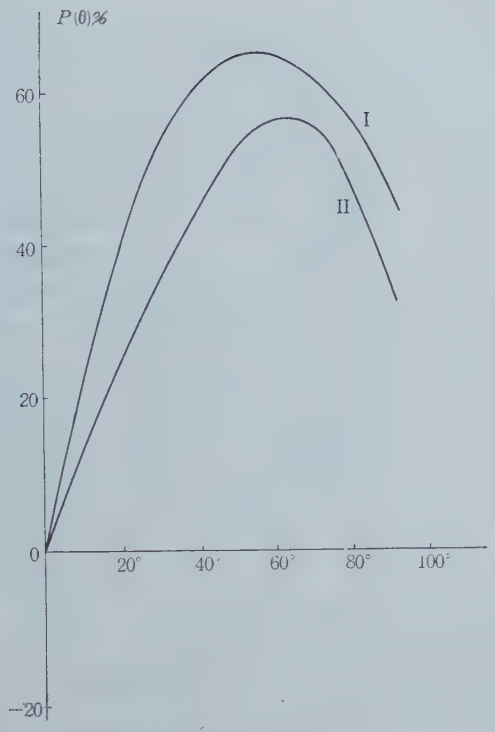

Fig. 6. The predicted value of the polarization of 90Mev *n-d* scattering
Curve I is calculated by using the meson theoretical phase shifts of the two-body scattering.
Curve II is calculated by using the phase shifts analysed by Gammel and Thaler.

This factor is independent of the sticking factor. When the interference is constructive, $Q(\theta)$ is greater than 1. When the interference is destructive, $Q(\theta)$ is smaller than 1. In Fig. 5 Curve I is the calculated value of $Q(\theta)$ by using the phases suggested by meson theory. Curve II shows the calculated value of $Q(\theta)$ by using the phases analysed by Gammel and Thaler. The shadowed region is the experimental result*. From this figure we see that Curve I is very close to the experimental result. On the other hand, Curve II is not so good as Curve I.

Then we can conclude that our numerical result expressed in Fig. 3 may be the best attainable result. Hence, the two body phase shifts and the deuteron wave function suggested by meson theory gives the best fit to the experiment.

(b) *The polarization*

Next we shall investigate the polarization. The polarization is defined by $P(\theta)$. Since both $P(\theta)\,I(\theta)$, expressed by (4·1), and $I(\theta)$, expressed by (3·20), have the form of the "dynamical" factor times the sticking factor, $P(\theta)$ is independent of the sticking factor. Hence in $P(\theta)$ the difference of the nature of chosen sets of phase shifts will appear explicitly. This is shown in Fig. 6.

This difference is exaggerated by the plot $P(\theta)/\sin\theta$ *vs* θ. See Fig. 7. At 30°, the difference of $P(\theta)\,I(\theta)$ for both sets of phase shifts is only 3 mb,[17] and the difference in the plot of $P(\theta)$ is 10%, but the difference in the plot $P(\theta)/\sin\theta$ *vs* θ amounts to about 40%.

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* We used $I(\theta)_{n-d}$ by Chamberlain et al.[26] As for $I(\theta)_{nn}$ and $I(\theta)_{n-p}$, we used the values cited in Gammel's paper[8]. Here we take our S_{S-S} as the sticking factor. This is the reason why we obtained so much different values for our "experimental" $Q(\theta)$ from Δ by Chamberlain et al.

In Fig. 8, we plot the quantity $T(\theta)=\dfrac{P(\theta)I(\theta)_{n-d}/[(16/9)S(\Delta k)]}{P(\theta)I(\theta)_{n-n}+P(\theta)I(\theta)_{n-p}}$. (6·2)

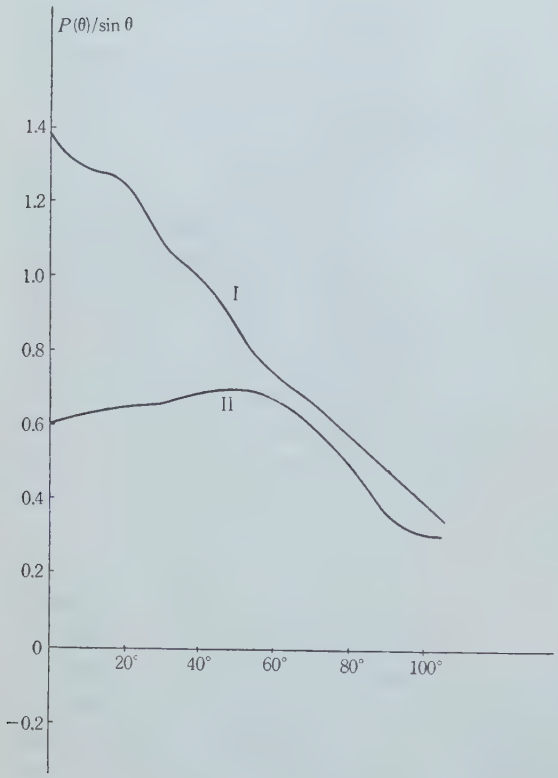

Fig. 7. The predicted value of the polarization divided by $\sin\theta$.
Curve I is calculated by using the meson theoretical phase shifts of the two-body scattering.
Curve II is calculated by using the phase shifts analysed by Gammel and Thaler.

We see that the polarizations of n-n and n-p scattering interfere constructively at all angles. This is common to both sets of phase shifts. Unfortunately, we have no experimental results to be compared with. The experimental results[27] at higher energies suggest that the impulse approximation will reproduce the experimental results well at small angles. But meager results are expected at large angles. Anyhow, the precise experiment on the polarization may serve as the further test of discriminating the chosen sets of phase shifts, as was seen in Figs. 6, 7 and 8.

§ 7. Conclusion

1) The impulse approximation can well reproduce the experimental result of the differential cross section in the region $\theta\lesssim 90°$ near 100 Mev neutron energy.

2) The phase shifts and the deuteron wave function derived by using meson theory may give the best fit among others.

3) The impulse approximation will be expected to reproduce the polarization at small angle also well. The experiment on the polarization may serve as the further test for discriminating various sets of phase shifts.

4) The examination of the off-energy shell contributions is left untouched. This effect is expected to be small at small angles. The examination of this effect is now in progress.

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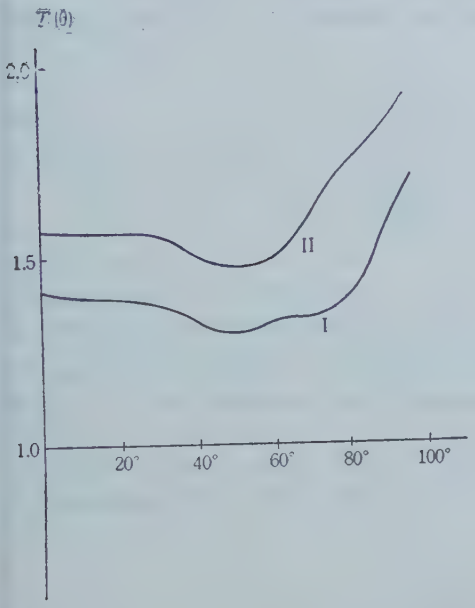

Fig. 8. Plot of the quantity $T(\theta)$ defined by (6·2) in the text. From this figure we see the polarization of n-n and n-p scatterings interfere constructively.
Curve I and Curve II correspond to the calculated results in Fig. 4. 5. 6. 7.

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On the Space-Time Formulation of Non-Relativistic Quantum Mechanics

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For the case of the quantization of the usual non-relativistic classical Lagrangian function quadratic in the velocity the validity is demonstrated of the non-canonical space-time formulation of quantum mechanics proposed recently by the author, which aims to evaluate, without appealing to the Schrödinger equation, the transformation function $K(x, t''; y, t')$ in the space representation on the basis of the composition rule

$$K(x, t''; y, t') = \int K(x, t''; z, t)\, dz\, K(z, t; y, t') \tag{1}$$

coupled with the supposition that it is approximated to zeroth order in the quantum of action h by the so-called semi-classical kernel

$$K_c(x, t''; y, t') = [(i/h)\partial^2 S/\partial x \partial y]^{1/2} \exp[(i/\hbar) S(x, t''; y, t')] \tag{2}$$

written in terms of the classical action $S(x, t''; y, t')$ alone.

In the first place the action function corresponding to the above Lagrangian is expanded in power of the interval of time $T=t''-t'$. Then the deviation of the semi-classical kernel (2) from the unitary transformation function is shown to be of the third order in T, and the corresponding correction term is evaluated by solving the integral equation (1). It is also shown that the semi-classical kernel is unitary for a free motion of a particle with its mass being a function in the space coordinate.

§ 1. Introduction and summary

In quantum mechanics the temporal development of a physical system is described essentially by the transformation function or the kernel $K(x, t; y, t')$, such that for $t=t'$

$$K(x, t; y, t) = \delta(x-y). \tag{1·1}$$

In the traditional formulation this is a solution of the Schrödinger equation

$$i\hbar(\partial/\partial t)K(x, t; y, t') = H(x, p, t)K(x, t; y, t'), \tag{1·2}$$

where p denotes the differential operator $-i\hbar(\partial/\partial x)$. If $\partial H/\partial t=0$, the solution reads simply with $T=t-t'$

$$K(x, y, T) = \exp[-(iT/\hbar)H(x, p)]\delta(x-y). \tag{1·3}$$

In this canonical formalism the Hamiltonian operator plays an essential part, which is constructed on the basis of a kind of formal correspondence with the Hamiltonian function $H(q, p, t)$ in classical mechanics, where q and p denote the space coordinate of a particle and its canonically conjugate momentum, respectively.

In contrast with this, Feynman's space-time version of quantum mechanics[1] aimed to establish, without the use of canonical variables, an equivalent and self-contained theoretical scheme in the usual space-time framework. The superiority of this approach starting from the classical action function to the usual one consists in securing a direct correspondence of quantum mechanical description with the classical mechanical one formulated originally in the form of Hamilton's principle of stationary action in the usual space-time.

Feynman's formulation suffered, however, from the drawback, as was pointed out by himself, that it required an unnatural and cumbersome infinite subdivision of the time interval T. This is not only true from the side of mathematical conceptions needed therein, but also from the practical point of view of the actual integration of the transformation function. The exact realization of Feynman's recipe has so far been limited to the simple and trivial cases of the free particle and the harmonic oscillator alone and no practical method of integration has been devised applicable to wider class of action functionals.[2]

In the present paper the following two things are shown in the non-relativistic quantum mechanics of a particle of mass m moving in one space dimension, whose classical Lagrangian and Hamiltonian functions are given respectively by

$$L(q, v) = (m/2)v^2 - V(q) \tag{1·4}$$

and by

$$H(q, p) = (1/2m)p^2 + V(q), \tag{1·5}$$

where v denotes the velocity $dq(t)/dt$. In the first place it is possible to construct a space-time formulation of quantum mechanics without appealing to an infinite subdivision of the interval of time. In the second place it becomes thus possible to evaluate the transformation function, when the potential $V(q)$ is given arbitrarily. In order to make clear the differences between the fundamental postulates underlying the present formulation and those in Feynman's, it is needed first to analyse the essentials of the so-called Feynman principle.

Let $q(t)$ represent a classical path of a particle connecting a pair of space-time points (x, t'') and (y, t'), then $x=q(t'')$ and $y=q(t')$. This function $q(t)$ is determined by the variation principle

$$\delta \int_{t'}^{t''} dt\, L[q(t)\ ,\ v(t),\ t]=0, \tag{1·6}$$

or equivalently by the Lagrangian equation of motion

$$(d/dt)\partial L/\partial v=\partial L/\partial q, \tag{1·7}$$

and the time integral of the Lagrangian function

$$S(x, t''\ ;\ y, t') = \int_{t'}^{t''} dt\, L[q(t),\ v(t),\ t] \tag{1·8}$$

along the actual classical path is called the classical action function. From the very

beginning of quantum mechanics the relation of this function to quantum mechanics has repeatedly been emphasized by various authors.[3] Especially Van Vleck showed that the transformation function could be approximated to zeroth order in the quantum of action h by the semi-classical kernel

$$K_c(x, t; y, t') = [(i/h)\partial^2 S/\partial x \partial y]^{1/2} \exp[(i/\hbar) S(x, t; y, t')] \qquad (1\cdot 9)$$

constructed from the classical action alone.[4] Of course this can not in general represent a unitary transformation. But in some favourable cases it happens that this is correct to the first order in the time interval $T=t-t'$, or more precisely that, supposing $\partial H/\partial t=0$,

$$K_c(x, y, T) = [1-(iT/\hbar) H(x, p)]\delta(x-y) + O(T^2). \qquad (1\cdot 10)$$

Now it is well known that any finite unitary transformation is a result of an infinite unfolding of successive infinitesimal unitary transformations. It was thus asserted by Feynman that, when and only when the semi-classical kernel is correct to the first order in T, the infinite unfolding of infinitesimal unitary transformations goes over to the functional integration method

$$K(x, t; y, t') = \int d(\text{paths}) \exp[(i/\hbar) S(\text{path})] \qquad (1\cdot 11)$$

over all possible imaginary paths in space-time with the common end points (x, t) and (y, t'), of which the result will be the same with that of the Schrödinger equation. The reason for this was stated as follows. If a given finite interval of time, T^*, is subdivided into equal steps T, the number of the factors of the form $(1\cdot 10)$ is T^*/T. If an error of order T^2 is made in each, the resulting error will not accumulate beyond the order $T^2(T^*/T)$ or T^*T, which vanishes in the limit $T\to 0$.

This is actually the case for the non-relativistic quantum mechanics corresponding to the classical system characterized by $(1\cdot 4)$ and $(1\cdot 5)$. Feynman showed this by establishing, with the aid of the approximation formula

$$S(x, y, T) = (m/2T)(x-y)^2 - TV(x) \qquad (1\cdot 12)$$

to the classical action function $(1\cdot 8)$, the relationship

$$\int dy\, K_c(x, y, T)\psi(y) = [1-(iT/\hbar) H(x, p)]\psi(x) + O(T^2), \qquad (1\cdot 13)$$

which is equivalent to the above $(1\cdot 10)$. In this connection the question arises to what extent the semi-classical kernel is a correct one in the non-relativistic case now under consideration. The following three sections will thus be devoted to the improvement of the approximation formula $(1\cdot 12)$ yielding the result

$$S(x, y, T) = (m/2T)(x-y)^2 - TF_1(x, y) - (T^3/2m)F_3(x, y) + O(T^5), \qquad (1\cdot 14)$$

wherein

$$F_1(x, y)=\int_0^1 du\, V[y+u(x-y)] \tag{1·15}$$

and the expression for $F_3(x, y)$ will be found in (3·22). It will then be shown in § 5 that (1·10) will accordingly be changed to the form

$$K_c(x, y, T)=K(x, y, T)-(i\hbar T^3/240m^2)[(d/dx)^4V(x)]\delta(x-y)+O(T^4) \tag{1·16}$$

with the $K(x, y, T)$ given by (1·3). The semi-classical kernel is, therefore, correct to the second order in T and the transformation function should be written in the form

$$\begin{aligned} K(x, y, T)&=K_c(x, y, T)[1+(i\hbar T^3/8m^2)D(x, y)+O(T^4)] \\ &=K_c(x, y, T)+(i\hbar T^3/8m^2)D(x, x)\delta(x-y)+O(T^4). \end{aligned} \tag{1·17}$$

Comparing this with the above (1·16), we get at once the condition

$$D(x, x)=(1/30)(d/dx)^4V(x). \tag{1·18}$$

The task of the space-time formulation of quantum mechanics will thus be to determine the unknown function $D(x, y)$ subjected to the condition (1·18) without relying upon the Schrödinger equation.

It is evident from the above that the Feynman principle is correct at least in principle so far as we are concerned with the non-relativistic Lagrangian (1·4). But being hindered by mathematical difficulties, no attempt has so far been made to obtain by this method the function $D(x, y)$ in the general case of an arbitrary potential $V(q)$. Apart from this difficulty the limitation is inherent in Feynman's method that it is valid only for the restricted class of action functionals, for which the corresponding semi-classical kernels are correct to the first order in the time interval. Therefore, as far as the correspondence with the classical mechanical picture is made via the semi-classical kernel written in terms of the classical action functional, the most general form of the space-time formulation of quantum mechanics applicable also to wider class of action functionals will essentially be at variance with an infinite subdivision of the interval of time.

Now in so far as the parameter h is concerned alone, the semi-classical kernel is correct to the zeroth order in it for any finite interval of time. The unitarity of the transformation function will thus be secured by putting

$$K(x, t; y, t')=K_c(x, t; y, t')U(x, t; y, t') \tag{1·19}$$

with the correction factor $U(x, t; y, t')$ such that

$$\lim_{\hbar\to 0} U(x, t; y, t')=1. \tag{1·20}$$

Then in our non-relativistic case we have from (1·17)

$$U(x, y, T)=1+i\hbar[(T^3/8m^2)D(x, y)+O(T^4)]+O(\hbar^2). \tag{1·21}$$

But for the purpose of determining the unknown factor $U(x, t; y, t')$ the use of the unitarity condition

$$\int dz\, K(x, t; z, t') K(y, t; z, t')^* = \delta(x-y), \tag{1·22}$$

where the asterisk denotes the complex conjugate, is inconvenient, because the delta-function in the right-hand side is mathematically hard to manipulate. Therefore, the author has recently proposed[5] to base the space-time formulation of quantum mechanics on the use of the recursive condition

$$K(x, t''; y, t') = \int K(x, t''; z, t)\, dz\, K(z, t; y, t'), \tag{1·23}$$

which yields the above (1·22) for $t''=t'$ according to (1·1). The final object of the present paper is thus to show that this recipe for the construction of the transformation function works well at least in the non-relativistic case wherein the action function is approximately given by (1·14). In § 6 the integral equation (1·23) is solved to determine the function $D(x, y)$ in (1·21) as

$$D(x, y) = (\partial^2/\partial x \partial y)^2 F_1(x, y) \tag{1·24}$$

in terms of $F_1(x, y)$ given in (1·15). This at once gives

$$D(x, x) = \int_0^1 du\, u^2 (1-u)^2 (d/dx)^4 V(x), \tag{1·25}$$

which is nothing but the condition (1·18). In addition to this it will be shown in § 7 that the semi-classical kernel corresponding to the classical Lagrangian function

$$L(q, v) = (m/2)[w(q)v]^2 \tag{1·26}$$

will itself satisfy the recursive condition (1·23). Therefore the semi-classical kernel is unitary and we can put in this case the function U in (1·19) equal to unity.

Now from the above one may safely expect that the space-time formulation of quantum mechanics based on the coupled use of the recursive condition (1·23) and the factorization (1·19) of the transformation function in terms of the semi-classical kernel (1·9) is valid in general for wider class of action functionals. In order to show that this is actually the case, it is needed first to devise some mathematical transformation of the condition (1·23) into a more tractable one and then to utilize it in quantizing classical systems different essentially from those treated in this paper. Moreover, in order that the present formulation should be a complete and self-contained one, or more precisely that it should be completely free from the influence of the Schrödinger equation, one has to be able to explain, in the framework of the recursive condition alone, the reason why one is to start from the factorization (1·19). This means, as has already been pointed out by the author,[5] that the principle of correspondence with the classical mechanical description is to be implied, in the limit of the quantum of action h tending to zero,

in the recursive condition representing the group property of the transformation function. In addition to these it will be of interest to investigate what condition is imposed on the classical Lagrangian function in order that the semi-classical kernel is correct to the first order in the time interval or accordingly that the passage to Feynman's path integral method of quantization is really possible. These are the problems left to be solved and the solutions will be given in succeeding papers.

§ 2. Evaluation of action function by the Hamilton-Jacobi equation

Since the Lagrangian function (1·4) does not depend explicitly on the time t, the action function (1·8) involves t'' and t' in the form of the difference $T=t''-t'$ and can accordingly be written as $S(x, y, T)$. In the case of free motion with vanishing potential it reads simply

$$S_f(x, y, T)=(m/2T)(x-y)^2. \tag{2·1}$$

When the particle is acted on by force, the explicit construction of action function is difficult to perform and it has so far been successful only for the restricted cases, where the potential is linear or quadratic in the coordinate. For

$$V(q)=kq \qquad \text{and} \qquad V(q)=(m\omega^2/2)q^2 \tag{2·2}$$

with the constants k and ω, the results are respectively

$$S(x, y, T)=S_f(x, y, T)-(k/2)T(x+y)-(k^2/24m)T^3 \tag{2·3}$$

and

$$\begin{aligned} S(x, y, T) &= (m\omega/2\sin\omega T)[(x^2+y^2)\cos\omega T-2xy] \\ &= S_f(x, y, T)-(m\omega^2/6)T(x^2+xy+y^2) \\ &\quad -(m\omega^4/90)T^3[x^2+(7/4)xy+y^2]-\cdots. \end{aligned} \tag{2·4}$$

It is common to these that the first term $S_f(x, y, T)$ is followed by the terms of odd power in T, so that we put in the general case of an arbitrary potential $V(q)$

$$S(x, y, T)=S_f(x, y, T)-TF(x, y, T) \tag{2·5}$$

with

$$F(x, y, T)=\sum_{k=0}^{\infty}(T^2/2m)^k F_{2k+1}(x, y). \tag{2·6}$$

The Hamilton-Jacobi equation derived from the Hamiltonian function (1·5) is

$$(\partial/\partial T)S(x, y, T)+(1/2m)[(\partial/\partial x)S(x, y, T)]^2+V(x)=0, \tag{2·7}$$

which is transcribed with $a=T^2/2m$ as

$$\sum_{n=0}^{\infty}a^n(x-y)^{-2n}(\partial/\partial x)[(x-y)^{2n+1}F_{2n+1}(x, y)]=\sum_{n=0}^{\infty}a^nG_{2n}(x, y), \tag{2·8}$$

where

$$G_0(x, y)=V(x) \tag{2·9}$$

and

$$G_{2n}(x, y)=\sum_{k=0}^{n-1}(\partial/\partial x)F_{2k+1}\cdot(\partial/\partial x)F_{2n-(2k+1)}, \qquad (n>0). \tag{2·10}$$

The solutions are given at once by

$$F_{2n+1}(x, y)=(x-y)^{-(2n+1)}\Big[\int_y^x dz(z-y)^{2n}G_{2n}(z, y)+C_{2n+1}(y)\Big]$$
$$=\int_0^1 du\, u^{2n}G_{2n}[g(u), y]+(x-y)^{-(2n+1)}C_{2n+1}(y), \tag{2·11}$$

where we have replaced z with $g(u)=y+u(x-y)$ and $C_{2n+1}(y)$ are the integration constants.

According to the definition (1·8) of the classical action, the term of the lowest order in T should be $TL(q, v)$, when the substitutions $y=q$ and $x-y=vT$ are made in it. But from the above we get

$$S(q+vT, q, T)=-\sum_{k=0}^{\infty}(2m)^{-k}v^{-(2k+1)}C_{2k+1}(q)+O(T), \tag{2·12}$$

so we must put $C_{2n+1}(y)=0$ in (2·11). Then for $n=0$ we get the solution (1·15) according to (2·9) and for $n=1$ we have from (2·10) and (2·11) respectively with the abbreviations $V_k(x)=(d/dx)^kV(x)$

$$G_2(x, y)=[(\partial/\partial x)F_1(x, y)]^2=\Big[\int_0^1 dw\, w\, V_1[g(w)]\Big]^2, \tag{2·13}$$

$$F_3(x, y)=\int_0^1 du\, u^2G_2[g(u), y]=\int_0^1 du\, u^2\Big[\int_0^1 dw\, w\, V_1[g(uw)]\Big]^2$$
$$=\int_0^1 du\,\{(1/u)\int_0^u dw\, w\, V_1[g(w)]\}^2. \tag{2·14}$$

The results in this section were obtained by the author[6] in 1950 and essentially the same results were also attained afterwards in 1955 by Choquard.[7] Instead of adopting a roundabout method based on the partial differential equation (2·7) we had, however, better to construct the classical action directly according to its original definition (1·8). The next section will thus be devoted to this purpose.

§ 3. Evaluation of action function by direct integration

The classical path $q(t)$ of a particle, whose Lagrangian function is given by (1·4), is determined by the Newtonian equation of motion

$$m(d/dt)^2q(t)=-V_1[q(t)] \tag{3·1}$$

derived from (1·7) and in addition to this by the initial conditions $x=q(t'')$ and $y=q(t')$. Then in terms of a real parameter u defined by $t=t'+u(t''-t')$ the classical path is represented by a function $q(u)$, such that $x=q(1)$ and $y=q(0)$, and at the same time the above equation is rewritten as

$$q''(u)=-2aV_1[q(u)], \tag{3·2}$$

where $a=T^2/2m$ and the prime denotes the differentiation with respect to the parameter u.

In the case of a free motion the solution $q(u)$ is at once given by $g(u)=y+u\xi$ with $g'(u)=\xi=x-y$ and $g''(u)=0$. Then putting in the general case of an arbitrary potential $V(q)$

$$q(u)=g(u)+f(u), \tag{3·3}$$

we get $f(1)=f(0)=0$ and from (3·2) the equation

$$\begin{aligned}(-1/2a)f''(u)&=V_1[g(u)+f(u)]\\&=V_1[g(u)]+V_2[g(u)]f(u)+(1/2)\,V_3[g(u)]f(u)^2+\cdots.\end{aligned} \tag{3·4}$$

The expansion of $f(u)$ in powers of a

$$f(u)=\sum_{k=1}^{\infty}(-2a)^k f_{2k}(u) \tag{3·5}$$

gives $f_{2k}(1)=f_{2k}(0)=0$ and transforms the above (3·4) into

$$f_2''(u)=V_1[g(u)]=(1/\xi)\,(d/du)\,V[g(u)], \tag{3·6}$$

$$f_4''(u)=V_2[g(u)]f_2(u), \tag{3·7}$$

$$f_6''(u)=V_2[g(u)]f_4(u)+(1/2)\,V_3[g(u)]f_2(u)^2, \tag{3·8}$$

and so on. After repeated differentiations (3·6) gives

$$f_2^{(n+1)}(u)=\xi^{n-1}V_n[g(u)] \tag{3·9}$$

and accordingly (3·7) and (3·8) are transformed respectively into

$$\begin{aligned}\xi f_4''(u)&=\xi V_2[g(u)]f_2(u)=f_2^{(3)}(u)f_2(u)\\&=[f_2'(u)f_2(u)]''-(3/2)[f_2'(u)^2]',\end{aligned} \tag{3·10}$$

$$\begin{aligned}2\xi f_6''(u)&=2f_2^{(3)}(u)f_4(u)+(1/\xi)f_2^{(4)}(u)f_2(u)^2\\&=[f_2(u)f_4'(u)-3f_2'(u)f_4(u)]''+5[f_2''(u)f_4(u)]'.\end{aligned} \tag{3·11}$$

Now (3·6) is at once integrated to give

$$f_2(u)=(1/\xi)[\int_0^u-u\int_0^1]\,dw\;V[g(w)] \tag{3·12}$$

$$=(1/\xi)u\int_0^1 dw\,\{V[g(uw)]-V[g(w)]\} \tag{3·13}$$

and in the same way we get from (3·10) and (3·11)

$$\xi f_4(u)=f_2'(u)f_2(u)-(3/2)[\int_0^u-u\int_0^1]\,dw\,f_2'(w)^2, \tag{3·14}$$

$$2\xi f_6(u)=f_2(u)f_4'(u)-3f_2'(u)f_4(u)+5[\int_0^u-u\int_0^1]dwf_2''(w)f_4(w). \tag{3·15}$$

According to the above results the Lagrangian function (1·4) is transcribed as

$$\begin{aligned}L(q,v)&=(m/2T^2)[g'(u)+f'(u)]^2-V[q(u)]\\&=(1/T)S_f(x,y,T)-D(u)+(d/du)C(u)\end{aligned} \tag{3·16}$$

with $$D(u)=V[q(u)]+(m/2T^2)f(u)f'(u)$$
$$=V[g(u)+f(u)]-(1/2)f(u)\,V_1[g(u)+f(u)] \tag{3·17}$$
and $$C(u)=(m/2T^2)f(u)[2(x-y)+f'(u)], \tag{3·18}$$
which gives $C(1)=C(0)=0$. Then we get from (3·17)

$$D(u)-V[g(u)]=(1/2)\sum_{n=1}^{\infty}(1/n!)\,(2-n)\,V_n[g(u)]f(u)^n$$
$$=(1/2)\sum_{n=1}^{\infty}(1/n!\,\xi^{n-1})\,(2-n)f_2^{(n+1)}(u)f(u)^n$$
$$=(1/2)f_2''(u)f(u)-(1/12\xi^2)f_2^{(4)}(u)f(u)^3+\cdots$$
$$=-af_2''(u)f_2(u)+2a^2f_2''(u)f_4(u)+O(a^3) \tag{3·19}$$

and accordingly

$$D(u)=V[g(u)]+a[f_2'(u)^2-(f_2f_2')'] \tag{3·20}$$
$$+a^2(2/\xi)[f_2'(u)^3+(f_2^2f_2''-f_2f_2'^2)'+\xi(f_2'f_4-f_2f_4')']+O(a^3).$$

Since (1·8) gives, with the aid of (3·16) and (3·18),

$$S(x,\,y,\,T)=T\int_0^1 du\,L[q(u),\,v(u)]=S_f(x,\,y,\,T)-T\int_0^1 du\,D(u), \tag{3·21}$$

one has, according to (2·5), (2·6), (3·20) and (3·12), in the first place the result (1·15) and then

$$F_3(x,\,y)=\int_0^1 du\,f_2'(u)^2$$
$$=(1/\xi)^2[\int_0^1 du\,V[g(u)]^2-\{\int_0^1 du\,V[g(u)]\}^2], \tag{3·22}$$
$$F_5(x,\,y)=(2/\xi)\int_0^1 du\,f_2'(u)^3. \tag{3·23}$$

Finally according to (3·6) one has

$$[(1/u)\int_0^u dw\,w\,V_1[g(w)]]^2=[(1/u)\int_0^u dw\,w\,f_2''(w)]^2$$
$$=[f_2'(u)-(1/u)f_2(u)]^2=f_2'(u)^2-(d/du)[(1/u)f_2(u)^2], \tag{3·24}$$

and since $(1/u)f_2(u)^2$ vanishes both for $u=0$ and for $u=1$ owing to (3·13), we see that (3·22) is identical with (2·14).

§ 4. Some properties of action function

Let a function $q(t)$ with $q(t'')=x$ and $q(t')=y$ represent a classical path $P(x,\,t'';\,y,\,t')$ of a particle and let an imaginary path, which is composed of two classical paths $P(x,\,t'';\,q^*,\,t^*)$ and $P(q^*,\,t^*;\,y,\,t')$ jointed together at $(q^*,\,t^*)$, be

represented by a varied function $q(t)+\delta q(t)$, then we have $\delta q(t'')=\delta q(t')=0$ and $\delta q(t^*)=q^*-q(t^*)$. The definition (1·8) of the classical action gives now

$$S(x, t''; q^*, t^*)+S(q^*, t^*; y, t')-S(x, t''; y, t')$$
$$=\int_{t'}^{t''}dt\, L(q+\delta q, v+\delta v, t)-\int_{t'}^{t''}dt\, L(q, v, t) \tag{4·1}$$

and taking up to the first order variation the right-hand side vanishes owing to (1·6), so that from the expansion of the left-hand side in $\delta q(t^*)$, we get, by putting $z=q(t)$, the following two conditions,

$$S(x, t''; y, t')=S(x, t''; z, t)+S(z, t; y, t') \tag{4·2}$$

and
$$(\partial/\partial z)[S(x, t''; z, t)+S(z, t; y, t')]=0. \tag{4·3}$$

In the case of a free motion the classical path $q(t)$ is given by $g(u)=ux+wy$ with $w=1-u$, and for the classical action (2·1) we have the relationship

$$S_f(x, z, wT)+S_f(z, y, uT)-S_f(x, y, T)$$
$$=(m/2uwT)[z-g(u)]^2=S_f[z, g(u), uwT], \tag{4·4}$$

where z is taken arbitrarily. Then for $z=g(u)$ the above conditions (4·2) and (4·3) are easily seen to be fulfilled. In the general case of an arbitrary potential $V(q)$ they are rewritten with the aid of (2·5) and (4·4) respectively as

$$wF(x, z, wT)+uF(z, y, uT)-F(x, y, T)=(1/4uwa)[z-g(u)]^2 \tag{4·5}$$

and
$$(\partial/\partial z)[wF(x, z, wT)+uF(z, y, uT)]=(1/2uwa)[z-g(u)], \tag{4·6}$$

where z is given according to (3·3) and (3·5) by $z=g(u)-2af_2(u)+O(a^2)$. Then in view of the expansion $F(x, y, T)=F_1(x, y)+aF_3(x, y)+O(a^2)$ given in (2·6) the condition (4·5) gives with the abbreviation $\langle\varphi(x, y, z)\rangle=\varphi[x, y, g(u)]$

$$\langle wF_1(x, z)+uF_1(z, y)\rangle-F_1(x, y)=0 \tag{4·7}$$

and
$$\langle w^3F_3(x, z)+u^3F_3(z, y)\rangle-F_3(x, y)$$
$$=2f_2(u)\langle(\partial/\partial z)[wF_1(x, z)+uF_1(z, y)]\rangle+(1/uw)f_2(u)^2. \tag{4·8}$$

In the same way the condition (4·6) yields

$$\langle(\partial/\partial z)[wF_1(x, z)+uF_1(z, y)]\rangle=-(1/uw)f_2(u), \tag{4·9}$$

which transforms (4·8) into

$$F_3(x, y)-\langle w^3F_3(x, z)+u^3F_3(z, y)\rangle=uw\langle(\partial/\partial z)[wF_1(x, z)+uF_1(z, y)]\rangle^2. \tag{4·10}$$

Finally (4·7) gives by repeated differentiations

$$(\partial/\partial x)^m(\partial/\partial y)^nF_1(x, y)=w\langle[(\partial/\partial x)+u(\partial/\partial z)]^m[w(\partial/\partial z)]^nF_1(x, z)\rangle$$
$$+u\langle[u(\partial/\partial z)]^m[w(\partial/\partial z)+(\partial/\partial y)]^nF_1(z, y)\rangle, \tag{4·11}$$

which reads more specifically

$$D_2(x, y) = (\partial^2/\partial x\partial y) F_1(x, y)$$
$$= \langle w^2 D_2(x, z) + u^2 D_2(z, y)\rangle + uw\langle (\partial/\partial z)^2 [wF_1(x, z) + uF_1(z, y)]\rangle \tag{4·12}$$

and $$D_3(x, y) = (\partial^2/\partial x\partial y)^2 F_1(x, y) = \langle w^3 D_3(x, z) + u^3 D_3(z, y)\rangle \tag{4·13}$$
$$+2uw\langle (\partial/\partial z)^2 [w^2 D_2(x, z) + u^2 D_2(z, y)]\rangle + (uw)^2\langle (\partial/\partial z)^4 [wF_1(x, z) + uF_1(z, y)]\rangle.$$

§ 5. Unitarity of the semi-classical kernel

Before entering into the main subject of the present section to show (1·16), we shall first evaluate the free kernel or the transformation function corresponding to the classical free motion, for which the action function is $S_f(x, y, T) = (m/2T) \times (x-y)^2$. According to the definition (1·9) the semi-classical kernel reads

$$K_f(x, y, T) = (m/2\pi i\hbar T)^{1/2} \exp[(im/2\hbar T)(x-y)^2]$$
$$= (1/2\pi)\int_{-\infty}^{+\infty} du \exp[-(i\hbar T/2m)u^2 + iu(x-y)]$$
$$= \exp[(i\hbar T/2m)(\partial/\partial x)^2]\delta(x-y)$$
$$= \exp[-(iT/\hbar)H(p)]\delta(x-y) \tag{5·1}$$

with $H(p) = (1/2m)p^2$ and $p = -i\hbar(\partial/\partial x)$, which is evidently a unitary one. Then we have two equivalent expressions

$$K_f(x, y, T) = \exp[\lambda\mu(\partial/\partial x)^2]\delta(x-y) \tag{5·2}$$

and
$$K_f(x, y, T) = \exp[\lambda\mu(\partial/\partial y)^2]\delta(x-y), \tag{5·3}$$

where we have put

$$\lambda = -iT/\hbar, \quad \mu = -\hbar^2/2m \tag{5·4}$$

and accordingly $a = T^2/2m = \lambda^2\mu$.

Proceeding to the general case of an arbitrary potential $V(q)$, the action function is given by

$$S(x, y, T) = S_f(x, y, T) - TF(x, y, T), \tag{2·5}$$

and we have according to (2·6)

$$F(x, y, T) = F_1(x, y) + \lambda^2\mu F_3(x, y) + O(\lambda^4). \tag{5·5}$$

Then since the above (2·5) gives

$$\partial^2 S(x, y, T)/\partial x\partial y = -(m/T)[1 + (T^2/m)\partial^2 F(x, y, T)/\partial x\partial y], \tag{5·6}$$

the semi-classical kernel (1·9) is rewritten according to (5·3) as

$$K_c(x, y, T) = K_f(x, y, T) W(x, y, T)$$
$$= W(x, y, T) \exp[\lambda\mu(\partial/\partial y)^2]\delta(x-y), \tag{5·7}$$

wherein $W(x, y, T)$ is defined according to (5, 5) by

$$\log[W(x, y, T)]=\lambda F(x, y, T)+(1/2)\log[1+2\lambda^2\mu\partial^2 F/\partial x\partial y]$$
$$=\lambda F_1(x, y)+\lambda^2\mu D_2(x, y)+\lambda^3\mu F_3(x, y)+O(\lambda^4). \quad (5\cdot 8)$$

We thus get the expansion

$$W(x, y, T)=\sum_{j=0}^{\infty}(\lambda^j/j!)\,W_j(x, y) \quad (5\cdot 9)$$

with

$$W_0(x, y)=1, \quad W_1(x, y)=F_1(x, y),$$
$$W_2(x, y)=F_1(x, y)^2+2\mu D_2(x, y), \quad (5\cdot 10)$$
$$W_3(x, y)=F_1(x, y)^3+6\mu[F_3(x, y)+F_1(x, y)D_2(x, y)].$$

With the aid of the abbreviation $\langle\varphi(x, y)\rangle=\varphi(x, x)$ one accordingly gets by repeated partial integrations

$$\int dy\, K_c(x, y, T)\psi(y)=\int dy\,\delta(x-y)\,\exp[\lambda\mu(\partial/\partial y)^2]\,W(x, y, T)\psi(y)$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(1/k!\,j!)\,(\lambda\mu)^k\lambda^j\langle(\partial/\partial y)^{2k}W_j(x, y)\psi(y)\rangle$$
$$=\sum_{n=0}^{\infty}(\lambda^n/n!)\sum_{k=0}^{n}{}_nC_k\mu^k\sum_{j=0}^{2k}{}_{2k}C_j\langle(\partial/\partial y)^j W_{n-k}(x, y)\rangle(d/dx)^{2k-j}\psi(x), \quad (5\cdot 11)$$

which at once yields

$$K_c(x, y, T)=\sum_{n=0}^{\infty}(\lambda^n/n!)\,M_n(x, \partial/\partial x)\delta(x-y) \quad (5\cdot 12)$$

with the differential operators

$$M_n(x, \partial/\partial x)=[\mu(\partial/\partial x)^2]^n$$
$$+\sum_{k=0}^{n-1}{}_nC_k\mu^k\sum_{j=0}^{2k}{}_{2k}C_j\langle(\partial/\partial y)^j W_{n-k}(x, y)\rangle(\partial/\partial x)^{2k-j}. \quad (5\cdot 13)$$

Since (1·15), (2·14) and (4·12) furnish the relationships

$$\langle(\partial/\partial y)^n F_1(x, y)\rangle=(n+1)^{-1}V_n(x),$$
$$\langle(\partial/\partial y)\,F_1(x, y)^2\rangle=V(x)\,V_1(x),$$
$$\langle(\partial/\partial y)^2 F_1(x, y)^2\rangle=(2/3)\,V(x)\,V_2(x)+(1/2)\,V_1(x)^2, \quad (5\cdot 14)$$
$$\langle F_3(x, y)\rangle=(1/12)\,V_1(x)^2,$$
$$\langle(\partial/\partial y)^n D_2(x, y)\rangle=[(n+2)\,(n+3)]^{-1}V_{n+2}(x),$$

we have according to (5·10)

$$\langle(\partial/\partial y)^n W_1(x, y)\rangle=(n+1)^{-1}V_n(x),$$
$$\langle W_2(x, y)\rangle=V(x)^2+(\mu/3)\,V_2(x),$$

$$\langle(\partial/\partial y)\,W_2(x,\,y)\rangle = V(x)\,V_1(x)+(\mu/6)\,V_3(x), \tag{5·15}$$
$$\langle(\partial/\partial y)^2 W_2(x,\,y)\rangle = (2/3)\,V(x)\,V_2(x)+(1/2)\,V_1(x)^2+(\mu/10)\,V_4(x),$$
$$\langle W_3(x,\,y)\rangle = V(x)^3+\mu[(1/2)\,V_1(x)^2+V(x)\,V_2(x)].$$

One obtains, therefore, from (5·13) the results: $M_0(x,\,\partial/\partial x)=1$,

$$M_1(x,\,\partial/\partial x)=\mu(\partial/\partial x)^2+\langle W_1(x,\,y)\rangle=\mu(\partial/\partial x)^2+V(x), \tag{5·16}$$
$$\begin{aligned} M_2(x,\,\partial/\partial x) &= \mu^2(\partial/\partial x)^4+2\mu[\langle(\partial/\partial y)^2 W_1\rangle+2\langle(\partial/\partial y)\,W_1\rangle(\partial/\partial x) \\ &\qquad +\langle W_1\rangle(\partial/\partial x)^2]+\langle W_2\rangle \\ &= \mu^2(\partial/\partial x)^4+\mu[V_2+2V_1(\partial/\partial x)+2V(\partial/\partial x)^2]+V^2 \\ &= [\mu(\partial/\partial x)^2+V(x)]^2, \end{aligned} \tag{5·17}$$
$$\begin{aligned} M_3(x,\,\partial/\partial x) &= \mu^3(\partial/\partial x)^6+\mu^2[(9/10)\,V_4+4V_3(\partial/\partial x)+7V_2(\partial/\partial x)^2 \\ &\qquad +6V_1(\partial/\partial x)^3+3V(\partial/\partial x)^4] \\ &\quad +\mu[3VV_2+2V_1^2+6VV_1(\partial/\partial x)+3V^2(\partial/\partial x)^2]+V^3 \\ &= [\mu(\partial/\partial x)^2+V(x)]^3-(\mu^2/10)\,V_4(x). \end{aligned} \tag{5·18}$$

Therefore, (5·4) and (1·5) tell us that the above (5·12) is identical with (1·16) up to the terms of the third order in T.

Only when $V_4(x)$ vanishes, it is thus possible that the semi-classical kernel is a unitary one. It will then be shown for the classical potential $V(q)$ given by (2·2) that the corresponding $K_c(x,\,y,\,T)$ is identical with the transformation function (1·3), or more precisely that

$$K_c(x,\,y,\,T)=\exp[\lambda\mu(\partial/\partial x)^2+\lambda V(x)]\delta(x-y). \tag{5·19}$$

For the linear potential $V(q)=kq$ we have according to (2·3), (5·2) and (5·4)

$$\begin{aligned} K_c(x,\,y,\,T) &= \exp[(\lambda k/2)\,(x+y)+(1/12)\lambda^3\mu k^2]\cdot K_f(x,\,y,\,T) \\ &= \exp(\lambda^3\mu k^2/12)\exp(\lambda k x/2)\exp[\lambda\mu(\partial/\partial x)^2]\exp(\lambda k x/2)\delta(x-y). \end{aligned} \tag{5·20}$$

This is transformed into (5·19) by putting $u=\lambda\mu$ and $w=k/2\mu$ in Equation (A·1) in the Appendix. Then for the quadratic potential $V(q)=(m\omega^2/2)q^2$ the classical action (2·4) is rewritten with $\omega T^*=\sin\omega T$ as

$$S(x,\,y,\,T)=S_f(x,\,y,\,T^*)-(m\omega/2)\,(x^2+y^2)\tan(\omega T/2), \tag{5·21}$$

so that one has by putting $u=\omega T/2$ and $w=i\hbar/m\omega$

$$\begin{aligned} K_c(x,\,y,\,T) &= \exp[(m\omega/2i\hbar)\,(x^2+y^2)\tan(\omega T/2)]\cdot K_f(x,\,y,\,T^*) \\ &= \exp[(1/2w)\,x^2\tan u]\exp[(w/2)\sin 2u(\partial/\partial x)^2]\exp[(1/2w)x^2\tan u]\delta(x-y). \end{aligned} \tag{5·22}$$

Then since $uw=\lambda\mu$ and $u/w=\lambda(m\omega^2/2)$, the identity (A·2) again yields the above (5·19). This furnishes an improvement of the operator calculus that was given in 1952 by the author.[8]

§ 6. Solution of the integral equation

The function $U(x, y, T)$ in (1·21) is rewritten according to (5·4) as

$$U(x, y, T)=1+(1/2)\lambda^3\mu^2 D(x, y)+O(\lambda^4) \tag{6·1}$$

and we have from (1·19) and (5·8)

$$K(x, y, T)=K_c(x, y, T)U(x, y, T)=K_f(x, y, T)Y(x, y, T) \tag{6·2}$$

with
$$Y(x, y, T)=W(x, y, T)U(x, y, T). \tag{6·3}$$

Then since (4·4) gives with $w=1-u$ and $g(u)=ux+wy$

$$K_f(x, z, wT)K_f(z, y, uT)=K_f(x, y, T)K_f[z, g(u), uwT], \tag{6·4}$$

the integral equation (1·23) is transformed into

$$\begin{aligned} K(x, y, T)&=\int K(x, z, wT)\,dz\,K(z, y, uT) \\ &=K_f(x, y, T)\int dz\,K_f[z, g(u), uwT]\,Y(x, z, wT)\,Y(z, y, uT) \\ &=K(x, y, T)\int dz\,\delta[z-g(u)]\exp[\lambda\mu\,uw(\partial/\partial z)^2] \\ &\qquad\times Y(x, z, wT)\,Y(z, y, uT)\,Y(x, y, T)^{-1}. \end{aligned} \tag{6·5}$$

In terms of the symbol $\langle\varphi(x, y, z)\rangle=\varphi[x, y, g(u)]$ we thus get the equation

$$1=\langle\exp[\lambda\mu\,uw(\partial/\partial z)^2]\,Y(x, z, wT)\,Y(z, y, uT)\,Y(x, y, T)^{-1}\rangle. \tag{6·6}$$

By putting

$$Y_1(x, y, z)=wF_1(x, z)+uF_1(z, y)-F_1(x, y) \tag{6·7}$$

we get from (5·8)

$$\begin{aligned} &\log[W(x, z, wT)\,W(z, y, uT)\,W(x, y, T)^{-1}]=\lambda Y_1(x, y, z) \\ &+\lambda^2\mu[w^2D_2(x, z)+u^2D_2(z, y)-D_2(x, y)] \\ &+\lambda^3\mu[w^3F_3(x, z)+u^3F_3(z, y)-F_3(x, y)]+O(\lambda^4), \end{aligned} \tag{6·8}$$

which yields together with the above (6·1)

$$Y(x, z, wT)\,Y(z, y, uT)\,Y(x, y, T)^{-1}=\sum_{j=0}^{\infty}(\lambda^j/j!)\,Y_j(x, y, z) \tag{6·9}$$

with $Y_0(x, y, z)=1$ and

$$Y_2(x, y, z)=Y_1(x, y, z)^2+2\mu[w^2D_2(x, z)+u^2D_2(z, y)-D_2(x, y)], \tag{6·10}$$

$$\begin{aligned} Y_3(x, y, z)&=Y_1(x, y, z)^3+6\mu[w^3F_3(x, z)+u^3F_3(z, y)-F_3(x, y)] \\ &\quad+6\mu Y_1(x, y, z)\cdot[w^2D_2(x, z)+u^2D_2(z, y)-D_2(x, y)] \\ &\quad+3\mu^2[w^3D(x, z)+u^3D(z, y)-D(x, y)]. \end{aligned} \tag{6·11}$$

Then we have according to (4·7), (4·12) and (4·10) the relationships

$$\langle Y_1(x, y, z)\rangle = 0,$$
$$\langle Y_2(x, y, z)\rangle = -2\mu uw\langle(\partial/\partial z)^2 Y_1(x, y, z)\rangle, \tag{6·12}$$
$$\langle Y_3(x, y, z)\rangle - 3\mu^2[\langle w^3 D(x, z) + u^3 D(z, y)\rangle - D(x, y)]$$
$$= -6\mu uw\langle(\partial/\partial z) Y_1(x, y, z)\rangle^2 = -3\mu uw\langle(\partial/\partial z)^2 Y_1(x, y, z)^2\rangle,$$

so that the above (6·6) and (6·9) give

$$0 = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(1/k!\,j!)\lambda^{k+j}(\mu uw)^k\langle(\partial/\partial z)^{2k} Y_j(x, y, z)\rangle - 1$$
$$= \sum_{n=1}^{\infty}(\lambda^n/n!)\sum_{k=0}^{n-1} {}_nC_k(\mu uw)^k\langle(\partial/\partial z)^{2k} Y_{n-k}(x, y, z)\rangle$$
$$= \lambda\langle Y_1\rangle + (\lambda^2/2)[\langle Y_2\rangle + 2\mu uw\langle(\partial/\partial z)^2 Y_1\rangle]$$
$$+ (\lambda^3/6)[\langle Y_3\rangle + 3\mu uw\langle(\partial/\partial z)^2 Y_2\rangle + 3(\mu uw)^2\langle(\partial/\partial z)^4 Y_1\rangle] + O(\lambda^4)$$
$$= (\lambda^3\mu^2/2)[\langle w^3 D(x, z) + u^3 D(z, y)\rangle - D(x, y)$$
$$+ (uw/\mu)\langle(\partial/\partial z)^2(Y_2 - Y_1^2)\rangle + (uw)^2\langle(\partial/\partial z)^4 Y_1\rangle] + O(\lambda^4). \tag{6·13}$$

Therefore the equation for determining the function $D(x, y)$ is

$$D(x, y) = \langle w^3 D(x, z) + u^3 D(z, y)\rangle$$
$$+ 2uw\langle(\partial/\partial z)^2[w^2 D_2(x, z) + u^2 D_2(z, y)]\rangle$$
$$+ (uw)^2\langle(\partial/\partial z)^4[wF_1(x, z) + uF_1(z, y)]\rangle, \tag{6·14}$$

which is identical in structure with (4·13). We thus get the solution (1·24).

§ 7. Quantization of the Lagrangian $L = (m/2)[w(q)v]^2$

If the classical Lagrangian is of the form

$$L(q, v) = L(R) \quad \text{with} \quad R = w(q)\,dq/dt \tag{7·1}$$

and d^2L/dR^2 does not vanish, the Lagrangian equation of motion (1·7) gives $dR/dt = 0$, which is integrated at once to give

$$R = (1/T)\int_y^x dq\, w(q). \tag{7·2}$$

When the Lagrangian is quadratic in R, as is given in (1·26), we get the classical action

$$S(x, y, T) = T(m/2)R^2 = (m/2T)[\int_y^x dq\, w(q)]^2$$
$$= (m/2T)(X - Y)^2 = S_f(X, Y, T) \tag{7·3}$$

with $X = \int_0^x dq w(q)$ and $Y = \int_0^y dq w(q)$. Then since

$$\partial^2 S(x, y, T)/\partial x \partial y = -(m/T) w(x) w(y), \tag{7·4}$$

the semi-classical kernel (1·9) reads

$$K_c(x, y, T) = [w(x) w(y)]^{1/2} K_f(X, Y, T), \tag{7·5}$$

which gives for $T=0$

$$K_c(x, y, 0) = [w(x) w(y)]^{1/2} \delta(X-Y) = \delta(x-y). \tag{7·6}$$

Now putting $G(u) = uX + wY$ with $w = 1-u$, we get according to (6·4)

$$K_f(X, Z, wT) K_f(Z, Y, uT) = K_f(X, Y, T) K_f[Z, G(u), uwT] \tag{7·7}$$

and then putting $Z = \int_0^z dq\, w(q)$ with $dZ = w(z)\, dz$, (7·5) and the relationship $\int dZ K_f[Z, G(u), uwT] = 1$ give at once

$$\int K_c(x, z, wT)\, dz\, K_c(z, y, uT) = K_c(x, y, T). \tag{7·8}$$

The semi-classical kernel $K_c(x, y, T)$ can thus be identified with the unitary transformation function $K(x, y, T)$.

According to (5·1) we get the integral representation

$$K(x, y, T) = (1/2\pi)[w(x) w(y)]^{1/2} \int dk \exp[-(iT/\hbar)(\hbar^2 k^2/2m) + ik(X-Y)], \tag{7·9}$$

which is nothing but the eigenfunction expansion of the transformation function and accordingly the normalized orthogonal eigenfunction $\psi_k(x)$ corresponding to the energy eigenvalue $E = \hbar^2 k^2/2m$ is

$$\psi_k(x) = [w(x)/2\pi]^{1/2} \exp[ik \int_0^x dq\, w(q)]. \tag{7·10}$$

Since this satisfies the equation

$$w(x)^{-1/2} \cdot p \cdot w(x)^{-1/2} \psi_k(x) = \hbar k \psi_k(x) \tag{7·11}$$

with $p = -i\hbar(d/dx)$, the Hamiltonian operator for this system is

$$H(x, p) = (1/2m)[w(x)^{-1/2} \cdot p \cdot w(x)^{-1/2}]^2. \tag{7·12}$$

Appendix

In this appendix we shall establish the identities

$$\exp[u(d/dx)^2 + 2uwx] = \exp(uwx) \exp[u(d/dx)^2] \exp(uwx) \exp[(1/3) u^3 w^2] \tag{A·1}$$

and

$$\exp[uw(d/dx)^2 + (u/w) x^2] = \exp[(1/2w) x^2 \tan u] \times \exp[(w/2) \sin 2u (d/dx)^2] \cdot \exp[(1/2w) x^2 \tan u]. \tag{A·2}$$

In order to prove (A·1) we first put its right-hand side in the form

$$=\exp[f(u)wx]\exp[g(u)(d/dx)^2]\exp[f(u)wx]\exp[w^2h(u)]$$

with the unknown functions $f(u)$, $g(u)$ and $h(u)$ in the parameter u, for which we must put $f(0)=g(0)=h(0)=0$. Then the differentiation of both sides with respect to u gives

$$\begin{aligned}(d/dx)^2+2wx=&f'(u)wx+g'(u)e^{f(u)wx}(d/dx)^2e^{-f(u)wx}\\&+f'(u)w\cdot e^{f(u)wx}e^{g(u)(d/dx)^2}\cdot x\cdot e^{-g(u)(d/dx)^2}e^{-f(u)wx}+w^2h'(u)\\=&f'(u)wx+g'(u)[(d/dx)-f(u)w]^2\\&+f'(u)w[x+2g(u)[(d/dx)-f(u)w]]+w^2h'(u),\end{aligned}$$

which at once yields $f'(u)=g'(u)=1$. So we have $f(u)=g(u)=u$ and $h'(u)=u^2$. Then we get $h(u)=(1/3)u^3$ and accordingly (A·1) is proved. The (A·2) can be established in quite the same way, so the proof is omitted here.

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A Study of Charge Independence in Terms of Kaluza's Five Dimensional Theory

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An attempt is made to give a foundation to the charge independence based on the five dimentional theory proposed by Kaluza. It is shown that his fifth co-ordinate can be identified with the angular variable of the charge space describing the rotation around its third axis, and that a transformation with respect to this co-ordinate proposed by Klein as a generalization of the gauge transformation is isomorphic to the rotation in a three dimensional Lorentz space. It is shown that this space has a close relation to the charge space.

§ 1. Introduction

The hypothesis of charge independence has played an important role in the study of strong interactions, and it seems that we can hardly talk about the theory of elementary particles without this hypothesis. In the present form, however, this is of a very conventional one. The purpose of this paper is to try to give it a foundation based on the five dimensional theory proposed by Kaluza.[1)]

Let us start from the gauge transformation

$$\psi \rightarrow e^{-ie/\hbar c \cdot f}\psi, \tag{1·1}$$

and

$$A_\mu \rightarrow A_\mu - \partial_\mu f. \tag{1·2}$$

It has been shown that it is very useful[2)] to interpret (1·1) as a rotation through f around the third axis in the charge space, ω_3. To make the corresponding interpretation in (1·2) also, let us decompose A_μ into the transverse part $\mathcal{A}_\mu$ and the longitudinal part $\partial_\mu \Lambda$:*

$$A_\mu = \mathcal{A}_\mu - \partial_\mu \Lambda.$$

Under the gauge transformation, $\mathcal{A}_\mu$ and Λ transform as follows

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* Strictly speaking, A_μ is given by

$$A_\mu = \mathcal{A}_\mu - \partial_\mu \Lambda + n_\mu n_\nu \partial/\partial x^\nu (\Lambda - \Lambda'),$$

where n_μ is an arbitrary time like unit vector. We drop the last term since it vanishes by the Lorentz condition.

$$\mathcal{A}_\mu \to \mathcal{A}_\mu, \tag{1·3}$$

and

$$\Lambda \to \Lambda + f. \tag{1·4}$$

That is, $\mathcal{A}_\mu$ is invariant, and Λ is added simply by f. In (1·1), however, gauge transformation was identified with a rotation around ω_3. This means that if we wish to preserve the same interpretation in (1·2) also, we must identify Λ itself with the variable describing the rotation around ω_3. We shall call this variable φ. This suggests that if we can find a space in which the charge independence is contained as its integral part, at least some parts of the electromagnetic field must appear in connection with its geometrical structure.

As an example of such spaces, let us consider the five dimensional space proposed by Kaluza and discussed by Klein[3] and by others. In Kaluza's theory, a fifth co-ordinate x^0 is introduced in addition to the usual space time co-ordinates $x^1, x^2, \cdots x^4$ which satisfies the following requirements:

i) all physical quantities should be periodic with respect to x^0,

ii) the momentum canonically conjugate to x^0 (quantum mechanically $-i\partial/\partial x^0$) is proportional to the electric charge.

With these variables, he introduced a five dimensional Riemanian space with the line element

$$d\sigma = \gamma_{\mu\nu} dx^\mu dx^\nu \quad (\gamma_{\mu\nu} = \gamma_{\nu\mu}), \tag{1·5}$$

and started from the variation principle

$$\delta \int \left(L + \frac{\beta^2}{2} M \right) \sqrt{\gamma}\, dx^0, \cdots dx^4 = 0, \tag{1·6}$$

where $\beta = \sqrt{2\kappa}$ (κ is the gravitational constant), $\gamma = |\det \gamma_{\mu\nu}|$ and L is the Lagrangian for the field:

$$L = \gamma^{\mu\nu} (\Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\sigma\rho}), \tag{1·7}$$

with

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} \gamma^{\rho\sigma} \left(\frac{\partial \gamma_{\mu\sigma}}{\partial x^\nu} + \frac{\partial \gamma_{\nu\sigma}}{\partial x^\mu} - \frac{\partial \gamma_{\mu\nu}}{\partial x^\sigma} \right). \tag{1·8}$$

M is the Lagrangian for the matter. Its explicit example will be given in § 4. From this variation principle we get following equation

$$\Gamma_{\mu\nu} = -\tfrac{1}{2} \beta^2 \Theta_{\mu\nu}, \tag{1·9}$$

where $\Theta_{\mu\nu}$ is a five dimensional tensor composed of the energy momentum tensor and the current vector which can be regarded as a generalization of the energy momentum tensor. $\Gamma_{\mu\nu}$ is given by

$$\Gamma_{\mu\nu} = P_{\mu\nu} - \tfrac{1}{2} \gamma_{\mu\nu} P, \tag{1·10}$$

with

$$P_{\mu\nu}=\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}+\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\log\sqrt{\gamma}-\frac{1}{\gamma}\frac{\partial}{\partial x^{\rho}}(\sqrt{\gamma}\,\Gamma^{\rho}_{\mu\nu}), \tag{1·11}$$

and

$$P=\gamma^{\rho\sigma}P_{\rho\sigma}. \tag{1·12}$$

In general, $\gamma_{\mu\nu}$'s are functions of $x^0\cdots\cdots x^4$. It can be shown, however, that it is always possible to choose a co-ordinate system in which $\gamma_{\mu\nu}$'s with μ or ν equal to 0 are independnt of x^0. Such co-ordinate system is called special co-ordinate system. Let us introduce A_m and g_{mn} in the special co-ordinate system by*

$$\begin{aligned}\gamma_{00}&=\alpha,\\ \gamma_{0m}&=\alpha\beta A_m,\\ \gamma_{mn}&=g_{mn}+\alpha\beta^2A_mA_n.\end{aligned} \tag{1·13}$$

Then it can be shown that among field equations derived from the variation principle (1·6), those with μ or ν equal to $1\cdots\cdots4$ correspond to Einstein's equation for the gravitational field, and those with one of μ or ν equal to 0 to Maxwell's equation in the gravitational field. (The one for $\mu=\nu=0$ is an equation determining α. It has not such counterpart.) The most general transformation which is allowed in the special coordinate system is

$$\begin{aligned}x^{1\prime}&=f_1(x^1, x^2,\cdots x^4),\\ x^{2\prime}&=f_2(x^1\ x^2,\cdots x^4),\\ &\cdots\cdots\cdots\cdots\\ x^{4\prime}&=f_4(x^1, x^2,\cdots x^4),\end{aligned} \tag{1·14}$$

and

$$x^{0\prime}=x^0+\beta f_0(x^1, x^2,\cdots x^4). \tag{1·15}$$

They correspond to the general co-ordinate transformation and the gauge transformation.

Now let us consider the relation of Kaluza's theory to the charge space. First, let us note that, as it is clear from (1·4) and (1·15), Λ and x^0 transform in exactly the same way under the gauge transformation. Therefore, we may suppose that there exists a simple relation between Λ and x^0. Λ was, however, identified with φ, an angular variable describing the rotation around ω_3. Therefore, it is natural to suppose that there would exist a simple relation between x^0 and φ. Let us assume most simply that x^0 is nothing but φ. Then we can see that this assumption fits very well with the requirements made for x^0 at the beginning. First, the requirement of periodicity is trivial if x^0 is an angular variable. Second, if x^0 is reinterpreted as an angular variable, $-i\partial/\partial x^0$ must correspondingly be reinterpreted as a component of an angular momentum along ω_3, and the requirement that it

* In the following we assume that greek indices run from 0 to 4, while latin ones from 1 to 4.

must be equal to the charge is nothing but the relation between the third component of the isotopic spin and the charge.* Thus, Kaluza's theory can be reinterpreted very naturally in terms of the charge space. From (1·1), (1·4) and (1·15) it is readily seen that the precise relation between x^0 and φ is given by

$$\varphi = x^0/l_0, \tag{1·16}$$

where l_0 is given by

$$l_0 = \left(\frac{e^2}{\hbar c}\right)^{-1/2} \sqrt{2\kappa\hbar c} \simeq 10^{-31}\text{cm}. \tag{1·17}$$

Recently, Klein[4] reexamined Kaluza's theory in connection with the parity problem. He showed that in the five dimensional theory, P (the usual space inversion) cannot be regarded as a meaningful operation, and instead we should use $P\cdot C$ (space inversion $\times$ charge conjugation).

In view of these promising features, it seems of interest to examine Kaluza's theory further more to see whether it is possible to get better understanding of the charge independence and of the parity non-conservation. At first sight this seems impossible, since Kaluza's theory contains only one extra variable corresponding to φ, while the charge space is at least three dimensional. It should be noted, however, that Kaluza's theory allows much more general transformations than the latter, which allows only linear transformations. Therefore, it is at least in principle possible that Kaluza's theory can contain the charge independence. The main purpose of this paper is to discuss this point, and in fact it is shown that a transformation proposed by Klein[5] as a generalization of the gauge transformation is isomorphic to the rotational group in a three dimensional Lorentz space. ((§2).) In §3, an attempt is made to identify this Lorentz space with the charge space. In § 4, a simple example of spin zero particles interacting with the electromagnetic field is treated.

§ 2. Analysis of Klein's transformation

As discussed in §1, Kaluza's theory contains only one extra variable describing the rotation around ω_3. Therefore, at first sight it seems that even if it can be related to the charge space, we can expect to get only I_3, the third component of the isotopic spin $\boldsymbol{I}$. What is important in the charge independence is, however, that the charge space is three dimensional, and $\boldsymbol{I}$ is conserved as a three dimensional vector. Therefore, it is important to clarify in what way such degrees of freedom are contained in Kaluza's theory.

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* At first, we consider the simplest case where the charge is given by I_3. For a more general case see §3.

For this purpose, let us consider a transformation proposed by Klein[5] as a generalization of the gauge transformation (1·15) :

$$x^{0\prime}=x^0+\varepsilon\beta g(x^0,\ x^1,\cdots x^4), \tag{2·1}$$

where ε is an infinitesimal constant.* Since we can assume that $g(x^0,x^1,\cdots x^4)$ is periodic in x^0, (2·1) can be rewritten as

$$x^{0\prime}=x^0+\varepsilon l_0\sum_{s=-\infty}^{\infty}\xi_s(x^1,\cdots x^4)e^{is x^0/l_0}. \tag{2·1}'$$

Let us investigate the transformation law of the wave function caused by (2·1)′. For simplicity we assume that the wave function $\psi(x^0,x^1,\cdots x^4)$ is a five dimensional scalar. $\psi(x^0,\ x^1,\cdots x^4)$ can be expanded as follows

$$\psi(x^0,\ x^1,\cdots x^4)=\sum_n\psi_n(x^1,\ x^2,\cdots x^4)U_n(x^0)/\sqrt[4]{\gamma}, \tag{2·2}$$

where $U(x^0)$ is the eigenfunction of the charge operator $-i\partial/\partial x^0$, and is given by

$$U_n(x^0)=\frac{1}{\sqrt{2\pi l_0}}e^{inx^0/l_0}. \tag{2·3}$$

Under the transformation (2·1),′ $\psi_n(x^1,\cdots x^4)$ is transformed into**

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* For simplicity we consider only infinitesimal transformations.

** In general co-ordinate systems the orthonormal set is defined by

$$\int\psi_n^*\psi_m\ \sqrt{\gamma}\ dx^0,\ \cdots,\ dx^4=\delta_{nm}.$$

Therefore, expansion of $\psi(x^0,\ x^1,\cdots x^4)$ in $x^{0\prime},\ x^1,\cdots x^4$ system takes the form

$$\psi'(x^{0\prime},\ x^1,\cdots x^4)=\sum_n\psi_n'(x^1,\cdots x^4)U_n(x^{0\prime})/\sqrt[4]{\gamma'}, \tag{*·1}$$

that is,

$$\psi'(x^{0\prime},\ x^1,\cdots x^4)\sqrt[4]{\gamma'}=\sum_n\psi_n'(x',\cdots x^4)U_n(x^{0\prime}). \tag{*·1}'$$

Multiplying $\sqrt[4]{\gamma'}$ on both sides of (2·2), and using the invariance of $\psi(x^0,\ x^1,\cdots x^4)$, we get

$$\psi'(x^{0\prime},\ x^1,\cdots x^4)\sqrt[4]{\gamma'}=\sum_n\psi_n(x^1,\cdots x^4)\bar{U}_n(x^{0\prime})/\sqrt[4]{\gamma}, \tag{*·2}$$

where

$$\bar{U}_n(x^{0\prime})=U_n(x^0)\sqrt[4]{\gamma'}.$$

In our case γ' is given by $(1-2\cdot\varepsilon\beta\cdot\partial g/\partial x^0)\gamma$. Therefore, $\bar{U}_n(x^{0\prime})$ is related to $U_n(x^{0\prime})$ by

$$\bar{U}_n(x^{0\prime})=\sum_{n'}(\delta_{nn'}+i\varepsilon Q_{n'n})U_{n'}(x^{0\prime})\sqrt[4]{\gamma}. \tag{*·3}$$

Substituting (*·3) into (*·2), and comparing with (*·1)′, we get

$$\psi_{n'}(x^1,\cdots x^4)=\sum_{n'}(\delta_{nn'}+i\varepsilon Q_{nn'})\psi_{n'}(x^1,\cdots x^4).$$

$$\psi'_n(x^1,\cdots x^4)=\sum_{n'}(\delta_{nn'}+i\varepsilon Q_{nn'})\psi_{n'}(x^1,\cdots x^4), \tag{2·4}$$

where $Q_{nn'}$ is given by

$$Q_{nn'}=-\tfrac{1}{2}(n+n')\xi_{n-n'}. \tag{2·5}$$

Further, if we assume that $g(x^0, x^1,\cdots x^4)$ is real, $\xi_s(x^1,\cdots x^4)$ satisfy

$$\xi_s^*(x^1, \cdots x^4)=\xi_{-s}(x^1, \cdots x^4). \tag{2·6}$$

Therefore, $Q_{nn'}$ is hermetic, and (2·4) is a unitary transformation.

As seen from (2·4), this transformation connects various states with different charge. In this respect, it corresponds to the rotation in the charge space.

To make clear the meaning of this transformation, let us consider the simplest case, where all $\xi_s(x^1, \cdots, x^4)$'s expect for those with $s=0$ and ± 1 vanish. Writing Q corresponding to this case explicitly, we get

$$Q=-\tfrac{1}{2}\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & -4\xi_0 & -3\xi_{-1} & & 0 & & \\ & -3\xi_1 & -2\xi_0 & -\xi_{-1} & & & \\ & & -\xi_1 & 0 & \xi_{-1} & & \\ & & & \xi_1 & 2\xi_0 & 3\xi_{-1} & \\ & 0 & & & 3\xi_1 & 4\xi_0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \tag{2·7}$$

Therefore, if we write

$$\begin{aligned} &\xi_{-1}=a+ib, \\ &\xi_1=a-ib, \\ &(a, b, \text{ real}), \end{aligned} \tag{2·8}$$

(2·7) can be written as

$$Q=-\xi_0 I_3-2aI_1-2bI_2, \tag{2·9}$$

where

$$I_1=\tfrac{1}{4}\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 0 & -3 & & 0 & & \\ & -3 & 0 & -1 & & & \\ & & -1 & 0 & 1 & & \\ & & & 1 & 0 & 3 & \\ & & & & \ddots & 0 & \ddots \\ & & 0 & & & & \ddots \end{pmatrix}, \tag{2·10}$$

$$(I_1)_{m,n}=\frac{m+n}{4}(\delta_{m-1,n}+\delta_{m+1,n}),$$

$$I_2=\frac{i}{4}\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 0 & -3 & & 0 & & \\ & 3 & 0 & -1 & & & \\ & & 1 & 0 & 1 & & \\ & & & -1 & 0 & 3 & \\ & 0 & & & -3 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{2·11}$$

$$(I_2)_{mn}=\frac{i}{4}(m+n)(-\delta_{m-1,n}+\delta_{m+1,n}),$$

and

$$I_3=\begin{pmatrix} \ddots & & & & & & \\ & -2 & & & & & \\ & & -1 & & 0 & & \\ & & & 0 & & & \\ & 0 & & & 1 & & \\ & & & & & 2 & \\ & & & & & & \ddots \end{pmatrix}, \tag{2·12}$$

$$(I_3)_{mn}=m\delta_{mn}.$$

Using these explicit forms, we can show easily that they satisfy following commutation relations

$$\begin{aligned}[I_1, I_2]&=-iI_3,\\ [I_2, I_3]&=iI_1,\\ [I_3, I_1]&=iI_2.\end{aligned} \tag{2·13}$$

This is noting but the commutation relation for generator of the infinitesimal rotation in the three dimensinal *Lorentz* space. Therefore, in this case, (2·4) is isomorphic to the rotation in the three dimensinal Lorentz space, and $-2\varepsilon a$, $-2\varepsilon b$ and $-\varepsilon\xi_0^5$ are nothing but the angles of rotation around the first, second and third axis.

The unitary representation of the rotation group in a three dimensional Lorentz space has been investigated by Bargmann[6] in detail. Analogous to the rotation group in three dimensional Euclid space,

$$Q=I_1^2+I_2^2-I_3^2 \tag{2·14}$$

is commutative with I_1, I_2 and I_3, and the representation can be classified by the

eigenvalue of one of $I_i (i=1, 2, 3)$, for example, I_3, and of Q. According to him, the following representations exist. (q and m are eigenvalues Q and I_3 respectively).

$$\begin{aligned}
&1)\quad C_q^{\,0}: && q \text{ is any positive number,}\\
& && m=0,\ \pm1,\ \pm2,\ \cdots\cdots,\\
&2)\quad C_q^{\,1/2}: && \tfrac{1}{4}<q<\infty,\\
& && m=\pm\tfrac{1}{2},\ \pm\tfrac{3}{2},\ \cdots\cdots,\\
&3)\quad D_k^{+}: && q=k(1-k),\ k=\tfrac{1}{2},\ 1,\ \tfrac{3}{2},\cdots\cdots,\\
& && m=k,\ k+1,\ k+2,\ \cdots\cdots,\\
&4)\quad D_k^{-}: && q=k(1-k),\ k=\tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ \cdots\cdots,\\
& && m=-k,\ -(k+1),\ -(k+2),\ \cdots\cdots.
\end{aligned} \tag{2·15}$$

In our case, q is easily calculated to be $\frac{1}{4}$. Therefore, (2·10), (2·11) and (2·12) correspond to $C_{1/4}^{0}$. In fact, if we calculate the explicit form of the matrix for this representation using Bargmann's result, we get exactly (2·10)—(2·12) except for a simple unitary transformation. (See the appendix.)

Next, let us consider the case in which all ξ_s's except for those with $s=0$ or $\pm r$ vanish. Defining I_1, I_2 and I_3 in exactly the same way as before, we get

$$\begin{aligned}
(I_1^{(r)})_{mn}&=\tfrac{1}{4}(m+n)\,(\delta_{m-r,n}+\delta_{m+r,n}),\\
(I_2^{(r)})_{mn}&=\frac{i}{4}(m+n)\,(-\delta_{m-r,n}+\delta_{m+r,n}),\\
(I_3^{(r)})_{mn}&=m\delta_{mn}.
\end{aligned} \tag{2·16}$$

(To distinguish from the former case, we wrote these as $I_i^{(r)}$ $(i=1, 2, 3)$. In this notion, the former ones should be written as $I_i^{(1)}$). Calculating their commutation relation from their explicit form, we get further

$$\begin{aligned}
[I_1^{(r)}, I_2^{(r)}]&=-irI_3^{(r)},\\
[I_2^{(r)}, I_3^{(r)}]&=irI_1^{(r)},\\
[I_3^{(r)}, I_1^{(r)}]&=irI_2^{(r)}.
\end{aligned} \tag{2·17}$$

Therefore, if we define $\mathcal{J}_i^{(r)}$ by

$$\mathcal{J}_i^{(r)}=\frac{1}{r}I_i^{(r)}, \tag{2·18}$$

they satisfy

$$\begin{aligned}
[\mathcal{J}_1^{(r)}, \mathcal{J}_2^{(r)}]&=-i\mathcal{J}_3^{(r)},\\
[\mathcal{J}_2^{(r)}, \mathcal{J}_3^{(r)}]&=i\mathcal{J}_1^{(r)},\\
[\mathcal{J}_3^{(r)}, \mathcal{J}_1^{(r)}]&=i\mathcal{J}_2^{(r)}.
\end{aligned} \tag{2·19}$$

This is again nothing but the commutation relation for the generator of the infinitesimal rotation in the three dimensional Lorentz space, and if we calculate q, we get $1/4$ independently of r. Therefore, all $\mathcal{J}_i^{(r)}$ with different r belong to the same representation, and they are connected by unitary transformations.

The most general transformation can be written as

$$1+i\varepsilon Q=(1-i\varepsilon\xi_0 I_3-2i\varepsilon a I_1-2i\varepsilon b I_2)\prod_{r=2}^{\infty}(1-2ir\varepsilon a^{(r)}\mathcal{J}_1^{(r)}-2ir\varepsilon b^{(r)}\mathcal{J}_2^{(r)}). \tag{2·20}*,**$$

This is also a rotation since each factor is a rotation in the same space. The only difference from the usual case is that each factor differs by unitary transformations. Therefore, the inclusion of higher terms introduces nothing new, and essential features of Klein's transformation is contained in the simplest case where all ξ_s's with $s\geq 2$ vanish. Thus, the content of Klein's transformation has become clear. It is completely isomorphic to the rotation in a three dimensional Lorentz space.***

§ 3. Relation of Kaluza's theory to charge independence

As shown in § 2, Klein's transformation is equivalent to the rotation in a three dimensional Lorentz space. Therefore, as far as we are concerned with the special co-ordinate transformation and Klein's transformation, Kaluza's theory is equivalent to introducing the three dimensional Lorentz space other than the usual four dimensional Minkowski space, and to assuming invariance under the rotation in this space. We shall call three axes of this space ω_1, ω_2 and ω_3. (x^0 is the variable describing the rotation around ω_3.) Then the charge is given by the third component of an angular momentum in this space, and under Klein's transformation, each eigenstate of the charge transforms according to an irreducible representation of the rotation group of this space. Therefore, putting aside for a while that this is not Euclidean, it would be very natural to identify this space with the charge space, and Klein's transformation with the rotation in the charge space.****

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* $a^{(r)}$ and $b^{(r)}$ are defined in exactly the same manner as in (2·9).

** There seems to exist an ambiguity in adopting $\mathcal{J}_i^{(r)}$ to construct covariants such as $\psi^*\mathcal{J}_i^{(r)}\psi$ when various $\mathcal{J}_i^{(r)}$ ($r=1, 2,\cdots$) appear. The five dimensional invariance requires, however, that $\boldsymbol{I}$ should appear in the Lagrangian only in the form I^2 if there is no electromagnetic field. For I^2 expressions with different r are the same. Therefore, in practice, no difficulty appears. The ambiguity in placing $i\varepsilon\xi_0 I_3$ in various factors of (2·20) is also irrelevant by the same reason. Thus the interpretation that Klein's transformation is the rotation in a three dimensional Lorentz space is possible as long as we confine ourselves to quantities restricted by the five dimensional invariance and neglect electromagnetic field.

*** The reason why such situation occurs is because all quantities are assumed to be periodic with respect to x^0, that is, x^0 is identified with the angular variable.

**** As seen from the derivation of (2·4), in the transformation law for ψ_n, the variation of $\sqrt[4]{\gamma}$ playas an important role. Therefore, in this scheme, the charge space appears as a very conventional concept which emerges when interpreting Klein's transformation in terms of uncurved co-ordinates.

In this scheme, seveal differences appear from the usual charge space concerning the transformation law of various quantities. First, we must transform the electromagnetic field also by the rotation in the charge space.* The reason is that the rotation in the charge space is a co-ordinate transformation in the original five dimensional space, and the electromagnetic field is not a scalar there. Second, special caution is needed to treat $\partial\psi_n(x^1, \cdots x^4)/\partial x^m$ and $n\psi_n$. In the special co-ordinate system, γ is given by α if we neglect terms containing β. Moreover, α is invariant under special co-ordinate transformations. Therefore, we can take α as constant in special co-ordinate systems. In this case, γ does not contain $x^1, \cdots x^4$, and we get from (2·2)

$$\sqrt[4]{\gamma}\,\frac{\partial\psi}{\partial x^m}=\sum_n \frac{\partial\psi_n(x^1, \cdots x^4)}{\partial x^m}\cdot U_n(x^0). \tag{3·1}$$

In general co-ordinate systems, however, γ contains x^0, x^1, $\cdots$, x^4. In these cases, we must *define* $\partial\psi_n(x^1, \cdots x^4)/\partial x^m$ by (3·1). Therefore, in general, it is not identical with the derivative of ψ_n. Similar caution is necessary in transforming $n\psi_n$.**

To obtain the transformation law of the electromagnetic field, we assume that the Lagrangian is always bilinear in the matter field. Then $\sqrt{\gamma}$ can always be absorbed in the matter field, and it is unnecessary to include its variation in the transformation law of the electromagnetic field. Thus we get as the transformation law from the special co-ordinate system

$$A'_{mn}(x^1, \cdots x^4)=(\delta_{n_0}+i\varepsilon n\xi_n)A_m(x^1, \cdots x^4)-\varepsilon\frac{\hbar c}{e}\frac{\partial\xi_n}{\partial x_m}, \tag{3·2}$$

where A_{mn} is defined by

$$A'_m(x^{0\prime}, x^1, \cdots x^4)=\sum_n A'_{mn}(x^1, \cdots x^4)e^{in x^{0\prime}/l_0}. \tag{3·3}$$

The transformation laws for $\partial\psi_n(x^1, \cdots x^4)/\partial x^m$ and $n\psi_n$ are obtained by a slight extension of (2·4). Assuming that α is constant in the special co-ordinate system, we get

$$\frac{\partial\psi_n'(x^1, \cdots x^4)}{\partial x^m}=\sum_{n'}\left\{(\delta_{nn'}+i\varepsilon Q_{nn'})\frac{\partial}{\partial x^m}-i\varepsilon n'\frac{\partial\xi_{n-n'}}{\partial x^m}\right\}\psi_{n'}(x^1, \cdots x^4), \tag{3·4}$$

and

$$n\psi_n'=\sum_{n'}\{(\delta_{nn'}+i\varepsilon Q_{nn'})-i\varepsilon(n-n')\xi_{n-n'}\}n'\psi_{n'}. \tag{3·5}$$

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* If α appears, this must also be transformed. See the example in § 4.

** The transformation law for ψ_n takes a simple form only when the variation of $\sqrt[4]{\gamma}$ is included. Therefore, the transformation law for $\sqrt[4]{\gamma}\,\partial\psi/\partial x^0 (\neq\partial/\partial x^0\cdot(\sqrt[4]{\gamma}\psi))$ is not simple, and we had better define $n\psi_n'(x^1, \cdots x^4)$ by

$$-i\sqrt[4]{\gamma'}\,\frac{\partial\psi'(x^{0\prime}, x^1, \cdots x^4)}{\partial x^{0\prime}}=\sum_n n\psi_n'(x^1, \cdots x^4)\cdot U_n(x^{0\prime}).$$

Notice that the last term in the right-hand side of (3·4) is identical with $i\varepsilon\partial Q_{nn'}/\partial x^m$ only when ξ_s's with $s\neq 0$ vanish (the case corresponding to the gauge transformation), and (3·5) is not equal to the rotation.* Therefore, when electromagnetic field exists, or when $\xi_s(x^1, \cdots x^4)$ depend upon $x^1, \cdots x^4$ and derivative with respect to x^m appears, it is impossible to identify Klein's transformation with the simple rotation in the charge space. In the following we shall confine ourselves to the case in which this identification is possible.** Namely,

a) there is no electromagnetic field,

b) α does not contain $x^1, \cdots x^4$ in the special co-ordinate system,

c) all ξ's except for ξ_0 do not depend upon $x^1, \cdots x^4$.

Under these restrictions, we can treat the Minkowski space and the charge space as a direct product. We assume that the concept of the charge space is a restricted one valid under these conditions. When all quantities are transformed according to (2·4), (3·2), (3·4) and (3·5), all Lagrangians remain invariant. But in general they are not simple rotations.

If no electromagnetic interaction exists, invariance under these rotations leads to the conservation of ***I***, since in this case invariant quantity that can enter the S-matrix is only I^2. Thus, in Kaluza's theory, the charge independence appears as a natural extension of the conservation law of the charge.

Thus far, we have discussed the *unitary* representation of the Lorentz group, $C^0_{1/4}$. It has, however, a disadvantage that its charge multiplet necessarily becomes infinite. To get finite multiplet, we must introduce non-unitary representation.*** This representation has also been obtained by Bargmann. According to him, there exist irreducible representations denoted by D_q with $q=-j(j+1)$, $j=0, \frac{1}{2}, 1, \cdots\cdots$. To each D_q belong $(2j+1)$ eigenvectors v_m with $m=-j, -j+1, \cdots\cdots, +j$. These multiplets are exactly the same with those of the usual rotation group in a three dimensional Euclid space. In this case, however, there arises another difficulty. Namely, the invariant in this case is not $\sum' v_m^* v_m$ but $\sum'(-1)^{j-m} v_m^* v_m$.

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* For example, in the simplest case in which all ξ_s's except for ξ_0 and $\xi_{\pm 1}$ vanish, (3·5) becomes

$$n\psi_n' = \sum_{n'}(\delta_{nn'}+i\varepsilon Q_{nn'})n'\psi_{n'} - i\varepsilon(n-1)\xi_1\psi_{n-1} + i\varepsilon(n+1)\xi_{-1}\psi_{n+1},$$

whereas the result of the rotation is

$$\sum_{n'}(\delta_{nn'}+i\varepsilon Q_{nn'})n'\psi_{n'} - i\varepsilon(n-\tfrac{1}{2})\xi_1\psi_{n-1} + i\varepsilon(n+\tfrac{1}{2})\xi_{-1}\psi_{n+1}.$$

** The case in which $\xi_s(x^1, \cdots x^4)$ depends upon $x^1, \cdots x^4$ is related to the rotation considered by Yang and Mills.[7] They pointed out that to assure the locality of the field concept, it is necessary to allow the rotation in the charge space which depends upon space-time co-ordinates. In that case it was necessary to introduce new field (which they named b-field). In our case, the invariance is guaranteed without introducing any new field. But in general it is not a simple rotation.

*** It has been shown by Wigner[8] that there does not exist unitary representation with finite dimensions except for a trivial one-dimensional one.

Therefore, as pointed out by Higgs[9], the Hamiltonian becomes non-positive definite. To evade this point, we must generalize Klein's transformation and assume that actual particles correspond to the rotation where a and b become imaginary.

Thus far, we have considered the simplest case in which the charge is given by I_3. Except for the case of pion, however, this is not the case, and in general it is given by

$$Q=\begin{cases} e\left(I_3+\frac{s}{2}+\frac{n}{2}\right) & \text{for fermion,} \\ e\left(I_3+\frac{s}{2}\right) & \text{for boson,} \end{cases} \tag{3·6}$$

where s is the strangeness quantum number, and n is the nucleonic number. In this case, our result suggests that Q is also the third component of a vector as a whole, and this vector will be conserved if no electromagnetic interaction exists. This suggests that $s/2$, $n/2$ or $(s/2+n/2)$ must also be the third component of some angular momenta, and these angular momenta will be conserved if $\boldsymbol{I}$ is conserved. Such pictures have been proposed by several authors. For example, Hara and Fujii[2] assumed that

$$s/2=J_3, \quad \text{and} \quad n/2=K_3 \quad (|\boldsymbol{K}|=\tfrac{1}{2}),$$

and assumed that

i) $\boldsymbol{I}+\boldsymbol{J}+\boldsymbol{K}$ is conserved in all processes, if electromagnetic interaction does not take part,

ii) In strong interactions, $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{K}$ are conserved separately.

When applied to the decay processes, (i) leads to the $\Delta I=\frac{1}{2}$ rule, while (ii) leads to several new predictions in antinucleon physics. Namely, in this scheme, nucleon and antinucleon correspond to $+\frac{1}{2}$ and $-\frac{1}{2}$ of $\boldsymbol{K}$, and $\boldsymbol{K}$ is conserved in strong interactions. This corresponds just to the charge independence of the two-nucleon system. In this case, proton and neutron correspond to $+\frac{1}{2}$ and $-\frac{1}{2}$ of $\boldsymbol{I}(|\boldsymbol{I}|=\frac{1}{2})$, and $\boldsymbol{I}$ is conserved in strong interactions. Therefore, it is expected that just as neutron-proton force is related to proton-proton force by the charge independence, some part of the antinucleon-nucleon force will be related to the nucleon-nucleon force by this assumption. Predictions from these assumptions can be compared with experiment. Therefore, it is possible to test the validity of our scheme experimentally, though not directly.*

§ 4. Simple example

As the simplest example, we shall treat in this section spin zero particles

* We shall not try to derive $\boldsymbol{J}$ and $\boldsymbol{K}$. It is outside the scope of the simple five-dimensional theory.

interacting with electromagnetic field. This is achieved by introducing for M*

$$M=-2\hbar^2c^2(\partial_\mu\psi^*\gamma^{\mu\nu}\partial_\nu\psi+\mu^2\psi^*\psi),^* \tag{4·1}$$

where ψ is a five dimensional scalar quantity, and $\gamma^{\mu\nu}$ is the contravariant component of the metric tensor given by

$$\begin{aligned}\gamma^{mn}&=g^{mn},\\ \gamma^{0m}&=-\beta g^{mn}A_n,\\ \gamma^{00}&=\frac{1}{\alpha}+\beta^2g^{mn}A_mA_n.\end{aligned} \tag{4·2}$$

The field equations can be deduced by the straight forward application of the variation principle. Since we are not interested in the gravitational effect, we shall take the limit $\beta\to0$, and put $g^{mn}=\delta_{mn}$. Then, for the matter field we get

$$\frac{\partial}{\partial x^\mu}\left(\gamma^{\mu\nu}\frac{\partial\psi}{\partial x^\nu}\right)=\mu^2\psi, \tag{4·3}$$

from which we get, using the fact that A_m and α do not depend on x^0 in the special co-ordinate system, as the equation of motion for the eigenstate with charge ne

$$\left(\frac{\partial}{\partial x^m}-\frac{ine}{\hbar c}A_m\right)^2\psi_n=\left\{\mu^2+\frac{1}{\alpha}\left(\frac{ne}{l_0}\right)^2\right\}\psi_n. \tag{4·4}$$

$\theta_\mu^{\ \nu}$ is given by

$$\theta_\mu^{\ \nu}=\partial_\mu\psi\frac{\partial M}{\partial(\partial_\nu\psi)}+\partial_\mu\psi^*\frac{\partial M}{\partial(\partial_\nu\psi^*)}-M\delta_\mu^{\ \nu}. \tag{4·5}$$

We subsitute the explicit form for (4·5), and integrate over x^0 from 0 to $2\pi l_0$. Then its $(0, m)$ component is given by

$$\begin{aligned}\overline{\theta_0^{\ m}}=\int_0^{2\pi l_0}\theta_0^{\ m}\sqrt{\gamma}\,dx^0=-\sum_n\frac{ine\hbar c}{\beta}&(\psi_n\partial_m\psi_n^*-\psi_n^*\partial_m\psi_n\\ &+2ine/\hbar c\cdot A_m\psi_n^*\psi_n).\end{aligned} \tag{4·6}$$

Also to the first order in β,**

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* More conveniently, this should be written as

$$M=-2\hbar^2c^2\{\tfrac{1}{2}(\partial_\mu\psi^*\gamma^{\mu\nu}\partial_\nu\psi+\partial_\mu\psi^*\gamma^{\nu\mu}\partial_\nu\psi)+\mu^2\psi^*\psi\}.$$

** If we neglect gravitational effects, and assume that α is constant,

$$L=-\tfrac{1}{4}\alpha\beta^2(\partial A_m/\partial x_n-\partial A_n/\partial x_m)^2.$$

This is just $2\alpha\kappa$ times the usual Lagrangian for the electromagnetic field. If we assume that $\int_0^{2\pi l_0}L\sqrt{\gamma}\,dx^0$ corresponds to the Lagrangian of the usual theory, an extra factor $(2\pi l_0\sqrt{\gamma})$ appears. Further, as shown in (4·11), $\int_0^{2\pi l_0}M\sqrt{\gamma}\,dx^0$ is twice the usual Lagrangian for the matter field. Therefore, to make the correspondence with the usual theory clear, it is necessary to multiply a factor $(2\pi l_0\alpha\sqrt{\gamma})^{-1}$ at first before L. (4·7) is a result obtained by supplying this factor.

$$\overline{P_0^m}=\int_0^{2\pi l_0} P_0^m\sqrt{\gamma}\,dx^0=-\frac{\beta}{2}\left(\Box A_m-\frac{\partial^2 A_r}{\partial x^m\partial x^r}\right). \tag{4·7}$$

Therefore, we get from (1·9)

$$\Box A_m-\frac{\partial^2 A_r}{\partial x^m\partial x^r}=-\frac{1}{c}j_m, \tag{4·8}$$

with

$$j_m=\sum_n\{-ine\hbar c^2(\psi_n^*\partial_m\psi_n-\psi_n\partial_m\psi_n^*)-2(ne)^2c\psi_n^*\psi_n A_m\}. \tag{4·9}$$

(4·4), (4·8) and (4·9) are nothing but the usual equations of motion for spin zero particles interacting with the electromagnetic field. (The second term in the left-hand side of (4·8) vanishes by the Lorentz condition.) The only difference is the second term in the right-hand side of (4·4). This term expresses that the rest mass depend on the charge. Usually such effect is not observed. Therefore, not to contradict with experience, α must be taken to be very large. If we assume that such mass difference is at most of the order of electron mass, it must be larger than about 10^{36}.

One point to be noted is that although (1·9) contains β on the right-hand side, it cancels out when written in terms of A_m. Therefore although Einstein's equation becomes trivial in the limit $\beta\to 0$, Maxwell's equation remains invariant. This means that there exists an essential difference between the gravitational field and the electromagnetic field when we consider their effects on the matter field.

We shall give also the expression

$$\overline{M}=\int_0^{2\pi l_0} M\sqrt{\gamma}\,dx^0, \tag{4·10}$$

which corresponds to the Lagrangian in the usual theory. Substituting (2·2) into (4·1), and neglecting gravitational effects, we get

$$\overline{M}=-2\hbar^2c^2\sum_n\left\{\left(\frac{\partial}{\partial x^m}+ine\,A_m/\hbar c\right)\psi_n^*\cdot\left(\frac{\partial}{\partial x^m}-ine\,A_m/\hbar c\right)\psi_n\right.$$
$$\left.+\left(\mu^2+\frac{1}{\alpha}\left(\frac{ne}{l_0}\right)^2\right)\psi_n^*\psi_n\right\}, \tag{4·11}$$

which is just the (twice) usual Lagrangian except for the last term. In this expression, we see explicitly that this is invariant under the transformation given by (2·4), (3·2), (3·4) and (3·5) together with*

$$\alpha_n'=(\delta_{n0}-2in\varepsilon\xi_n)\alpha, \tag{4·12}$$

* Combining (3·5) with (3·2), we get

$$nA_m'\psi_n'=\sum_{n'}\left\{A_m(\delta_{nn'}+i\varepsilon Q_{nn'})-\varepsilon\frac{\hbar c}{e}\frac{\partial\xi_{n-n'}}{\partial x^m}\right\}n'\psi_{n'}.$$

where α_n' is given by

$$\alpha(x^{0\prime}, x^1, \cdots x^4) = \sum_n \alpha_n'(x^1, \cdots x^4) e^{inx^{0\prime}/l_0}.$$

§ 5. Concluding remarks

Thus far, we have shown that Kaluza's theory can be reinterpreted in terms of the charge space. It may be said that by this way of approach, we could expect to get better understanding of the charge space or the charge independence. In this connection, however, we must emphasize that the concept of charge space is a very conventional one. Therefore, it would not be right to give it too realistic a meaning.

Here it would be necessary to consider more seriously the necessity of introducing the fifth co-ordinate and the five-dimensional invariance. We usually say that the world is four-dimensional. This may be said as an expression of the standpoint that to describe the motion of a mass point it is sufficient to trace its spatial co-ordinates as a function of the time. If we assume, however, that the knowledge of the interaction is very essential, this standpoint is insufficient, and we have to require in addition that it is necessary to trace at the same time *the way in which the interaction is taking part.* If we assume most simply that this is determined by a single parameter, it means that this parameter must also be traced as a function of time, and the relativity of the physical laws must be satisfied including this parameter. We may say that the five-dimensional theory is an example that has realized this standpoint.

In this paper we have limited ourselves to the special co-ordinate transformation and Klein's transformation. In these cases, it was possible to treat the Minkowski space and the charge space as a direct product.* There exist, however, much more general transformations in the original Kaluza's theory. One example is

$$x^{\mu\prime} = x^\mu + \varepsilon_{\mu\nu} x^\nu, \tag{5·1}$$

where $\varepsilon_{\mu\nu}$'s are constants. The transformation of x^0 given by (5·1) is Klein's transformation given by (2·1). But the transformations of $x^1, \cdots x^4$ are different from the usual Lorentz transformation since they contain x^0. Therefore, if we allow (5·1), it is impossible to treat these spaces as a direct product, and rather we must consider their *amalgamation.* Whether we can use the Minkowski space in this case in an approximate sense depends upon whether we can neglect x^0 in the transformation of $x^1, \cdots x^4$. From (1·16) we have $\varphi = x^0/l_0$. From this, and since φ is defined to mod(2π) only, we can restrict x^0 to be of the order of l_0. This means that in a region smaller than about l_0, or in phenomena with energy higher than about $\hbar c/l_0$, it would be inadmissible to use the Minkowski space. On the other hand, in the limit $\kappa \to 0$, $x^0 = 0$. Therefore (5·1) reduces to the usual co-

* Strictly speaking, under the restriction mentioned in § 3.

ordinate transformation. In this case, we can use the Minkowski space in an exact sence without restriction, and the concept of length loses its meaning in the charge space. This corresponds just to the usual picture.*

We shall finally review the representation of the group given by (5·1). It is known that the proper rotation in a five-dimensional Lorentz space (linear transformation which leaves $(x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2-(x^4)^2$ invariant and with determinant $+1$) can be represented by a four row and four columed matrix of the form**

$$S=\begin{pmatrix} \alpha & \beta & \lambda & \mu \\ \gamma & \delta & \nu & \sigma \\ \bar{\sigma} & -\bar{\nu} & \bar{\delta} & -\bar{\gamma} \\ -\bar{\mu} & \bar{\lambda} & -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{5·2}$$

where

$$\begin{aligned} &\alpha\delta-\beta\gamma-\lambda\sigma+\mu\nu=1, \\ &-\alpha\bar{\nu}-\beta\bar{\sigma}+\bar{\gamma}\lambda+\bar{\delta}\mu=0, \\ &\lambda\bar{\alpha}+\mu\beta=\text{real}, \\ &\gamma\bar{\nu}+\delta\bar{\sigma}=\text{real}. \end{aligned} \tag{5·3}$$

Therefore, usual four component Dirac spinors are, in terms of a five-dimensional theory, bases of the *proper* rotation, and the usual inversion ($x^{1\prime}=-x^1$, $x^{2\prime}=-x^2$, $x^{3\prime}=-x^3$, $x^{4\prime}=+x^4$) which can be represented by

$$\rho_1=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{5·4}$$

corresponds to the proper rotation in this space. Indeed, if we calculate the

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* There seems to exist an analogy between this situation and the passing from the Galilei transformation to the Lorentz transformation. In the latter case it was the light velocity c that connects spatial co-ordinate with the time co-ordinate. In our case it is the gravitational constant κ that connects the Minkowski space with the charge space.

** If we assume that $x^{0\prime}=x^0$, λ, μ, ν and σ must vanish. In this case (5·2) and (5·3) reduces to

$$S=\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \bar{\delta} & -\bar{\gamma} \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

and

$$\alpha\delta-\beta\gamma=1,$$

which are nothing but the representation of the Lorentz group.

transformation of x^0 caused by (5·4), we easily get $x^{0\prime}=-x^0$.* This means that so long as we regard the five dimensional view-point as essential, and want to include the inversion, the Dirac spinor cannot be regarded as a suitable base. Therefore Klein's result that ρ_1 cannot be regarded as a suitable operation is a very general feature of the five-dimensional theory.

The scheme described here is insufficient in several points. For example, the relation of the rotation with imaginary a and b to the original five dimensional theory is not clear, or the derivation of $\boldsymbol{J}$ and $\boldsymbol{K}$ have not been given. On the other hand, it would be expected that the new situations in the inversion theorems and the amalgamation of the Minkowski space and the charge space suggested here will undoubtedly be related to the non-conservation of parity and the violation of charge independence in the weak interaction. In view of these promising results, it would be expected that the five-dimensional theory may be used as a starting point for a more profound theory of elementary particles.

In conclusion, the author wishes to express his sincere gratitude to Prof. H. Yukawa for his interest taken in this work.

Appendix

According to Bargmann, let us introduce X defined by

$$X=\begin{pmatrix} x^0 & x^1-ix^2 \\ x^1+ix^2 & x^0 \end{pmatrix}, \tag{A·1}$$

and consider the transformation

$$Y=WXW^+, \quad (+ \text{ means hermite conjugate}) \tag{A·2}$$

where W is given by

$$W=\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{A·3}$$

with

$$\alpha\bar{\alpha}-\beta\bar{\beta}=1. \tag{A·4}$$

From Y we can define y^0, y^1 and y^2 by

$$Y=\begin{pmatrix} y_0 & y^1-iy^2 \\ y^1+iy^2 & y^0 \end{pmatrix}. \tag{A·5}$$

---

* The transformation $x^{0\prime}=-x^0$ corresponds to $\varphi'=-\varphi$. This is not the inversion in the ω-space. It seems difficult to relate the inversion in the charge space which was considered by d'Espangnat and Prentki[10] to the five-dimensional model. Under $\varphi\to-\varphi'$, $ne\to-ne$, $A_m\to-A_m$ and $\psi_n\to\psi_n$. Therefore all Lagrangians remain invariant since ne always appears as a product with A_m.

Then we can easily see that the transformation $x \to y$ is a Lorentz transformation.

With these α and β, the explicit matrix element belonging to C_η^0 is given by

$$v_{mn}=\begin{cases}\theta_{mn}(q)\,\alpha^{m+n}\beta^{m-n}F(\frac{1}{2}+m+\sigma,\ \frac{1}{2}+m-\sigma,\ 1+m-n,\ -\beta\bar{\beta}) & (m\geq n)^* \\ \theta_{mn}(q)\,\alpha^{m+n}\bar{\beta}^{n-m}F(\frac{1}{2}+n+\sigma,\ \frac{1}{2}+n-\sigma,\ 1+n-m,\ -\beta\bar{\beta}) & (m\leq n),\end{cases} \tag{A·6}$$

where θ_{mn} is given by

$$\theta_{mn}(q)=\begin{cases}1 & \text{for} \quad m=n, \\ \dfrac{1}{(m-n)!}\prod_{j=1}^{m-n}\{q+(n+j)\,(n+j-1)\}^{1/2} & \text{for} \quad m>n, \\ (-1)^{n-m}\theta_{nm}(q) & \text{for} \quad m<n.\end{cases} \tag{A·7}$$

We shall show that infinitesimal operator given by (A·6) and (A·7) agree just (2·10)–(2·12) except for a simple unitary transformation. For infinitesimal transformations, we can put $\alpha=1+i\varepsilon$, $\beta=\tilde{\xi}+i\eta$, where ε, $\tilde{\xi}$ and η are infinitesimal real numbers, and the transformation is given by

$$\left.\begin{aligned}x^{1\prime}&=x^1+2\varepsilon x^2+2\tilde{\xi}x^0 \\ x^{2\prime}&=x^2-2\varepsilon x^1+2\eta x^0 \\ x^{0\prime}&=x^0+2\varepsilon x^1+2\eta x^2\end{aligned}\right\}, \tag{A·8}$$

where we have neglected terms of order ε^2. At first we see that we can put the hypergeometric function as 1, since $\beta\bar{\beta}$ is of order ε^2.

Let us calculate I_3 at first. For rotation in the (1, 2) plane, $\tilde{\xi}=\eta=0$. Therefore I_3 has only diagonal terms, and for diagonal terms we get

$$(I_3)_{mm}=\frac{1}{2i}\frac{dv_{mm}}{d\varepsilon}=m.$$

Therefore,

$$(I_3)_{mn}=m\delta_{mn}. \tag{A·9}$$

Next let us calculate I_1. To induce rotation in the (2, 3) plane, we must take $\varepsilon=\tilde{\xi}=0$. Therefore, only terms which satisfy $m-n=\pm 1$ survive. For these terms, we get from (A·6) and (A·7)

$$(I_1)_{n+1,n}=\frac{1}{2i}\theta_{n+1,n}\frac{d\beta}{d\eta}=\tfrac{1}{4}|2n+1|,$$

$$(I_1)_{n-1,n}=\frac{1}{2i}\theta_{n-1,n}\frac{d\bar{\beta}}{d\eta}=\tfrac{1}{4}|2n-1|.$$

---

* $q=\frac{1}{4}-\sigma^2$.

Therefore,

$$(I_1)_{mn}=\tfrac{1}{4}|m+n|\,(\delta_{m-1,n}+\delta_{m+1,n}). \tag{A·10}$$

By exactly the same way, we get

$$(I_2)_{mn}=\frac{i}{4}|m+n|\,(-\delta_{m-1,n}+\delta_{m+1,n}). \tag{A·11}$$

After a unitary transformation with U given by

$$U_{mn}=\begin{cases}(-1)^m\delta_{mn} & \text{for} \quad m<0,\\ \delta_{mn} & \text{for} \quad m\geq 0,\end{cases}$$

they go to

$$(I_1')_{mn}=\tfrac{1}{4}(m+n)\,(\delta_{m-1,n}+\delta_{m+1,n}),$$

$$(I_2')_{mn}=\frac{i}{4}(m+n)\,(-\delta_{m-1,n}+\delta_{m+1,n}),$$

$$(I_3')_{mn}=m\delta_{mn},$$

which are nothing but (2·10)–(2·12).

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Note added in proof From the explicit form of $\mathscr{D}^{(r)}_i$, we get easily

$$Q=\mathscr{D}_1{}^{(r)2}+\mathscr{D}_2{}^{(r)2}-\mathscr{D}_3{}^{(r)2}=\frac{1}{4}\begin{pmatrix}1 & & & 0\\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots\end{pmatrix}$$

independently of r. This is the reason why the most general transformation (2.20) reduces to the simple rotation in a three dimensional space.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures ar tables.

The Influence of Interelectronic Collision on Conduction in Non-Polar Crystals

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February 5, 1959

Up to this time, there have been many calculations carried out on the effect of electron-electron scattering in metals, which is very small as we know.

But we do not know much about it in non-metalic crystal. Yamashita[3] calculated the distribution function of electrons in non-polar crystals without considering interelectronic collisions. But Herring and Debye-Conwell[4] pointed out that in the strong electric field the interelectronic collision is dominant and in this case the distribution function of electrons is the Maxwell-Boltzmann function. Fröhlich[5] considered that this function must be characterized by electronic temperature. Recently Stratton[1] has proposed a method to calculate the effect of interelectronic collisions in connection with the dielectric breakdown in ionic crystals. He introduced two parameters i.e. electron temperature T and average momentum P_0, into the electronic distribution function, and calculated jF and neF, and eliminated these parameters. So we know the mobility as a function of electric field F and lattice temperature T_0.

The present author calculated mobility at $F\to 0$ and obtained the result,

$$\mu_i=\lim_{F\to 0}\cdot p_0/mF=(e/m)\cdot \frac{\pi^{3/2}M\hbar^4u_0}{8VC^2(2m)^2(kT_0)\cdot 1\Big/\left\{\exp\left(2u_0\sqrt{\frac{2m}{\pi kT_0}}\right)-1\right\}}$$
$$\div[\{\exp(2u_0/\sqrt{\pi}\cdot\sqrt{2m/kT_0})+1\}$$
$$\cdot\{\tfrac{1}{4}(2mu_0^2/kT_0)^3+93/64(2mu_0^2/kT_0)^2$$
$$+27/16(2mu_0^2/kT_0)+15/16\}$$
$$+\{\exp(2u_0\sqrt{\pi}\cdot\sqrt{2m/kT_0})-1\}$$
$$\cdot\exp(mu_0^2/4kT_0)\{3/32\cdot K_2(mu_0^2/4kT_0)$$
$$+\tfrac{1}{2}K_1(mu_0^2/4kT_0)+\tfrac{1}{8}K_0(mu_0^2/4kT_0)\}$$
$$\times(u_0\sqrt{2m/kT_0})^5+(u_0\sqrt{2m/kT_0})^7$$
$$\cdot\{\tfrac{1}{4}K_1(mu_0^2/4kT)+\tfrac{1}{8}K_0(mu_0^2/4kT)\}].$$

In the limit of $mu_0^2/kT\to 0$

$$\mu_i=\frac{\sqrt{2}}{15}\left(\frac{e}{m}\right)\frac{\pi^{1/2}M\hbar^4u_0^2}{VC^2m^{3/2}(kT_0)^{3/2}},$$

and in the same limit the usual theory gives

$$\tau_{e\text{-a.p.}}=\frac{3\pi^{1/2}}{16\sqrt{2}}\frac{M\hbar^4u_0^2}{VC^2m^{3/2}(kT_0)^{3/2}}$$

$$\mu_i/\mu_{e\text{-a.p.}}=0.711.$$

In the limit of $mu_0^2/kT_0 \to \infty$

$$\mu_i = \left(\frac{e}{m}\right) \frac{\pi M \hbar^4 (kT_0)^2}{64 V C^2 (2m)^5 u_0^5} .$$

In Ge, $(kT)^{1/2}/m^{1/2}u_0 = 6.21$, so we can use high temperature approximation at room temperature, but in Si, $(kT)^{1/2}/m^{1/2}u_0 = 2.41$ and involving all terms $\mu_i/\mu_{e\text{-a.p.}} \simeq 0.1$.

Thus we know that electron-acoustic phonon scattering is negligible in Si if the interelectronic collision is dominant. We assumed that the interelectronic collision is always dominant, and mobility is independent of electronic number.

But since mobility is, in fact, dependent on electronic number, we must calculate at what concentration of electrons interelectronic collisions become predominant. This is obtained by equations of energy and momentum balance.

According to Stratton's[1] preliminary consideration, the interelectronic collision is dominant if electronic number n is greater than

$$n_a \cdot kT_0/mu_0^2$$

where

$$n_a = \frac{1}{4\pi} \frac{E^{3/2} m^{3/2} u_0^2 \kappa^4}{kT_0 e^4 \tau_a(E)}$$

and κ = dielectric constant.

$n \simeq 10^{16}/cm^3$ at room temperature, and this is always satisfied.

Harrison[2] proposed that in order to explain the temperature variation of electron and hole mobility in Ge and Si, we must assume that electrons couple with acoustic phonons at the ratio of 4 : 1 compared to optical phonons, and holes couple with acoustic phonons at the ratio of 1 : 4 in Ge, and electrons seldom couple with acoustic phonons. This can be understood on the basis of our theory.

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T-Nt Diagram for Fusion Researches

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As explained in our previous letter,[1] the basic equations common to any thermonuclear reactor system, energy balancing " in the narrow and the wide sense ", are commanded only by the temperature T and τ the reduced time $\tau \equiv Nt$ besides other parameters of technical nature, where N is the number density of ions and t is the time. By the narrow sense we mean the balancing of the

produced power with the loss at each instance, while by the wide sense we mean that of the integrated produced energy with integrated loss for the whole duration of operation. Therefore on the T-τ space the energy balancing conditions can be plotted to form the regions. As a standard case, we assume that energy is injected only at the initial stage, $\tau \simeq 0$, and T is kept constant for $\tau > 0$.

If we plot on the T-τ space $\tau_{\min}$ and $\tau_{\max}$, at each T, the critical values at which the energy balancing in the narrow sense is violated, they make a contour (the broken curve in Fig. 1), which borders the "energy balancing region in the narrow sense" denoted by the region II. The temperature on the contour at a given τ has two values, $T_{\min}$ and $T_{\max}$. The value of $T_{\min}$ at $\tau = 0$ corresponds to Post's criteria.[2)]

For $T > T_{\max}$, the radiation loss, especially due to an additional contribution from the electron-electron bremsstrahlung, overcomes the energy generation rate. Similarly, if we plot $\bar{\tau}_{\min}$ and $\bar{\tau}_{\max}$ at each temperature, the critical values of τ at which the balancing in the wide sense is violated, they make a contour bordering the energy balancing region in the wide sense on the T-τ space (the solid line in Fig. 1) and we can define $\bar{T}_{\min}$ and $\bar{T}_{\max}$ at a given τ. The region III in Fig. 1, which satisfies the energy balancing in both senses, is the very region at which fusion reactions could be energetically economical. For pure deuterium fuel, the energy balancing region is so narrow* that the fuel

* This is strongly dependent on the electron-electron bremsstrahlung, which is being studied in more detail than Thompson's[3)] and will be published in the near future.

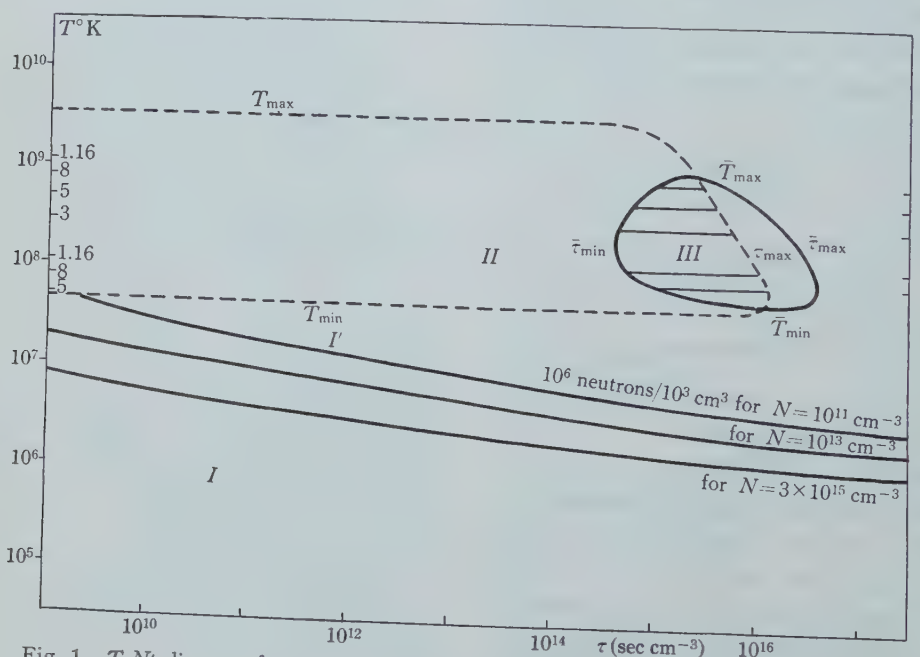

Fig. 1. T-Nt diagram for mixed fuel of deuterium with 50% tritum. Here the efficiencies of energy carried by all particles and radiations except neutrons are assumed to be ~1/3, and the efficiency for the energy injected initially into plasma is also 1/3.

may be of no practical use, unless one would consider some cycling processes and the production of T^3 by the neutrons produced through DD reactions.

The T-τ space with various technical criteria as well as the energy balancing regions may be a strategic chart for fusion researches, which we name the T-Nt diagram. For example, neutrons more than 10^6 yielded by the thermal reactions from a plasma of $10^3 cm^3$ should be observable for the duration of time and the temperature corresponding to any point in the I' on Fig. 1. All fusion researches so far reported have been on the way from I to I'. The values of T and τ imply respective conditions for the heating and stability of the confined plasma. The projects of simple pinch experiments try to climb the diagonal path to reach the region III, since in the pinch processes both heating and confinement of the plasma are included together. The simple pinch process, however, may not have the possibility of the stable confinement. Thus in the recent researches, heating and confinement technique are strengthened, often separated artificially from each other. The variety of the direction of fusion researches may be effectively discussed on the $T-Nt$ diagram. In the case of low tritium ratio, i.e. $\lesssim 1\%$, the region III, if reaction products are removed out of the furnace, lies more rightward and is more difficult to reach than the one if reaction products are left unremoved, while, for mixed fuel with tritium more than $\gtrsim 10\%$, the region III lies more leftward and is narrower in the former case than in the latter case as corresponding to the results in our previous letter.[1]

In cases of thermonuclear bombs and energy generating stars, the region III is much wider than that for reactors, because high densities of bombs and stars result in their high efficiencies and the large interval of τ corresponding to small t.

Details of this work will be published in the Journal of Nuclear Energy by one of us (M.I).

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Note on the "Toron" Formalism Developed by Montroll and Ward

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March 26, 1959

Montroll and Ward[1] have shown that the correlation energy of electron gas at high density limits obtained by Gell-Mann and Brueckner[2] is derived from another point of view, that is, by the method of toron. In this note we shall show that the toron formalism can be greatly simplified if we use the linked cluster expansion[3,4,5] of the grand

partition function in terms of the Feynman graphs.

It is shown that the quantum statistical equation of state based on the perturbation method is given for the interacting Fermi particles by

$$PV/\kappa T=\log Z_0(\alpha,\beta) + \sum_{n=1}^{\infty}(-1)^n \sum_{\Gamma_n}^{\text{linked}} Z(\alpha\beta,\Gamma_n), \tag{1}$$

$$Z(\alpha\beta,\Gamma_n)=(-1)^{\nu(\Gamma_n)}/g(\Gamma_n) \sum_{s_i,s_{i'}} \int_0^\beta dx_1\cdots dx_n' \prod_{(\Gamma_n)} S(\sigma_i,\sigma_i') \prod_{i=1}^{n} v(x_i-x_i'), \tag{2}$$

where $Z_0(\alpha,\beta)$ is the grand partition function of free particles, Γ_n indicates the class of Feynman graphs of order n,[4] $\nu(\Gamma_n)$ the number of closed polygons, $g(\Gamma_n)$ the symmetrization factor[4] defined to be the number of ways in which the topological property of the Feynman graph is unchanged by $2^n n!$ permutations $(12\cdots n)\to(i_1\cdots i_n)$, $(x_i, x_i')\rightleftarrows(x_i', x_i)$, $S(\sigma_i,\sigma_j')$ a propagator, $v(\mathbf{x}_i-\mathbf{x}_i')$ the interaction potential between particles and dx_i a space-temperature integration. The summation s_i, s_i' implies that over the spin states.

If we perform a partial summation over the graphs of ring type shown in Fig. 1, it is shown that

$$PV/\kappa T=\log Z_0(\alpha,\beta) + \tfrac{1}{2}\sum_{\mathbf{q},j}\{\nu(\mathbf{q})\lambda_j(\mathbf{q})-\log[1+\nu(\mathbf{q})\lambda_j(\mathbf{q})]\} \tag{3}$$

where $\nu(\mathbf{q})$ is the Fourier component of potential function and λ_j is given by

$$\lambda_j(\mathbf{q})=\frac{1}{V}\sum_{\mathbf{k}} \cdot \frac{\{f(\mathbf{k}+\mathbf{q})-f(\mathbf{k})\}\{\varepsilon(\mathbf{k})-\varepsilon(\mathbf{k}+\mathbf{q})\}}{[\varepsilon(\mathbf{k})-\varepsilon(\mathbf{k}+\mathbf{q})]^2+(2\pi j/\beta)^2}. \tag{4}$$

In eq. (4) $\varepsilon(\mathbf{k})$ is given by $\hbar^2k^2/2m$, and $f(\mathbf{k})$ is the distribution function defined by

$$f(\mathbf{k})=\frac{1}{e^{\beta\varepsilon(\mathbf{k})-\alpha}+1}.$$

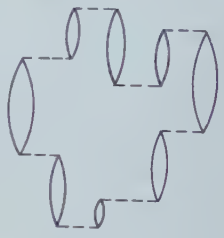

Fig. 1.

If we expand the right-hand side of eq. (4) in powers of fugacity, it is verified that the result is identical with that given by Montroll and Ward.[1]

The summation procedure is not only restricted to the graphs of ring type but can be extended to the more complicated graphs. As a matter of fact, there is a close relation between the Feynman graphs and the bond diagrams in classical theory, and it is possible to develop the expansion scheme corresponding to Mayor's one in terms of pratotype graphs.[6] The detail of this article will be published elsewhere.

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Indirect Nuclear Spin-Spin Couplings in Saturated Hydrocarbon Molecules

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April 9, 1959

Previously we calculated a coupling constant $J_{H_1H_2}$ of electron-coupled interaction between two protons H_1 and H_2 in a saturated hydrocarbon molecule, by taking account of two important electronic structures, and obtained the results in fairly good agreement with experiment.[1] In the following we shall again calculate $J_{H_1H_2}$ by the use of a more accurate wave function including all possible structures. Recently Karplus and Anderson[2] have made a similar calculation with the assumption of orthogonality of atomic orbitals, while our calculation will be made without this assumption.

Using the valence bond method, we express the ground state wave function Ψ as a linear combination of independent canonical structures Ψ_ξ:

$$\Psi=C_0\Psi_0+\sum_{\xi=1}^{\tau} C_\xi\Psi_\xi\,, \tag{1}$$

where Ψ_0 is the ground canonical structure, namely all of the bonds in Ψ_0 are actual chemical bonds. Substituting (1) in the expression for $J_{H_1H_2}$ given by Ramsey[3] and McConnell,[4] we obtain

$$J_{H_1H_2}=-2/3h\cdot(8\beta h/3)^2\gamma_p^2(\Delta E)^{-1}\cdot\sum_{\xi,\eta} C_\xi C_\eta(\Psi_\xi|\sum_{i,j}\delta(\boldsymbol{r}_{iH_1})\delta(\boldsymbol{r}_{jH_2})\boldsymbol{S}_i\cdot\boldsymbol{S}_j|\Psi_\eta), \tag{2}$$

where ΔE is the average value of energy differences between the ground state (singlet) and the excited states (triplet), and the other letters have usual meanings.

We divide the structures into two groups: group A contains the structures involving H_1-H_2 bond, while group B the structures not involving H_1-H_2 bond. We denote the structures in A and B by Ψ_a and Ψ_b, and their coefficients in Ψ by C_a and C_b respectively. Then we can show

$$\mathrm{T}^2(\Psi_a|\sum_{i,j}\delta(\boldsymbol{r}_{iH_1})\delta(\boldsymbol{r}_{jH_2})\boldsymbol{S}_i\cdot\boldsymbol{S}_j|\Psi_0)\approx(\Psi_b|\sum_{i,j}\delta(\boldsymbol{r}_{iH_1})\delta(\boldsymbol{r}_{jH_2})\boldsymbol{S}_i\cdot\boldsymbol{S}_j|\Psi_0), \tag{3}$$

where T is the order of overlap integrals between atomic orbitals which are not chemically bonded to each other and $T\lesssim 10^{-1}$.

The dominant terms in (2) are

$$J_{H_1H_2}=-4/3h\cdot(8\beta h/3)^2\gamma_p^2(\Delta E)^{-1}\cdot\sum_a C_aC_0(\Psi_a|\sum_{i,j}\delta(\boldsymbol{r}_{iH_1})\delta(\boldsymbol{r}_{jH_2})\boldsymbol{S}_i\cdot\boldsymbol{S}_j|\Psi_0). \tag{4}$$

The neglected terms in (2) are 1) $\xi=\eta=0$, 2) $\xi=b$ $(\neq 0)$, $\eta=0$ and 3) ξ, $\eta\neq 0$. The error of the neglect of 1), by taking account of (3), is about $\frac{1}{2}T^2C_0/\max|C_a|$, which is less than 10% when $\max|C_a|\gtrsim 1/20$. A detailed consideration shows that this error does not exceed the limit even if $\max|C_a|\gtrsim 1/40$. The error of the neglect of 2) is about T^2(number of b)/(number of a), which increases with the number of bonds in the molecule. We can show that this error is less than 10% (20%) when the number of bonds is less than seven (ten). It can be shown that the neglect of 3), though its error is not necessarily

* *nee* Aihara

small, causes no effect on the below-mentioned conclusions.

Carrying out the calculation of (4), we obtain the following results. (It is to be noted that the overlap integrals between chemically bonded atomic orbitals appearing in (4) are not neglected.)

1) When the electronegativities of atoms chemically bonded to H_1 and H_2 are large, $J_{H_1H_2}$ is small, while $J_{H_1H_3}$ is almost independent of the electronegativities of the other atoms.

2) When the distance between H_1 and H_2 is large, $J_{H_1H_2}$ is very small. Both results agree with the experimental fact.[5]

Application of (4) to methane molecule (when we assume $\Delta E \sim 10$ eV and use the exchange integrals between atomic orbitals given by Van Vleck[6]) leads to a numerical value, 11.9 cps for $J_{H_1H_2}$. This is in good agreement with the experimental value, 12.4 ± 0.6 cps[7] measured in CH_3D.

Details of this calculation will be published in the Journal of Physical Society of Japan.

The author wishes to express her appreciation to Professor K. Niira for his valuable discussions.

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Errata

A Note on the Fundamental Symmetry of Elementary Particles

Takao OKABAYASHI

Prog. Theor. Phys. **21** (1959), 653

"$V-A$" should be read as "V-A" in all places except for the 19th line from the top on the page 654.

Errata

The Imaginary Part of the Optical Potential

Atsushi SUGIE

Prog. Theor. Phys. **21** (1959), 681

1. p. 683, Fig. 2: The curve labelled b is incorrect. It should be a monotonic upward convex curve passing the point $E_i/E_F=1.5$, $\bar{\sigma}/\sigma=0.6$. Two end points are correct. This correction does not alter the conclusion in the text.
2. p. 683, Caption of Fig. 2: σ in the first line should be $\bar{\sigma}$.
3. p. 684, Eq. (4): $f_n{}^A(\boldsymbol{r}_0)$ should be $f_n{}^s(\boldsymbol{r}_0)$.
4. p. 685, lines 4–6 from below: The condition given there to cause the mentioned cancellation is necessary but not sufficient.
5. p. 686, Eq. (13): There should be P_{0i} between $(-)^{P_{0i}}$ and ϕ.
6. p. 687, line 6 from below: "single particle model" should read "independent particle model".
7. p. 688, Eq. (16a): $\langle c_n{}^{\sigma p} c_n{}^{s p'}\rangle I$ should be $\langle c_n{}^{\sigma p} c_{ns}{}^{p'}\rangle_I$.
8. p. 693, the first line of the expression above Eq. (27): $K_{0s}(\boldsymbol{r}, \boldsymbol{r}')\varphi^{sp}(\boldsymbol{r}')\}d\boldsymbol{r}'$ should be $K_{0s}(\boldsymbol{r}, \boldsymbol{r}')\}\varphi^{sp}(\boldsymbol{r}')d\boldsymbol{r}'$.
9. p. 694, line 2 "the single particle strength function" should be "the pole strength function".
10. p. 689, the last paragraph of the text: There was a misunderstanding about Frantz et al.'s work. Their expression for the optical potential corresponds to Eq. (26) in our case. Although Eq. (26) is very complicated and it is rather difficult to grasp, it ensures the orthogonality of the scattering state to the occupied bound states owing to the exchange integral K. So, Eq. (26) corresponds to Frantz et al.'s potential.

$$PWP$$

where P is the projection operator to the unoccupied states. In the second order approximation the imaginary part of their potential becomes

$$PwP'wP$$

where P' is the projection operator to the unoccupied states below the incident energy and w is the interaction matrix element. If, and only if, w and P' are commutable, the above expression becomes

$$P'w^2=P'W$$

as given in our text. Generally, however, w and P' are not commutable.

I am grateful to Dr. L. M. Frantz for sending me the preprint of the full paper by him and Dr. L. Mills.

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